Computability Theory

Thursday, March 18, 2021 11:32 AM

Strings, numbers, data structures, lists etc can all be encoded one in the other : so it does not matter which data type I am using.

I will use numbers $\mathbb{N} = \{0, 1, 2, \cdots\}$

or strings with letters a, b, ...

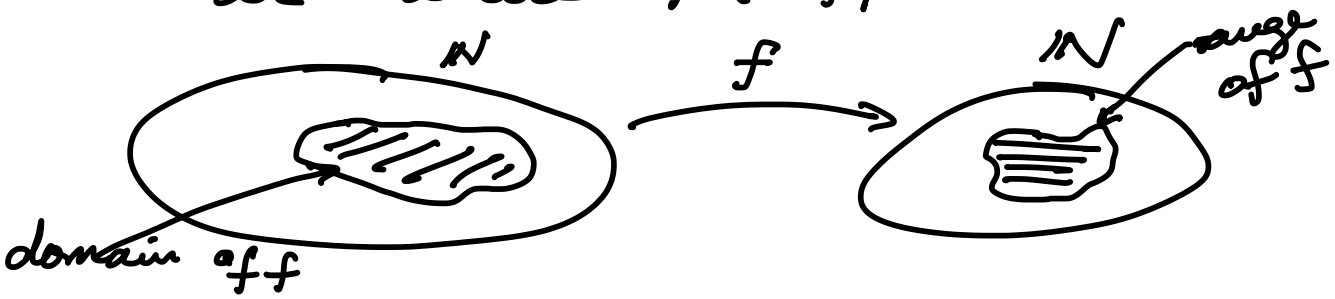
These computations are modelled as functions. BUT we have seen some computations may not halt. So we have to use partial fns

$f : \mathbb{N} \longrightarrow \mathbb{N}$

Given input $n \in \mathbb{N}$

$f(n)$ may halt and produce output m: we write $f(n) = m$

or $f(n)$ may not be defined:
we write $f(n)\uparrow$



f may only be defined on a (proper) subset of $\mathbb{N}$

$$dom(f) = \{n \in \mathbb{N} \mid f(n) \text{ is defined}\}$$

$f(n)\downarrow$ means $f(n)$ is defined.

When we write $f = g$ we mean

$\forall n \quad f(n)\downarrow \iff g(n)\downarrow$ and
$\forall n \in dom(f) \quad f(n) = g(n)$.

What should I use to describe algs?

I will use natural language pseudocode.

A set X of natural numbers is called computable or decidable if its characteristic function is

total computable.

Given X; its characteristic fn

$$\chi_X(n) = \begin{cases} 1 \text{ if } n \in X \\ 0 \text{ if } n \notin X \end{cases}$$

A function is computable if there is an algorithm to compute it.

A function f is total if $dom(f) = \mathbb{N}$

A function is total computable if the algorithm to compute it halts on every input.

A set is decidable means we have an algorithm that always terminates and answers the membership question.

Ex (1) The set of odd numbers

(2) The set of prime numbers

Given a TM M we write

$\langle M \rangle$ for the encoding of M as an integer.

$\langle M, x \rangle$: encoding of M and input string x as an integer.

Ex $S = \{ \langle M, x \rangle \mid M$ halts on input $x \}$

We cannot have an algorithm to decide membership in S.

We call this set H_{TM}

$A_{TM} = \{ \langle M, x \rangle \mid M$ accepts $x \}$

H_{TM} is not decidable but it is semi decidable.

There is an algorithm that will answer YES every time if $n \in H_{TM}$ but if $n \notin H_{TM}$ it may or may not answer.

THM An infinite set X of natural numbers is computable (decidable) if

and only if it is the range of some total non-decreasing computable function f.

PROOF Suppose we have an f as described and A is an algorithm that computes it. I will give you a decision procedure for X. Given $x \in \mathbb{N}$, Run A with 0, 1, 2, ... as inputs and check if the output is ever x. If the output is $< x$, move on to the next number; if output $= x$ STOP and say "YES"; if output $> x$ STOP and say "NO".

Suppose we know X is decidable and that B is a decision procedure for membership in X. We define f as follows. We run B on 0, 1, 2, ... every time we find a number in X we store it in a list. Then define $f(n)$ to be the n^{th} element on this list.

