Answer Set Programming[1]

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COMP4418

[1] Slides designed by Christoph Schwering

Non-Monotonic Reasoning

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$$\forall x\,(\mathrm{Car}(x) \rightarrow \neg \mathrm{Entry}(x))$$
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Non-Monotonic Reasoning

$$\left.\begin{array}{l}\forall x\,(\mathrm{Car}(x) \to \neg\mathrm{Entry}(x)) \\ \forall x\,(\mathrm{Car}(x) \wedge \mathrm{Auth}(x) \to \mathrm{Entry}(x))\end{array}\right\} \models \mathrm{Car}(C) \wedge \mathrm{Auth}(C) \to \neg\mathrm{Entry}(C)$$

ASP at a Glance

- ASP $=$ Answer Set Programming
  - ASP $\neq$ Microsoft's Active Server Pages

- ASP belongs to logic programming
  - Like Prolog: $Head \leftarrow Body$ or $Head$:- $Body$.
  - Like Prolog: negation as failure
  - Unlike Prolog: $Head$ may be empty $\Rightarrow$ constraints

- Declarative programming
  - Unlike Prolog: no procedural control
  - Order has no impact on semantics

- ASP programs compute *models*
  - Unlike Prolog: not query-oriented, no resolution
  - Unlike Prolog: not Turing-complete
  - Tool for problems in NP and NP^{NP}

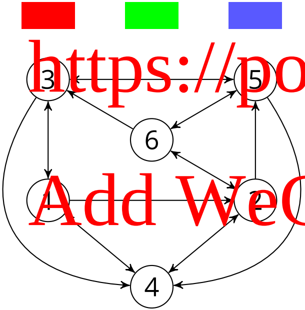
Motivation for ASP and this Lecture

- Very useful in practice!
  - Declarative problem solving
  - Very fast to write
  - Very fast to run
  - Few experts
- Interesting case study
  - Small, simple core language
  - Great expressivity by reduction to core language
- Knowing the theory is essential

Example: Graph Colouring

Definition: graph colouring problem

Input: graph with vertices V and edges $E \subseteq V \times V$, set of colors C.
Is there a mapping $m : V \to C$ with $m(x) \neq m(y)$ for all $(x, y) \in E$?



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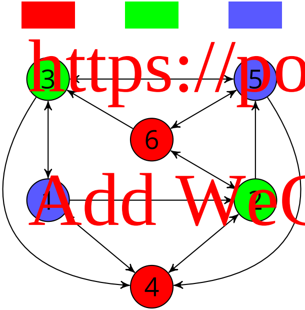

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- Graph Coulouring is NP-complete
  - NP: guess solution, verify in polynomial time
  - NP-complete: among hardest in NP

- Many applications:
  - Mapping (neighbouring countries to different colors)
  - Compilers (register allocation)
  - Scheduling (e.g., conflicting jobs to different time slots)
  - Allocation problems, Sudoku, ...

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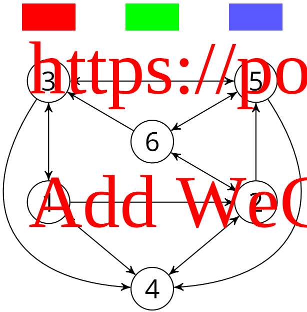

$c(r)\,.\; c(g)\,.\; c(b)\,.$
$v(1)\,.\; \ldots\; v(6)\,.$
$e(1,2)\,.\; e(1,3)\,.\; e(1,4)\,.$
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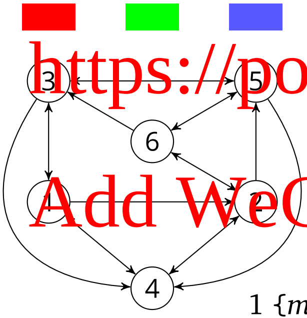

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$\text{:- } e(X,Y), m(X,C), m(Y,C)\,.$ verify $m(X) \neq m(Y)$

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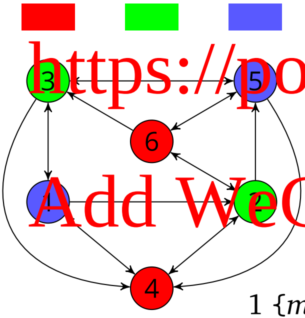

