# COMP90038



# Algorithms and Complexity



Lecture 16: Time/Space Tradeoffs – String Search Revisited



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# Recap

- BST have optimal performance when they are balanced.

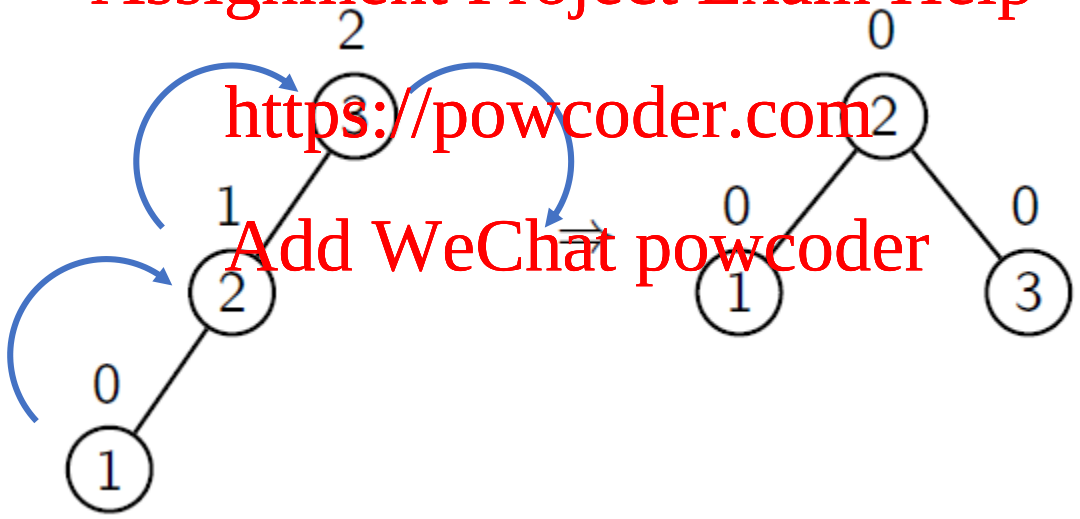


- AVL Trees:
  - Self-balancing trees for which the balance factor is –1, 0, or 1, for every sub-tree.
  - Rebalancing is achieved through rotations.
  - It guarantees depth of a tree with $n$ nodes to be $O(\log n)$

- 2–3 trees:
  - Trees that allow more than one item to be stored in a tree node.
  - This allows for a simple way of keeping search trees perfectly balanced.
  - Insertions, splits and promotions are used to grow and balance the tree.

