COMP90038
Algorithms and Complexity

Lecture 10: Decrease-and-Conquer-by-a-Factor

(with thanks to Harald Søndergaard)

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Decrease-and-Conquer

- **Last lecture:** to solve a problem of size n, try to express the solution in terms of a solution to the same problem of size n–1.

- A simple example was sorting: To sort an array of length n, just:

  1. sort the first n – 1 items, then

  2. locate the cell A[j] that should hold the last item, right-shift all elements to its right, then place the last element in A[j].

- This led to an $O(n^2)$ algorithm called **insertion sort**. We can implement the idea either with recursion or iteration (we chose iteration).

Decrease-and-Conquer by-a-Factor

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- We now look at better utilization of the approach, often leading to methods with logarithmic time behaviour or better!

- **Decrease-by-a-constant-factor** is exemplified by binary search.

- **Decrease-by-a-variable-factor** is exemplified by interpolation search.

- Let us look at these and other instances.

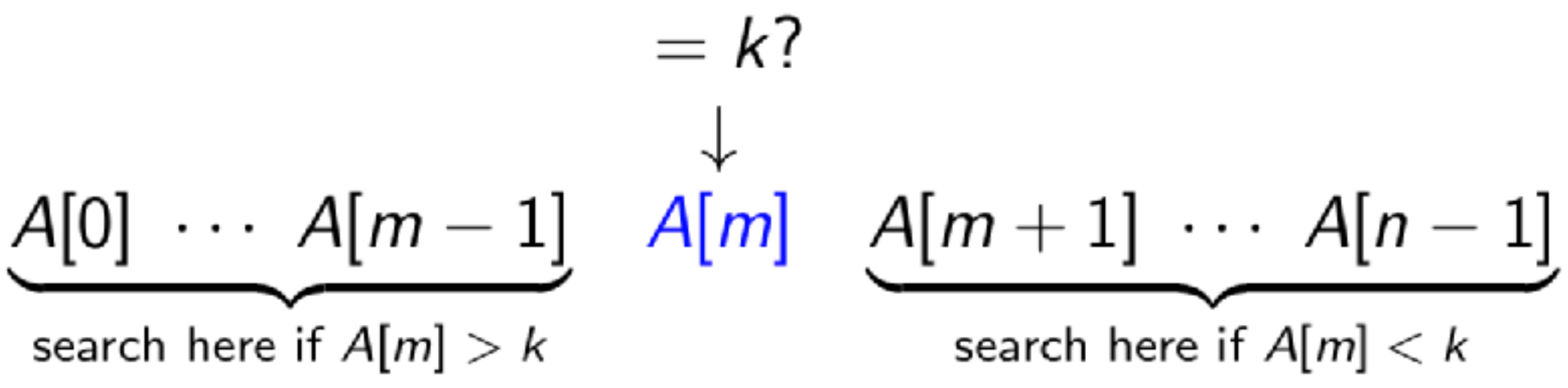
Binary Search

- This is a well-known approach for searching for an element *k* in a sorted array.

- Start by comparing against the array's middle element A[*m*]. If A[*m*] = *k* we are done.

- If A[m] > k, search the sub-array up to A[m – 1] recursively.

- If A[m] < k, search the sub-array from A[m + 1] recursively.



Binary Search

- We have already seen a recursive formulation in Lecture 4. Here is an iterative one.

```
function BinSearch(A[·], n, k)
    lo ← 0
    hi ← n − 1
    while lo ≤ hi do
        m ← ⌊(lo + hi)/2⌋
        if A[m] = k then
            return m
        if A[m] > k then
            hi ← m − 1
        else
            lo ← m + 1
    return −1
```

Example:
Binarv Search in Sorted Array

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```


k: 41

lo: 0

hi: 6

m: 3

A:

4	9	13	22	41	83	96
0	1	2	3	4	5	6

BinSearch(A,7,41)

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Complexity of Binary Search

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- Worst-case input to binary sarch:
  - When *k* is not in the array

- In that case, its complexity is given by the following recursive equation:

$$C(n) = \begin{cases} 1 & \text{if } n = 1 \\ C(\lfloor n/2 \rfloor) + 1 & \text{if } n > 1 \end{cases}$$

- A closed form is: $C(n) = \lfloor \log_2 n \rfloor + 1$

- In the worst case, searching for k in an array of size 1,000,000 requires 20 comparisons.

- The average-case time complexity is also Θ(log n)

Russian Peasant Multiplication

- A way of doing multiplication.

- For even *n*:

$$n \cdot m = \frac{n}{2} \cdot 2m$$

- For odd *n:*

$$n \cdot m = \frac{n-1}{2} \cdot 2m + m$$

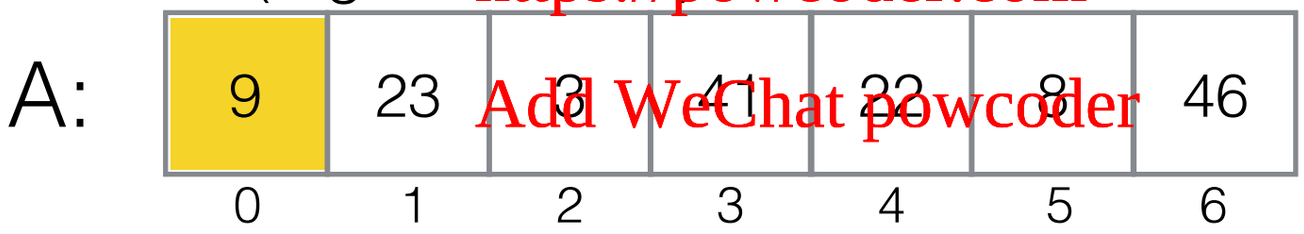
- Thus, ~halve *n* repeatedly, until n = 1. Add up all odd values of *m*

n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472
2	2944	
1	5888	5888
		= 7452

Finding the Median

- Given an array, an important problem is how to find the **median**, that is, an array value which is no larger than half the elements and no smaller than half.

- More generally, we would like to solve the problem of finding the kth smallest element. (e.g. when k=3)

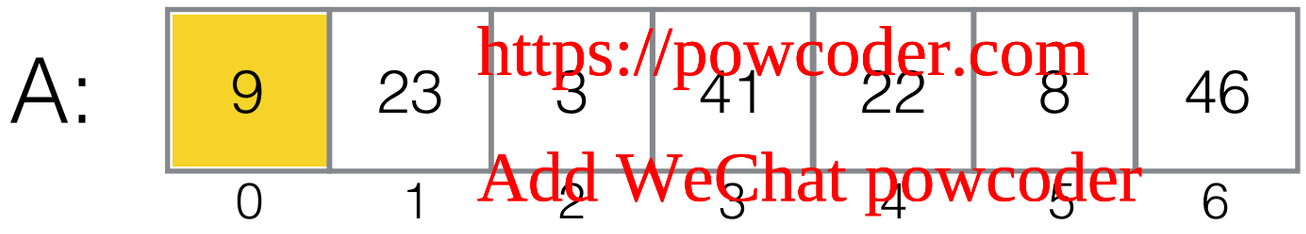


- If the array is sorted, the solution is straight-forward, so one approach is to start by sorting (as we'll soon see, this can be done in time O (n log n)).

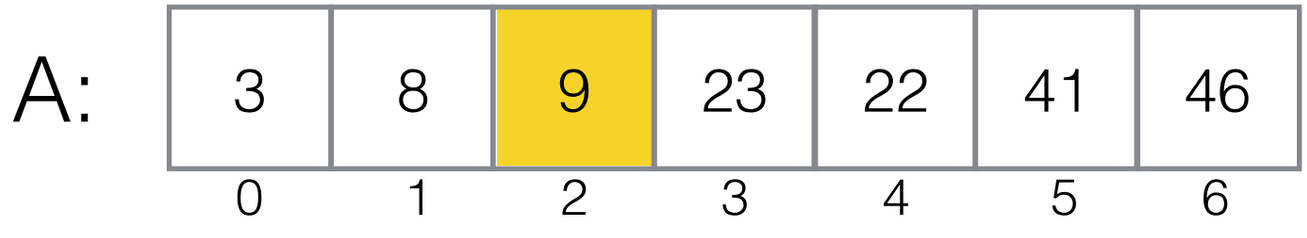
- However, sorting the array seems like overkill.

A Detour via Partitioning

- Partitioning an array around some pivot element p means reorganizing the array so that all elements to the left of p are no greater than p, while those to the right are no smaller.



