THE UNIVERSITY OF MELBOURNE

COMP90038
Algorithms and Complexity

Lecture 4: Analysis of Algorithms

(with thanks to Harald Søndergaard)

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Last Time: Time Complexity

- Measure **input size** by natural number n
- Measure **execution time** as number of **basic operations** performed

- **Time complexity** $t(n)$ for an algorithm: number of **basic operations** as a function of n

- How to **compare** different $t(n)$?
  - Asymptotic growth rate
  - $O(g(n))$, $\Omega(g(n))$, $\Theta(g(n))$

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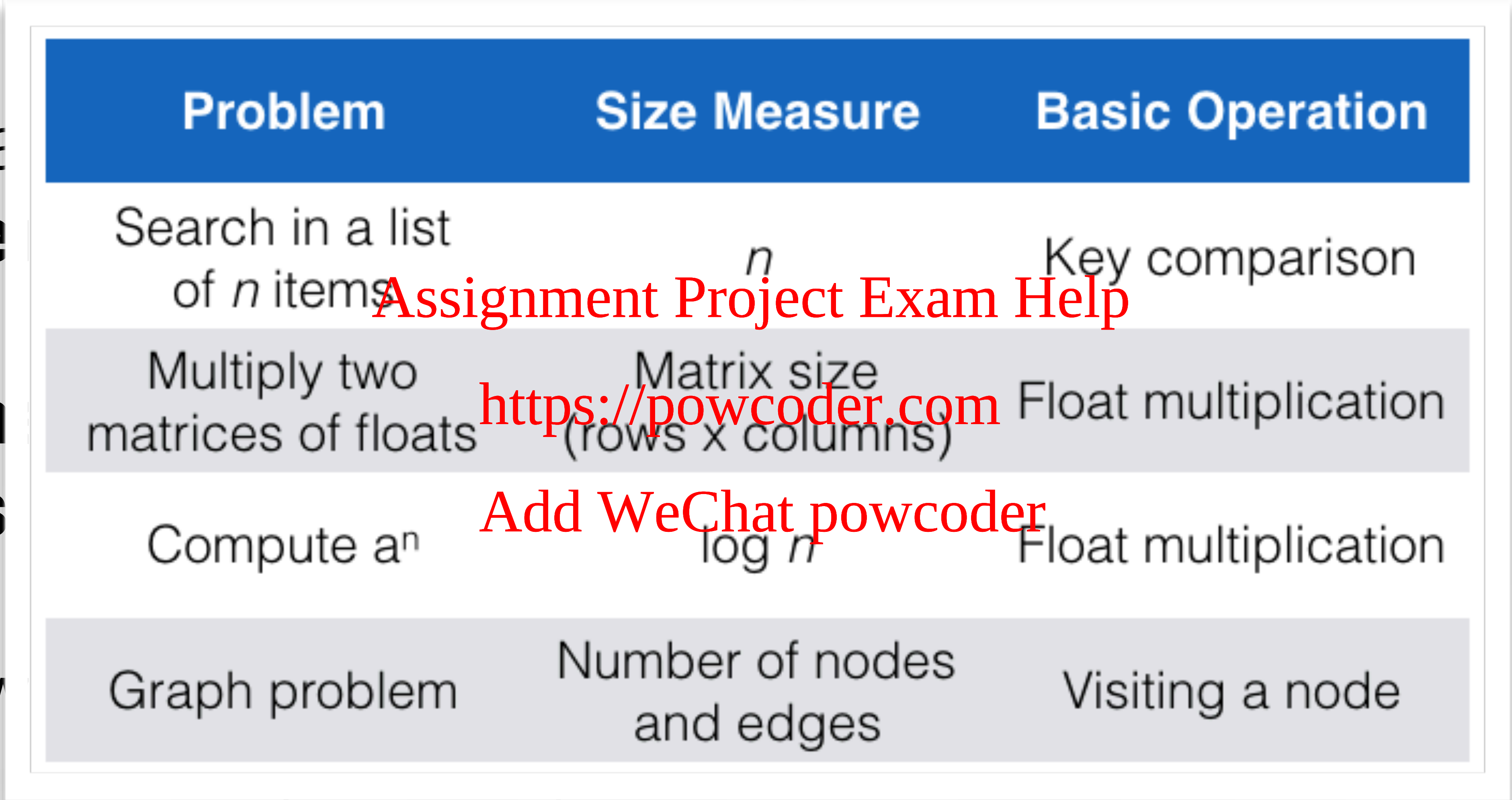