DEF A set $X \subseteq \mathbb{N}$ is called computably enumerable (CE) if there is an algorithm that lists all the members of X in some order; not necessarily increasing order.

REMARKS: Producing the whole list is an infinite process, but any given member of the list is produced at some finite stage.

THM A set X is CE iff any of the following equivalent conditions hold:

(i) X is the domain of a computable function

(ii) the semi-characteristic function of X

$$\mathcal{S}_X(n) = \begin{cases} 1 \text{ if } n \in X \\ \text{undefined if } n \notin X \end{cases}$$

is computable

(iii) X is the range of a computable f^n.

PROOF (ii) ⇒ (i) immediately.

(CE) ⇒ (ii) Suppose A enumerates X.
To compute $\delta_X(n)$; run A and wait for it to produce n. It may or may not do so; if A does produce n we set $\delta_X(n) = 1$ otherwise we wait forever.

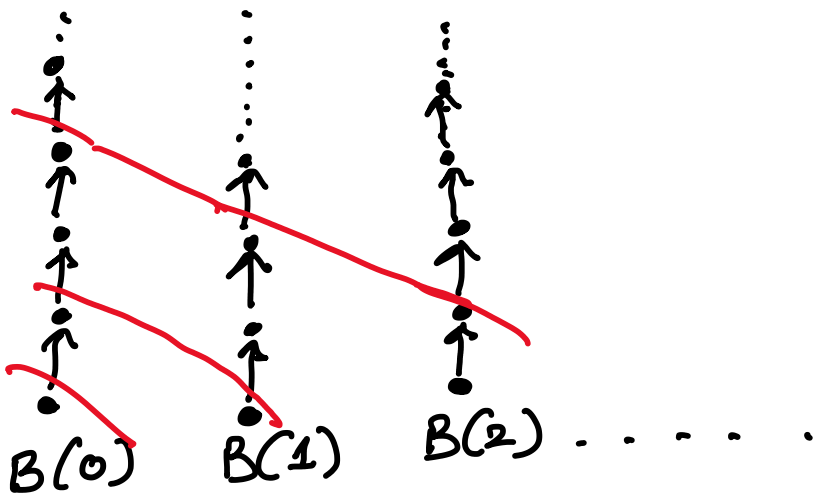
(i) ⇒ (CE) Suppose B is an algorithm to compute $f_{enum(X)} = X$.

Idea (BAD): Run B on 0, 1, 2, 3, ...

whenever B halts on an n we output n as our enumeration process.

BUT this will not work because perhaps B(0) does not halt.

NEW IMPROVED IDEA: DOVETAILING



RUN B(0) for one step; then store the state
RUN B(1) for one step; then store
RUN B(0) for 2 steps, then store
RUN B(1) for 2 steps, then store
RUN B(2) for 2 steps then store

RUN B(0) . . . 3 . . . . . .
RUN B(1) . . . 3 . . . . . .
RUN B(2) . . . 3 . . . .
RUN B(3) . . . . 3 . . . . . .

For any $n \in dom(f)$ this procedure will eventually run long enough to find out that B terminates on n.

This is an enumerator for X

THM The union of 2 CE sets is CE
The intersection of 2 CE sets is CE.

THM (POST)

(a) If X is a computable (decidable) set then X is CE. [TRIVIAL]

(b) If X is CE and $\overline{X}$ is also CE

then X is computable (decidable).

PROOF (6): Run the enumerators for X and $\overline{X}$ in parallel; any $n \in \mathbb{N}$ must lie in one of these sets so if they are both CE one of them will eventually enumerate it.

THM $X \subseteq \mathbb{N}$ is CE iff $\exists$ a computable set Y of pairs of natural numbers s.t. $\forall x,\ x \in X \Leftrightarrow \exists y \in \mathbb{N}\ (\langle x, y\rangle \in Y)$.

PROOF Suppose X is CE, A enumerates it we define Y as follows

$$Y = \{\langle x, n\rangle \mid A \text{ enumerates } x \text{ within } n \text{ steps}\}$$

Clearly Y is decidable but clearly

$$\forall x,\ x \in X \Leftrightarrow \exists n \in \mathbb{N}.\ (\langle x, n\rangle \in Y).$$
