COMP90038
Algorithms and Complexity

Lecture 13: Priority Queues, Heaps and Heapsort

(with thanks to Harald Søndergaard)

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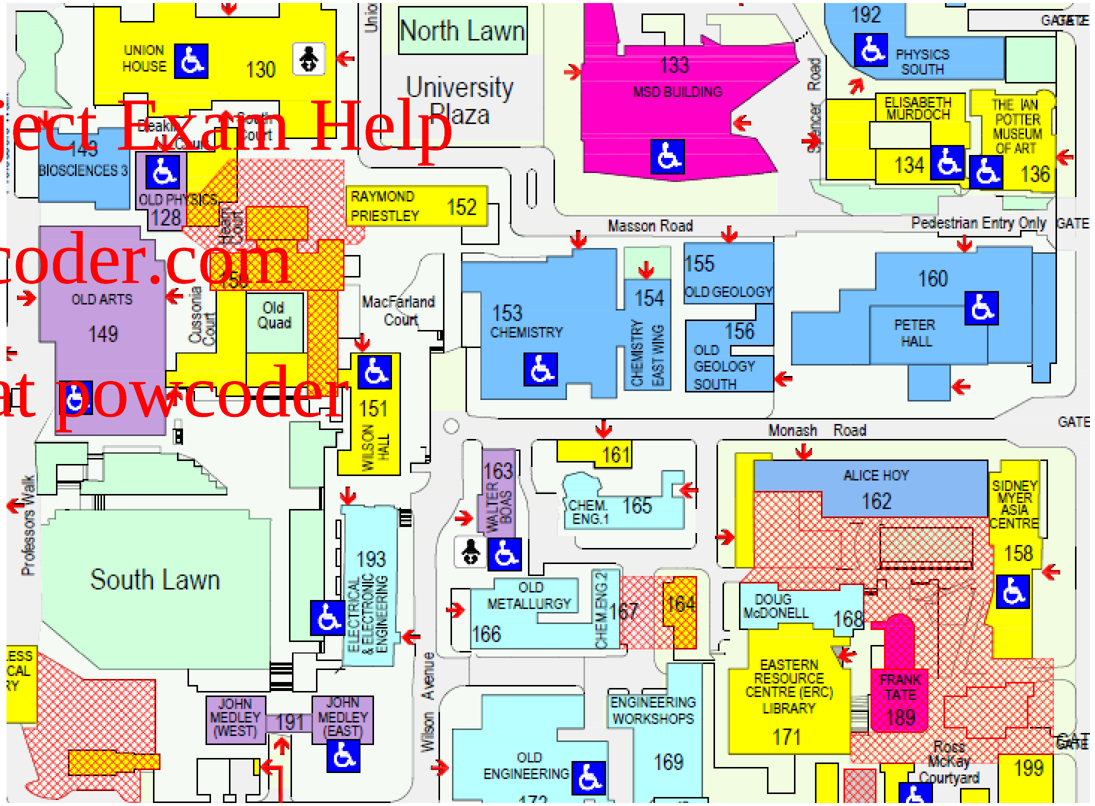
Peter Hall Building G.83

Where to find me?

- My office is at the Peter Hall building (Room G.83)