Problem	Size Measure	Basic Operation
Search in a list of n items	n	Key comparison
Multiply two matrices of floats	Matrix size (rows x columns)	Float multiplication
Compute a^n	$\log n$	Float multiplication
Graph problem	Number of nodes and edges	Visiting a node

- Mea
  ope
- **Tim** of
  bas
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  - Asymptotic growth rate
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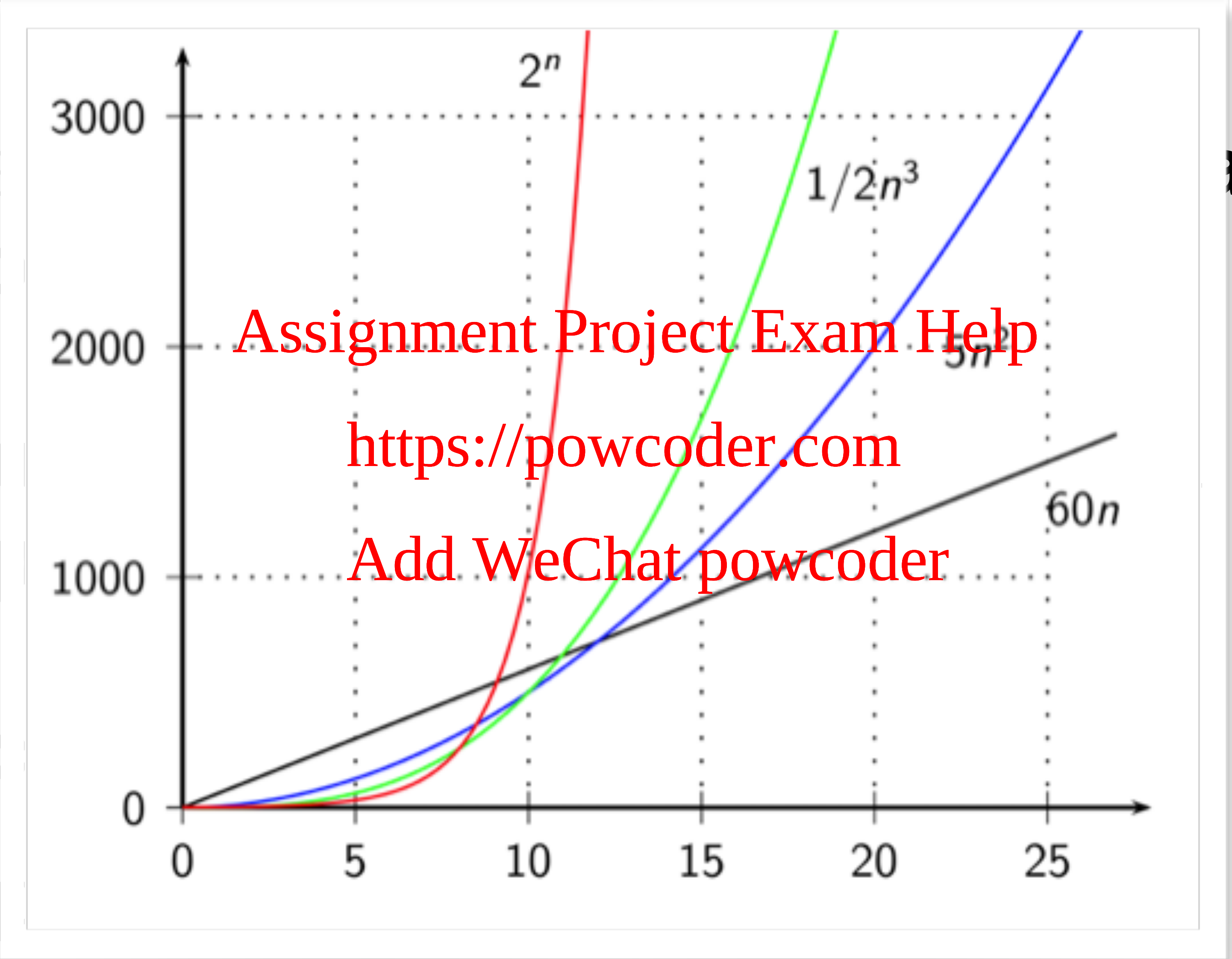