Partitioning around the pivot 9



Lomuto Partitioning

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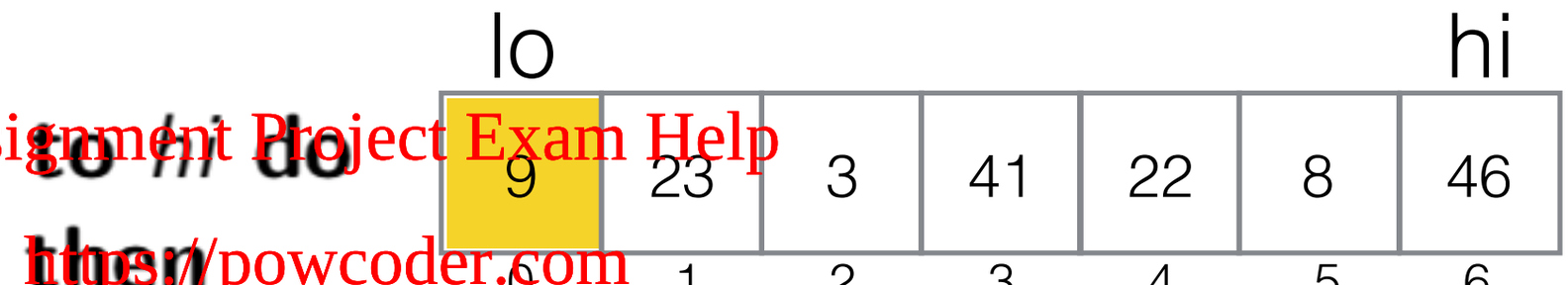

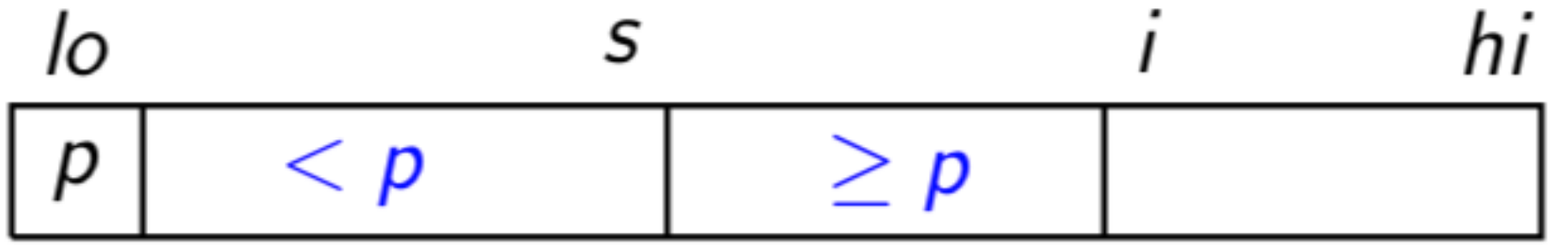

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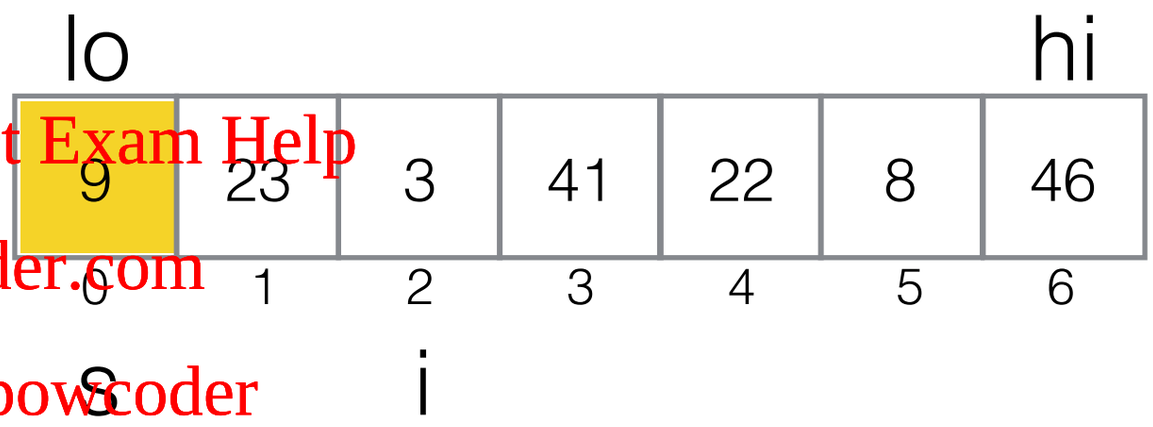

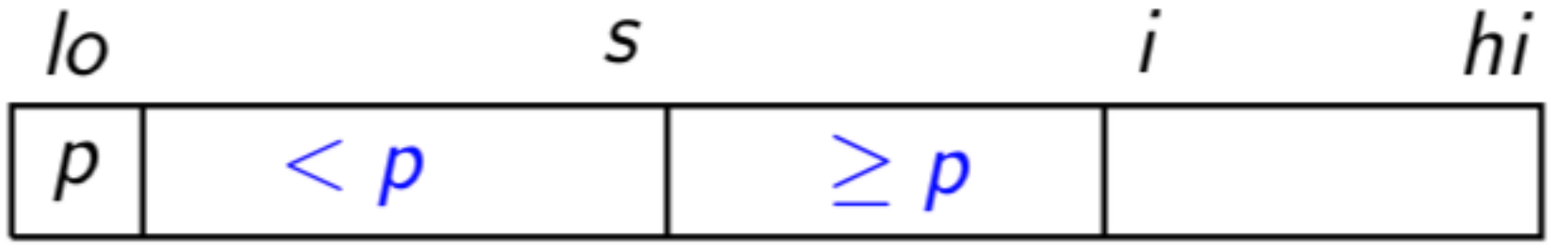

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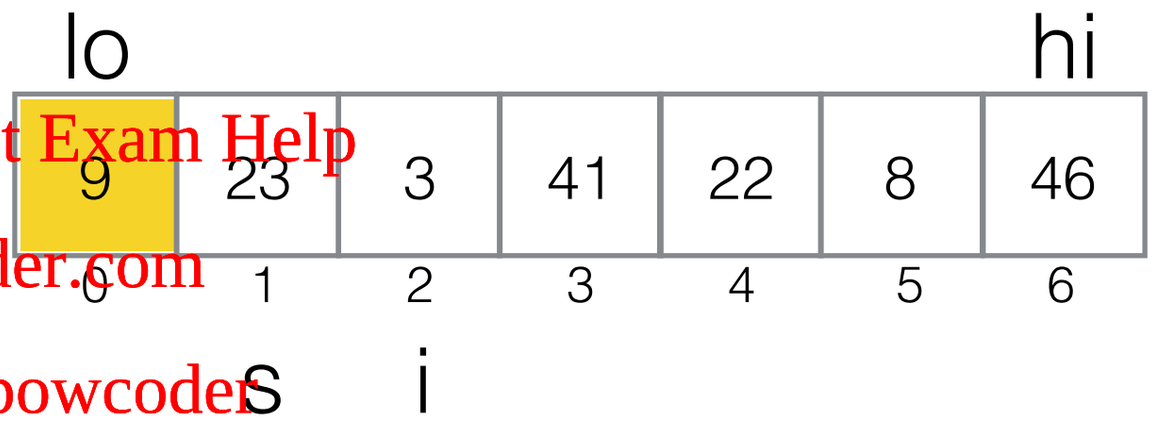

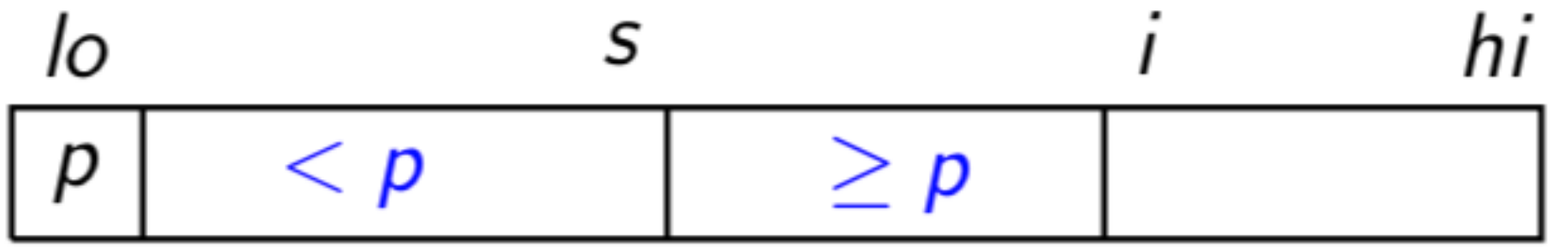

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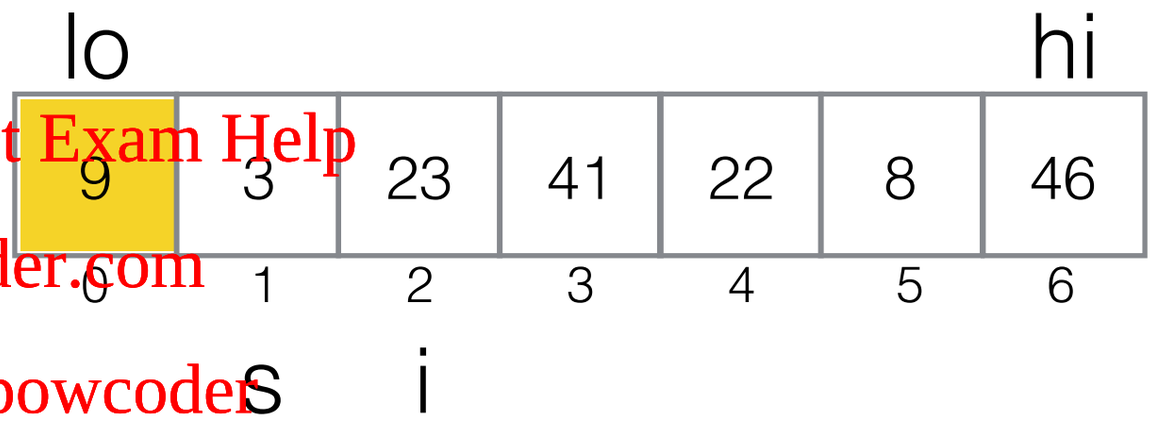

