The pumping lemma for context-free languages

Tuesday, March 9, 2021 11:21 AM

Context - free languages

$S \to aSb \mid \varepsilon$

$L(G) = \{a^n b^n \mid n \geq 0\}$

$(ab)^*$ is a regular language with equal # of a's & b's.

$\{a^n b^n c^n \mid n \geq 0\}$ is <u>NOT</u> context-free.

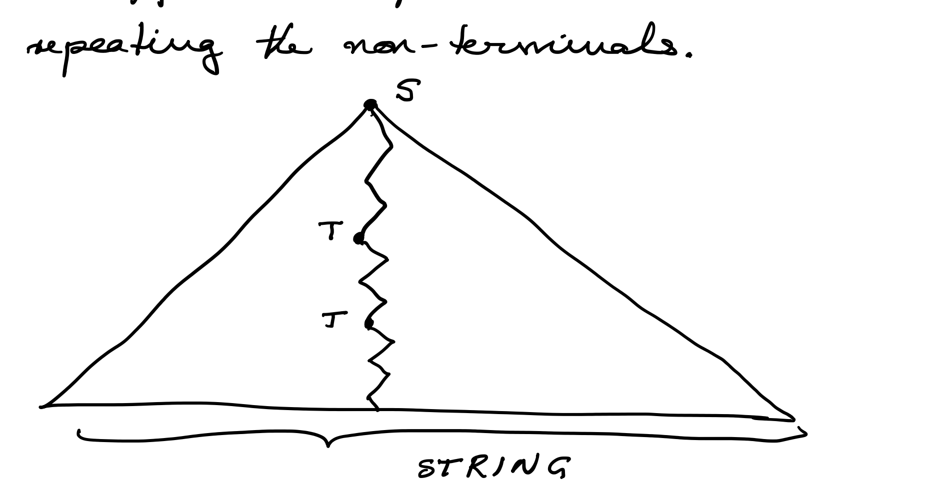
<u>Recall</u> every CFL has a CFG in Chomsky Normal Form

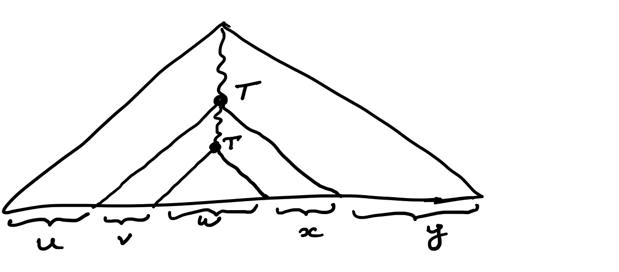
$A \to BC$

$A \to a$

When we are using a CNF grammar the parse <u>tree</u> is binary.

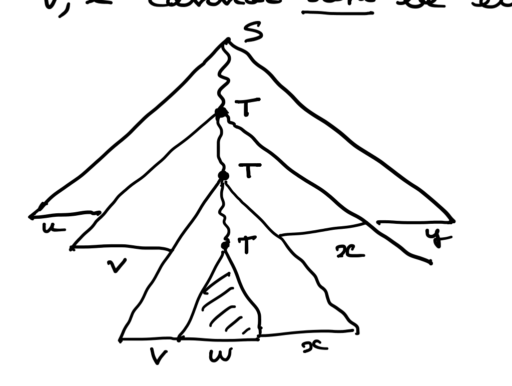
A grammar has only <u>finitely</u> many non-terminals. So if we generate a sufficient <u>deep</u> tree we must start repeating the non-terminals.



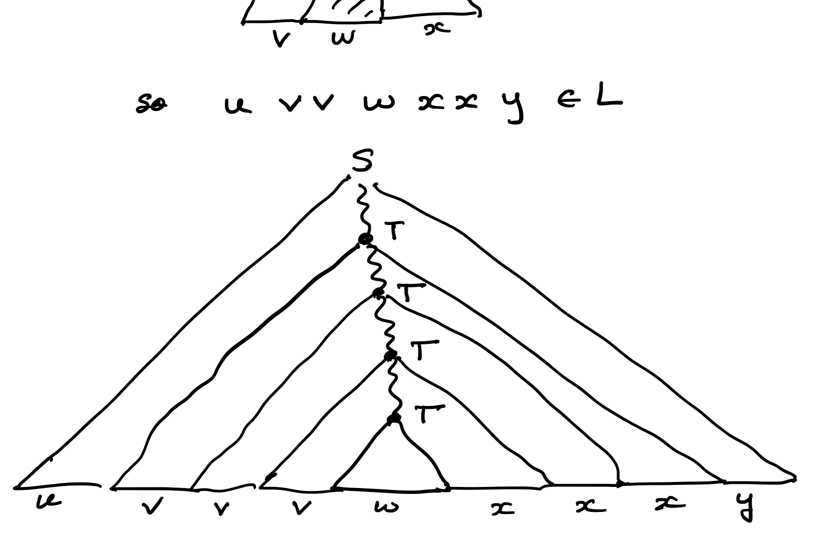


$S \xrightarrow{*} uvwxy \qquad T \xrightarrow{*} vwx \qquad T \xrightarrow{*} w$

v, x cannot <u>both</u> be empty (CNF)