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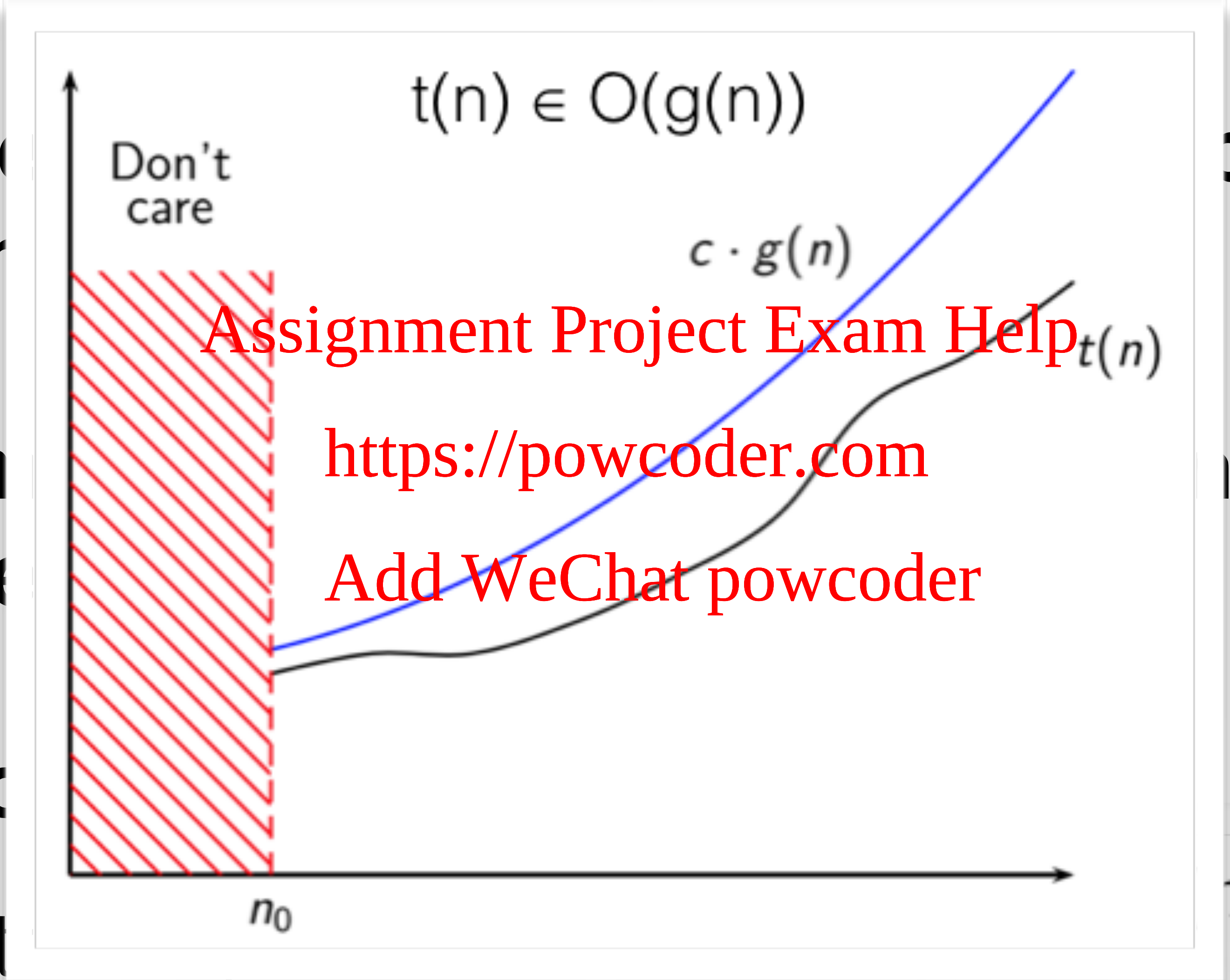