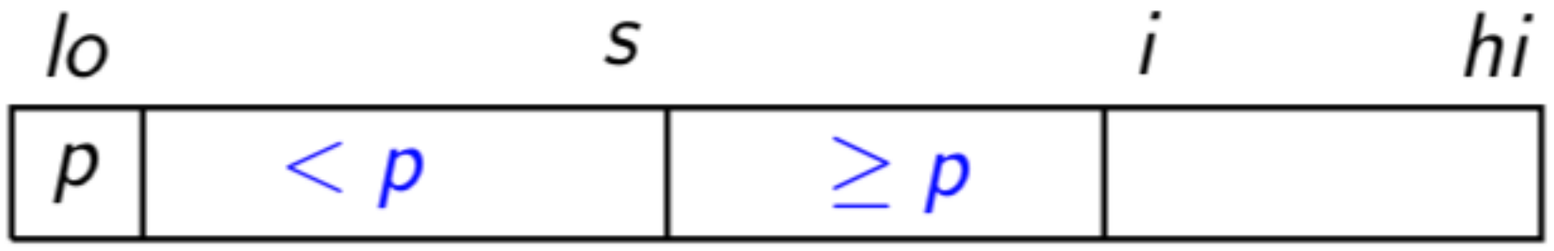

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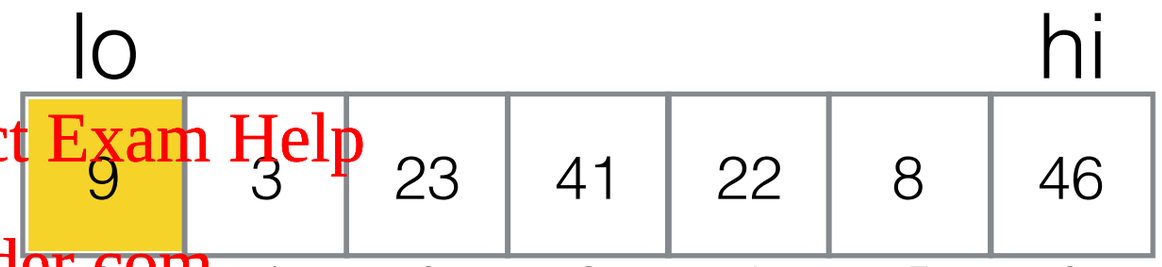

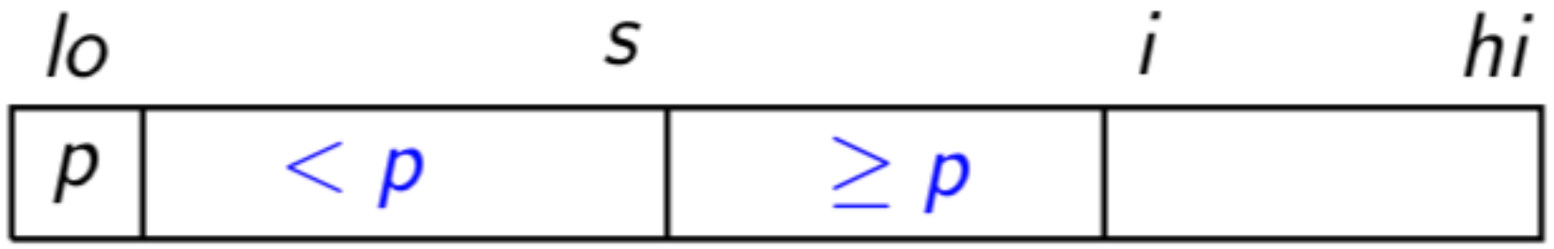

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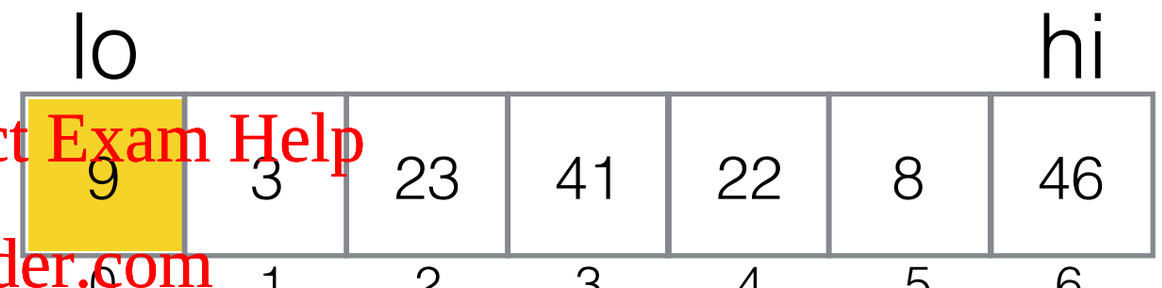

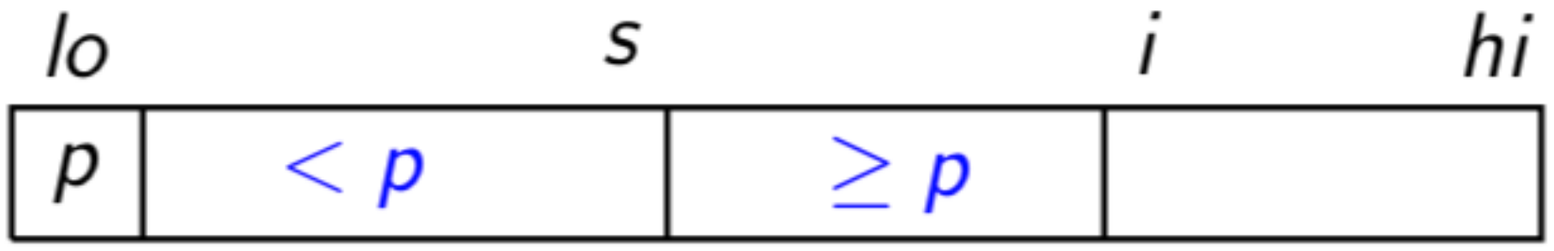

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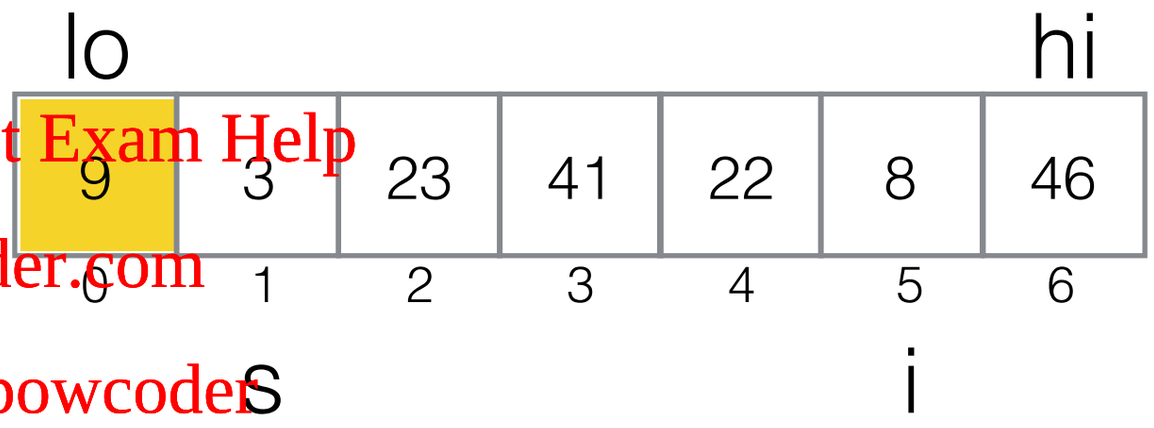