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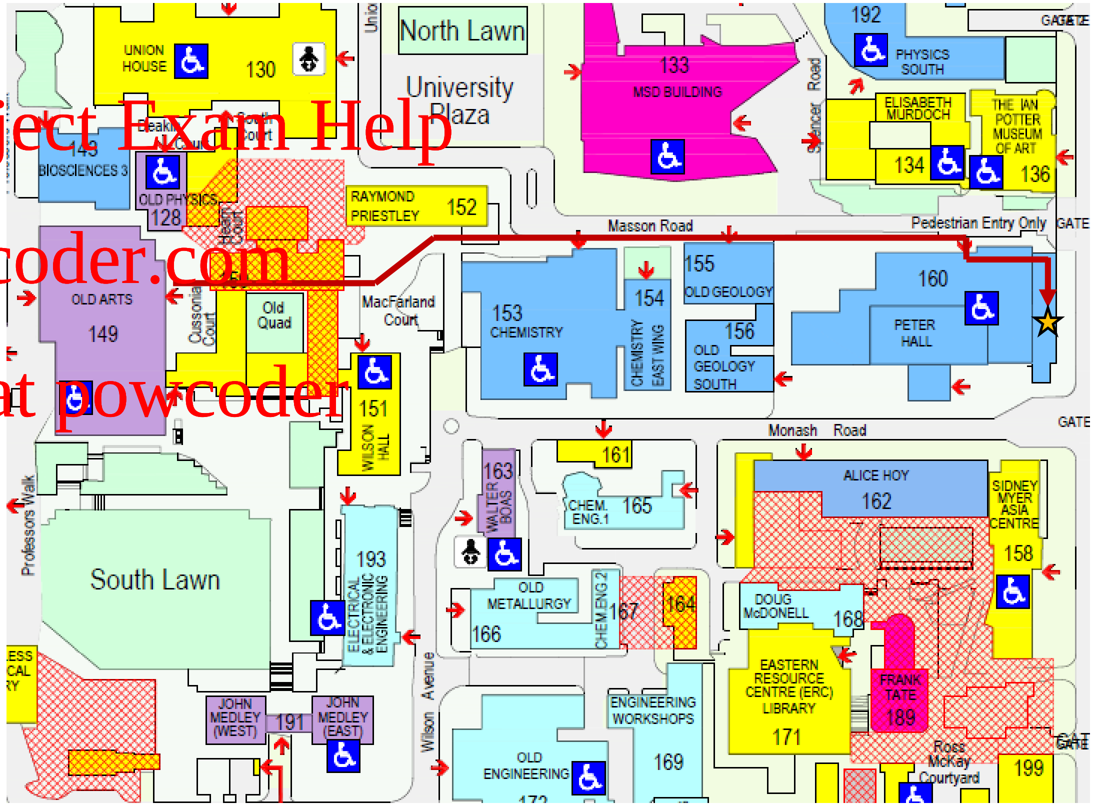
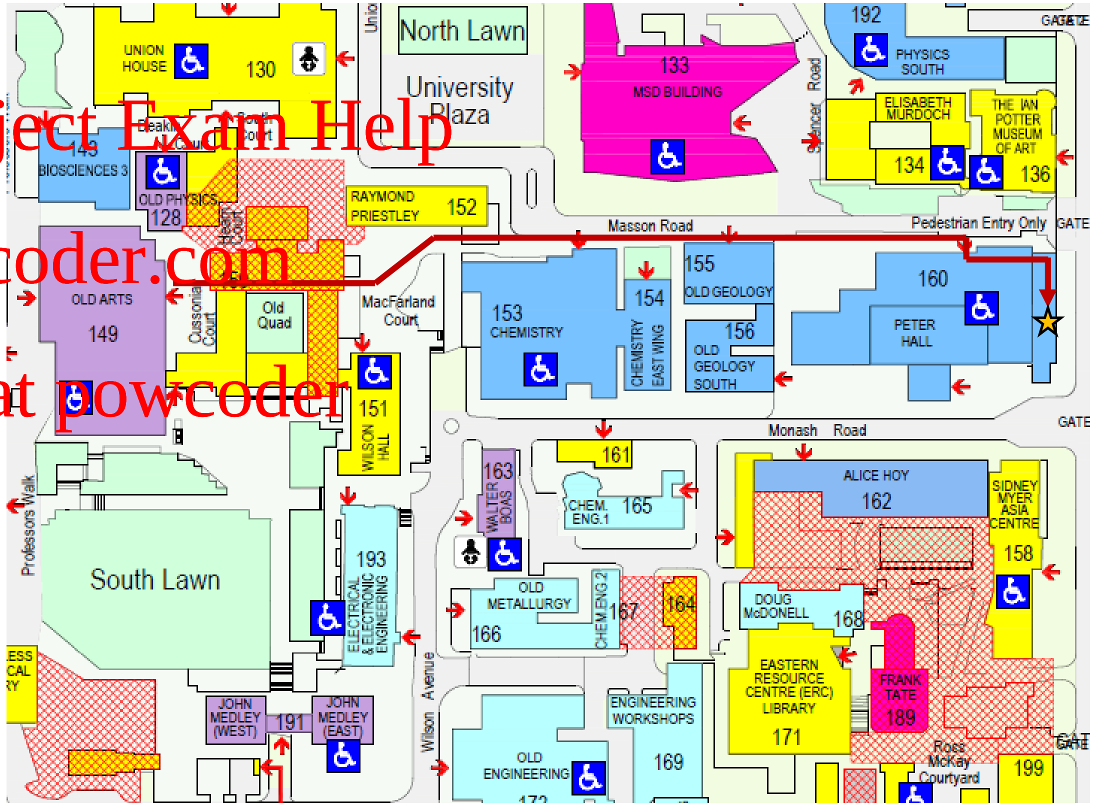


Where to find me?

- My office is at the Peter Hall building (Room G.83)
- Consultation hours:
  - Wednesdays 10:00am-11:00am
  - By appointment on Monday/Friday (limited slots)



Heaps and Priority Queues

- The **heap** is a very useful data structure for **priority queues**, used in many algorithms.

- A priority queue is a **set** (or **pool**) of elements.

- An element is injected into the priority queue together with a **priority** (often the key value itself) and elements are ejected according to priority.

- We think of the heap as a **partially ordered binary tree**.

- Since it can easily be maintained as a **complete** tree, the standard implementation uses an array to represent the tree.

The Priority Queue

- As an abstract data type, the priority queue supports the following operations on a “pool” of elements (ordered by some linear order):

  - **find** an item with maximal priority
  - **insert** a new item with associated priority
  - test whether a priority queue is empty
  - **eject** the **largest** element

- Other operations may be relevant, for example:

  - **replace** the maximal item with some new item
  - **construct** a priority queue from a list of items
  - **join** two priority queues

Some Uses of Priority Queues

- **Job scheduling** done by your operating system. The OS will usually have a notion of "importance" of different jobs.

- (Discrete event) **simulation** of complex systems (like traffic, or weather). Here priorities are typically event times.

- Numerical computations involving floating point numbers. Here priorities are measures of computational "error".

- Many sophisticated algorithms make essential use of priority queues (Huffman encoding and many shortest-path algorithms, for example).

Stacks and Queues as Priority Queues

- Special instances are obtained when we use **time** for priority:

  - If "large" means "late" we obtain the **stack**.
  - If "large" means "early" we obtain the **queue**.

Possible Implementations of the Priority Queue

- Assume priority = key.

	INJECT(e)	EJECT()
Unsorted array or list		
Sorted array or list		
Heap	$O(\log n)$	$O(\log n)$

- How is this accomplished?

The Heap

- A **heap** is a complete binary tree which satisfies the **heap condition**:

