RICE'S THEOREM

Let <u>Prog</u> be the set of all programs in some Turing-complete programming language. The set of such programs can be enumerated effectively. For simplicity, assume all programs take a natural number as input and produce a natural number as output. Every $P \in$ <u>Prog</u> defines a CE set (or a Turing-recognizable set) of pairs

$$[\![P]\!] = \{ (x,y) \mid P(x) = y \}$$

$P(x)=y$ means: P executed with input x terminates with output y. Two different programs may compute exactly the same set: $P_1 \neq P_2$ but $[\![P_1]\!] = [\![P_2]\!]$ may occur.

We say $P_1 \sim P_2$ if $[\![P_1]\!] = [\![P_2]\!]$: clearly an equivalence relation.

In terms of Turing machines viewed as acceptors we can say $M_1 \sim M_2$ if $L(M_1) = L(M_2)$.

We say $Q : \underline{Prog} \to \{T, F\}$ is a <u>property</u> of programs.

We say Q is an <u>extensional</u> <u>property</u> of programs if

$$P_1 \sim P_2 \Rightarrow Q(P_1) \Leftrightarrow Q(P_2)$$

for TM's $\quad M_1 \sim M_2 \Rightarrow Q(M_1) \Leftrightarrow Q(M_2)$

Examples (i) This program runs in $O(n^2)$: <u>NOT</u> extensional
(ii) This program sorts its input : extensional
(iii) This program is 100K lines of code : <u>NOT</u> extensional.

An extensional property is only sensitive to IO behaviour. It ignores the actual text or running time or any performance characteristics. In terms of CE sets we say it is a property of the CE set & not of the TM. In software engineering we call it a <u>functional</u> <u>specification</u>.

Two TRIVIAL PROPERTIES: $Q_F(P) = F$ for every P and $Q_T(P) = T$ for every P.

THM (RICE) Every non-trivial extensional property (i.e. a property of CE sets) is undecidable.

<u>PROOF</u> Let Q be a non-trivial property of CE sets i.e. $\exists P$ s.t. $Q(P) = T$ & $\exists P'$ s.t. $Q(P') = F$.

Assume $EMPTY = \{\langle M\rangle \mid L(M) = \emptyset\}$ does not satisfy Q i.e.
$\forall M \quad L(M) = \emptyset \implies \neg Q(M)$ [HARMLESS ASSUMPTION]

Let M_0 be such that $Q(M_0) = T$. Of course $L(M_0) \neq \emptyset$.

I will show $A_{TM} \leq_m L_Q = \{\langle M\rangle \mid Q(M) = T\}$.

$\longrightarrow$ INPUT: $\langle M, w\rangle$ HAVE GADGET to SOLVE (DECIDE) $x \in L_Q$?

CONSTRUCT M' from $\langle M, w\rangle$ as follows

M' on input x:
1. Simulate M on w
2. If M accepts w then simulate M_0 on x.

FEED $\langle M'\rangle$ to L_Q? GADGET.
If M accepts w $\quad L(M') = L(M_0)$
otherwise $\quad L(M') = \emptyset$
Since Q is an extensional property
$L(M') = L(M_0) \implies Q(M') = T; \quad L(M') = \emptyset \implies Q(M') = F$
So my L_Q gadget <u>decides</u> whether M accepts w.

$$A_{TM} \leq L_Q?$$

Thus L_Q must be undecidable. END OF PROOF