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Applications of ASP

- Automated product configuration
- Linux package manager
- Decision-support system for space shuttle
- Bioinformatics (diagnosis, inconsistency detection)
- General game playing

- Several implementations are available
- For this lecture: **Clingo** `www.potassco.org`

Overview of the Lecture

- **Semantics of ASP programs**

- Extensions of ASP programs

- Handling of variables in ASP

- ASP as modelling language

Prolog vs ASP

Consider the following logic program:

- $a.$ $a\,.$
$c \leftarrow a, b.$ $c\ \text{:-}\ a\,,\,b\,.$
$d \leftarrow a, \text{not}\, b.$ $d\ \text{:-}\ a\,,\,\text{not}\ b\,.$

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  - Models: $M_1 =$

a	b	c	d
1	0	0	1

$M_2 =$

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1	1	1	0

$\ldots$

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  - Models: $M_1 = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline 1 & 0 & 0 & 1 \\ \hline\end{array}$ $\quad M_2 = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline 1 & 1 & 1 & 0 \\ \hline\end{array}$ $\quad \ldots$
  - M_1 corresponds to Prolog, what is special about M_1?

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  - M_1 corresponds to Prolog, what is special about M_1?
  - M_1 is a **stable model** a.k.a. **answer set**:

    M_1 only satisfies *justified* propositions

  ASP gives **semantics** to **logic programming**

Intuition

The motivating guidelines behind stable model semantics are:

- A stable model satisfies all the rules of a logic program
- The reasoner shall not believe anything they are not forced to believe — the **rationality principle**

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Next: formalisation of this intuition

For now: only ground programs, i.e., no variables

Syntax

Definition: normal logic program (NLP)

A **normal logic program** P is a set of (normal) rules of the form

$$A \leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n.$$

where A, B_i, C_j are atomic propositions.
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For such a rule r, we define:

- $\text{Head}(r) = \{A\}$
- $\text{Body}(r) = \{B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n\}$

In code, r is written as A `:-` $B_1, \ldots, B_m,$ `not` $C_1, \ldots,$ `not` C_n `.`

Semantics: Interpretation

Definition: interpretation, satisfaction

An **interpretation** S is a set of atomic propositions.

S **satisfies** $A \leftarrow B_1, \ldots, B_m, \mathrm{not}\, C_1, \ldots, \mathrm{not}\, C_n$ iff
$A \in S$ or some $B_i \notin S$ or some $C_j \in S$.

In English:

- S satisfies rule iff S satisfies the head or falsifies the body
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Semantics without Negation

Definition: stable model for programs without negation

For P without negated literals:
S is a **stable model** of P iff
S is a minimal set (w.r.t. $\subseteq$) that satisfies all $r \in P$.

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Theorem: unique-model property

If P is negation-free (i.e., contains no $(\mathrm{not}\, C)$), then there is exactly one stable model, which can be computed in linear time.

Semantics without Negation – Examples

Compute stable model of a negation-free P by *unit propagation*:

- $S^0 = \{\}$
- $S^{i+1} = S^i \cup \bigcup_{r \in P : S \text{ satisfies } \mathrm{Body}(r)} \mathrm{Head}(r)$ until $S^{i+1} = S^i$

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Ex.: $P_3 = \{a \leftarrow b. \quad b \leftarrow a. \quad a.\}$
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Semantics with Negation

Definition: reduct

The **reduct** P^S of P relative to S is the least set such that

if $A \leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n \in P$ and $C_1, \ldots, C_n \notin S$

then $A \leftarrow B_1, \ldots, B_m \in P^S$.

In English: for each rule r from P,

- if $(\text{not}\, C) \in \text{Body}(r)$ for some $C \in S$: drop the rule
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- else: remove all negated literals and add to P^S

Ex.: $P = \{a. \quad c \leftarrow a, b. \quad d \leftarrow a, \text{not}\, b.\}$

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Semantics with Negation

Definition: reduct

The **reduct** P^S of P relative to S is the least set such that

if $A \leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n \in P$ and $C_1, \ldots, C_n \notin S$

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Definition: stable model for programs with negation

For P with negated literals:

S is a **stable model** of P iff S is a stable model of P^S.

