Lecture 5

Thursday, January 21, 2021   11:17 AM

The algebra of regular expressions

LAWS of regular expression:

1. $R + \phi = \phi + R = R$
2. $R + S = S + R$
3. $R + (S+T) = (R+S) + T$
4. $R + R = R$ (idempotence)
5. $R \cdot \phi = \phi \cdot R = \phi$
6. $R \cdot \varepsilon = \varepsilon \cdot R = R$
7. $R \cdot (S \cdot T) = (R \cdot S) \cdot T$
8. $R \cdot (S+T) = R \cdot S + R \cdot T$
9. $(S+T) \cdot R = S \cdot R + T \cdot R$
10. $\varepsilon + R R^* = \varepsilon + R^* R = R^*$

BASIC PROOF PATTERN

To show $X = Y \Leftrightarrow X \subseteq Y$ and $Y \subseteq X$.

e.g (5) If $w \in R \cdot \phi$

then $w = w_1 w_2$ with $w_1 \in R$ & $w_2 \in \phi$.

But there is no element of ϕ so such a decomposition cannot exist & hence there is no word in $R \cdot \phi$ i.e $R \cdot \phi = \phi$.

(8) $w \in R \cdot (S+T)$

i.e. $w = w_1 w_2$ where $w_1 \in R$ & $w_2 \in S+T$

if $w_2 \in S+T$ we have 2 cases

(a) $w_2 \in S$ so then $w_1 w_2 \in R \cdot S$

(b) $w_2 \in T$ so then $w_1 w_2 \in R \cdot T$

i.e $w_1 w_2 \in R \cdot S + R \cdot T$

reverse similarly

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other valid equations:

$$R^{**} = R^*$$

suppose $w \in R^{**}$ then

$(R^*)^*$

$w = w_1 \cdots w_n$ where each $w_i \in R^*$

what does it mean to say $w_i \in R^*$?

$w_i = w_i^{(1)} w_i^{(2)} \cdots w_i^{(k_i)}$ for some $k_i \in \mathbb{N}$.

where $w_i^{(j)} \in R$

$$w = w_1^{(1)} \cdots w_1^{(k_1)} w_2^{(1)} \cdots w_2^{(k_2)} \cdots w_n^{(1)} \cdots w_n^{(k_n)}$$

But this is just a sequence of words from R i.e $w \in R^*$.

$R^{**} \subseteq R^*$

clearly $R^* \subseteq R^{**}$ so $R^* = R^{**}$

self-test exercise: Show

$$(R^* S)^* R^* = (R+S)^*$$

The point of algebra: to solve equations!

$$x = Sx + T$$

x: unknown S, T regular expressions.

We want to solve this for x.

ASSUME: $T \neq \phi$ $\varepsilon \notin S$

Then the solution is $S^* T$ and in fact this is the only solution.

$Sx + T = S(S^* T) + T$

$= S S^* T + \varepsilon \cdot T = (SS^* + \varepsilon) \cdot T$

$= S^* T = x$

$5x + 2 = 17 \Rightarrow x = 3$

$5x + 3 = 17 \Rightarrow x = ?$ $14/5 = 2\frac{4}{5}$

$2x + 7 = 3 \Rightarrow x = ?$ -2

→ negative numbers.

→ ?? fractions (rational numbers)

p/q

$\alpha x + \beta = \gamma$

α, β, γ rationals

The equation can be solved and the solutions are rationals.

$\alpha x^2 + \beta x + \gamma = 0$

$a x^2 + b x + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What if $b^2 - 4ac = 2$ for example?

$\sqrt{2}$ is not rational!

You have to expand the concept of number.

Algebraic numbers

Cubic equation: Tartaglia, Cardano, Scipio...

They solved cubic & biquadratic eqns.

They came across $\sqrt{-1}$!!

One has to expand the concept of number again!

COMPLEX numbers.

How to solve the quintic (5^{th} order)?

Abel   NORWAY (died at age 26)

Galois   FRANCE (died at age 22)

GAUSS: Fundamental theorem of algebra any polynomial equation of order n with complex coefficients has n solutions that are complex numbers

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CLOSURE PROPERTIES of REGULAR LANGUAGES

Suppose L_1, L_2 are regular languages:

(1) L_1^* is also regular

(2) $L_1 \cdot L_2$ is also regular

(3) $L_1 \cup L_2$ is also regular

I will formalize the proof of (2) in terms of automata. Fix Σ as the alphabet

$M_1 = (S_1, s_1, \delta_1, F_1)$ $L(M_1) = L_1$

$M_2 = (S, s_2, \delta_2, F_2)$ $L(M_2) = L_2$

I want to define an NFA+ε moves N st $L(N) = L_1 \cdot L_2$.

$N = (Q, Q_0, \Delta, F)$

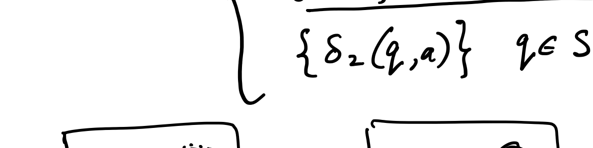
$Q = S_1 \cup S_2$ $S_1 \cap S_2 = \phi$

$Q_0 = \{s_1\}$

$F = F_2$

$a \in \Sigma \cup \{\varepsilon\}$

$$\Delta(q,a) = \begin{cases} \{\delta_1(q,a)\} & q \in S_1 \setminus F_1,\ a \in \Sigma \\ \{\delta_1(q,a)\} & q \in F_1,\ a \neq \varepsilon \\ \{s_2\} & q \in F_1\ \ a = \varepsilon \\ \{\delta_2(q,a)\} & q \in S_2,\ a \neq \varepsilon,\ a \in \Sigma \end{cases}$$



L is a regular language so is $\overline{L}$. why?

$L_1 \cap L_2$ is also regular

since $L_1 \cap L_2 = \overline{(\overline{L_1} \cup \overline{L_2})}$

But I want to show a direct construction:

$M_1 = (S_1, s_1, \delta_1, F_1)$ $L(M_1) = L_1$

$M_2 = (S_2, s_2, \delta_2, F_2)$ $L(M_2) = L_2$

New DFA (S, s_0, δ, F)

$S = S_1 \times S_2 = \{(s,t) \mid s \in S_1, t \in S_2\}$

$s_0 = (s_1, s_2)$

$F = F_1 \times F_2$

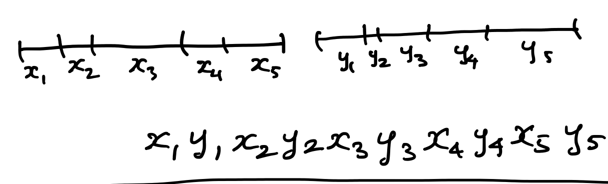
$\delta((s,t), a) = (\delta_1(s,a), \delta_2(t,a))$

This m/c keeps track of both sets of transitions.

Self-study exercise:

L_1, L_2 are regular languages then $L_1 \| L_2$ is regular.

$$L_1 \| L_2 = \left\{ x_1 y_1 x_2 y_2 \cdots x_n y_n \;\middle|\; \begin{matrix} x_1 x_2 \cdots x_n \in L_1 \\ y_1 y_2 \cdots y_n \in L_2 \end{matrix} \right\}$$



$x_1 y_1 x_2 y_2 x_3 y_3 x_4 y_4 x_5 y_5$
