Lecture 8 Minimization

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In NFA's the transition relation is

$$\Delta : Q \times \Sigma \to 2^Q$$

$\phi \in 2^Q$ so $\Delta(Q_i, a) = \phi$ is possible

NOTATION REMINDERS : DFA

$\delta : S \times \Sigma \to S$

$\delta^* : S \times \Sigma^* \to S$

$\delta^*(s, \varepsilon) = s$

$\delta^*(s, aw) = \delta^*(\delta(s,a), w)$

$\delta^*(s, wa) = \delta(\delta^*(s,w), a)$

MINIMIZATION of DFA's:

Lump together states that behave exactly the same way.

Def Given a DFA $M = (S, s_0, \delta, F)$ over alphabet Σ we say, $p, q \in S$ are equivalent and write $p \approx q$ if

$\forall x \in \Sigma^* \quad \delta^*(p,x) \in F \Leftrightarrow \delta^*(q,x) \in F$

Remark When are p, q not $\approx$?

$\exists x \in \Sigma^* \quad (\delta^*(p,x) \in F \;\&\; \delta^*(q,x) \notin F)$
OR $(\delta^*(p,x) \notin F \;\&\; \delta^*(q,x) \in F)$.

we call such an x a distinguishing string.

OBSERVATION $\approx$ is an equivalence relation.

We write $[p] = \{ q \mid p \approx q \}$
if $p \approx q$ then $[p] = [q]$.

LEMMA A $\quad p \approx q \Rightarrow \forall a \in \Sigma$
$\delta(p,a) \approx \delta(q,a)$.

PROOF Suppose $\delta^*(\delta(p,a), x) \in F$
$= \delta^*(p, ax) \in F$
But we assumed $p \approx q$ so
$\delta^*(q, ax) \in F$
i.e. $\delta^*(\delta(q,a), x) \in F$
similarly for the other direction
so the proof is complete ☺

REMARK $p \approx q$ can be written as
$[p] = [q]$. So what we have shown
is $[p] = [q] \Rightarrow [\delta(p,a)] = [\delta(q,a)] \; \forall a \in \Sigma$.

We can construct a new machine:
$M' = (S', s_0', \delta', F')$

S': equivalence classes of $\approx$

s_0' : $[s_0]$

$\delta'([s], a) = [\delta(s,a)]$ [Well defined by Lemma A]

$F' = \{ [s] \mid s \in F \}$

LEMMA B $\quad p \in F \;\&\; p \approx q \Rightarrow q \in F$ (DIY)

LEMMA C $\quad \forall w \in \Sigma^*$
$\delta'^*([p], w) = [\delta^*(p,w)]$

PROOF Induction on w

BASE $w = \varepsilon$
$\delta'^*([p], \varepsilon) = [p] = [\delta^*(p, \varepsilon)]$ ✓

Induction step: Assume true for w
$\delta'^*([p], w) = [\delta^*(p,w)]$

Calculate:
$\delta'^*([p], wa) = \delta'(\delta'^*([p], w), a)$ Def of δ^*
$= \delta'([\delta^*(p,w)], a)$ Ind. hyp.
$= [\delta(\delta^*(p,w), a)]$ Def of δ'
$= [\delta^*(p, wa)]$ Done.

THM $\quad L(M') = L(M)$

Proof $\quad x \in L(M') \Leftrightarrow \delta'^*([s_0], x) \in F'$
$\Leftrightarrow [\delta^*(s_0, x)] \in F'$
$\Leftrightarrow \delta^*(s_0, x) \in F \Leftrightarrow x \in L(M)$ ∎

The machine with equivalent states lumped together recognizes the same language as the original machine.

Idea : Assume initially all states are equivalent. Then start splitting the states as you examine the transitions.

Def $p \bowtie q$ if $\exists w \in \Sigma^*$ s.t.
$\delta^*(p,w) \in F \;\&\; \delta^*(q,w) \notin F$
OR
$\delta^*(p,w) \notin F \;\&\; \delta^*(q,w) \in F$.
i.e. $p \bowtie q$ is $\neg(p \approx q)$

If $p \bowtie q$ then p & q are distinguishable

FACT If $\exists a \in \Sigma$ s.t. $\delta(p,a) \bowtie \delta(q,a)$
then $p \bowtie q$. [DIY]