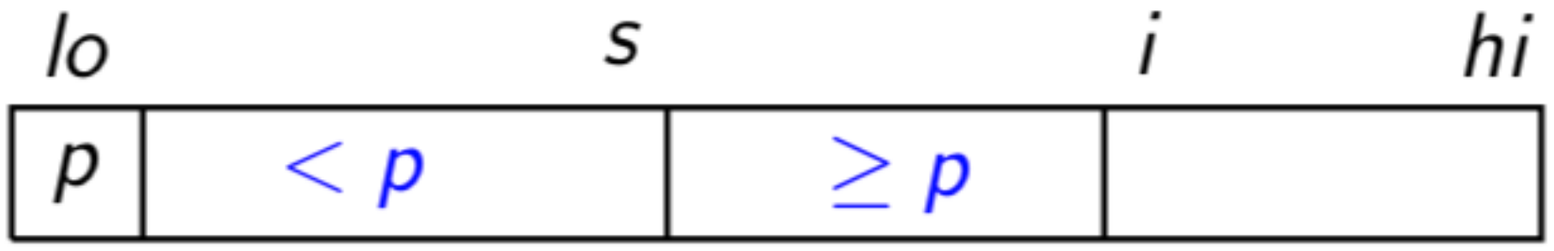

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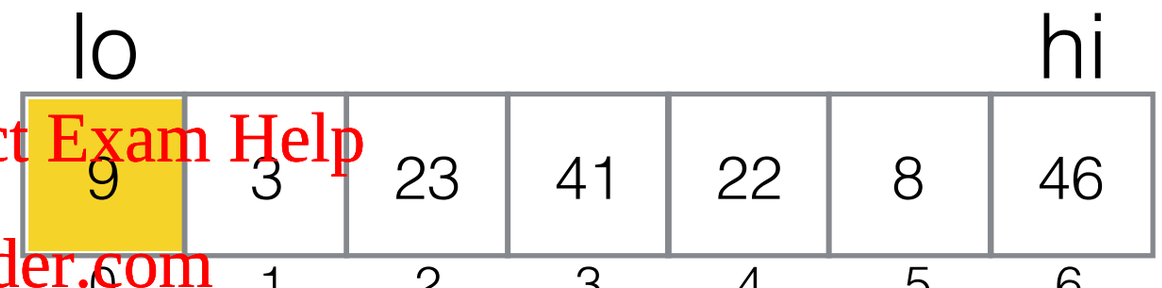

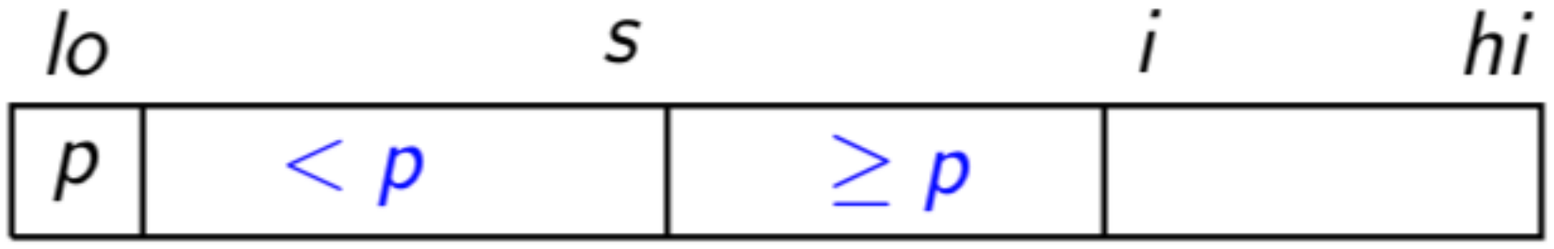

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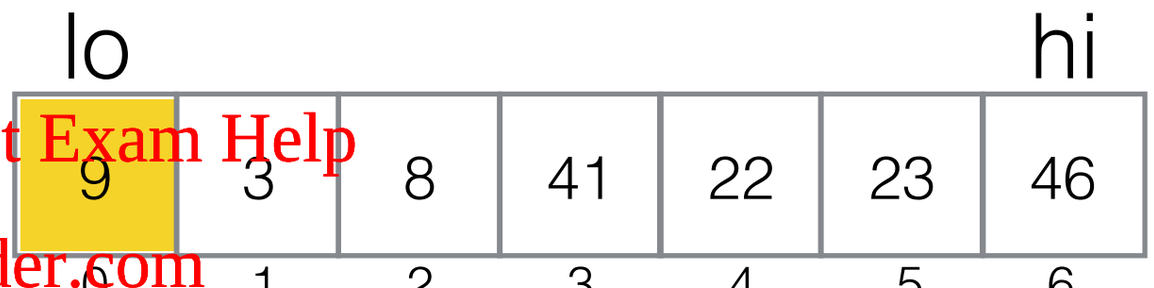

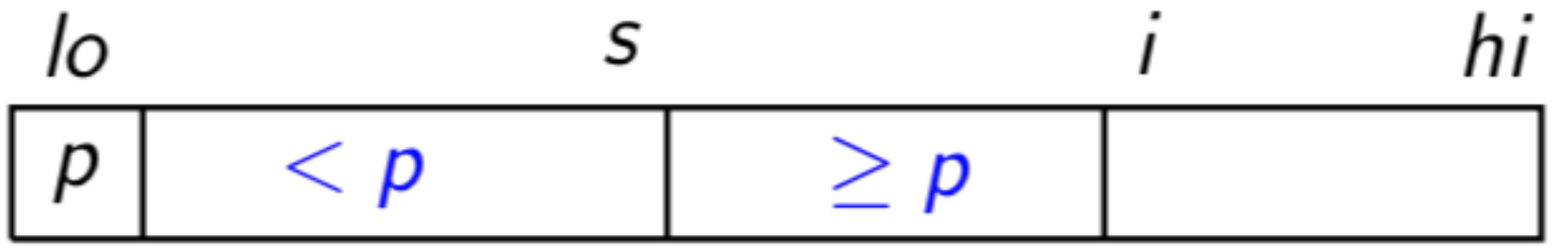

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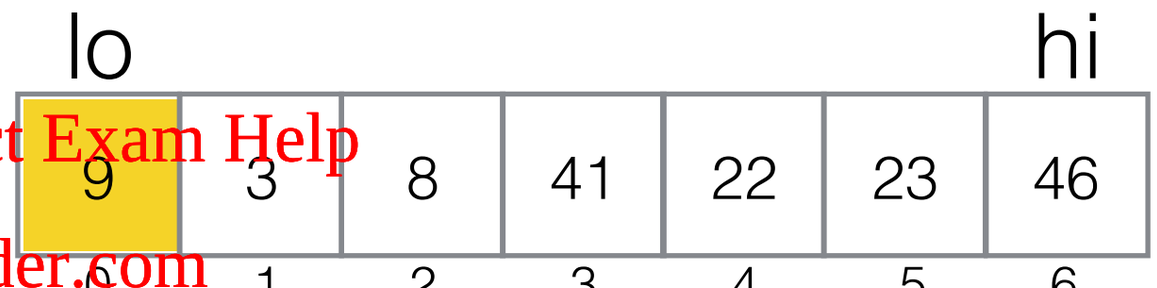