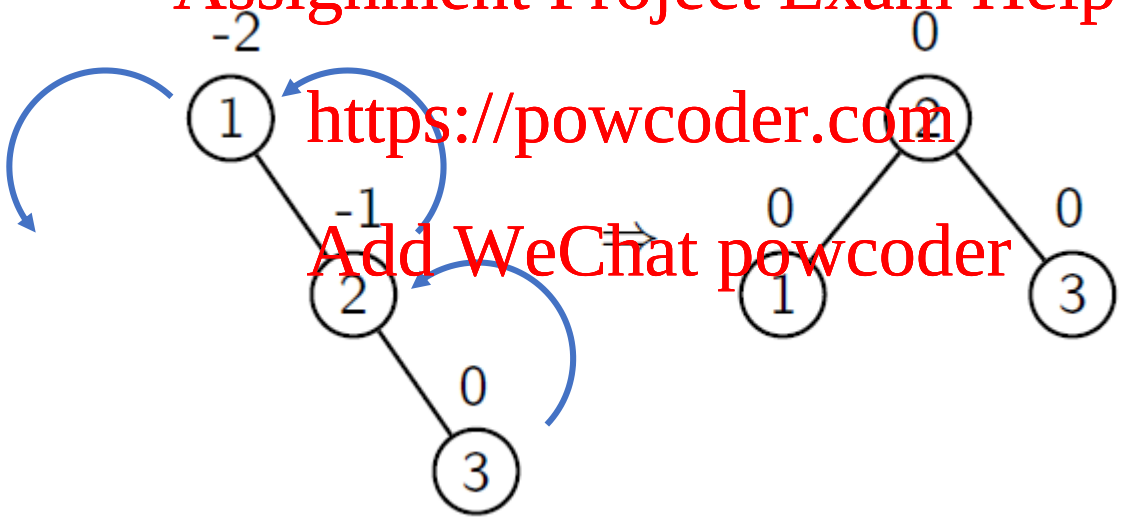
# AVL Trees: R-Rotation





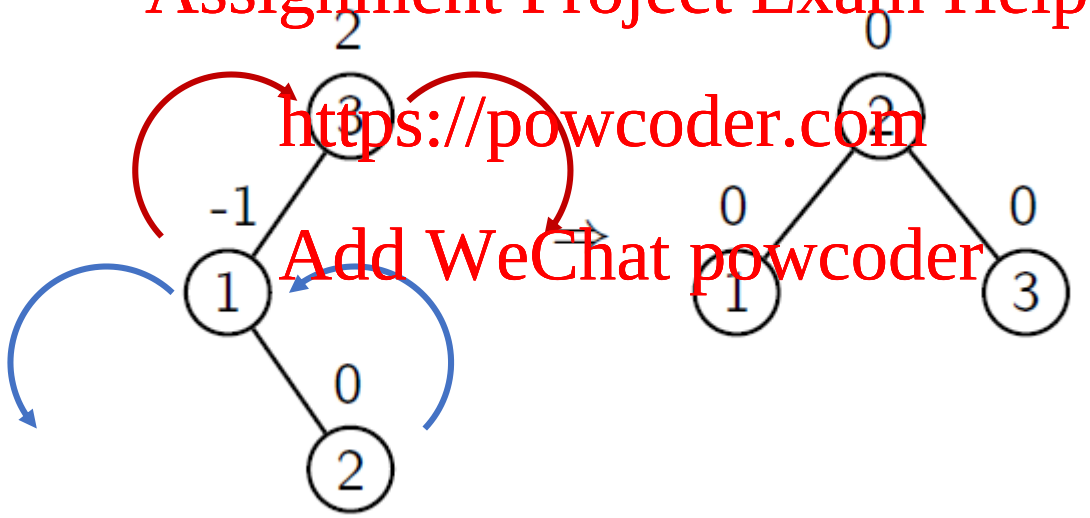
# AVL Trees: L-Rotation





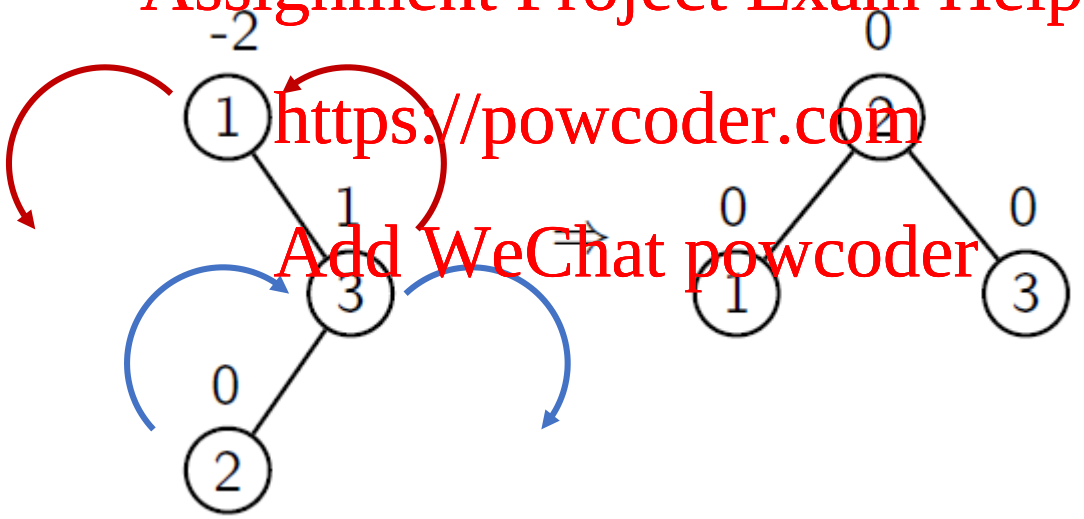
# AVL Trees: LR-Rotation





# AVL Trees: RL-Rotation

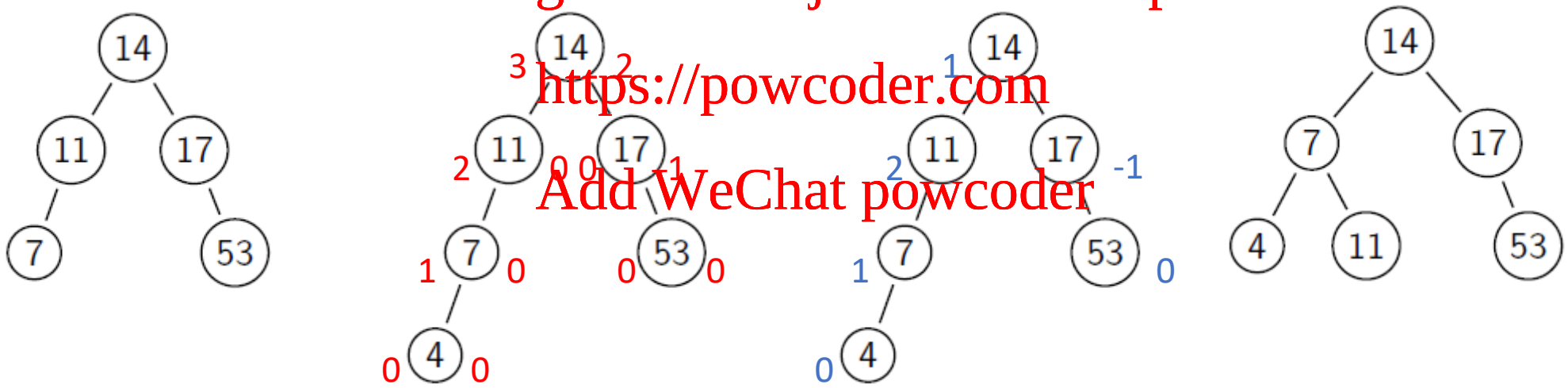




# Example

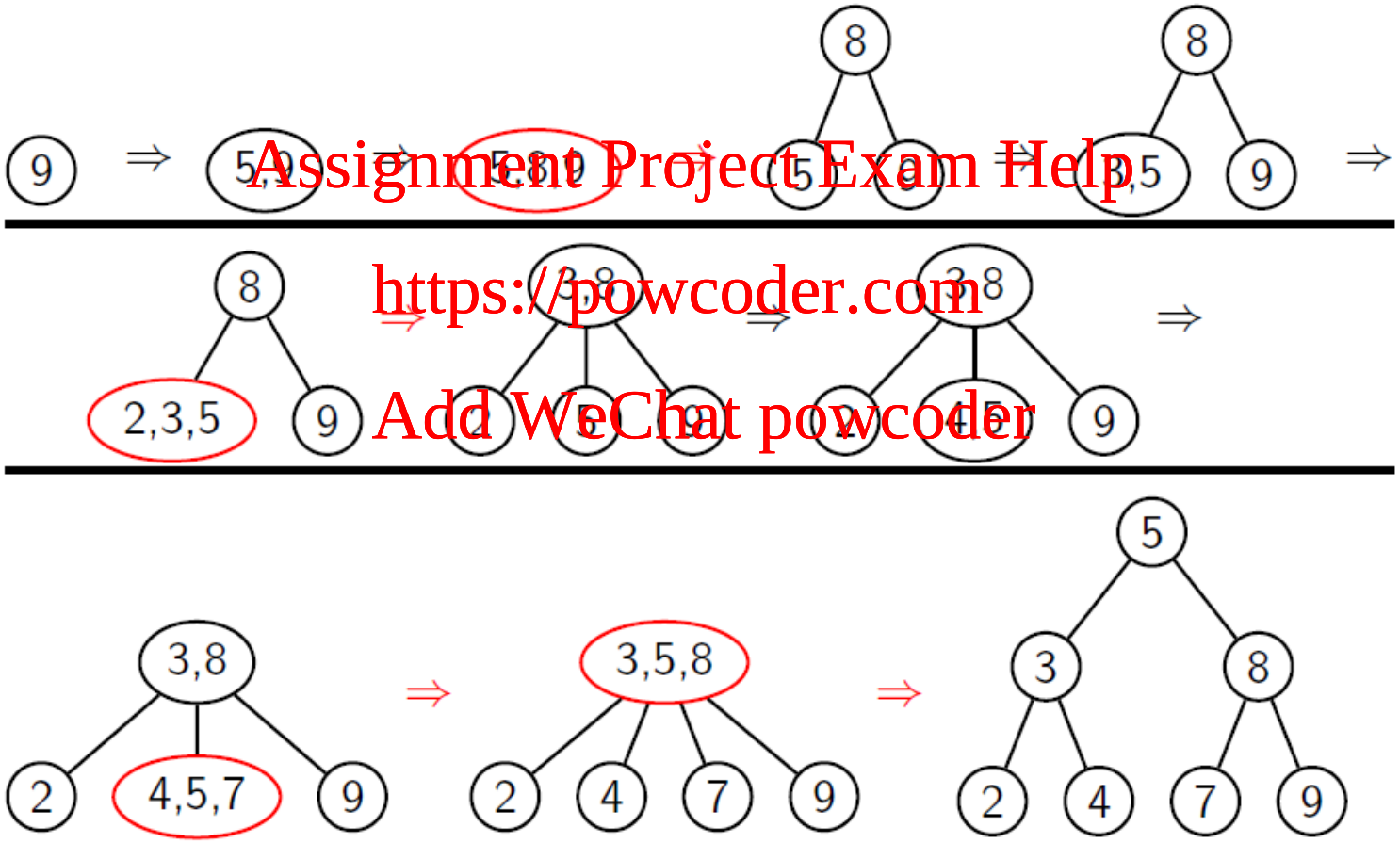
- On the tree below, insert the elements {4, 13, 12}






- https://www.cs.usfca.edu/~galles/visualization/AVLtree.html

# Example: Build a 2–3 Tree from {9, 5, 8, 3, 2, 4, 7}



# 2–3 Tree Analysis

- Worst case search time results when all nodes are 2-nodes. The relation between the number *n* of nodes and the height *h* is:

$$n = 1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$$



- That is, $\log_2(n+1) = h+1$.