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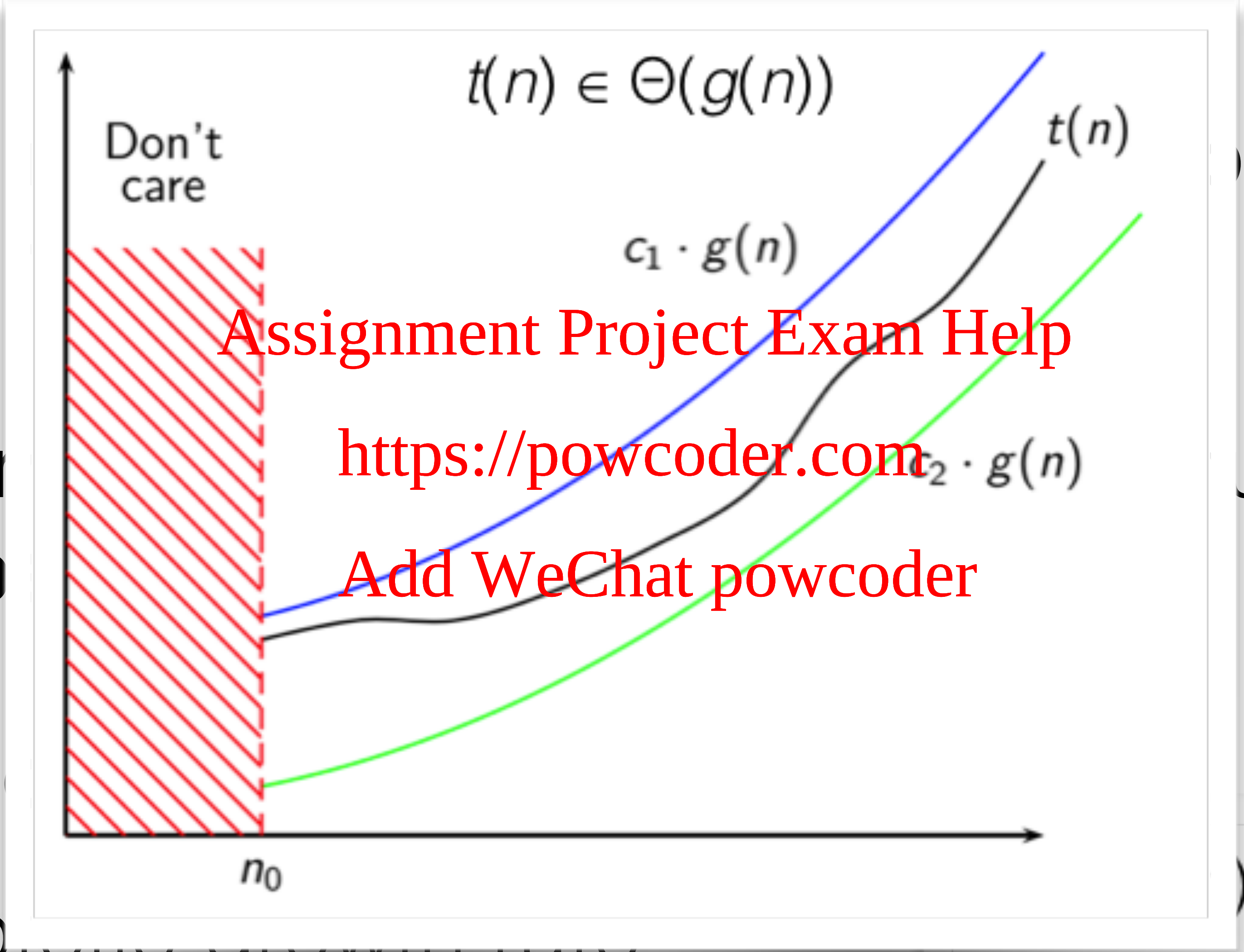

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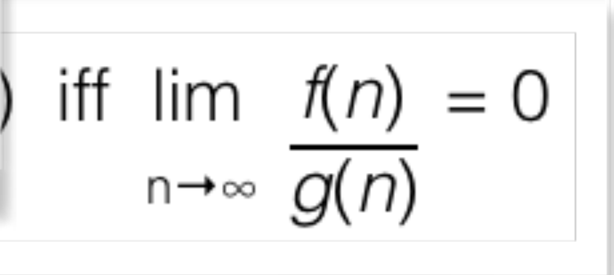

