Lecture 9 The Myhill-Nerode Theorem

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Σ : alphabet finite set of symbols or letters

Σ^* : collection of all strings

an operation <u>concatenation</u>

$$\underbrace{x}_{\in \Sigma^*} \cdot \underbrace{y}_{\in \Sigma^*} = \underbrace{xy}_{\in \Sigma^*}$$

a <u>binary</u> operation

a unit ε $x \cdot \varepsilon = \varepsilon \cdot x = x$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$ ASSOCIATIVITY

A <u>monoid</u>

Another example S : finite set

$S \to S$ all functions from S to itself

This is a monoid

operation is function composition

Equivalence relations on a set define a <u>partition</u> of the set into equivalence

classes. The <u>index</u> of an eq. rel is the number of equivalence classes.
Let's consider eq. rel. on Σ^* and see how concatenation interacts with the concept.

<u>Def</u> An equivalence relation R on Σ^* is said to be **right invariant** if whenever $x R y$ then $\forall z \in \Sigma^*$ we have

$$xz \, R \, yz$$

Suppose we have a DFA

$$M = (S, \Sigma, s_0, \delta, F)$$

$$\delta^* : S \times \Sigma^* \longrightarrow S$$

$$\delta^*(s, xy) = \delta^*(\delta^*(s, x), y)$$

<u>Def</u> $x R_M y$ if and only if
$$\delta^*(s_0, x) = \delta^*(s_0, y)$$

<u>FACT</u> This is an example of a

right invariant relation.

<u>def</u> $L \subseteq \Sigma^*$, L not <u>necessarily</u> regular. Define R_L

$x \, R_L \, y$ iff $\forall z \;\; xz \in L \Leftrightarrow yz \in L$.

FACT This is also right invariant.

THEOREM (Myhill-Nerode)
The following are equivalent:

1. The language L is accepted by a DFA (i.e. L is regular)

2. L is the union of <u>some</u> of the equivalence classes of <u>some</u> right invariant equivalence relation of finite index

3. The equivalence relation R_L has finite index. Any relation satisfying (2) will <u>refine</u> R_L.

<u>PROOF</u> $(1) \Rightarrow (2)$

$$M = (S, s_0, \delta, F)$$

We know R_M is right-invariant.

How many equivalence classes does R_M have ?

Ans: one for every state

$q \in S$ $\quad S_q := \{ x \mid \delta^*(s_0, x) = q \}$

$\therefore$ R_M has finite index.

$$L = \bigcup_{q \in F} S_q$$

$(2) \Rightarrow (3)$ $\quad R_1$ refines R_2 means

if $x R_1 y$ then $x R_2 y$.

Let R be any right-invariant equivalence relation of finite index such that L is the union of some of the equivalence classes of R.

Suppose $x R y$ $\qquad x, y \in \Sigma^*$

Suppose $xz \in L$ then $yz \in L$. Why?

[illegible]

$xz \, R \, yz$ since R is right invariant so xz, yz are in one equivalence class of R. That equivalence class is a subset of L so

$$xz \in L \Rightarrow yz \in L$$

we can reverse this easily

$$yz \in L \Rightarrow xz \in L$$

ie $\quad xz \in L \Leftrightarrow yz \in L \quad \forall z \in \Sigma^*$

But this means $x R_L y$

So $x R y \Rightarrow x R_L y$.

(3) $\Rightarrow$ (1) We construct a m/c from R_L

$$M' = (S', s_0', \delta', F')$$

S' : the equivalence classes of R_L

$$s_0' = [\varepsilon]$$

$$\delta'([x], a) = [xa]$$

$$F' = \{ [x] \mid x \in L \}$$

EXERCISE for you: Prove that the

$L(M') = L.$ END of the PROOF

ISOMORPHISM of MACHINES

$M_1 = (S_1, s_1, \delta_1, F_1)$ $M_2 = (S_2, s_2, \delta_2, F_2)$

We say M_1 and M_2 are <u>isomorphic</u> if there is a function $\varphi : S_1 \to S_2$ such that:

(1) φ is a bijection

(2) $\varphi(\delta_1(s,a)) = \delta_2(\varphi(s), a)$

$$\begin{tikzcd} s \arrow[r, "a"] \arrow[d] & \delta_1(s,a) \arrow[d] \\ & \varphi(\delta_1(s,a)) \arrow[d, phantom, "\parallel"] \\ \varphi(s) \arrow[r, "a"'] & \delta_2(\varphi(s), a) \end{tikzcd}$$

(3) $s \in F_1$ iff $\varphi(s) \in F_2$.

It is easy to see that if we have such an isomorphism $L(M_1) = L(M_2)$.

<u>PROP</u> The machine constructed in the last part of the MN Thm proof is the unique (up to isomorphism) minimal

the unique (up to isomorphism) minimal machine recognizing L.

With NFA's you can have distinct minimal versions. There is an algorithm for finding a minimal NFA but it is very complex & expensive.