- In the best case, all nodes are 3-nodes:

$$n = 2 + 2\times3 + 2\times3^2 + \ldots + 2\times3^h = 3^{h+1} - 1$$



- That is, $\log_3(n+1) = h+1$.

- Hence we have $\log_3(n+1) - 1 \le h \le \log_2(n+1) - 1$.

- Useful formula: $\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1} \text{ for } a \neq 1$

# Spending Space to Save Time

- Often we can find ways of decreasing the time required to solve a problem, by using additional memory in a clever way.



- For example, in **Lecture 6 (Recursion)** we considered the simple recursive way of finding the *n*-th Fibonacci number and discovered that the algorithm uses exponential time.



```
function FIB(n)
    if n = 0 then
        return 1
    if n = 1 then
        return 1
    return FIB(n − 1) + FIB(n − 2)
```

# Spending Space to Save Time



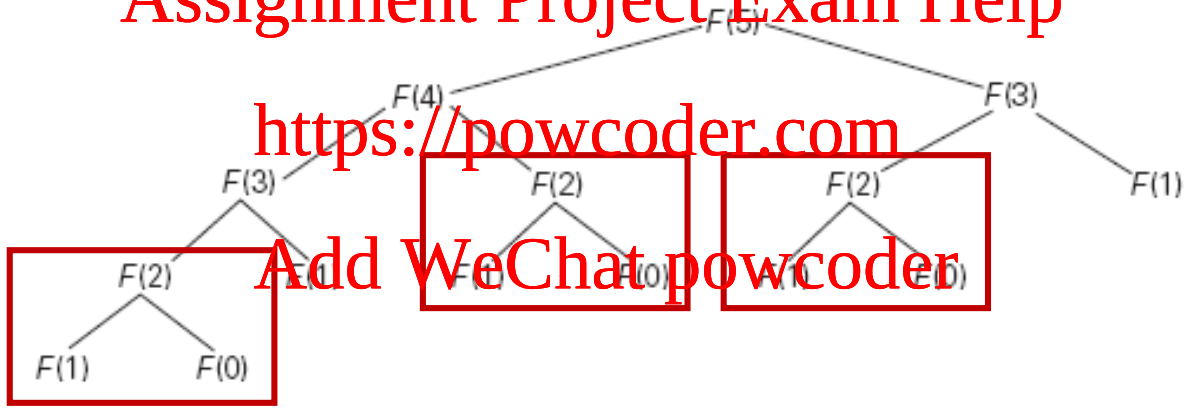



**FIGURE 2.6** Tree of recursive calls for computing the 5th Fibonacci number by the definition-based algorithm.

# Spending Space to Save Time

- However, suppose the same algorithm uses a table to **tabulate** the function FIB() as we go: As soon as an intermediate result FIB($i$) has been found, it is not simply returned to the caller; the value is first placed in slot $i$ of a table (an array). Each call to FIB() first looks in this table to see if the required value is there, and only if it is not, the usual recursive process kicks in.

# Fibonacci Numbers with Tabulation

- We assume that, from the outset, all entries of the table *F* are 0.

```
function FIB(n)
    if n = 0 or n = 1 then
        return 1
    result ← F[n]
    if result = 0 then
        result ← FIB(n − 1) + FIB(n − 2)
        F[n] ← result
    return result
```



- (I show this code just so that you can see the principle; in **Lecture 6** we already discovered a different linear-time algorithm, so here we don't really need tabulation.)

# Sorting by Counting

- Suppose we need to sort large arrays, but we know that they will hold keys taken from a **small**, **fixed** set (so lots of duplicate keys).
- For example, suppose all keys are single digits:



6 3 3 8 1 0 8 7 9 2 5 3 5 3 1 8 7 6 5 1 2 1 5 3



- Then we can, in a single linear scan, count the occurrences of each key in array *A* and store the result in a small table:



| *key* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| *Occ* | 1 | 4 | 2 | 5 | 0 | 4 | 2 | 2 | 3 | 1 |

- Now use a second linear scan to make the counts **cumulative**:

| *key* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| *Occ* | 1 | 5 | 7 | 12 | 12 | 16 | 18 | 20 | 23 | 24 |

-

# Sorting by Counting

- We can now create a sorted array $S[1]...S[n]$ of the items by simply slotting items into pre-determined slots in $S$ (a third linear scan).



6 3 3 8 1 8 7 9 2 5 3 5 1 8 7 6 5 1 2 1 5 3

| *key* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| *Occ* | 1 | 5 | 7 | 12 | 12 | 16 | 18 | 20 | 23 | 24 |





- Place the last record (with key 3) in $S[12]$ and decrement $Occ[3]$ (so that the next `3' will go into slot 11), and so on.