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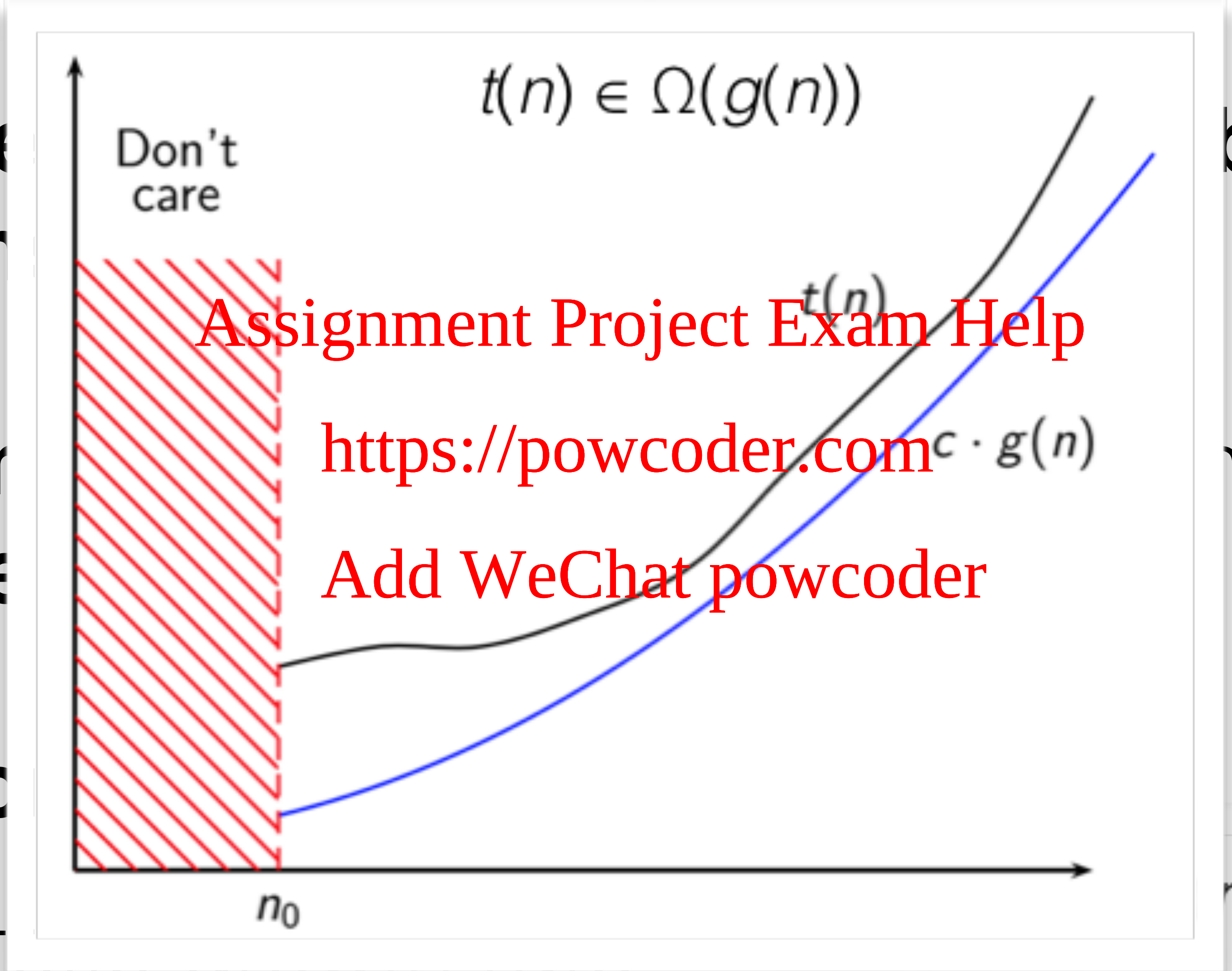

