Lecture 6 - The Pumping Lemma

Tuesday, January 26, 2021 11:19 AM

$$P \Rightarrow Q$$

converse $Q \Rightarrow P$ is NOT the same

$\underbrace{\neg Q \Rightarrow \neg P}$ is the same

↳ the contrapositive

— x —

NEGATING QUANTIFIED STATEMENTS

$$\neg \forall x \exists y \forall z \exists w \; \phi(x,y,z,w)$$

push the negation inside & flip the quantifiers

$$\exists x \forall y \exists z \forall w \; \neg \phi(x,y,z,w)$$

$$\neg \forall x \, \varphi(x) \Longleftrightarrow \exists x \, \neg \varphi(x)$$

PIGEON HOLE PRINCIPLE

If I put n objects in m boxes and $n > m$ then at least one box must get 2 or more objects.

Claim $L = \{ a^n b^n \mid n \geq 0 \}$ **no** DFA can do this.

Assume you have a DFA that can recognize L. This has k states.

Choose the string $a^{k'} b^{k'}$ where $k' > k$.

Let us analyze what happens as **this** string is processed: some state must repeat

Why can't I go around the loop twice?

l a's in the loop from s back to s.

$s_0 \xrightarrow{m} s \xrightarrow{l} s \xrightarrow{l'} s' \cdots \circledcirc$

$m+l+l'$, $m+l+l'$ ...

$a \qquad b \qquad \in L$

$$s_0 \xrightarrow{m} s \xrightarrow{l} s \xrightarrow{l} s \xrightarrow{l'} s' \cdots \to \bigcirc$$

$$a^{m+l+l+l'}\, b^{m+l+l'} \in L$$

but this string is not supposed to be accepted. So the m/c <u>fails</u> to reject some strings that it should reject.

We can easily generalize this.

FORMAL STATEMENT

If L is a regular language then

$\exists p \in \mathbb{N}$ $p > 0$ s.t. $\forall w \in L$ with $|w| \geq p$

$\exists x, y, z \in \Sigma^*$ s.t. $w = xyz$ & $|xy| \leq p$ & $|y| > 0$

$\forall i \in \mathbb{N}$ $xy^iz \in L$

$\exists p$ some number which depends on L

$$\overbrace{\qquad\qquad}^{x}\;\overbrace{\qquad\qquad}^{y}\;\overbrace{\qquad\qquad}^{z}$$

$$|w| \geq p$$

$$|xy| \leq p \ \& \ |y| > 0$$

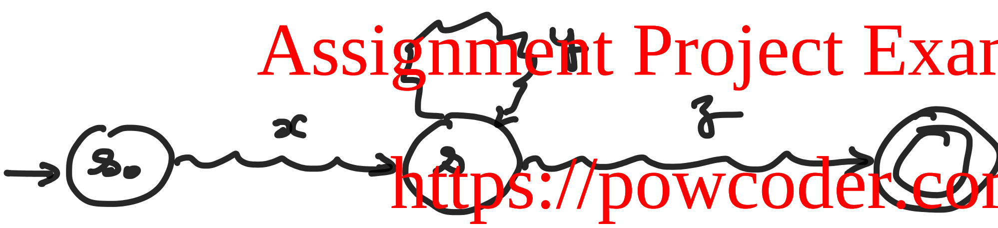
$$\underline{\quad x \quad}\ \underline{\ y\ }\ \underline{\ y\ }\ \underline{\quad z \quad} \in L$$

$$\vdots$$

$$\underline{\quad x \quad}\ \underline{\ y\ }\ \underline{\ y\ }\ \underline{\ y\ }\ \underline{\quad z \quad} \in L$$

$$\vdots$$

$$\underline{\quad x \quad}\ \underline{\quad z \quad} \in L$$



L regular $\Longrightarrow$ L can be pumped

CONTRAPOSITIVE

L <u>cannot</u> be pumped $\Longrightarrow$ L not regular

Suppose $L \subseteq \Sigma^*$ such that

$\forall p > 0 \ \exists w \in L \ |w| \geq p$ s.t.

$\forall x, y, z \in \Sigma^*$ s.t. $w = xyz$ and

$|xy| \le p$ and $|y| > 0$

$\exists i \quad xy^i z \notin L$

then L is NOT regular.

GAMES : 2-player games

$\forall$: demon $\qquad \exists$: you (angel)

(1) Demon chooses p

(2) You choose w with $|w| \ge p$

(3) Demon chooses x, y, z all conditions must be respected

$$w = xyz$$

$$|xy| \le p$$

$$|y| > 0$$

(4) You choose i and verify that with all these choices $xy^i z \notin L$

You must have a winning strategy.

EXAMPLE $\quad \{a^n b^n \mid n \ge 0\}$

(1) Demon chooses p

(2) I choose $a^p b^p$

(3) Demon has to choose x, y, z but $|xy| \leq p$ so this part must consist exclusively of a's.

$|y| > 0$ so let $|y| = \ell > 0$

(4) I pick $i = 2$

$$|\underbrace{xy}\underset{=}{y}\underbrace{z}| = 2p + \ell$$

How many a's in this string?

$p + \ell$

How many b's?

p

$a^{p+\ell} b^p \notin L$

Thus L is not regular.

<u>EXAMPLE 2</u> $\Sigma = \{a, b\}$

$L = \{ w \in \Sigma^* \mid \#\text{ of } a\text{'s in } w \neq \#\text{ of } b\text{'s in } w \}$

aab, babaabbb, ...

(1) Demon picks p

(2) I pick $a^p b^{p+p!}$

(3) Demon has to pick y consisting of a's only, $|y| > 0$ say $|y| = \ell > 0$. $\ell \leq p$

(4) I pick $i = (p!/\ell) + 1$

xy^iz : I have added y^{i-1}

$$xy^iz = a^{p+(i-1)\ell}\, b^{p+p!}$$
$$= a^{p+p!}\, b^{p+p!} \notin L$$

MUCH SIMPLER WAY:

Suppose we are wondering whether L is regular. If we know L' is regular then $L \cap L'$ should be regular if L is regular. So if it turns out that $L \cap L'$ is <u>not</u> regular then L cannot be regular. Perhaps $L \cap L'$ is easier to show not regular.

In our example: if $\bar{L}$ is not regular

[illegible]

then L cannot be regular. So

$$L \cap a^* b^* = \{a^n b^n \mid n \geq 0\}$$