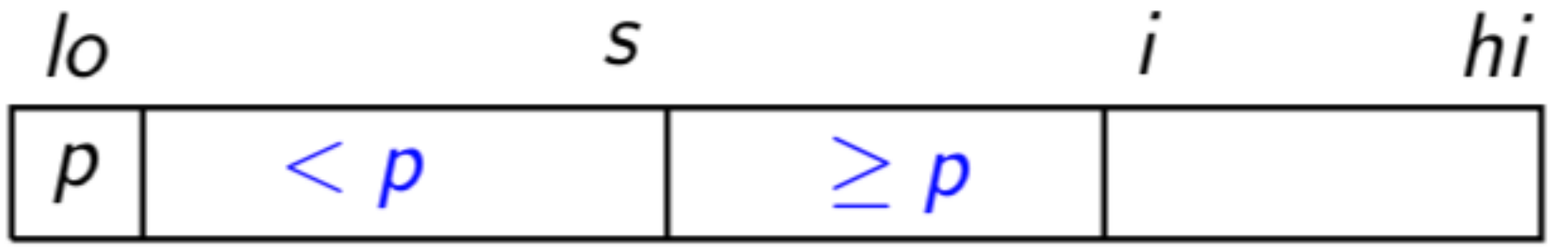

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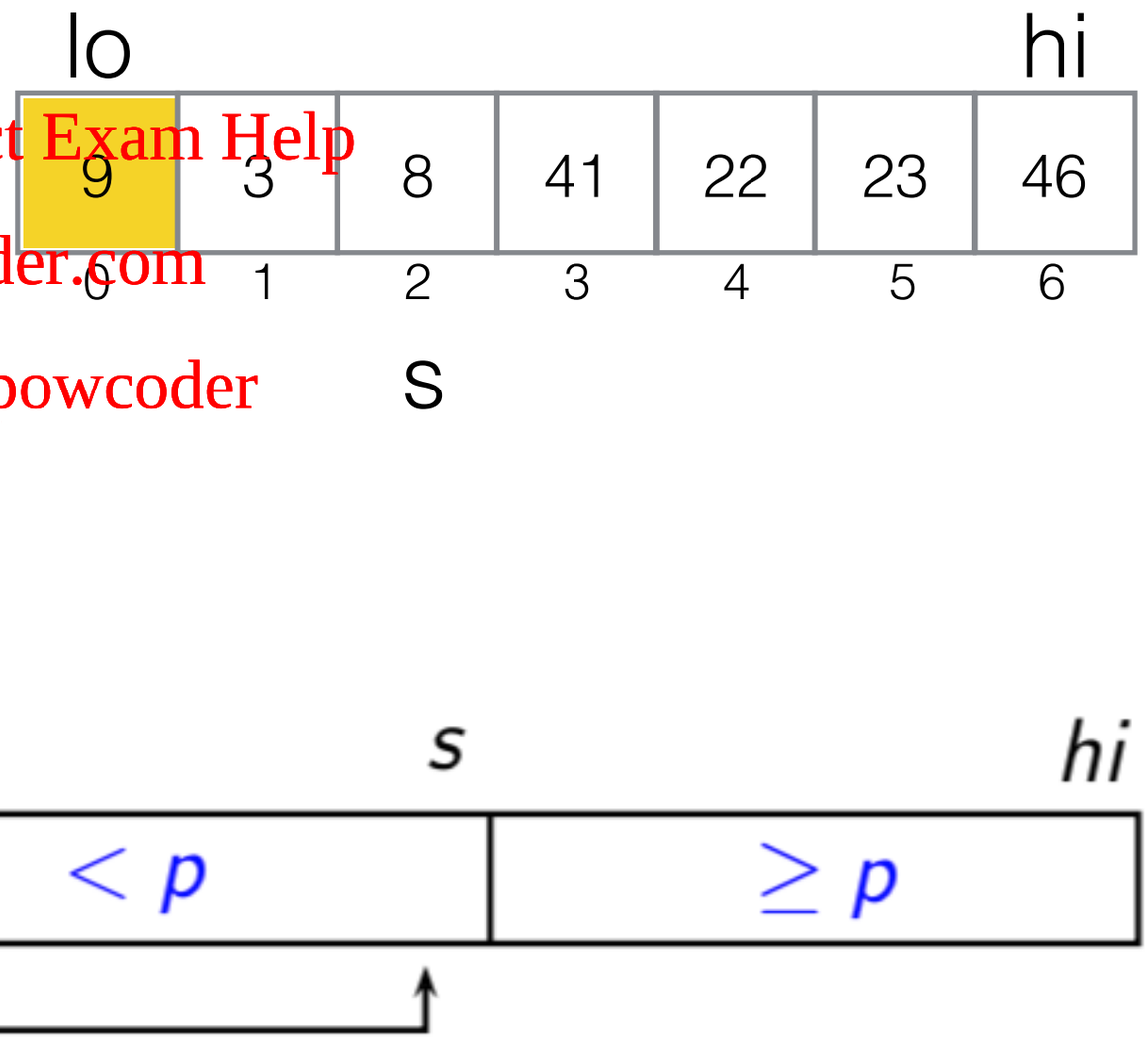

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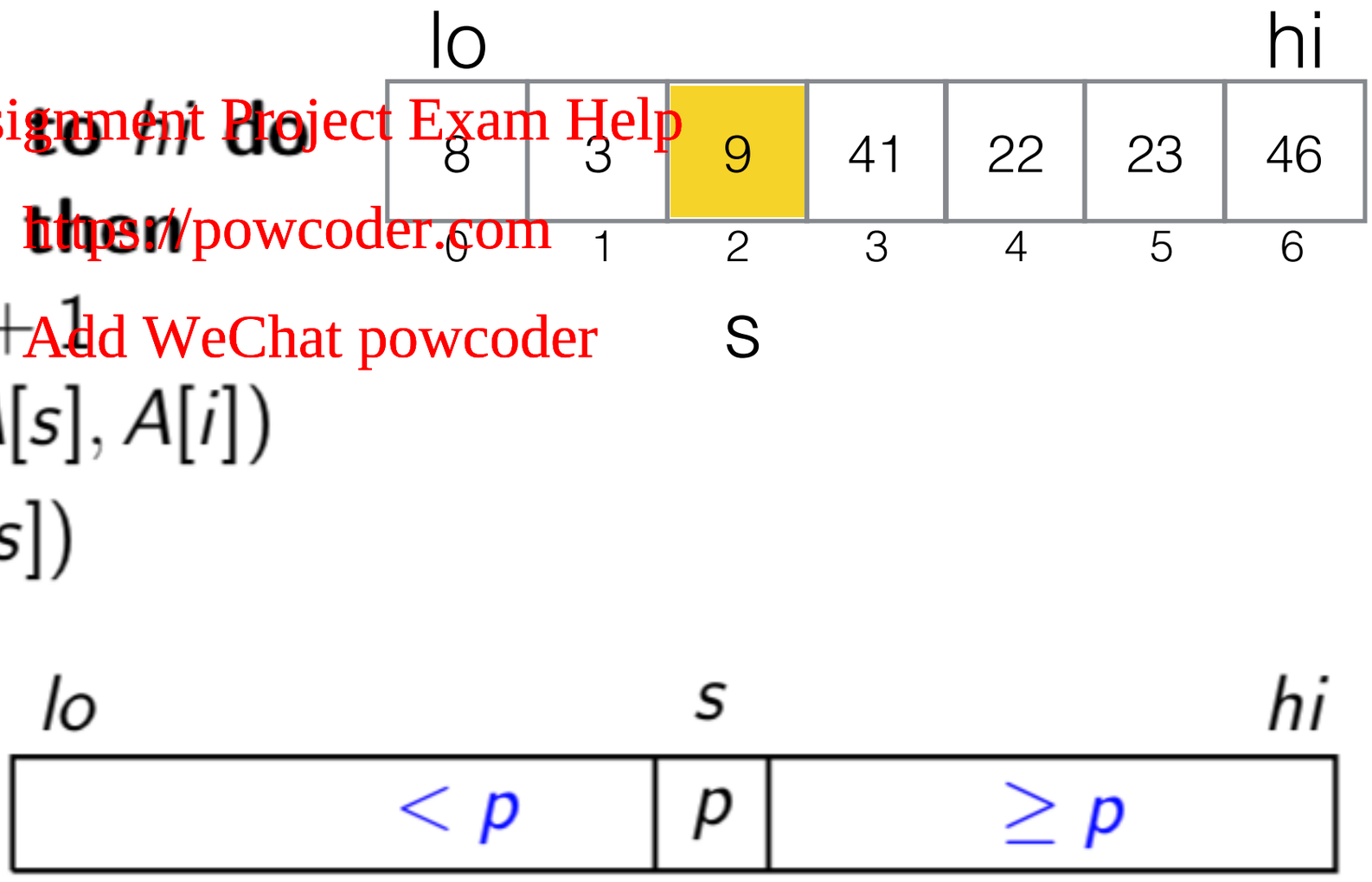

Finding the *k*th-smallest Element

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function QUICKSELECT($A[\cdot], lo, hi, k$)
    $s \leftarrow$ LOMUTOPARTITION(A, lo, hi)
    if $s - lo = k - 1$ **then**
        return $A[s]$
    else
    if $s - lo > k - 1$ **then**
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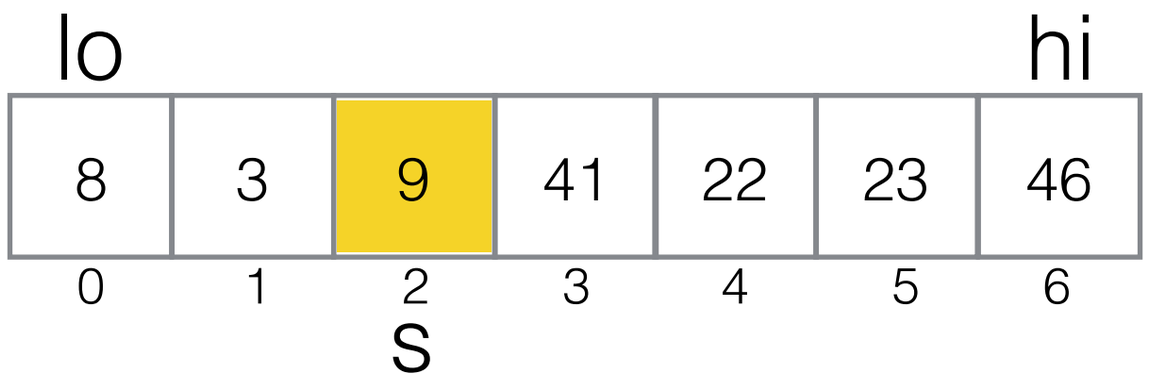