**for** $i \leftarrow n$ **to** 1 **do**
    $S[Occ[A[i]]] \leftarrow A[i]$
    $Occ[A[i]] \leftarrow Occ[A[i]] - 1$

-

# Sorting by Counting

- Note that this gives us a **linear-time** sorting algorithm (for the cost of some extra space).



- However, it only works in situations where we have a small range of keys, known in advance.





- The method never performs a key-to-key comparison.

- The time complexity of **key-comparison based sorting** has been proven to be in $\Omega(n \log n)$.

# String Matching Revisited

- In **Lecture 5 (Brute Force Methods)** we studied an approach to string search.

```
for i ← 0 to n − m do
    j ← 0
    while j < m and p[j] = t[i + j] do
        j ← j + 1
    if j = m then
        return i
return −1
```

# String Matching Revisited




**FIGURE 3.3** Example of brute-force string matching. The pattern's characters that are compared with their text counterparts are in bold type.

# String Matching Revisited

- “Strings” are usually built from a small, pre-determined alphabet.



- Most of the better algorithms rely on some pre-processing of strings before the actual matching process starts.





- The pre-processing involves the construction of a small table (of predictable size).

- Levitin refers to this as “input enhancement”.

# Horspool's String Search Algorithm

- Comparing from right to left in the pattern.
- Very good for random text strings.



S T R I N G S E A R C H E X A M P
E X A M



- We can do better than just observing a mismatch here.
- Because the pattern has **no occurrence of I**, we might as well slide it 4 positions along.
- This decision is based only on knowing the pattern.

# Horspool's String Search Algorithm

```
S T R I N G S E A R C H E X A M P
E X A M
      E X A M
```

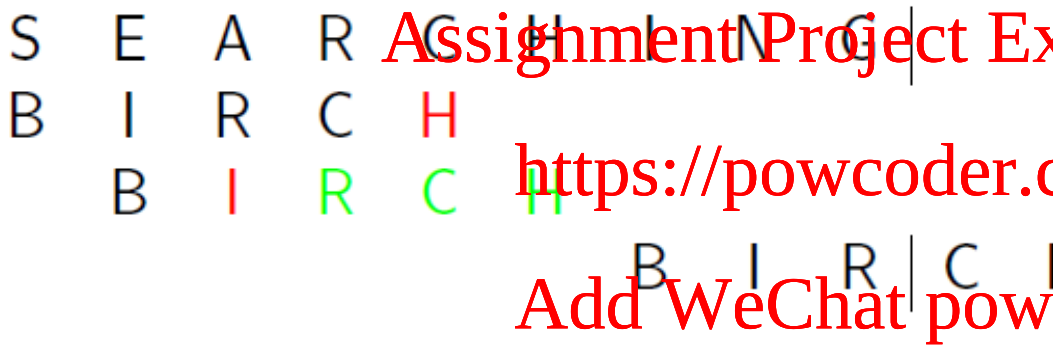


- Here we can slide the pattern 3 positions, because the last occurrence of E in the pattern is its first position.

```
S T R I N G S E A R C H E X A M P
E X A M
      E X A M
            E X A M
                  E X A M
                        E X A M
                           E X A M
```

# Horspool's String Search Algorithm

- What happens when we have longer partial matches?






- The shift is determined by the last character in the pattern.

- Note that this is the same as the character in the text that we first matched against. Hence the skip is **always** determined by that character, whether it matched or not.

| Char | Shift |
|---|---|
| A | 5 |
| B | 4 |
| C | 1 |
| ⋮ | ⋮ |
| H | 5 |
| I | 3 |
| ⋮ | ⋮ |
| R | 2 |
| S | 5 |
| ⋮ | ⋮ |
| Z | 5 |

# Horspool's String Search Algorithm

- Building (calculating) the shift table is easy.
- We assume indices start from 0.
- Let *alphasize* be the size of the alphabet.



```
function FindShifts(P[·], m)        ▷ Pattern P has length m
    for i ← 0 to alphasize − 1 do
        Shift[i] ← m
    for j ← 0 to m − 2 do
        Shift[P[j]] ← m − (j + 1)
```

# Horspool's String Search Algorithm

```
function HORSPOOL(P[·], m, T[·], n)
    FINDSHIFTS(P, m)
    i ← m − 1
    while i < n do
        k ← 0
        while k < m and P[m − 1 − k] = T[i − k] do
            k ← k + 1
        if k = m then                          ▷ We have a match
            return i − m + 1                   ▷ Start of the match
        else
            i ← i + Shift[T[i]]                ▷ Slide the pattern along
    return −1
```

# Horspool's String Search Algorithm

- We can also consider posting a sentinel: Append the pattern *P* to the end of the text *T* so that a match is guaranteed.



```
function HorspoolSearch(P[·], m, T[·], n)
    FindShifts(P, m)
    i ← m − 1
    while True do
        k ← 0
        while k < m and P[m − 1 − k] = T[i − k] do
            k ← k + 1
        if k = m then
            if i ≥ n then
                return −1
            else
                return i − m + 1
        i ← i + Shift[T[i]]
```

# Horspool's String Search Algorithm



- Unfortunately the worst-case behaviour of Horspool's algorithm is still $O(m \times n)$, like the brute-force method.





