Reductions — I

Tuesday, March 23, 2021 11:18 AM

P, Q are problems

P <u>reduces</u> <u>to</u> Q

$\equiv$ If I can solve Q then I can solve P

$\equiv$ Q is "harder than" P

$$P \leq Q$$

Q is <u>at</u> <u>least</u> <u>as</u> <u>hard</u> <u>as</u> P

$\rightarrow$ If I can prove $P \leq Q$ and I know P is undecidable then Q must be undecidable.

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If I know Q is decidable then P is also decidable

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When proving $P \leq Q$ I do NOT have to explain how I propose to solve Q: $P \leq Q$ is a <u>CONDITIONAL</u> statement.

RECALL: $\langle M, w \rangle$ means encoding of a TM M and a word w.

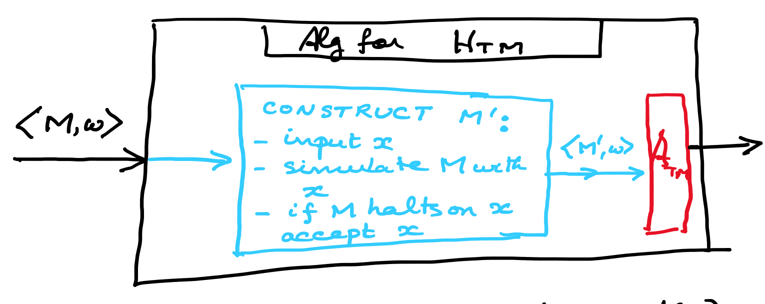
$H_{TM} = \{ \langle M, w \rangle \mid M \text{ halts on } w \}$

We showed membership in H_{TM} is undecidable.

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ accepts } w \}$

Very easy to see A_{TM} is also undecidable.

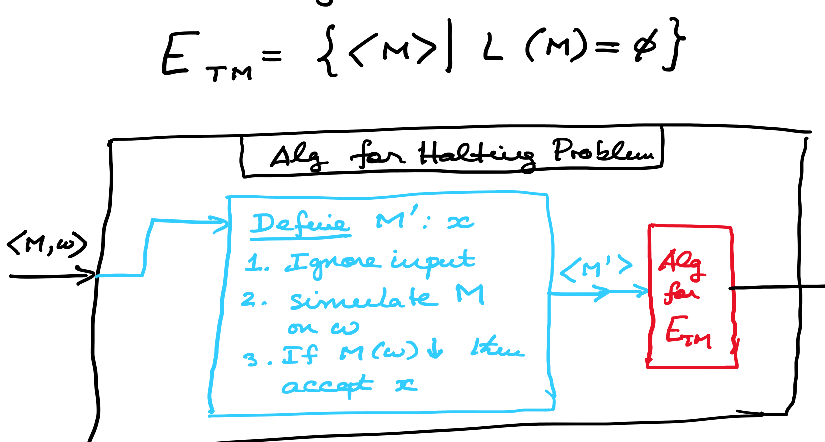
We will prove $H_{TM} \leq A_{TM}$



If $M(w)$ halts then $M'(w)$ halts and accepts and any hypothetical algorithm for deciding A_{TM} will be able to dectect this. So we will know if $M(w)$ halts or not.

Halting Problem $\leq$ Emptiness Problem

$E_{TM} = \{ \langle M \rangle \mid L(M) = \emptyset \}$



$L(M') = \emptyset$ iff $M(w)\uparrow$

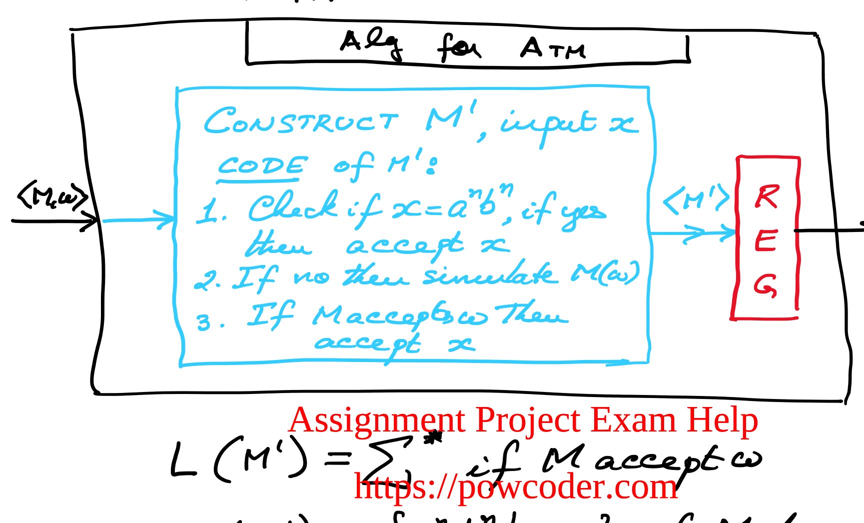
$L(M') \neq \emptyset \, (= \Sigma^*)$ iff $M(w)\downarrow$