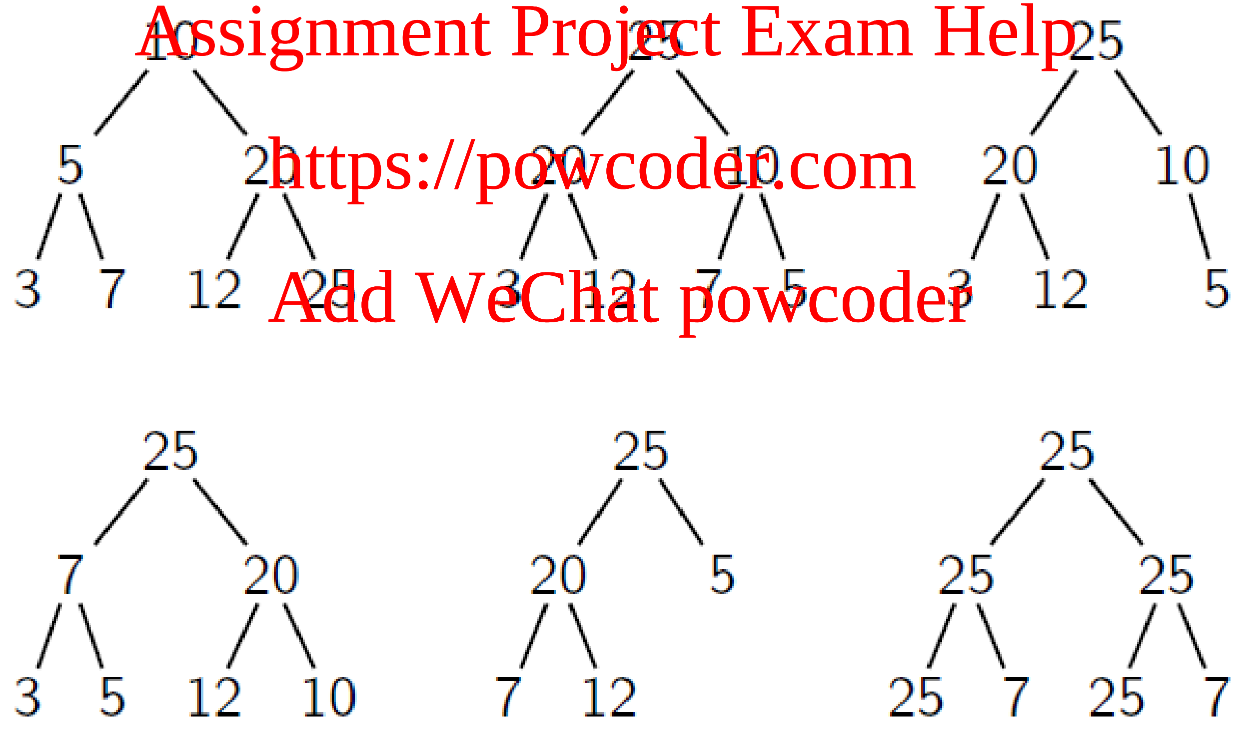
Each child has a priority (key) which is no greater than its parent's.

- This guarantees that the root of the tree is a maximal element.

- (Sometimes we talk about this as a **max-heap** – one can equally well have min-heaps, in which each child is no smaller than its parent.)

Heaps and Non-Heaps

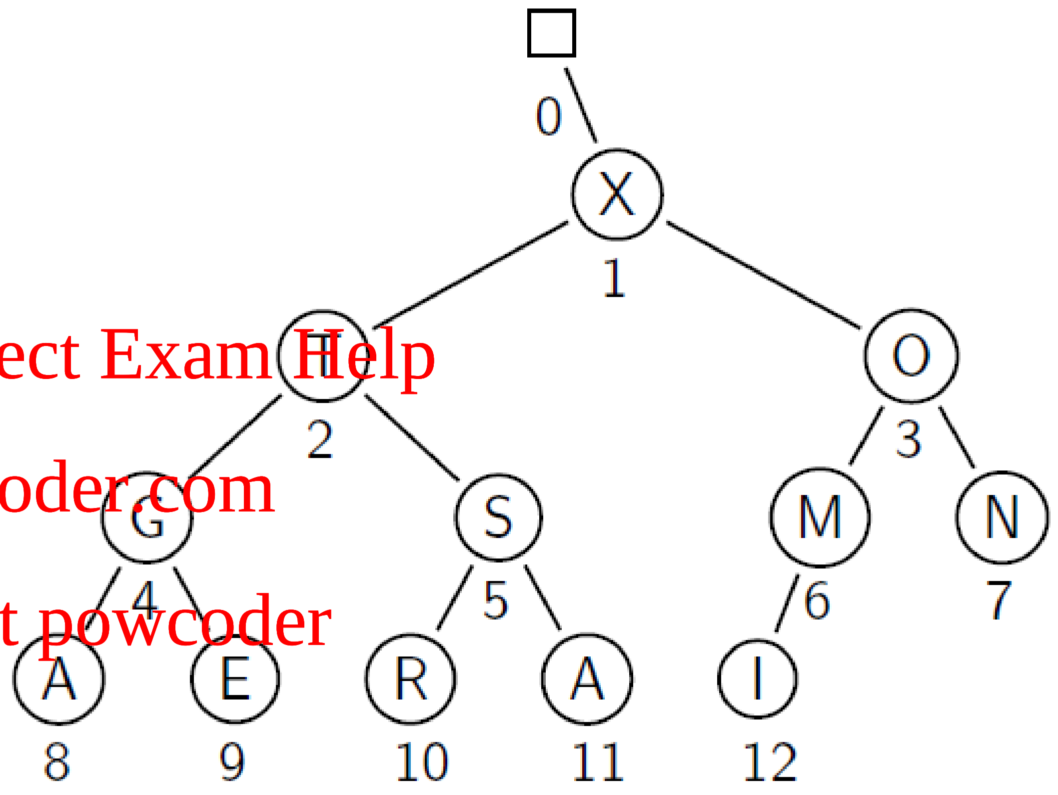
- Which of these are heaps?



Heaps as Arrays

- We can utilise the completeness of the tree and place its elements in level-order in an array *H*.

- Note that the children of node *i* will be nodes 2*i* and 2*i* + 1.