- However, in practice, for example, when used on English texts, it is linear-time, and fast.

# Other Important String Search Algorithms

- Horspool's algorithm was inspired by the famous **Boyer-Moore** algorithm (**BM**), also covered in Levitin's book. The BM algorithm is very similar, but it has a more sophisticated shifting strategy, which makes it $O(m+n)$.



- Another famous string search algorithm is the **Knuth-Morris-Pratt** algorithm (**KMP**), explained in the remainder of these slides. KMP is very good when the alphabet is small, say, we need to search through very long bit strings.





- Also, we shall soon meet the **Rabin-Karp** algorithm (**RK**), albeit briefly.

- While very interesting, **the BM, KMP, and RK algorithms are not examinable.**

# Knuth-Morris-Pratt (Not Examinable)

- Suppose we are searching in strings that are built from a small alphabet, such as the binary digits 0 and 1, or the nucleobases.
- Consider the brute-force approach for this example:



Text: 1 0 0 0 1 0 0 0 0

Pattern: 1 0 0 0 0

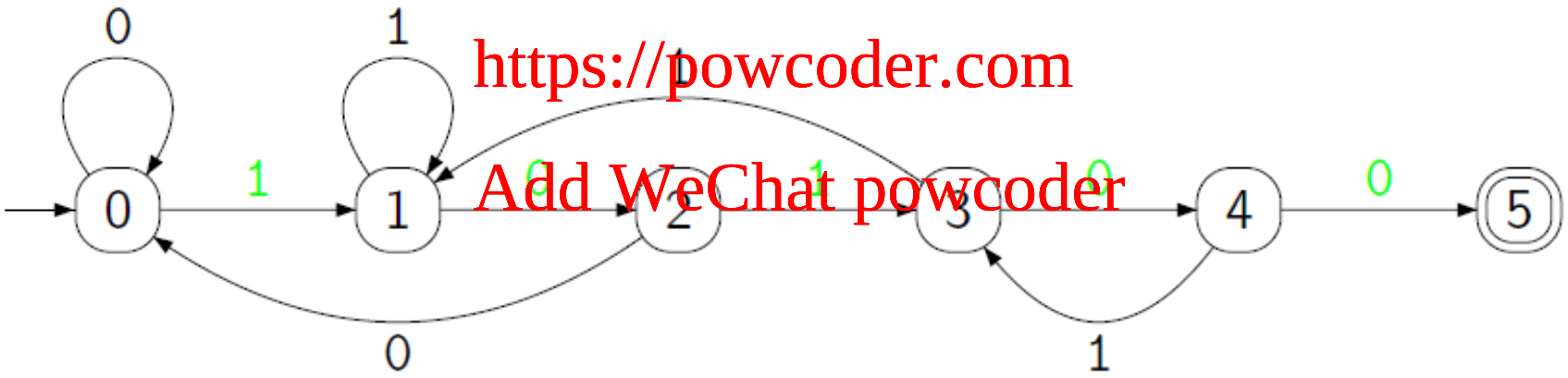


- Every “false start” contains a lot of information.
- Again, we hope to **pre-process** the pattern so as to find out when the brute-force method's index *i* can be incremented by more than 1.
- Unlike Horspool's method, KMP works by comparing from left to right in the pattern.

# Knuth-Morris-Pratt as Running an FSA

- Given the pattern [1 0 1 0 0] we want to construct the following **finite-state automaton**:





