Lecture 1 Introduction

Thursday, January 7, 2021 11:12 AM

Equivalence relations

A binary relation R on a set X is a subset of $X \times X$ i.e. it is a set of pairs

[NOTATION: $(a,b) \in R$ as aRb]

R must satisfy:

(i) Reflexivity $\forall x \in X \quad xRx$

(ii) Symmetry $\forall x, y \in X \quad xRy \Rightarrow yRx$

(iii) Transitivity $\forall x, y, z \in X \quad xRy \,\&\, yRz \Rightarrow xRz$

Eg $n \equiv m \pmod 7$

means remainder after dividing n by 7 and m by 7 is the same.

Partial order $\leqslant$

"think less than or equal to"

A binary relation R is a partial order on X if:

(i) $\forall x \in X \quad xRx$

(ii) $\forall x, y \in X \quad xRy \,\&\, yRx \Rightarrow x = y$ → ANTISYMMETRY

(iii) $\forall x, y, z \quad xRy \,\&\, yRz \Rightarrow xRz$

e.g. numerical inequality

e.g. 2 set inclusion

$\{a, b, c\}$

$\{a, b\} \subseteq \{a, b, c\}$ $\quad \subseteq$ is a partial order

$\{b, c\} \subseteq \{a, b, c\}$

$\{a\} \subseteq \{a, b\}$

$\{a, b\} \not\subseteq \{a, c\}$

$\{a, c\} \not\subseteq \{a, b\}$

It is not necessarily true that $\forall x, y$ xRy OR yRx.

If every pair of elements can be compared then we have a total order also called a linear order.

If we have a partial order $\leqslant$ we say $a < b$ if $a \leqslant b$ AND $a \neq b$.

a is strictly less than b

BASIC FACTS:

Suppose X is a set and R is an equivalence relation $\forall x \in X$ we define

$[x] = \{ y \in X \mid xRy \}$

↳ the equivalence class of x.

1. If xRy then $[x] = [y]$ (why?)

2. If $x, y \in X$ then either $[x] = [y]$ OR $[x] \cap [y] = \phi$ ← empty set (why?)

The equivalence classes divide X into disjoint "clumps".

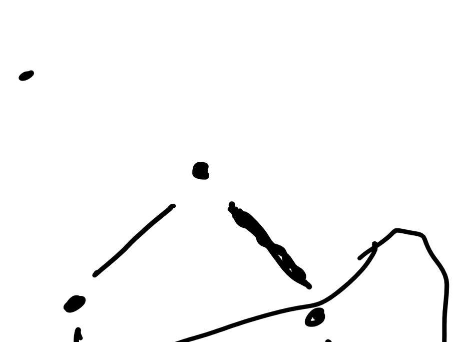
X

THIS IS CALLED A PARTITION

WELL-FOUNDED ORDERS

A partial order $\leqslant$ on S is well founded if every non-empty subset $U \subseteq S$ has a minimal element.

Remark We say $u \in U$ is minimal if there is nothing else in u strictly less than u.



2 minimal elements

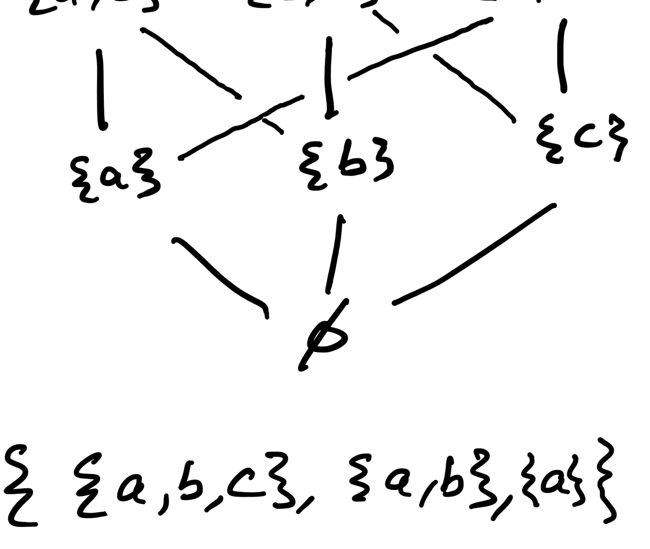
WHO CARES?

The non-negative integers are a well-founded order with the usual numerical order

All the integers do not form a well founded set.

Thm The principle of induction can be used if and only if the order is well founded.

AFTER CLASS QUESTIONS:



$\{ \{a,b,c\}, \{a,b\}, \{a\} \} \longrightarrow \{a\}$ ← MINIMAL?

$\{ \underline{\{a,b\}}, \underline{\{a,c\}}, \{b\} \} \longrightarrow$ BOTH ARE MINIMAL.

X

$\{a,b,c\}$ MINIMAL → MEANINGLESS.

$\{ \{a,b,c\} \} \longrightarrow \{a,b,c\}$

$\{ \{a,b,c\}, \{a,b\}, \{a\}, \phi \} \longrightarrow \phi$ ✓