Is $L(M)$ a regular language?

UNDECIDABLE PROBLEM

$REG = \{ \langle M \rangle \mid L(M) \text{ is regular} \}$

$A_{TM} \leq REG$



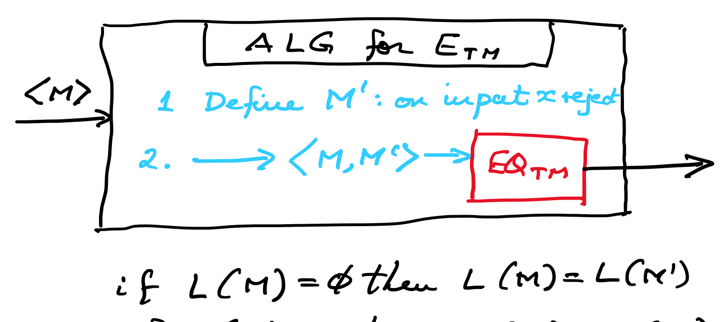
$L(M') = \Sigma^*$ if M accept w

$L(M') = \{ a^n b^n \mid n \geq 0 \}$ if M does not accept w

$L(M_1) = L(M_2)$?

$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$

$E_{TM} = EMPTY_{TM} \leq EQ_{TM}$



if $L(M) = \emptyset$ then $L(M) = L(M')$

if $L(M) \neq \emptyset$ then $L(M) \neq L(M')$

The stuff in BLUE is always computable.

EXERCISES:

1. $L(M) = L(M')$ where M' always halts
2. $L(M)$ is context free
3. $|L(M)| < \infty$
4. $L(M) = \Sigma^*$

MAPPING REDUCTION:

Suppose $L_1, L_2 \subseteq \Sigma^*$

$$L_1 \leq_m L_2$$

if there is a TOTAL, COMPUTABLE function $f : \Sigma^* \longrightarrow \Sigma^*$ such that $\forall w \in \Sigma^*$

$w \in L_1$ if and only if $f(w) \in L_2$.

f is called a mapping reduction

<u>NOTE</u> (1) $L_1 \leq_m L_2$ then $\overline{L_1} \leq_m \overline{L_2}$

(2) $\leq_m$ has a definite direction it is NOT the same as $_m\geq$.

The function f has a <u>direction</u>: from L_1 to L_2.

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1. If $P \leq_m Q$ and P is undecidable then Q is undecidable
2. If $P \leq_m Q$ and Q is decidable then P is decidable
3. If $P \leq_m Q$ and Q is CE then P is CE
4. If $P \leq_m Q$ and P is <u>NOT</u> CE then Q is not CE
5. If $P \leq_m Q$ and P is not coCE then Q cannot be coCE.

CE: L is CE if there is an alg which can be run on any $w \in \Sigma^*$ and if $w \in L$ the alg will terminate and say "yes". But if $w \notin L$, the alg may never give an answer.

coCE: L is coCE if there is an alg. which can be run on <u>every</u> $w \in \Sigma^*$. If $w \notin L$ the alg will eventually terminate and say "no" but if $w \in L$ the alg may never give an answer.

$\overline{A_{TM}}$, $\overline{H_{TM}}$ are coCE

L is coCE $\Leftrightarrow \overline{L}$ is CE

$A_{TM} \leq EMPTY_{TM}$

BUT NOT

$\cancel{A_{TM} \leq_m EMPTY_{TM}}$

if there were such a reduction

then $\overline{A_{TM}} \leq_m \overline{EMPTY_{TM}}$

but $\overline{A_{TM}}$ is not CE

whereas $\overline{E_{TM}}$ <u>is</u> CE.