Example: Find the Sixth Smallest Element

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```
function QuickSelect(A[·], lo, hi, k)
    s ← LomutoPartition(A, lo, hi)
    if s − lo = k − 1 then
        return A[s]
    else
        if s − lo > k − 1 then
            QuickSelect(A, lo, s − 1, k)
    else
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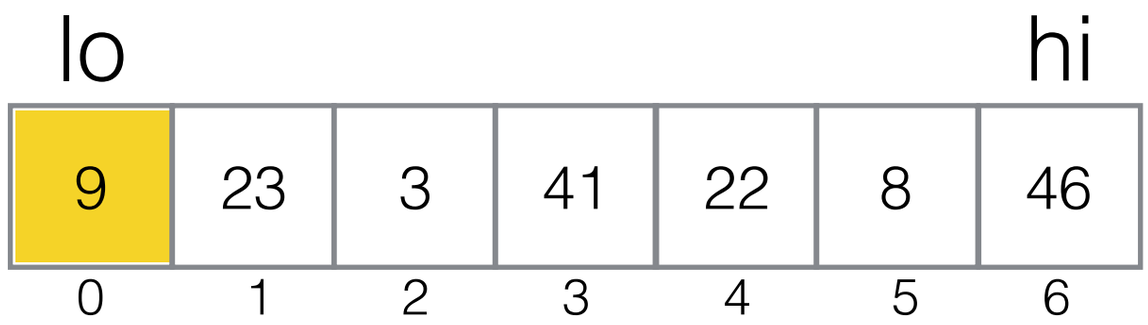
k: 6



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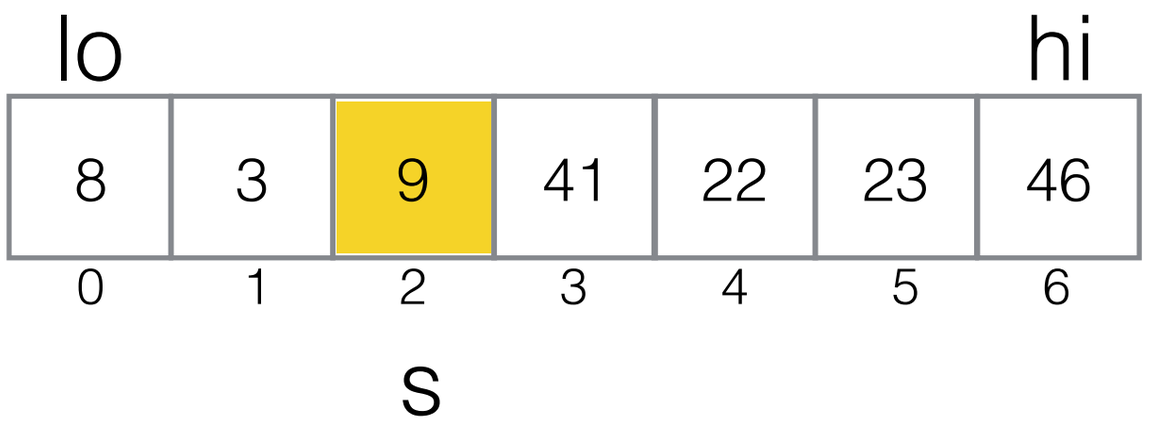

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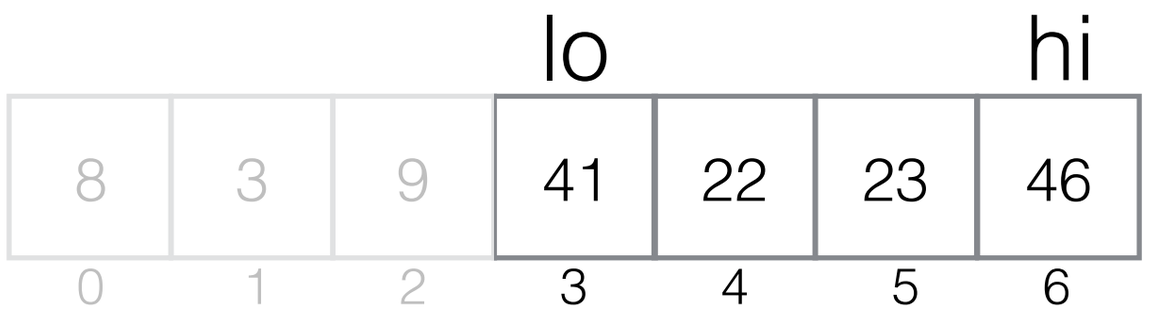

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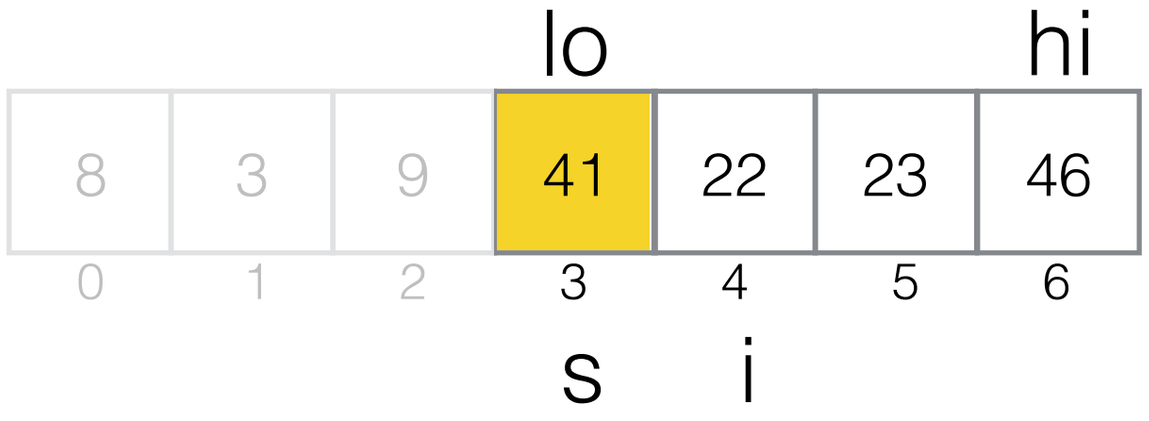

Example: Find the Fifth Smallest Element

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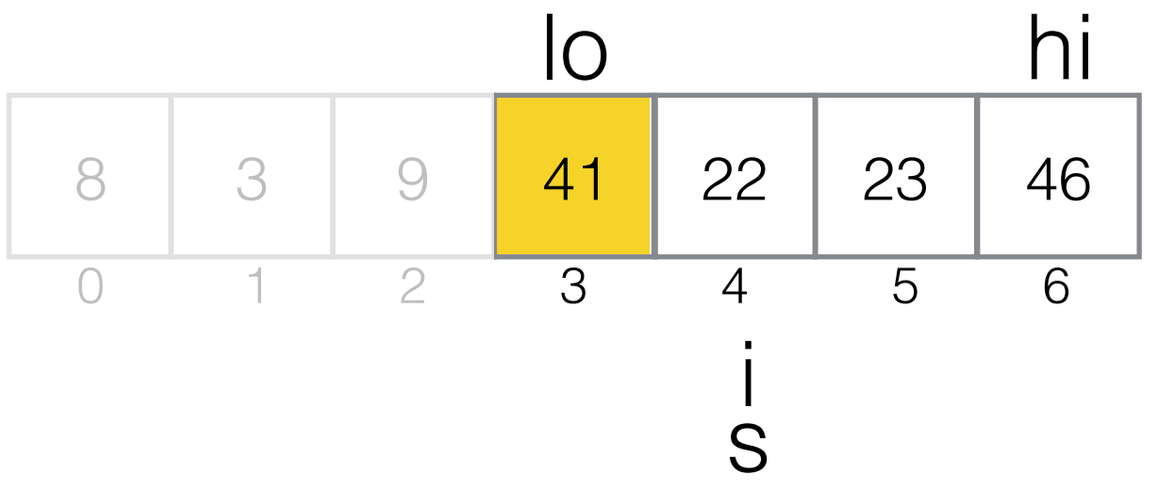