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Ex.: $P = \{a \leftarrow \text{not}\, b. \quad b \leftarrow \text{not}\, a.\}$

Semantics with Negation – Examples

<ins>Ex.</ins>: $P = \{a \leftarrow \text{not}\, b. \quad b \leftarrow \text{not}\, a.\}$

$S_1 = \{\} \quad \rightarrow \quad P^{S_1} =$

Semantics with Negation – Examples

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No stable model!

Semantics: Overview

Definition: reduct

The **reduct** P^S of P relative to S is the least set such that

if $A \leftarrow B_1, \ldots, B_m, \text{not } C_1, \ldots, \text{not } C_n \in P$ and $C_1, \ldots, C_n \notin S$

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Definition: stable model

If P contains no (not C):

S is a **stable model** of P iff

S is a minimal set (w.r.t. $\subseteq$) that satisfies all $r \in P$.

If P contains (not C):

S is a **stable model** of P iff S is a stable model of P^S.

Theorem: necessary satisfaction condition

If S is a stable model and $A \in S$,

then S satisfies some $r \in P$ with $A \in \mathrm{Head}(r)$.

Semantics – Examples

Ex.: $P = \{a \leftarrow a. \quad b \leftarrow \text{not}\, a.\}$

S P^S Stable model?

Ex.: $P = \{a \leftarrow \text{not}\, b. \quad b \leftarrow \text{not}\, c.\}$

S P^S Stable model?

Example on paper

Overview of the Lecture

- Semantics of ASP programs
- **Extensions of ASP programs**

- Handling of variables in ASP
- ASP as modelling language

Integrity Constraints

Definition: integrity constraint

An **integrity constraint** is a rule r of the form

$$\leftarrow B_1, \ldots, B_m, \text{not } C_1, \ldots, \text{not } C_n$$

S **satisfies** r iff some $B_i \notin S$ or some $C_j \in S$.

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P^S contains $\leftarrow B_1, \ldots, B_m$ iff P contains r and $C_1, \ldots, C_n \notin S$.

Theorem: reduction to normal rules

Let P' be like P except that every integrity constraint

$$\leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n$$

is replaced with

$$dummy \leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n, \text{not}\, dummy$$

for some new atom $dummy$.
Then P and P' have the same stable models.

Choice Rules

Definition: choice rule

A **choice rule** is a rule the form

$$\{A_1, \ldots, A_k\} \leftarrow B_1, \ldots, B_m, \text{not } C_1, \ldots, \text{not } C_n$$

which allows any subset of $\{A_1, \ldots, A_k\}$ in a stable model.

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Theorem: reduction to normal rules

A choice rule can be encoded by $2k + 1$ normal rules using $2k + 1$ new atoms.

Further extensions:

- Conditional literals: $\{A : B\}$
  Ex.: $\{m(v, C) : c(C)\}$ expands to $\{m(v, r), m(v, g), m(v, b)\}$
- Cardinality constraints: $min\ \{A_1, \ldots, A_k\}\ max$
  Ex.: $1\ \{m(v, r), m(v, g), m(v, b)\}\ 1$

Negation in the Rule Head

Definition: rules with negated head

A rule with **negated head** is of the form

$$\text{not } A \leftarrow B_1, \ldots, B_m, \text{not } C_1, \ldots, \text{not } C_n$$

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Theorem: reduction to normal rules

Let P' be like P except that every rule with negated head

$$\text{not}\, A \leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n$$

is replaced with

$$\leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n, \text{not}\, \mathit{dummy}$$

and

$$\mathit{dummy} \leftarrow \text{not}\, A$$

for some new atom dummy.

Then P and P' have the same stable models (modulo dummy propositions).

