Lecture 6 - The Pumping Lemma

Tuesday, January 26, 2021 11:19 AM

$P \Rightarrow Q$

converse $Q \Rightarrow P$ is NOT the same

$\neg Q \Rightarrow \neg P$ is the same

↳ the contrapositive

—— x ——

NEGATING QUANTIFIED STATEMENTS

$\neg \ \forall x \exists y \forall z \exists w \ \phi(x,y,z,w)$

push the negation inside & flip the quantifier

$\exists x \forall y \exists z \forall w \ \neg \phi(x,y,z,w)$

$\neg \forall x \ \varphi(x) \Leftrightarrow \exists x \ \neg \varphi(x)$

PIGEON HOLE PRINCIPLE

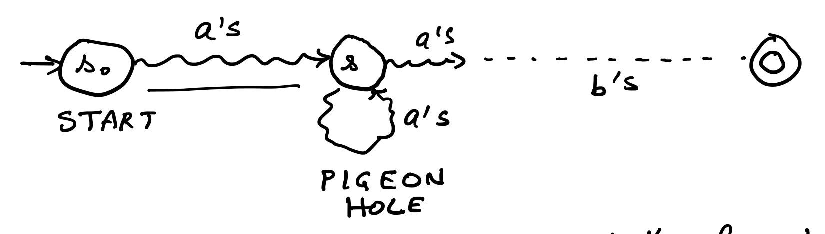
If I put n objects in m boxes and $n > m$ then at least one box must get 2 or more objects.

Claim $L = \{a^n b^n \mid n \geq 0\}$ no DFA can do this.

Assume you have a DFA that can recognize L. This has k states.

Choose the string $a^{k'} b^{k'}$ where $k' > k$.

Let us analyze what happens as this string is processed: some state must repeat



Why can't I go around the loop twice?

l a's in the loop from s back to s.

$s_0 \xrightarrow{m} s \xrightarrow{l} s \xrightarrow{l'} s' \dashrightarrow \circledcirc$

$a^{m+l+l'} b^{m+l+l'} \in L$

$s_0 \xrightarrow{m} s \xrightarrow{l} s \xrightarrow{l} s \xrightarrow{l'} s' \dashrightarrow \circledcirc$

$a^{m+l+l+l'} b^{m+l+l'} \in L$

but this string is not supposed to be accepted. So the m/c fails to reject some strings that it should reject.

We can easily generalize this.

FORMAL STATEMENT

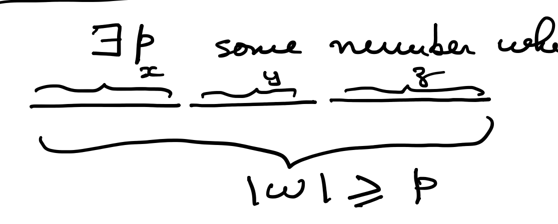
If L is a regular language then

$\exists p \in \mathbb{N} \ p > 0$ s.t. $\forall w \in L$ with $|w| \geq p$

$\exists x, y, z \in \Sigma^*$ s.t. $w = xyz$ & $|xy| \leq p$ & $|y| > 0$

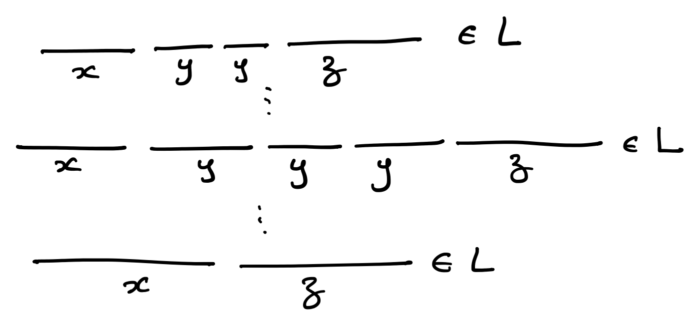
$\forall i \in \mathbb{N} \quad xy^iz \in L$

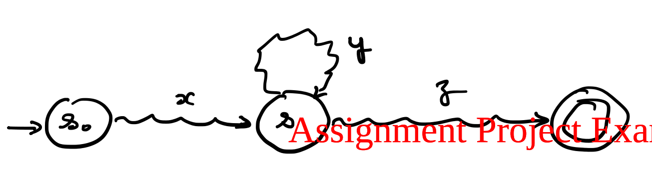
$\exists p$ some number which depends on L



$|w| \geq p$

$|xy| \leq p$ & $|y| > 0$





L regular $\Rightarrow L$ can be pumped

CONTRAPOSITIVE

L cannot be pumped $\Rightarrow L$ not regular

Suppose $L \subseteq \Sigma^*$ such that

$\left\{ \begin{array}{l} \forall p > 0 \ \exists w \in L \ |w| \geq p \ \text{s.t.} \\ \forall x, y, z \in \Sigma^* \ \text{s.t.} \ w = xyz \ \text{and} \\ \quad |xy| \leq p \ \text{and} \ |y| > 0 \\ \exists i \quad xy^iz \notin L \end{array} \right.$

then L is NOT regular.

GAMES : 2-player games

$\forall$: demon $\qquad \exists$: you (angel)

(1) Demon chooses p

(2) You choose w with $|w| \geq p$

(3) Demon chooses x, y, z all conditions must be respected

$w = xyz$

$|xy| \leq p$

$|y| > 0$

(4) You choose i and verify that with all these choices $xy^iz \notin L$

You must have a winning strategy.

EXAMPLE $\{a^n b^n \mid n \geq 0\}$

(1) Demon chooses p

(2) I choose $a^p b^p$

(3) Demon has to choose x, y, z but $|xy| \leq p$ so this part must consist exclusively of a's.

$|y| > 0$ so let $|y| = l > 0$

(4) I pick $i = 2$

$|xyyz| = 2p + l$

How many a's in this string?

$p + l$

How many b's?

p

$a^{p+l} b^p \notin L$

Thus L is not regular.

EXAMPLE 2 $\Sigma = \{a, b\}$

$L = \{w \in \Sigma^* \mid \#\text{ of a's in } w \neq \#\text{ of b's in } w\}$

aab, babaabbb, ...

(1) Demon picks p

(2) I pick $a^p b^{p+p!}$

(3) Demon has to pick y consisting of a's only, $|y| > 0$ say $|y| = l > 0$. $l \leq p$

(4) I pick $i = (p!/l) + 1$

xy^iz : I have added y^{i-1}

$xy^iz = a^{p+(i-1)l} b^{p+p!}$

$= a^{p+p!} b^{p+p!} \notin L$

MUCH SIMPLER WAY:

Suppose we are wondering whether L is regular. If we know L' is regular then $L \cap L'$ should be regular if L is regular. So if it turns out that $L \cap L'$ is not regular then L cannot be regular. Perhaps $L \cap L'$ is easier to show not regular.

In our example: if $\overline{L}$ is not regular then L cannot be regular. So

$\overline{L} \cap a^* b^* = \{a^n b^n \mid n \geq 0\}$