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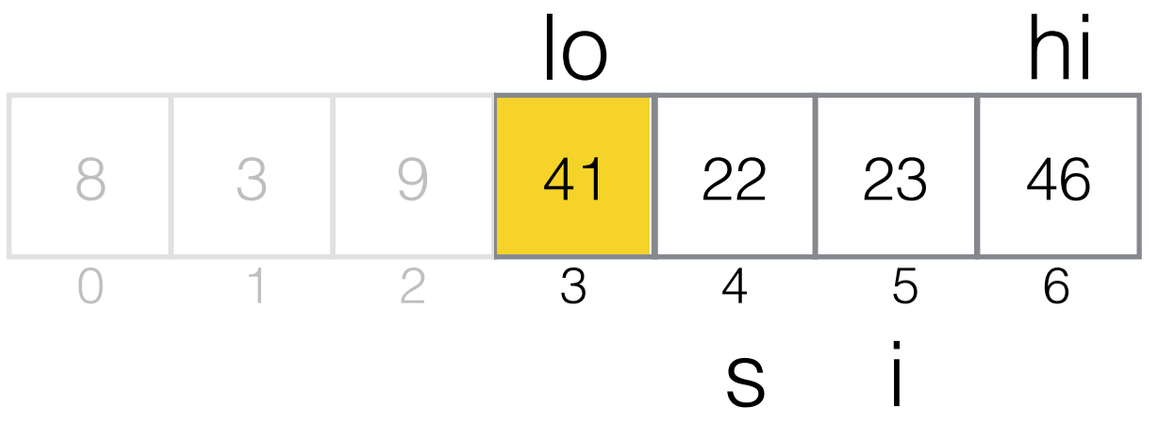

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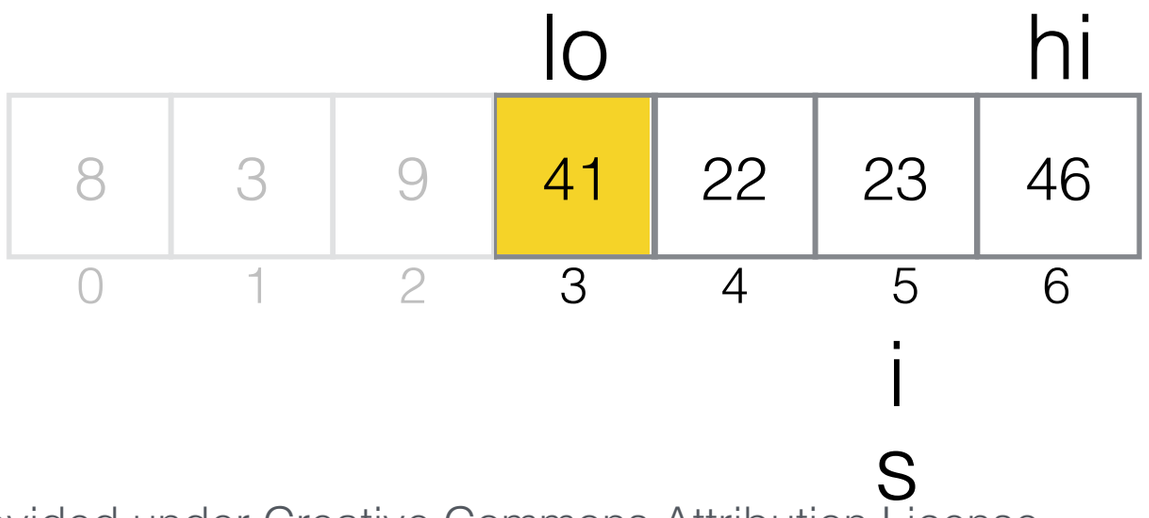

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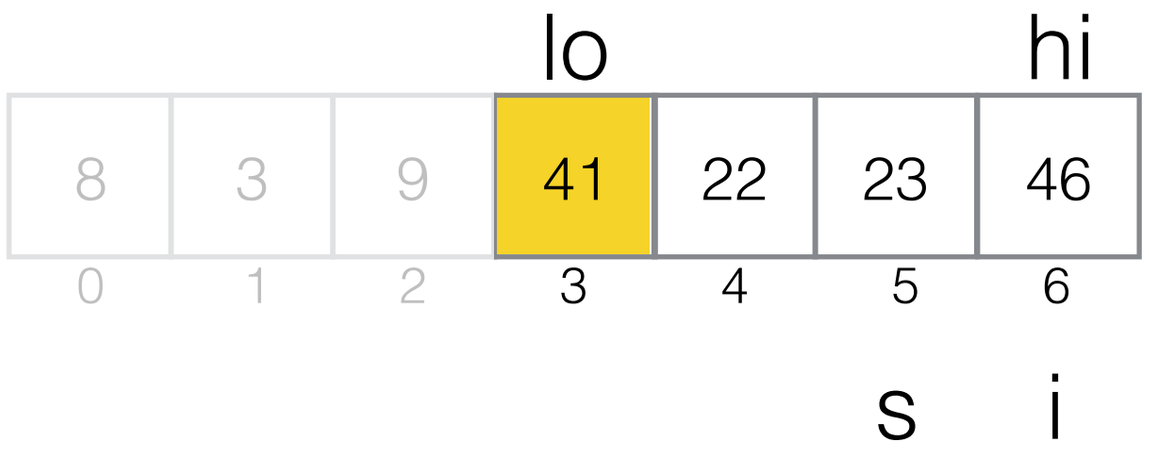

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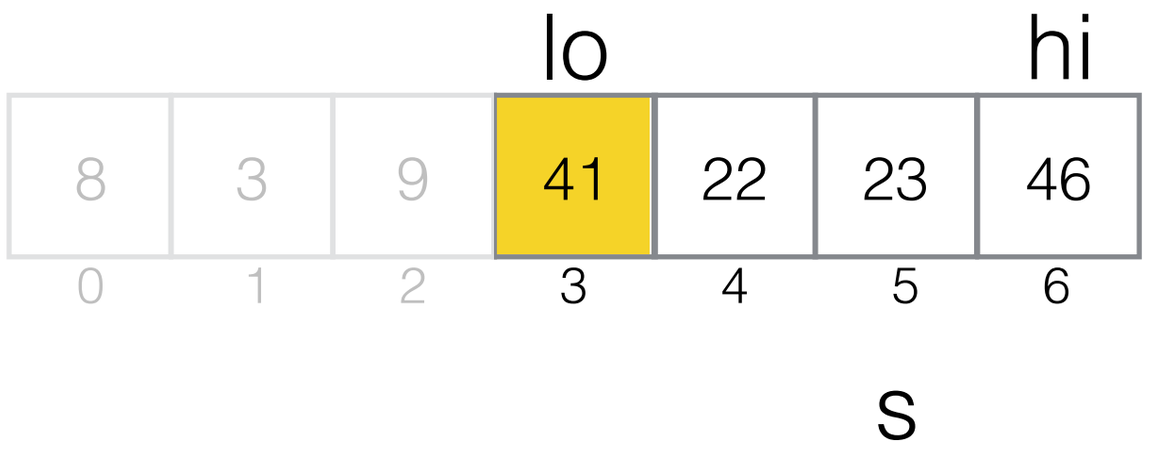

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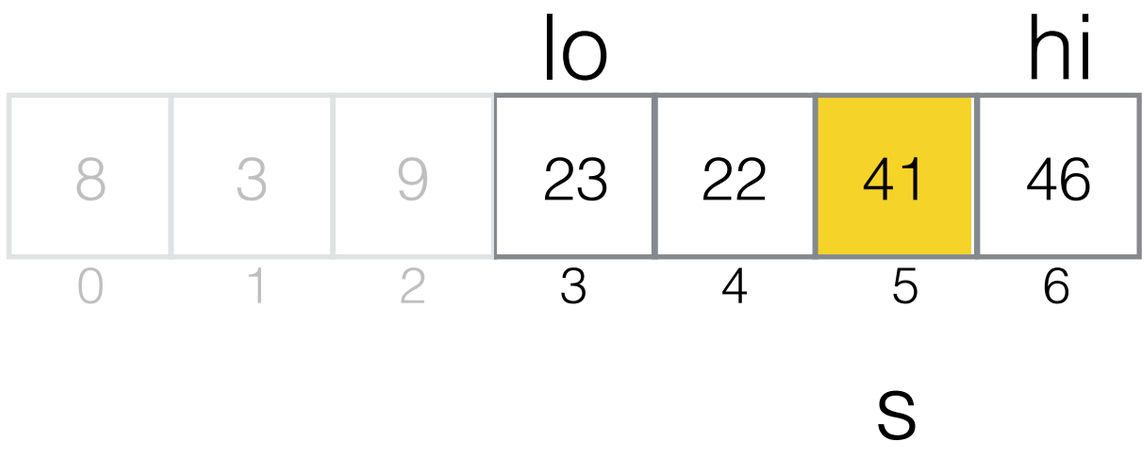