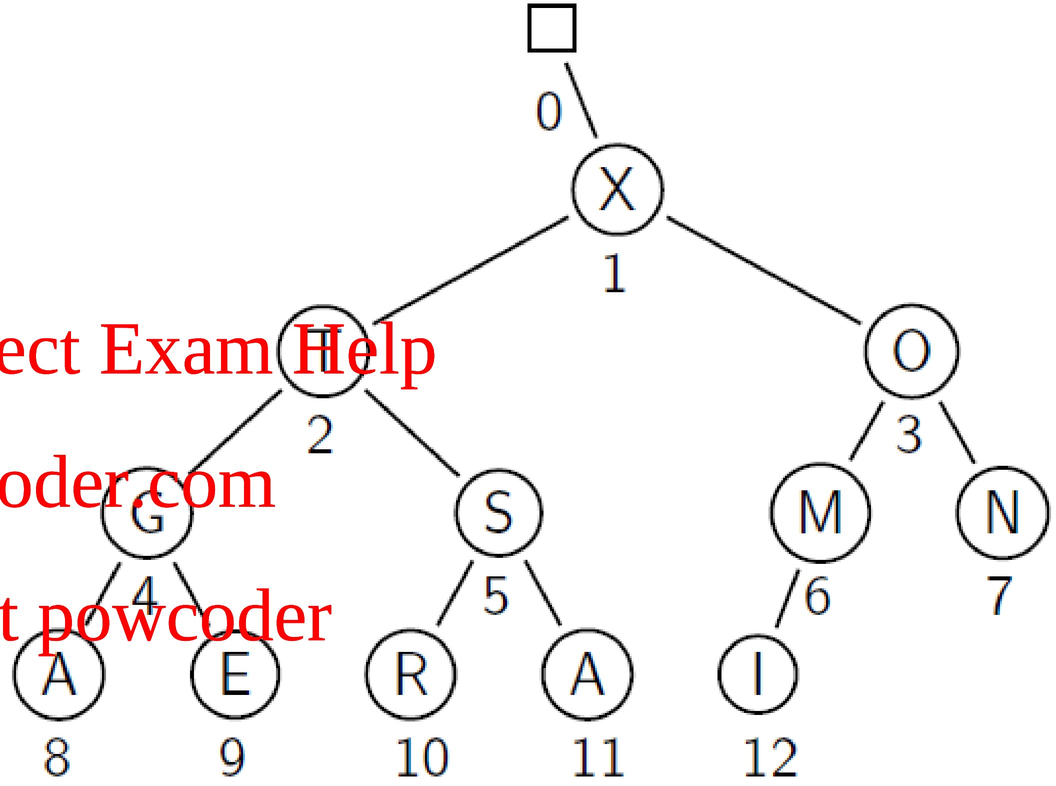


H:		X	T	O	G	S	M	N	A	E	R	A	I
	0	1	2	3	4	5	6	7	8	9	10	11	12

Heaps as Arrays

- This way, the heap condition is very simple:

- For all $i \subset \{0,1,\ldots,n\}$, we must have **H[i] ≤ H[i/2]**.



H:													
		X	T	O	G	S	M	N	A	E	R	A	I
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Properties of the Heap

- The root of the tree $H[1]$ holds a maximal item; the cost of EJECT is $O(1)$ plus time to restore the heap.

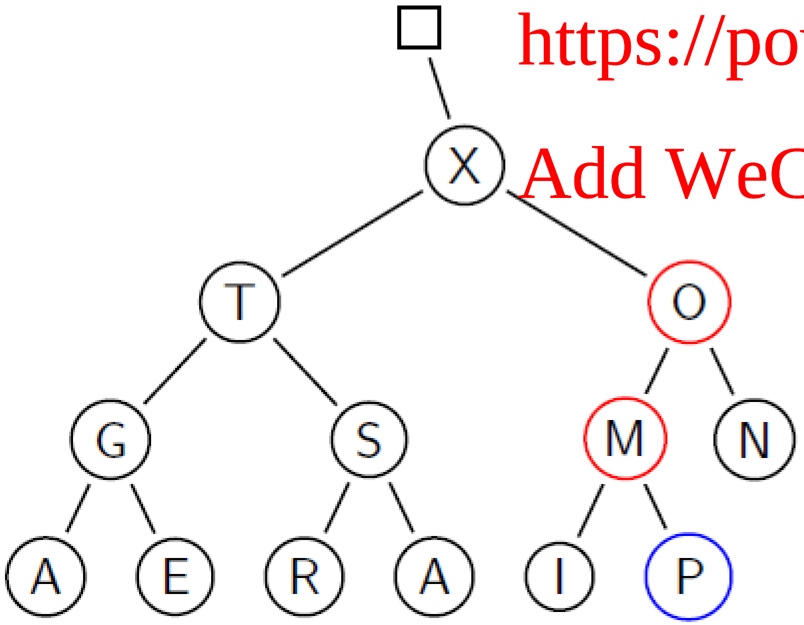
- The height of the heap is $\lfloor \log_2 n \rfloor$.

- Each subtree is also a heap.

- The children of node i are $2i$ and $2i+1$.
- The nodes which happen to be parents are in array positions 1 to $\lfloor n/2 \rfloor$.
- It is easier to understand the heap operations if we think of the heap as a tree.

Injecting a New Item

- Place the new item at the end; then let it "climb up", repeatedly swapping with parents that are smaller:



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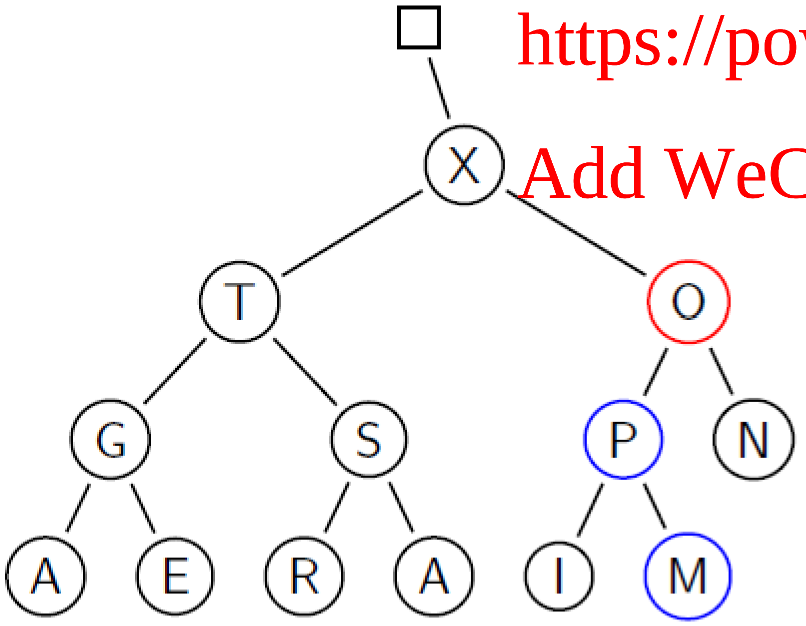