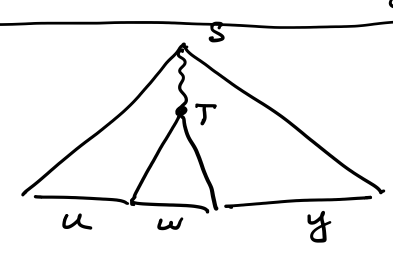


so $u\,vv\,w\,xx\,y \in L$



$u\,vvv\,w\,xxx\,y \in L$

$$\boxed{\forall i \geq 0 \quad uv^i w x^i y \in L}$$



$uwy \in L$

Full statement of the pumping lemma:

$\forall$ CFLs L $\exists p > 0$

$\forall s \in L$ $|s| \geq p$

$\exists u, v, w, x, y \in \Sigma^*$ such that

- $s = uvwxy$
- $|vx| \neq 0$
- $|vwx| \leq p$

$\forall i \geq 0 \quad uv^i w x^i y \in L$

CFL $\Rightarrow$ every string can be pumped

CONTRA POSITIVE:

if there is <u>even</u> <u>one</u> <u>string</u> that cannot be pumped $\Rightarrow$ not a CFL

Fix L some language

<u>If</u>
$\forall p > 0$
$\exists s \in L$ s.t. $|s| > p$
$\forall u, v, w, x, y \in \Sigma^*$ s.t.
- $s = uvwxy$
- $|vx| > 0$

$\to$ $\boxed{- \ |vwx| \leq p}$

$\exists i \geq 0 \quad uv^i w x^i y \notin L$

<u>then</u> L is NOT a CFL.

EXAMPLES

(1) $L = \{a^n b^n a^n \mid n \geq 0\}$

- Demon chooses $p > 0$
- I choose $a^p b^p a^p$

Have to analyze how the demon may break up the string into u, v, w, x, y:

$\underbrace{a^p}\ \underbrace{b^p}\ \underbrace{a^p}$

v v x

- v (or x) crosses a boundary between two blocks: $i = 2$, now a's and b's are out of order
- v, x both contain only a's they have to be in the <u>same</u> block: now choose $i = 2$ (or 3 (2)) the 3 blocks do not have the same length.
- same argument if both v, x contain only b's
- v contains only a's and x contains only b's: now one block is untouched by pumping: $i = 2$ and the block lengths cannot match

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EXAMPLE 2

Σ: arbitrary $|\Sigma| \geq 2$

$\to L = \{ww \mid w \in \Sigma^*\}$ $L' = \{ww^{REV} \mid w \in \Sigma^*\}$

L is not context free

<u>FACT</u>: if L is a CFL & R is <u>regular</u> then $L \cap R$ is a CFL.

$\hat{L} = L \cap a^* b^* a^* b^*$

$\hat{L} = \{\underline{a^i} b^j \underline{a^i} b^j \mid i, j \geq 0\}$

Demon picks $p > 0$

I pick $a^p b^p a^p b^p$

$\underbrace{a^p}\ \underbrace{b^p}\ \underbrace{a^p}\ \underbrace{b^p}$

v

- v or x straddles a block boundary: $i = 2$ letters are out of order
- If v, x are in the same block: choose $i = 2$. Either 1st & 3rd blocks are mismatched or 2nd & 4th are mismatched
- If v is in the first block, x is in the 2nd block: choose $i = 2$, now 1st & 3rd no longer match / also 2nd & 4th blocks don't match

Note: v, x must be in the same block or in adjacent blocks.

$\bar{L}$ is context-free: we have seen a PDA already and there is a grammar in the <u>supplementary</u> <u>notes</u>.

CFL's are not closed under complement.

- $L_1 = \{a^n b^n c^m \mid n, m \geq 0\}$ $L_2 = \{a^m b^n c^n \mid n, m \geq 0\}$

L_1, L_2 both CFL's.

$L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$

$\hookrightarrow$ NOT a CFL

CFL's are <u>NOT</u> closed under intersection.

If L is a CFL then lefthalf(L) may not be a CFL.

EXAMPLE ???

<u>FACT</u> Over a one-letter alphabet a CFL is regular.

<u>EXAMPLE</u> $L = \{a^i b^j c^k \mid 0 < i < j < k\}$

Adversary $\longrightarrow p$

I $\longrightarrow a^p b^{p+1} c^{p+2}$

Case analysis:

(a) If v or x straddle block boundaries then $i = 2$ will put letters out or order

(b) If v, x are both in the same block

(i) both in a's or both in b's then choose $i = 2$ & the inequality no longer holds.

(ii) if they are both in the c's then choose $i = 0$, strict inequality no longer holds

(c) $v = a^k$ $x = b^j$, $k > 0$ (OR) $j > 0$

choose $i = 2$ $\underline{p + 1 + j \not< p + 2}$ if $j \neq 0$

similarly if $k \neq 0$.

(d) v in the b's and x in the c's then pump down i.e. choose $i = 0$, now we have too many a's.

v w x $\leq p$