Example: Find the Fifth Smallest Element

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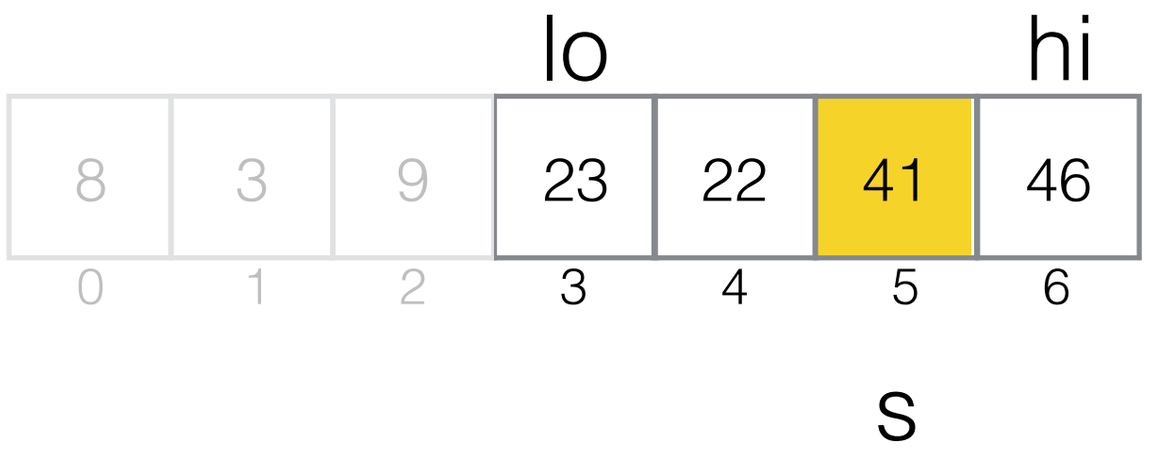
```
function QuickSelect(A[·], lo, hi, k)
    s ← LomutoPartition(A, lo, hi)
    if s − lo = k − 1 then
        return A[s]
    else
        if s − lo > k − 1 then
            QuickSelect(A, lo, s − 1, k)
        else
            QuickSelect(A, s + 1, hi, (k − 1) − (s − lo))
```

k: 3



Example: Find the Fifth Smallest Element

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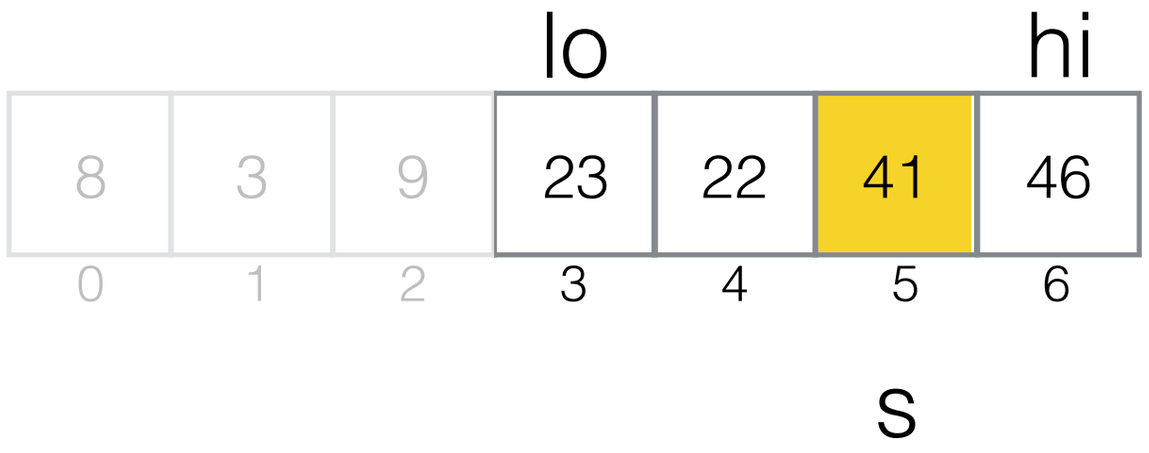
function QUICKSELECT($A[\cdot], lo, hi, k$)
  $s \leftarrow$ LOMUTOPARTITION(A, lo, hi)
  if $s - lo = k - 1$ **then**
    return $A[s]$
  else
    if $s - lo > k - 1$ **then**
      QUICKSELECT($A, lo, s - 1, k$)
    else
      QUICKSELECT($A, s + 1, hi, (k - 1) - (s - lo)$)

k: 3

returns 41!



QuickSelect Complexity

- **Worst case** complexity for QuickSelect is quadratic,

- **Average-case** complexity is linear.

Interpolation Search

- If the elements of a sorted array are distributed reasonably evenly, we can do better than binary search!

- Think about how you search for an entry in the telephone directory. If you look for 'Zobel', you make a rough estimate of where to do the first probe—very close to the end of the directory.

- This is the idea in interpolation search.

- When searching for k in the array segment A[lo] to A[hi], take into account where k is, relative to A[lo] and A[hi].

Interpolation Search

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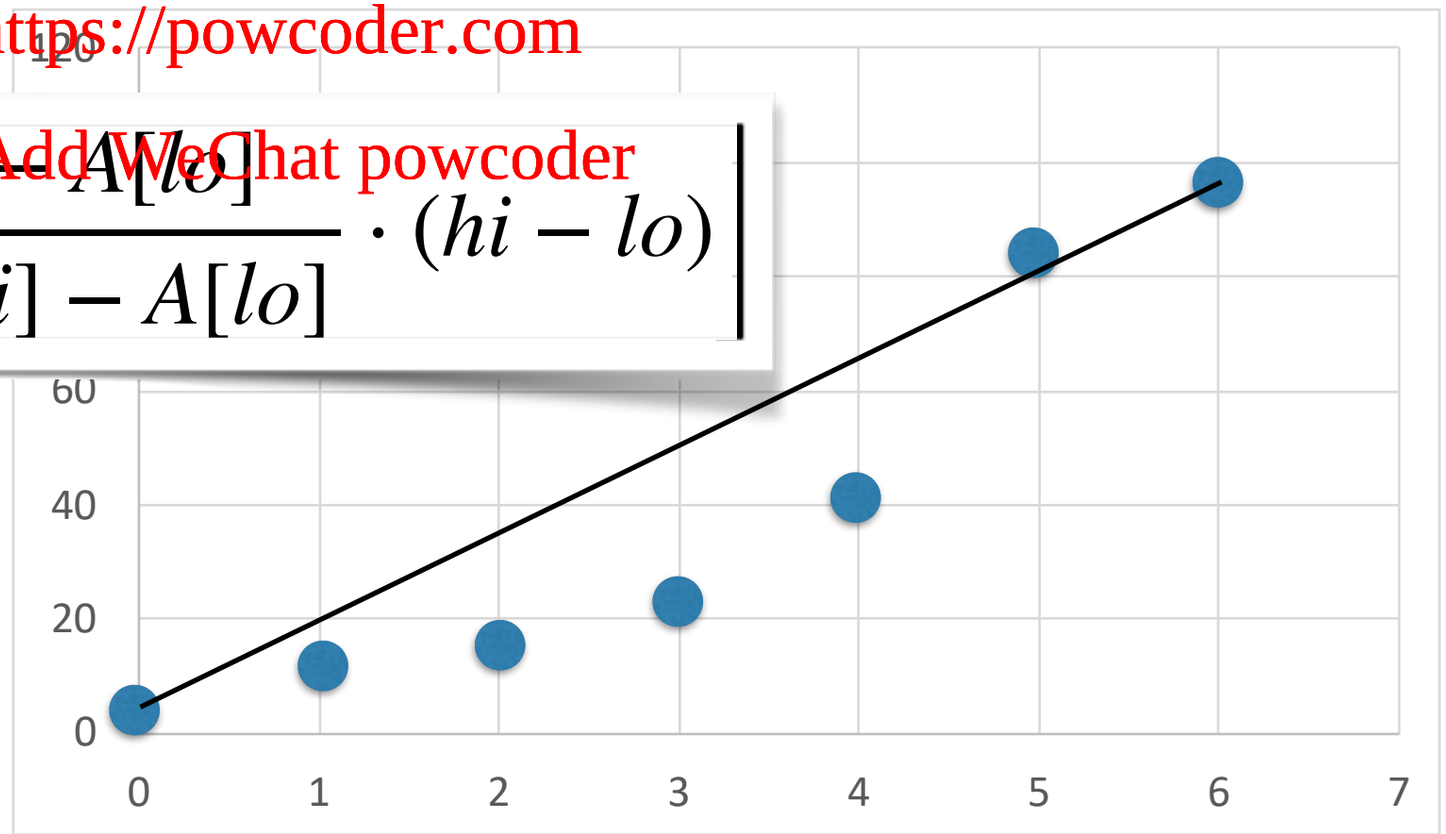
A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

Suppose we are searching for k = 83

$$mid = A[lo] + \left| \frac{k - A[lo]}{A[hi] - A[lo]} \cdot (hi - lo) \right|$$



Interpolation Search

- Instead of computing the mid-point *m* as in binary search:

$$m \leftarrow \lfloor (lo + hi)/2 \rfloor$$

we instead perform linear interpolation between the points (lo,A[lo]) and (hi,A[hi]). That is, we use:

$$m \leftarrow lo + \left\lfloor \frac{k - A[lo]}{A[hi] - A[lo]} (hi - lo) \right\rfloor$$

- Interpolation search has average complexity O(log log n)
- It is the right choice for large arrays when elements are **uniformly distributed**

Next Week

- Learn to divide and conquer!
- Read Levitin Chapter 5, but skip 5.4.