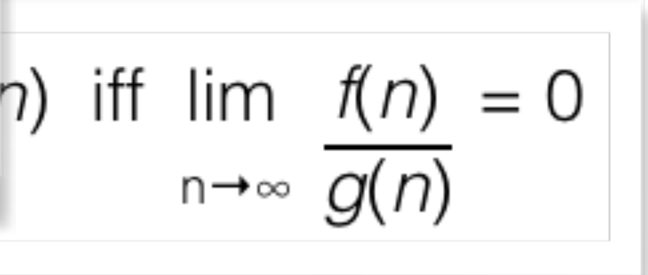

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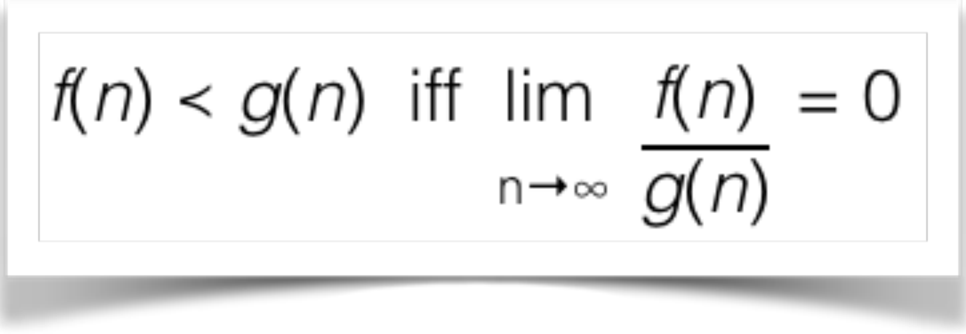

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Establishing Growth Rate

- In the last lecture we proved $t(n) \in O(g(n))$ for some cases of t and g, using the definition of O directly:

  $n > n_0 \Rightarrow t(n) < c \cdot g(n)$ for some c and n_0.

- A more common approach uses

- $$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f \text{ grows asymptotically slower than } g \\ c & f \text{ and } g \text{ have same order of growth} \\ \infty & f \text{ grows asymptotically faster than } g \end{cases}$$

- Use this to show that $1000n \in O(n^2)$

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$$\lim_{n \to \infty} \frac{1000n}{n^2}$$

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$1000n \in O(n^2)$

$$\lim_{n\to\infty} \frac{1000\cancel{n}}{n^{\cancel{2}}}$$

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$$= 0$$

So $1000n$ grows asymptotically slower than n^2

$1000n \in O(n^2)$

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So $1000n$ grows asymptotically slower than n^2

Thus $1000n \in O(n^2)$

O, Ω, Θ

What this tells us about how $f(n)$ relates to
$O(g(n))$, $\Omega(g(n))$ and $\Theta(g(n))$

- $$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f \text{ grows asymptotically slower than } g \\ c & f \text{ and } g \text{ have same order of growth} \\ \infty & f \text{ grows asymptotically faster than } g \end{cases}$$

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- $$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 \\ c \\ \infty \end{cases}$$

$f(n) \in O(g(n))$

f and g have same order of growth

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L'Hôpital's Rule

$$\lim_{h \to \infty} \frac{t(n)}{g(n)} = \lim_{h \to \infty} \frac{t'(n)}{g'(n)}$$

where t' and g' are the derivatives of t and g

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$$\lim_{n\to\infty}\frac{\log_2 n}{\sqrt{n}} = \lim_{n\to\infty}\frac{(\log_e 2)\frac{1}{n}}{\frac{1}{2\sqrt{n}}}$$

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Example:
Finding Largest Element in an Array

```
function MaxElement($A[\cdot], n$)
    $max \leftarrow A[0]$
    for $i \leftarrow 1$ to $n - 1$ do
        if $A[i] > max$ then
            $max \leftarrow A[i]$
    return $max$
```

(where n is length of the array)

A:	23	12	42	6	69	18	3
	0	1	2	3	4	5	6

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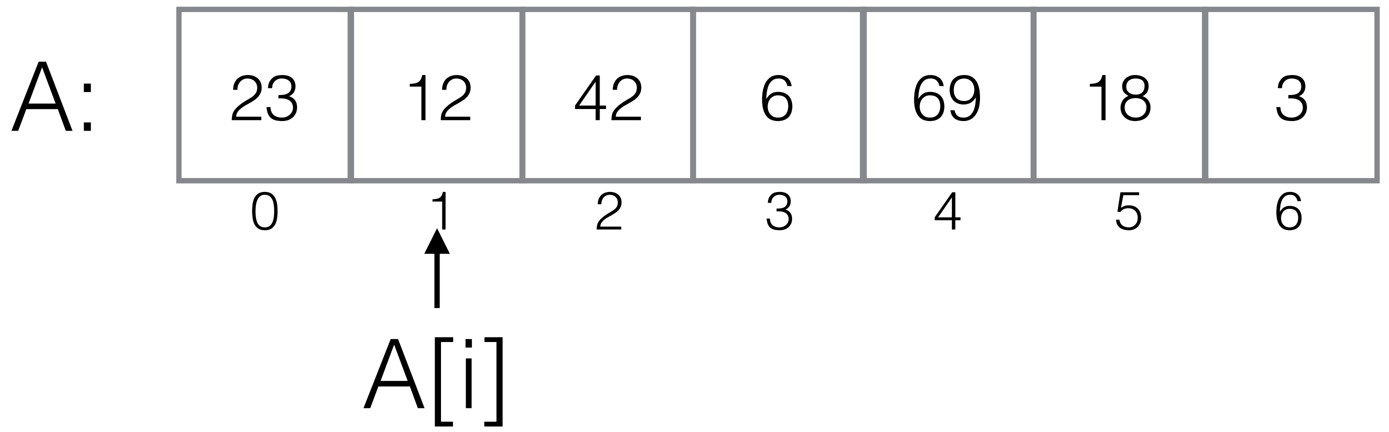