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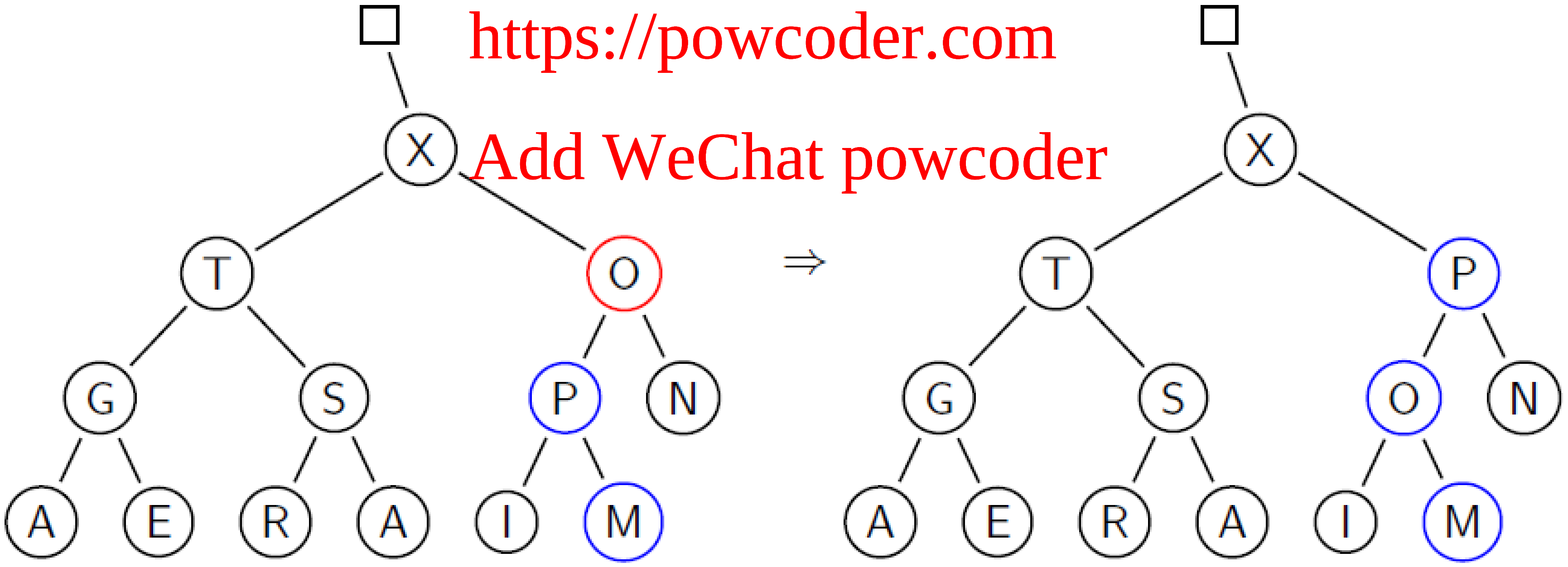
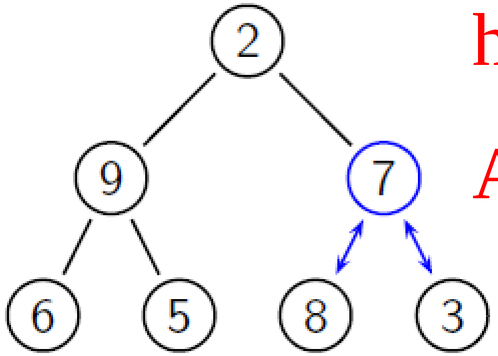


Building a Heap Bottom-Up

- To construct a heap from an arbitrary set of elements, we can just use the inject operation repeatedly. The construction cost will be *n* log *n*. But there is a better way:



- Start with the last parent and move backwards, in level-order. For each parent node, if the largest child is larger than the parent, swap it with the parent.

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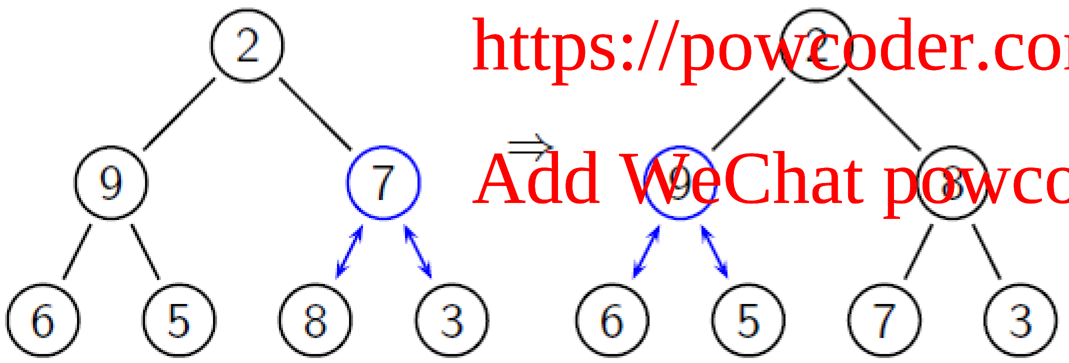