Complexity

Theorem: complexity of NLPs without negations

Is S a stable model of a negation-free P? – **Linear time**

Does a negation-free P have a stable model? – **Constant** (yes, one)

Theorem: complexity of NLPs with negations

Is S a stable model of P? – **Linear time**

Does P have a stable model? – **NP-complete**

<u>Note</u>: integrity constraints, choice rules, negation in heads **preserve complexity** (program grows only polynomially)

Overview of the Lecture

- Semantics of ASP programs

- Extensions of ASP programs
- **Handling of variables in ASP**
- ASP as modelling language

Programs with Variables

- Atomic propositions may now contain variables, e.g., $p(X,Z) \leftarrow e(X,Y), p(Y,Z)$

Programs with Variables

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  - U contains all constants from P
  - U contains all $f(t_1, \ldots, t_k)$ from P if f is a k-ary function in P and U contains $t_1, \ldots, t_k$

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  - U contains all $f(t_1, \ldots, t_k)$ from P if f is a k-ary function in P and U contains $t_1, \ldots, t_k$
- ASP **grounds** variables with Herbrand universe
  - Unlike Prolog: instantiation instead of unification
  - Caution: the ground program may grow exponentially
  - Caution: function symbols make grounding Turing-complete
  - If P is finite and mentions only constants, grounding is finite

Programs with Variables

- $f(X) \leftarrow b(X), \text{not}\, a(X).$
  $a(X) \leftarrow p(X).$
  $b(\text{sam}).$
  $b(\text{tweety}).$
  $p(\text{tweety}).$

- $f(\text{sam}) \leftarrow b(\text{sam}), \text{not}\, a(\text{sam}).$
  $f(\text{tweety}) \leftarrow b(\text{tweety}), \text{not}\, a(\text{tweety}).$
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Overview of the Lecture

- Semantics of ASP programs

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ASP Modelling

Typical ASP structure:

- Problem **instance**: a set of facts

```
c(r). c(g). c(b).
v(1). ... v(6).
e(1,2). e(1,3). e(1,4).
e(2,4). e(2,5). e(2,6).
e(3,1). e(3,4). e(3,5).
e(4,1). e(4,2).
e(5,3). e(5,4). e(5,6).
e(6,2). e(6,3). e(6,5).
```

- Problem **class**: a set of rules
  - Generator rules: often choice rules
  - Test rules: often integrity constraints

```
1 {m(X,C) : c(C)} 1 :- v(X).
:- e(X,Y), m(X,C), m(Y,C).
```


Ideal modeling is **uniform**: problem class encoding fits all instances

Semantically equivalent encodings may differ immensely in performance!

Example: Non-monotonic Reasoning

Tweety the penguin:

- (Normal) Birds fly.
- Penguins are abnormal.
- Tweety is a bird. So Tweety flies.
- Tweety is a penguin. So Tweety doesn't fly.

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$U = \{f(X) \leftarrow b(X), \text{not}\, a(X). \quad a(X) \leftarrow p(X). \quad b(t).\}$
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$S_1 = \{b(t), f(t)\} \quad \Rightarrow \quad P^{S_1} = \{f(t) \leftarrow b(t), \cancel{\text{not}\, a(t)}. \quad a(t) \leftarrow p(t). \quad b(t).\}$ ✓
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Example: Non-monotonic Reasoning

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Tweety flies!

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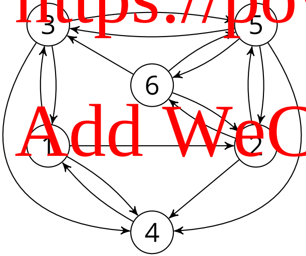
Tweety doesn't fly.

Example: Hamilton Cycle

Definition: Hamilton cycle problem

Input: graph with vertex set V and edges $E \subseteq V \times V$.
Is there a cycle that visits every vertex exactly once?



$$\{p(X,Y)\} \leftarrow e(X,Y).$$
$$r(X) \leftarrow p(1,X).$$
$$r(Y) \leftarrow r(X), p(X,Y).$$
$$\leftarrow 2\,\{p(X,Y)\}, v(X).$$
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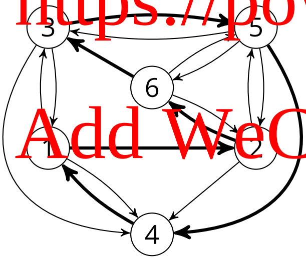

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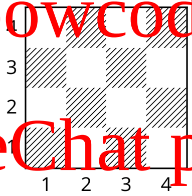
Example: N-Queens

Definition: N-queens problem

Place N queens on a $N \times N$ chessboard so that they do not attack each other, i.e., share no row, column, or diagonal.



Program on paper

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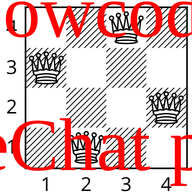

Program on paper