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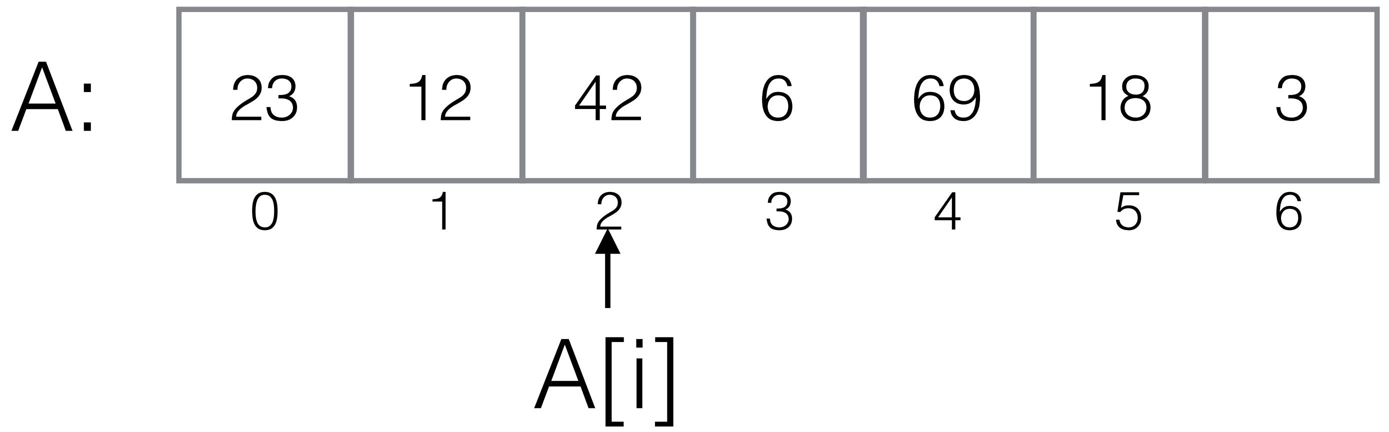

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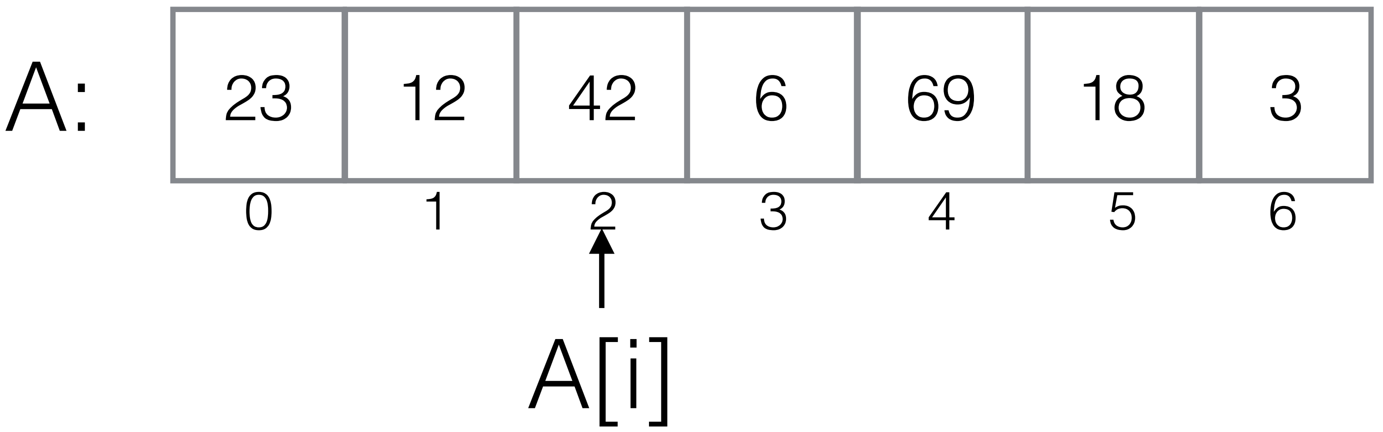