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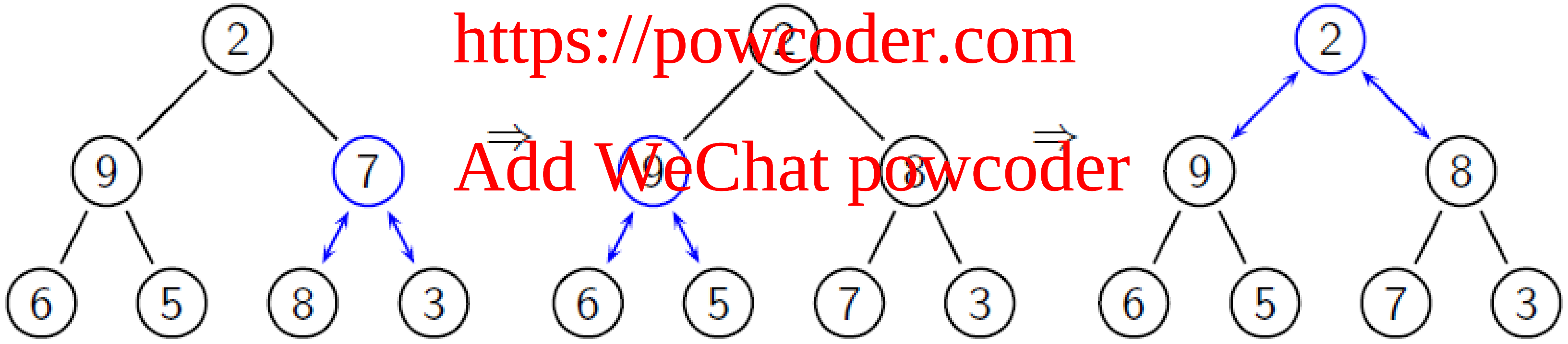

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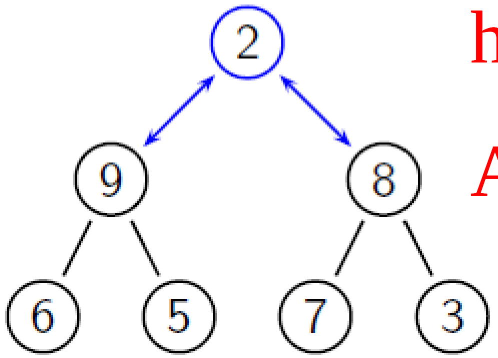
Building a Heap Bottom-Up: Sifting Down

- Whenever a parent is found to be out of order, let it "sift down" until both children are smaller:



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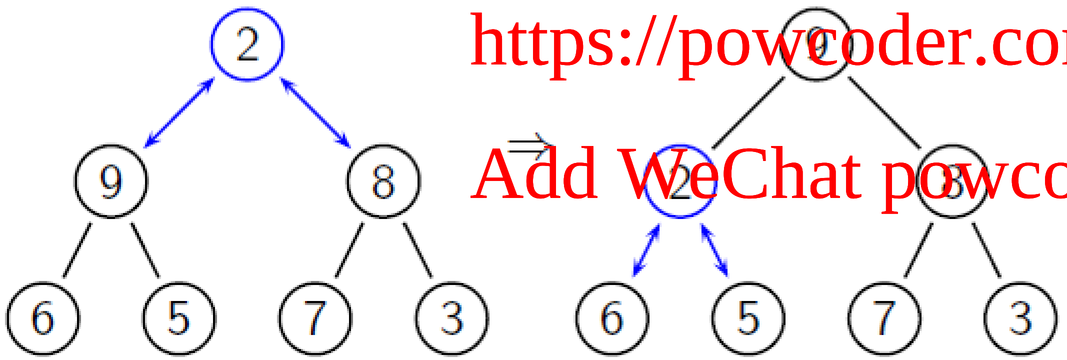

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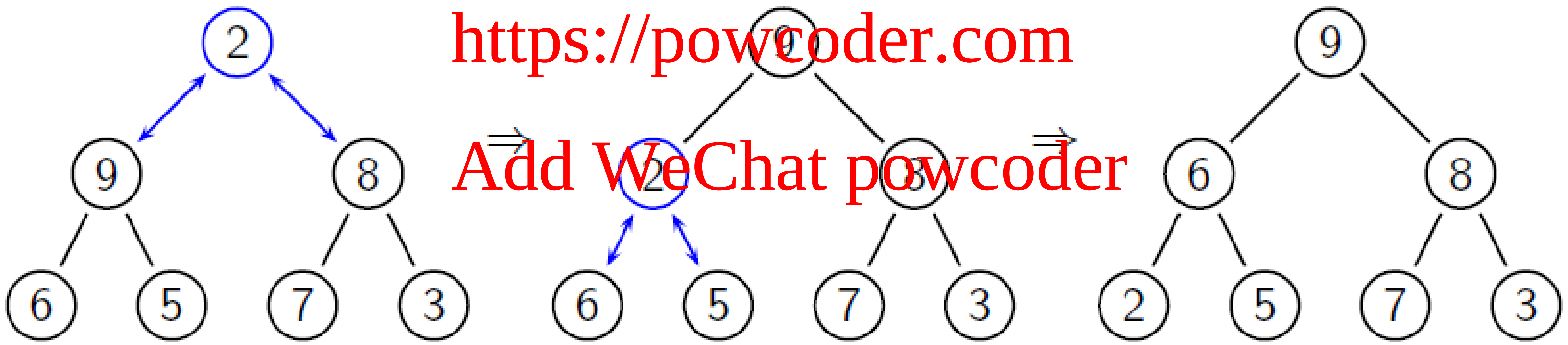


Turning $H[1] \ldots H[n]$ into a Heap, Bottom-Up

for $i \leftarrow \lfloor n/2 \rfloor$ downto 1 **do**
    $k \leftarrow i$
    $v \leftarrow H[k]$
    $heap \leftarrow False$
    while not $heap$ **and** $2 \times k \leq n$ **do**
        $j \leftarrow 2 \times k$ $\triangleright$ j is k's left child
        if $j < n$ **then**
            if $H[j] < H[j+1]$ **then**
                $j \leftarrow j + 1$ $\triangleright$ j is k's largest child
        if $v \geq H[j]$ **then**
            $heap \leftarrow True$
        else $\triangleright$ Promote $H[j]$
            $H[k] \leftarrow H[j]$
            $k \leftarrow j$
    $H[k] \leftarrow v$

