Software System Design and Implementation

More on the Curry Howard Isomorphism

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- In classical logic more can be proven but less can be expressed.
- Intuitionistic proof of an existence statement gives a witness for the statement.

Example of Existence in the Classical Sense

- Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{I}$ be the set of irrational numbers.
- Consider the statement $\exists x, y.(x \in \mathbb{I}) \wedge (y \in \mathbb{I}) \wedge (x^y \in \mathbb{Q})$.

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     - Pick $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$
     - Then $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ so $x^y \in \mathbb{Q}$

Recall: The Curry-Howard Isomorphism

This correspondence goes by many names, but is usually attributed to Haskell Curry and William Howard.

It is a *very deep* result.

Logic	Programming
Propositions	Types
Proofs	Programs
Proof Simplification	Evaluation

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It turns out, no matter what logic you want to define, there is always a corresponding λ-calculus, and vice versa.

Constructive Logic	Typed λ-Calculus
Classical Logic	Continuations
Modal Logic	Monads
Linear Logic	Linear Types, Session Types
Separation Logic	Region Types

Translating

We can translate logical connectives to types and back:

Conjunction ($\wedge$)	Tuples
Disjunction ($\vee$)	`Either`
Implication	Functions
True	`()`
False	`Void`

We can also translate our *equational reasoning* on programs into *proof simplification* on proofs!

Constructors and Eliminators for Sums


```
data TrafficLight = Red | Amber | Green
```

Example (Traffic Lights)

$$\texttt{TrafficLight} \simeq \texttt{Either}\ ()\ (\texttt{Either}\ ()\ ())$$

$$\begin{array}{lcl} \texttt{Red} & \simeq & \texttt{Left}\ () \\ \texttt{Amber} & \simeq & \texttt{Right}\ (\texttt{Left}\ ()) \\ \texttt{Green} & \simeq & \texttt{Right}\ (\texttt{Right}\ (\texttt{Left}\ ())) \end{array}$$

Type Correctness

$$\frac{}{\Gamma \vdash () :: ()}\,() \qquad \frac{\Gamma \vdash e :: A}{\Gamma \vdash \texttt{Left}\ e :: \texttt{Either}\ A\ B}\,\mathsf{S}_L \qquad \frac{\Gamma \vdash e :: B}{\Gamma \vdash \texttt{Right}\ e :: \texttt{Either}\ A\ B}\,\mathsf{S}_R$$

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Examples

```
prop_or_false :: a -> (Either a Void)
prop_or_false a = Left a

prop_or_true :: a -> (Either a ())
prop_or_true a = Right ()

prop_and_true :: a -> (a, ())
prop_and_true a = (a, ())

prop_double_neg_intro :: a -> (a -> Void) -> Void
prop_double_neg_intro a f = f a

prop_triple_neg_elim ::
  (((a-> Void) -> Void) -> Void) -> a -> Void
prop_triple_neg_elim f a = f (\g ->  g a)
```

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6. Fill in the myExperience reports, it is important for us to receive your feedback.

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- Will be in the Thursday lecture slot, 9am to 11am on Blackboard Collaborate.
- Make sure to join the queue on Hopper. Be ready to share your screen with REPL (`ghci` or `stack repl`) and editor set up.