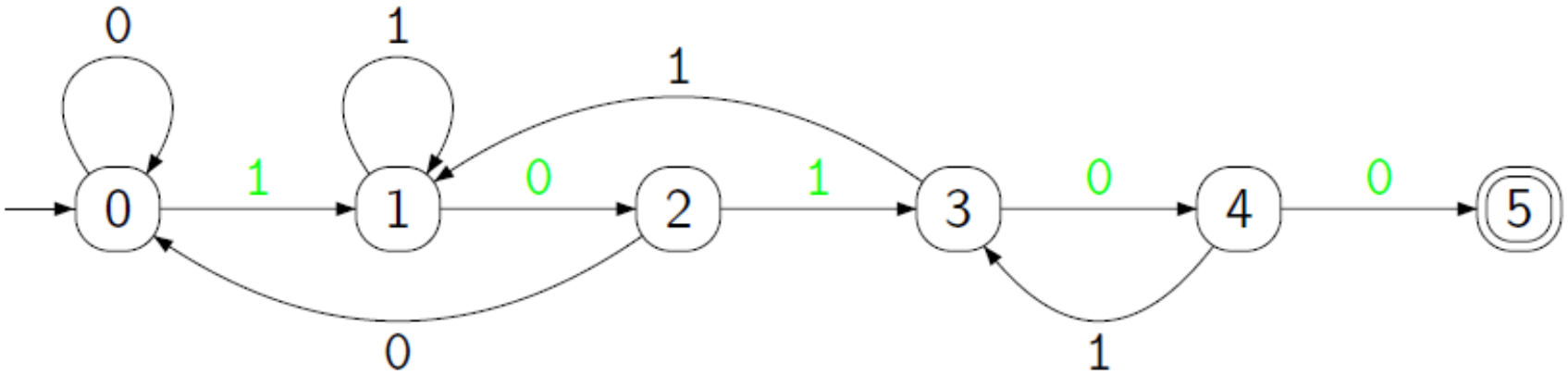

- We can capture the behaviour of this automaton in a table.

# Knuth-Morris-Pratt Automaton

- We can represent the finite-state automaton as a 2-dimensional "transition" array $T$, where $T[c][j]$ is the state to go to upon reading the character $c$ in state $j$.

| $j$ | $T['0'][j]$ | $T['1'][j]$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 2 | 1 |
| 2 | 0 | 3 |
| 3 | 4 | 1 |
| 4 | 5 | 3 |

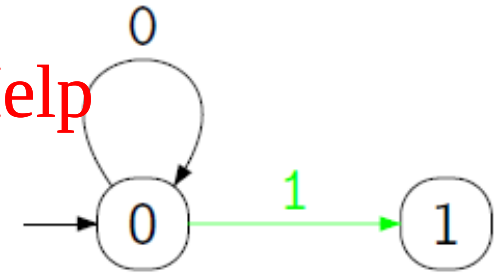




# Constructing the Automaton

- The automaton (or the table *T*) can be constructed step-by-step:



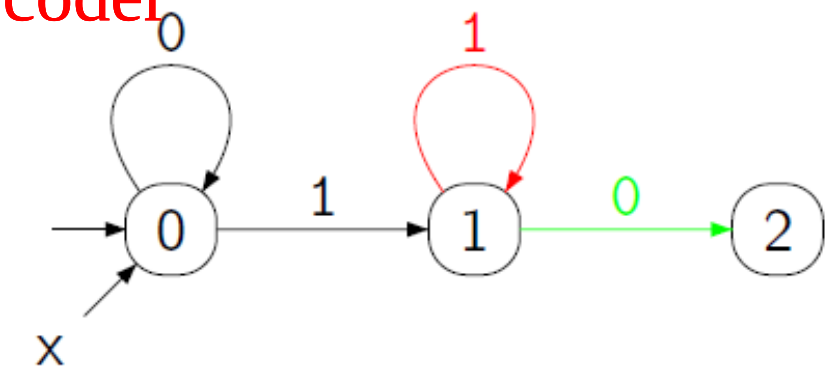





- Somewhat tricky but fast.
- *x* is a “backtrack point”.

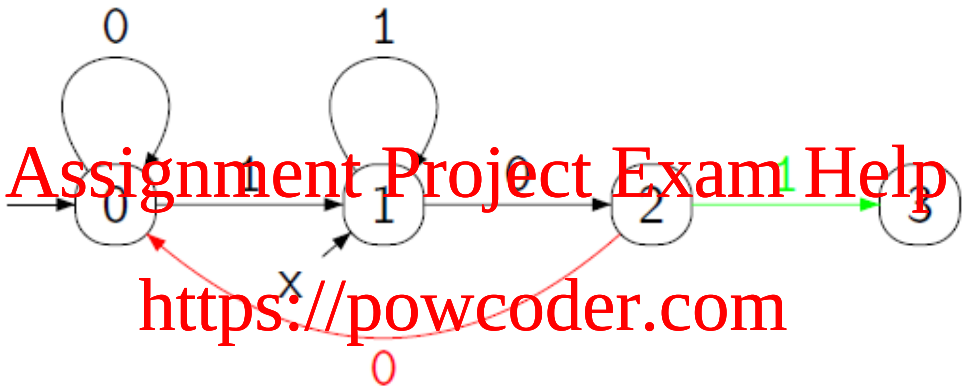


- For next state *j*:
  - First *x*'s transitions are copied (in red).
  - Then the success arc is updated, determined by *P*[*j*] (in green).