Analysis of Bottom-Up Heap Creation

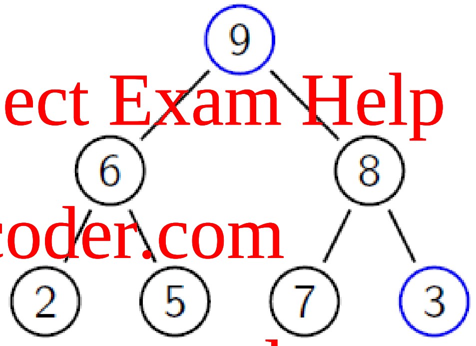
- For simplicity, assume the heap is a full binary tree: $n = 2^{h+1} - 1$. Here is an upper bound on the number of "down-sifts" needed (consider the root to be at level h, so leaves are at level 0):

$$\sum_{i=1}^{h} \sum_{\text{nodes at level } i} i = \sum_{i=1}^{h} i \cdot 2^{h-i} = 2^{h+1} - h - 2$$

- The last equation is easily proved by mathematical induction.
- Note that $2^{h+1} - h - 2 < n$, so we perform at most a linear number of down-sift operations. Each down-sift is preceded by two key comparisons, so the number of comparisons is also linear.
- Hence we have a **linear-time** algorithm for heap creation.

Ejecting a Maximal Element from a Heap

- Here the idea is to swap the root with the last item *z* in the heap, and then let *z* "sift down" to its proper place.



- After this, the last element (here shown in green) is no longer considered part of the heap, that is, *n* is decremented.

- Clearly ejection is $O(\log n)$.

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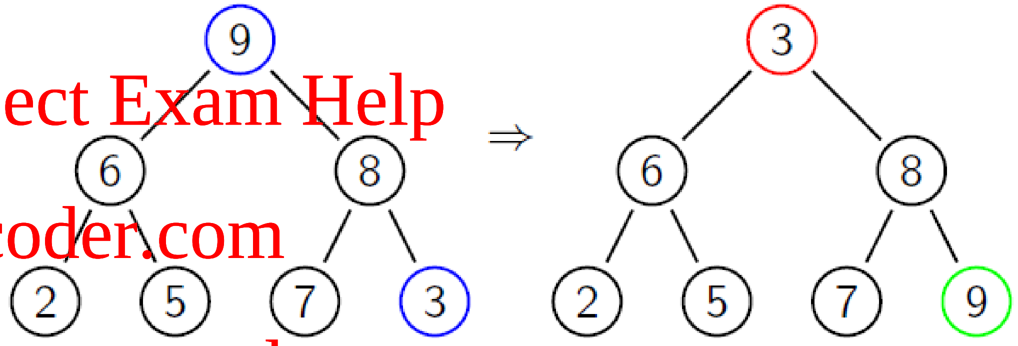

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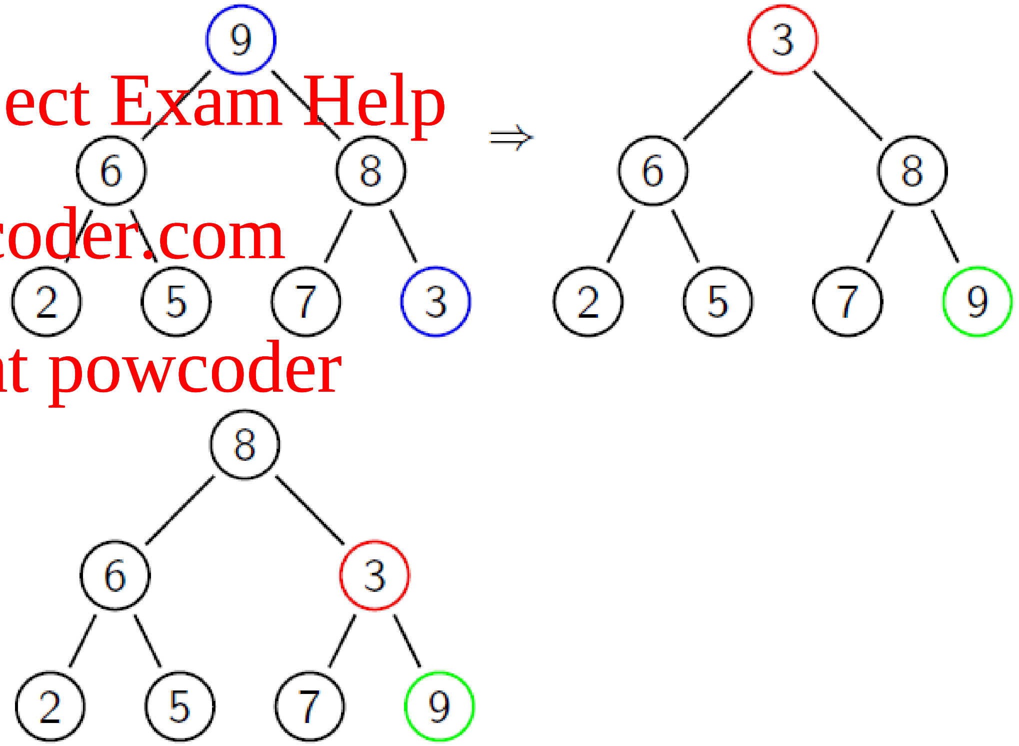

