Software System Design and Implementation

More on the Curry Howard Isomorphism

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What is Intuitionistic Logic?

- Classical logic is the logic that most people know about.
- Intuitionistic logic does not contain the axiom of excluded middle $p \vee \neg p$ or equivalently $\neg\neg p \to p$.
- In classical logic more can be proven but less can be expressed.
- Intuitionistic proof of an existence statement gives a witness for the statement.

Example of Existence in the Classical Sense

- Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{I}$ be the set of irrational numbers.
- Consider the statement $\exists x, y.(x \in \mathbb{I}) \wedge (y \in \mathbb{I}) \wedge (x^y \in \mathbb{Q})$.
- Proof:
  - Consider the number $\sqrt{2}^{\sqrt{2}}$.

1. If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$
2. Otherwise, if $\sqrt{2}^{\sqrt{2}} \in \mathbb{I}$

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     - Pick $x = \sqrt{2}$ and $y = \sqrt{2}$
     - Then $x^y = \sqrt{2}^{\sqrt{2}}$ so $x^y \in \mathbb{Q}$

  2. Otherwise if $\sqrt{2}^{\sqrt{2}} \in \mathbb{I}$
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  2. Otherwise if $\sqrt{2}^{\sqrt{2}} \in \mathbb{I}$
     - Pick $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$
     - Then $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ so $x^y \in \mathbb{Q}$

Recall: The Curry-Howard Isomorphism

This correspondence goes by many names, but is usually attributed to Haskell Curry and William Howard.

It is a *very deep* result:

Logic	Programming
Propositions	Types
Proofs	Programs
Proof Simplification	Evaluation

It turns out, no matter what logic you want to define, there is always a corresponding λ-calculus, and vice versa.

Constructive Logic	Typed λ-Calculus
Classical Logic	Continuations
Modal Logic	Monads
Linear Logic	Linear Types, Session Types
Separation Logic	Region Types

Translating

We can translate logical connectives to types and back:

Conjunction ($\wedge$)	Tuples
Disjunction ($\vee$)	`Either`
Implication	Functions
True	`()`
False	`Void`

We can also translate our *equational reasoning* on programs into *proof simplification* on proofs!

Constructors and Eliminators for Sums


```haskell
data TrafficLight = Red | Amber | Green
```

Example (Traffic Lights)

$$\texttt{TrafficLight} \simeq \texttt{Either () (Either () ())}$$

$$\begin{array}{lcl} \texttt{Red} & \simeq & \mathsf{Left}\ () \\ \texttt{Amber} & \simeq & \mathsf{Right}\ (\mathsf{Left}\ ()) \\ \texttt{Green} & \simeq & \mathsf{Right}\ (\mathsf{Right}\ (\mathsf{Left}\ ())) \end{array}$$

Type Correctness

$$\frac{}{\Gamma \vdash () :: ()}\,() \qquad \frac{\Gamma \vdash e :: A}{\Gamma \vdash \texttt{Left}\ e :: \texttt{Either}\ A\ B}\,\mathsf{S}_L \qquad \frac{\Gamma \vdash e :: B}{\Gamma \vdash \texttt{Right}\ e :: \texttt{Either}\ A\ B}\,\mathsf{S}_R$$

$$\cfrac{\cfrac{\cfrac{???}{\texttt{Left ()} :: ()}}{\texttt{Right (Left ())} :: \texttt{Either () ()}}\,\mathsf{S}_R}{\texttt{Right (Right (Left ()))} :: \texttt{Either () (Either () ())}}\,\mathsf{S}_R$$

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$$\dfrac{\dfrac{\dfrac{}{() :: ()}\,()}{\texttt{Right}\ () :: \texttt{Either}\ ()\ ()}\,\texttt{S}_R}{\texttt{Right}\ (\texttt{Right}\ ()) :: \texttt{Either}\ ()\ (\texttt{Either}\ ()\ ())}\,\texttt{S}_R$$

Examples


```haskell
prop_or_false :: a -> (Either a Void)
prop_or_false a = Left a

prop_or_true :: a -> (Either a ())
prop_or_true a = Right ()

prop_and_true :: a -> (a, ())
prop_and_true a = (a, ())

prop_double_neg_intro :: a -> (a -> Void) -> Void
prop_double_neg_intro a f = f a

prop_triple_neg_elim ::
  (((a-> Void) -> Void) -> Void) -> a -> Void
prop_triple_neg_elim f a = f (\g ->  g a)
```

Wrap-up

1. Assignment 2 is before my next lecture (5th August).
2. There is a quiz for this week, but no exercise.
3. Next week's lectures consist of an **extension** on dependent type systems and a **revision lecture** on Wednesday.

4. There will be a survey on Piazza for revision topics, comment on the poll with specific questions
5. If you enjoyed the course and want to do more in this direction, ask us for thesis topics, taste of research projects, and consider attending COMP3161 and COMP4161.

6. Fill in the myExperience reports, it is important for us to receive your feedback.

Consultations

- Consultations will be made on request. Ask on piazza or email `cs3141@cse.unsw.edu.au`.
- If there is a consultation it will be announced on Piazza with a link a room number for Hopper.

- Will be in the Thursday lecture slot, 9am to 11am on Blackboard Collaborate.
- Make sure to join the queue on Hopper. Be ready to share your screen with REPL (`ghci` or `stack repl`) and editor set up.