<u>Def</u> An NFA is a 4-tuple (fix Σ as the alphabet)

$$N = (Q, Q_0, \Delta, F)$$

Q : set of states, $Q_0 \subseteq Q$ (start <u>states</u>; note plural)

$\Delta : Q \times \Sigma \to 2^Q$ (2^Q is the powerset of Q)

[$\Delta \subseteq Q \times \Sigma \times Q$ or $\forall a \in \Sigma$ Δ_a is a binary relation on Q]

Given Δ we can define $\Delta^* : 2^Q \times \Sigma^* \to 2^Q$

<u>Def</u> $\Delta^*(A, \varepsilon) = A$ ‖ $A \subseteq Q,\ a \in \Sigma,\ \omega \in \Sigma^*$

$\Delta^*(A, \omega a) \stackrel{?}{=} \Delta(A, \Delta^*(A,\omega), a)$ ‖ ⟸ NOT QUITE RIGHT

$$= \bigcup_{q \in \Delta^*(A,\omega)} \Delta(q, a)$$

<u>FACT</u> (1) $\Delta^*(A \cup B, \omega) = \Delta^*(A,\omega) \cup \Delta^*(B,\omega)$

(2) $\Delta^*(A, xy) = \Delta^*(\Delta^*(A,x), y)$

<u>DEF</u> $L(N) \stackrel{def}{=} \{\omega \in \Sigma^* \mid \Delta^*(Q_0, \omega) \cap F \neq \phi\}$

<u>Thm</u> Given an NFA N there is a DFA M such that $L(M) = L(N)$.

<u>Proof</u> Let $M = (S, s_0, \delta, \hat{F})$; we will describe it explicitly: $S = 2^Q$; $s_0 = Q_0$ [Do the types make sense?]

$\hat{F} = \{A \subseteq Q \mid A \cap F \neq \phi\}$

$\delta(A, a) = \bigcup_{q \in A} \Delta(q, a) = \Delta^*(A, a)$.

Now we must prove $L(M) = L(N)$.

<u>Lemma</u> $\Delta^*(A, \omega) = \delta^*(A, \omega)$ $\forall \omega \in \Sigma^*$

<u>Proof</u> By induction on $|\omega|$

<u>Base</u> $\omega = \varepsilon$. $\Delta^*(A, \varepsilon) = A = \delta^*(A, \varepsilon)$

<u>Ind. Case</u> Let $\omega = xa$ & <u>assume</u> $\forall A \subseteq Q$

$$\Delta^*(A, x) = \delta^*(A, x)$$

$$\begin{aligned}
\delta^*(A, xa) &= \delta(\delta^*(A,x), a) && [\text{Def. of } \delta^*] \\
&= \delta(\Delta^*(A,x), a) && [\cancel{\text{Def of } \delta} \text{ Ind. Hyp}] \\
&= \Delta^*(\Delta^*(A,x), a) && [\text{Def of } \delta] \\
&= \Delta^*(A, xa) && [\text{Fact (2)}]
\end{aligned}$$

Lemma is proved.

Completion of the proof of the theorem:

$$\begin{aligned}
L(N) &= \{\omega / \Delta^*(Q_0, \omega) \cap F \neq \phi\} \\
&= \{\omega / \Delta^*(Q_0, \omega) \in \hat{F}\} && [\text{Def of } \hat{F}] \\
&= \{\omega / \delta^*(Q_0, \omega) \in \hat{F}\} && \text{by Lemma} \\
&= \{\omega / \delta^*(s_0, \omega) \in \hat{F}\} && \text{by def. of } s_0 \\
&= L(M). \quad \blacksquare
\end{aligned}$$

NFA with ϵ-moves

$N = (Q, Q_0, \Delta : Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q, F)$

<u>Def</u> ϵ-closure of $q \in Q \stackrel{def}{=}$

$\{q' /$ there is an ϵ-path from q to $q'\}$.

We modify Δ^* to $\hat{\Delta} : 2^Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q$

$$\hat{\Delta}(A, \epsilon) = \epsilon\text{-closure}(A) = \bigcup_{q \in A} \epsilon\text{-closure}(q).$$

$$\hat{\Delta}(A, xa) = \epsilon\text{-cl}(\Delta(\hat{\Delta}(A,x), a))$$

Define $N' = (Q, Q_0, \Delta', F')$

$$\Delta'(q, a) = \hat{\Delta}(\{q\}, a)$$

$$F' = \begin{cases} F \cup \{q_0\} & \text{if } \epsilon\text{-closure}(q_0) \cap F \neq \phi \\ F & \text{otherwise} \end{cases}$$

Not too hard to see $L(N) = L(N')$

Therefore DFA, NFA & NFA with ϵ-moves all have the same power.

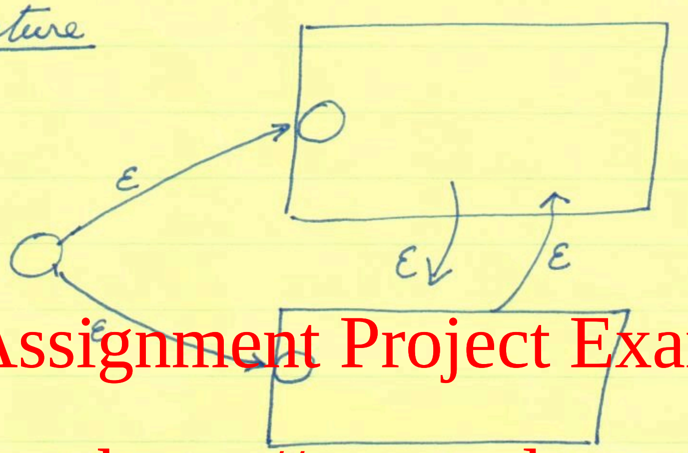
Example Suppose L_1, L_2 are regular languages

$$L_1 \| L_2 = \{ x_1 y_1 x_2 y_2 \cdots x_k y_k \mid x_1 x_2 \cdots x_k \in L_1 \ \& \ y_1 y_2 \cdots y_k \in L_2 \}$$

The shuffle of two languages is also regular

How do we prove this?

Picture



M_1 to recognize L_1

M_2 to recognize L_2

Use ε-transitions to go back and forth.

We need to remember where we were.

$M_1 = (S_1, s_1, \delta_1, F_1) \quad M_2 = (S_2, s_2, \delta_2, F_2)$

New NFA + ε m/c (Q, Q_0, Δ, F)

$Q = (S_1 \times S_2 \times \{1\}) \cup (S_1 \times S_2 \times \{2\}) \cup \{q_0\}$

$Q_0 = \{q_0\}$

$\Delta(q_0, \varepsilon) = \{ (s_1, s_2, 1), (s_1, s_2, 2) \}$

$\Delta((s, s', 1), a) = \{ (\delta_1(s, a), s', 1) \}$

$\Delta((s, s', 2), a) = \{ (s, \delta_2(s', a), 2) \}$

$\Delta((s, s', 1), \varepsilon) = \{ (s, s', 2) \}$

$\Delta((s, s', 2), \varepsilon) = \{ (s, s', 1) \}$

$F = \{ (s, s', 1) \mid s \in F_1 \ \& \ s' \in F_2 \} \cup \{ (s, s', 2) \mid s \in F_1 \ \& \ s' \in F_2 \}$