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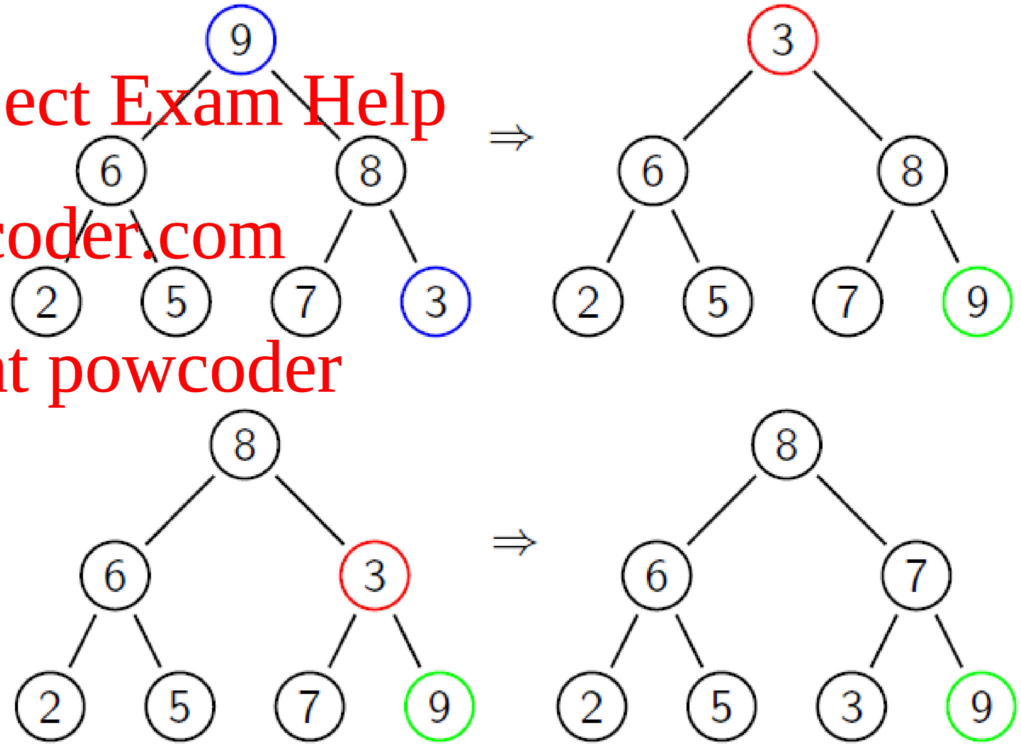
- Clearly ejection is *O*(log *n*).



Exercise: Build and Then Deplete a Heap

- First build a heap from the items S, O, R, T, I, N, G.

- Then repeatedly eject the largest, placing it at the end of the heap.

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- First build a heap from the items S, O, R, T, I, N, G.

- Then repeatedly eject the largest, placing it at the end of the heap.

- Anything interesting to notice about the tree that used to hold a heap?

Heapsort

- Heapsort is a $\Theta(n \log n)$ sorting algorithm, based on the idea from this exercise.

- Given an unsorted array $H[1] \ldots H[n]$:

- **Step 1:** Turn H into a heap.
- **Step 2:** Apply the eject operation n-1 times.

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 <u>8</u>

Stage 2 (maximum deletions)

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 <u>8</u>

2 **9** 8 <u>6 5</u> 7

Stage 2 (maximum deletions)

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 8

2 **9** 8 6 5 7

2 9 8 6 5 7

Stage 2 (maximum deletions)

Heapsort

Stage 1 (heap construction)

2	9	**7**	6	5	<u>8</u>
2	**9**	8	<u>6</u>	<u>5</u>	7
2	<u>9</u>	<u>8</u>	6	5	7
9	**2**	8	<u>6</u>	<u>5</u>	7

Stage 2 (maximum deletions)

Heapsort

Stage 1 (heap construction)

Stage 2 (maximum deletions)

2	9	**7**	6	5	8
2	**9**	8	6	5	7
2	9	8	6	5	7
9	**2**	8	6	5	7
9	6	8	2	5	7

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 8

2 **9** 8 6 5 7

2 9 8 6 5 7

9 **2** 8 6 5 7

9 6 8 2 5 7

Stage 2 (maximum deletions)

9 6 8 2 5 7

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 8

2 **9** 8 6 5 7

2 9 8 6 5 7

9 **2** 8 6 5 7

9 6 8 2 5 7

Stage 2 (maximum deletions)

9 6 8 2 5 7

7 6 8 2 5 | **9**

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 8

2 **9** 8 6 5 7

2 9 8 6 5 7

9 **2** 8 6 5 7

9 6 8 2 5 7

Stage 2 (maximum deletions)

9 6 8 2 5 7

7 6 8 2 5 | **9**

8 6 7 2 5 | **9**

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 8

2 **9** 8 6 5 7

2 9 8 6 5 7

9 **2** 8 6 5 7

9 6 8 2 5 7

Stage 2 (maximum deletions)

9 6 8 2 5 7

7 6 8 2 5 | **9**

8 6 7 2 5 | **9**

5 6 7 2 | **8** **9**

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 8
2 **9** 8 6 5 7
2 9 8 6 5 7
9 **2** 8 6 5 7
9 6 8 2 5 7

Stage 2 (maximum deletions)

9 6 8 2 5 7
7 6 8 2 5 | **9**
8 6 7 2 5 | **9**
5 6 7 2 | **8** **9**
7 6 5 2 | **8** **9**
2 6 5 | **7** **8** **9**

Heapsort

Stage 1 (heap construction)

2 9 **7** 6 5 8

2 **9** 8 6 5 7

2 9 8 6 5 7

9 **2** 8 6 5 7

9 6 8 2 5 7

Stage 2 (maximum deletions)

9 6 8 2 5 7

7 6 8 2 5 | **9**

8 6 7 2 5 | **9**

5 6 7 2 | **8** **9**

7 6 5 2 | **8** **9**

2 6 5 | **7** **8** **9**

6 2 5 | **7** **8** **9**

5 2 | **6** **7** **8** **9**

Heapsort

Stage 1 (heap construction)

2	9	**7**	6	5	8
2	**9**	8	6	5	7
2	9	8	6	5	7
9	**2**	8	6	5	7
9	6	8	2	5	7

Stage 2 (maximum deletions)

9	6	8	2	5	7
7	6	8	2	5 |	**9**
8	6	7	2	5 |	**9**
5	6	7	2 |	**8**	**9**
7	6	5	2 |	**8**	**9**
2	6	5 |	**7**	**8**	**9**
6	2	5 |	**7**	**8**	**9**
5	2 |	**6**	**7**	**8**	**9**
5	2 |	**6**	**7**	**8**	**9**
2 |	**5**	**6**	**7**	**8**	**9**
2	**5**	**6**	**7**	**8**	**9**

Properties of Heapsort

- On average slower than quicksort, but stronger performance guarantee.

- Truly in-place.

- Not stable.

Next lecture

- Transform-and-Conquer
  - Pre-sorting (Levitin Section 6.1)