ALGORITHM

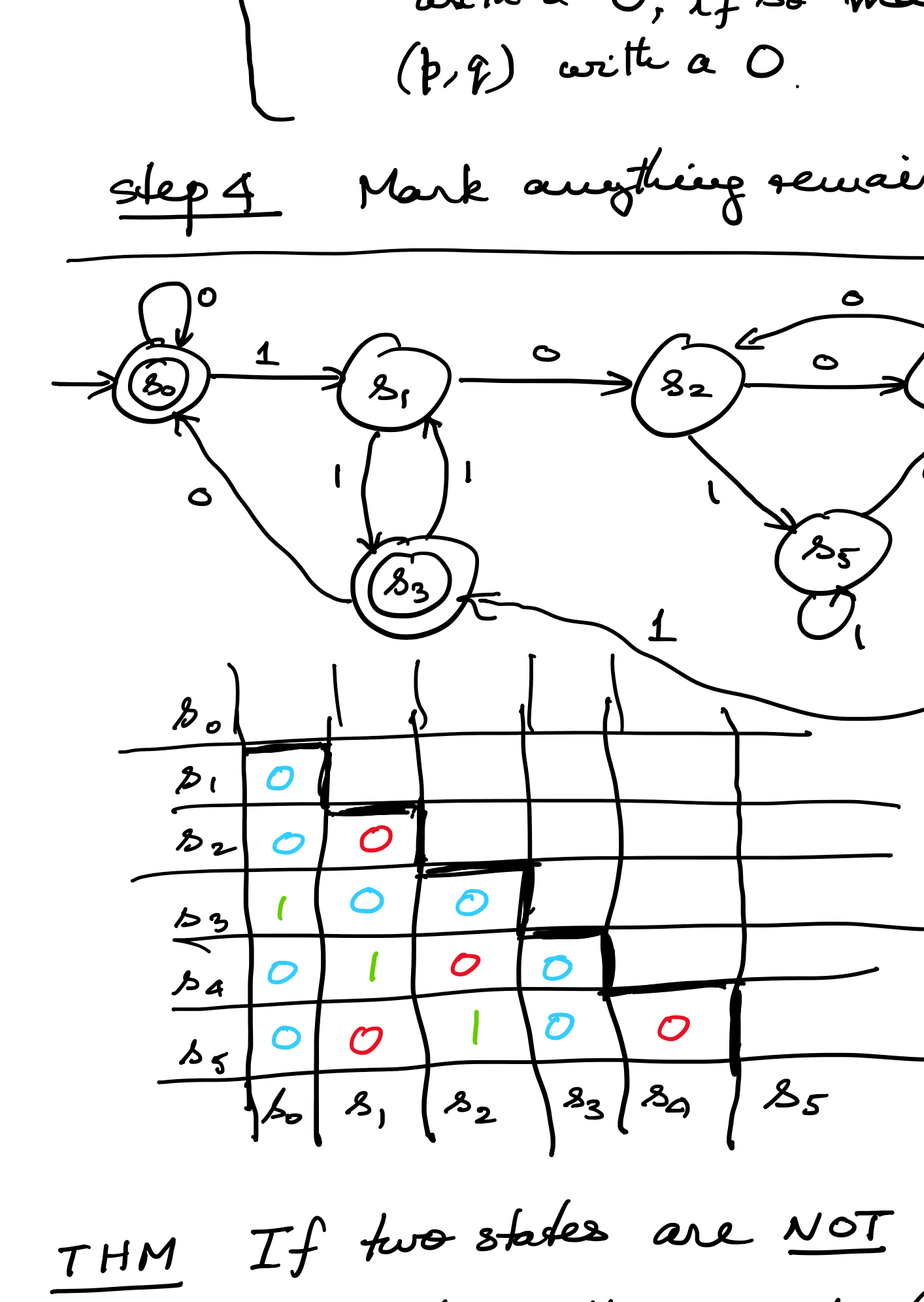
Step 0 Get rid of unreachable states.

step 1 Define an $S \times S$ array of booleans

step 2 For every pair $(p,q) \in S \times S$ such that $p \in F$ and $q \notin F$ put a 0 in the (p,q) cell of the array.

step 3 Repeat until there are no more changes:
- For each pair (p,q) that is not marked with a 0 check if $\exists a \in \Sigma$ s.t. $(\delta(p,a), \delta(q,a))$ is marked with a 0, if so mark (p,q) with a 0.

step 4 Mark anything remaining 1



	s_0	s_1	s_2	s_3	s_4	s_5
s_0						
s_1	0					
s_2	0	0				
s_3	1	0	0			
s_4	0	1	0	0		
s_5	0	0	1	0	0	

INITIAL
NEXT
EQUIVALENT

THM If two states are NOT labelled by a 0 then they must be equivalent

PROOF Assume theorem is false. So there are some pairs of states s.t. $p \bowtie q$ but they are not marked. Call such a pair a bad pair. Each such bad pair must have a distinguishing string. Choose a bad pair with the shortest distinguishing string. $x = x_1 \cdots x_n \quad x_i \in \Sigma$

Note $x \neq \varepsilon$ [why not?]

Suppose (s,t) is a bad pair and x is the dist. string for this pair

$\delta^*(s,x) \bowtie \delta^*(t,x)$ since
$\delta^*(\delta(s,x_1), x_2 \cdots x_n) \in F$
$\delta^*(\delta(t,x_1), x_2 \cdots x_n) \notin F$

so $\delta(s,x_1) \bowtie \delta(t,x_1)$
& $x_2 \cdots x_n$ is the distinguishing string.

So $\delta(s,x_1)$ & $\delta(t,x_1)$ cannot be a bad pair so the algorithm did mark them. But then according to the code s,t would be marked in the next step.
This is a contradiction. ∎

So this algorithm finds all the equivalent states.

RUNNING TIME: $O(n^4)$ $\quad n$ # of states
each round is $O(n^2)$
$O(n^2)$ rounds

IMPROVED alg $\quad O(n^2 k) \rightarrow k$ # of trans

HOPCROFT'S ALG: $\quad O(n \log n) \leftarrow$

BRZOZOWSKI: $\quad O(2^n)$!