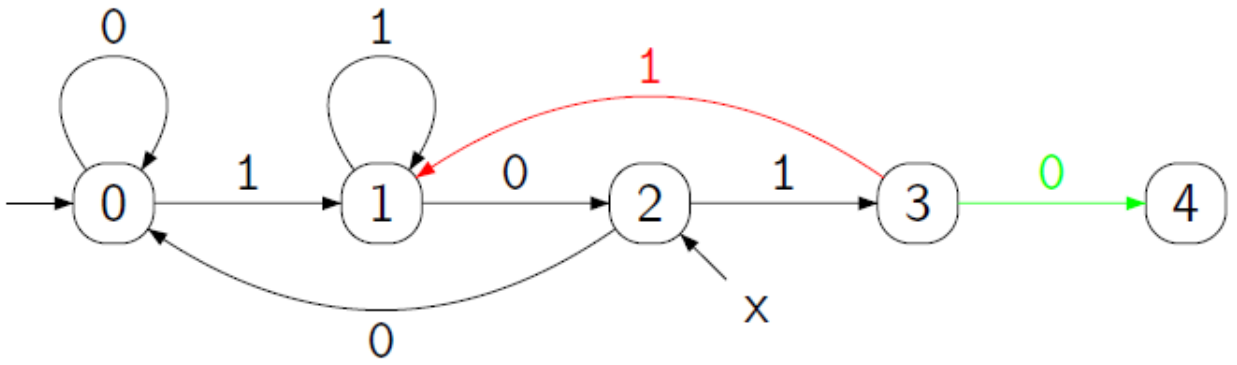



- Finally *x* is updated based on *P*[*j*].

# Constructing the Automaton

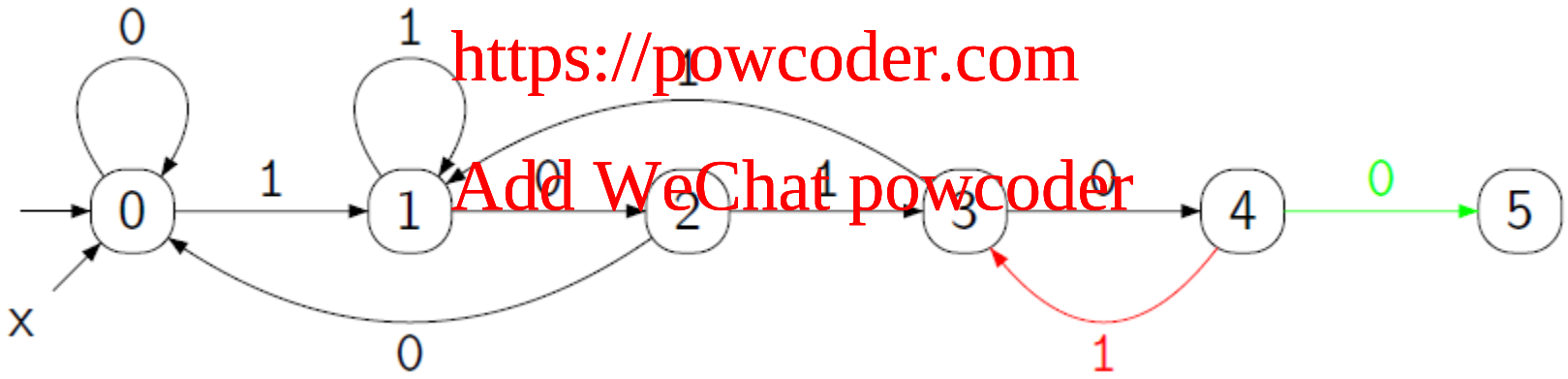




# Constructing the Automaton





# Constructing the Automaton

$T['0'][0] \leftarrow 0$
$T['1'][0] \leftarrow 0$
$T[P[0]][0] \leftarrow 1$
$x \leftarrow 0$
$j \leftarrow 1$
**while** $j < m$ **do**
    $T['0'][j] \leftarrow T['0'][x]$
    $T['1'][j] \leftarrow T['1'][x]$
    $T[P[j]][j] \leftarrow j + 1$
    $x \leftarrow T[P[j]][x]$
    $j \leftarrow j + 1$

# Pattern Compilation: Hard-Wiring the Pattern

- Even better, we can directly produce code that is specialised to find the given pattern. As a C program, for the example `p = 1 0 1 0 0`:



```
int kmp(char *s) {
        int i = -1;
        s0: i++; if (s[i] == '0') goto s0;
        s1: i++; if (s[i] == '1') goto s1;
        s2: i++; if (s[i] == '0') goto s0;
        s3: i++; if (s[i] == '1') goto s1;
        s4: i++; if (s[i] == '1') goto s3;
        s5: return i-4;
}
```

- Again, this assumes that we have posted a sentinel, that is, appended `p` to the end of `s` before running `kmp(s)`.

# Next week



- We look at the hugely important technique of **hashing**, a standard way of implementing a "dictionary".





- Hashing is arguably the best example of how to gain speed by using additional space to great effect.