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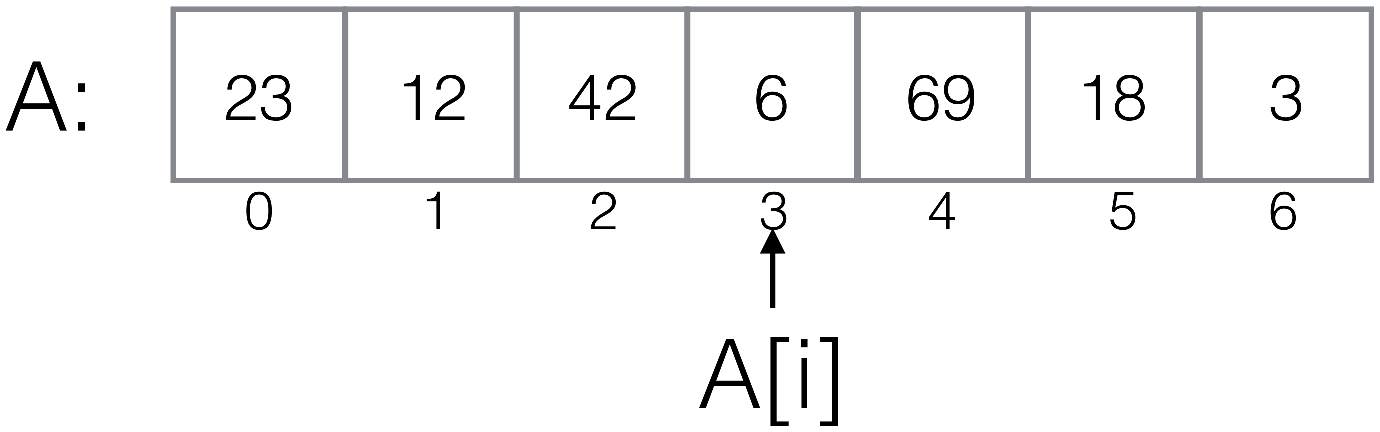

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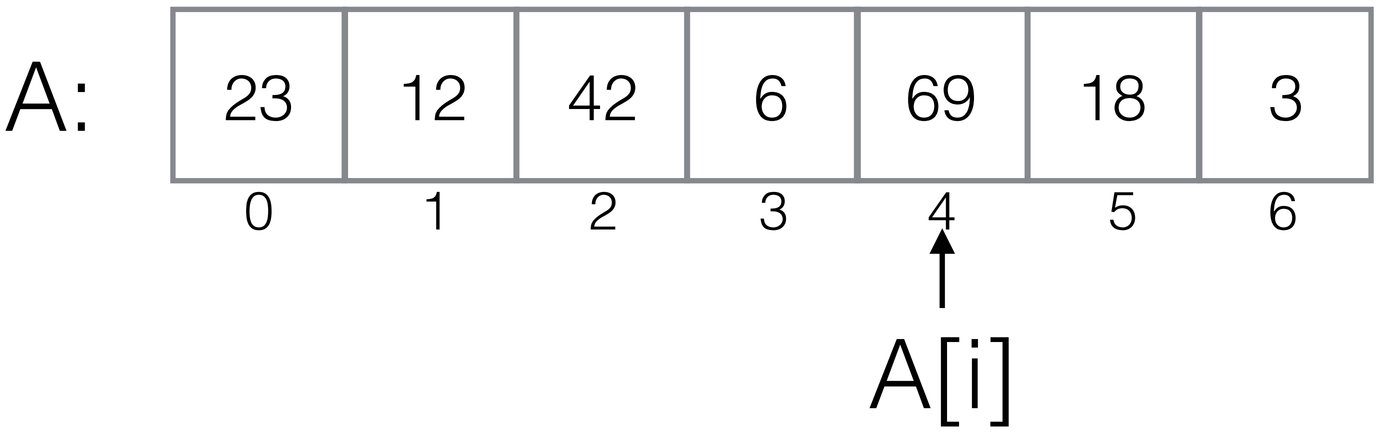

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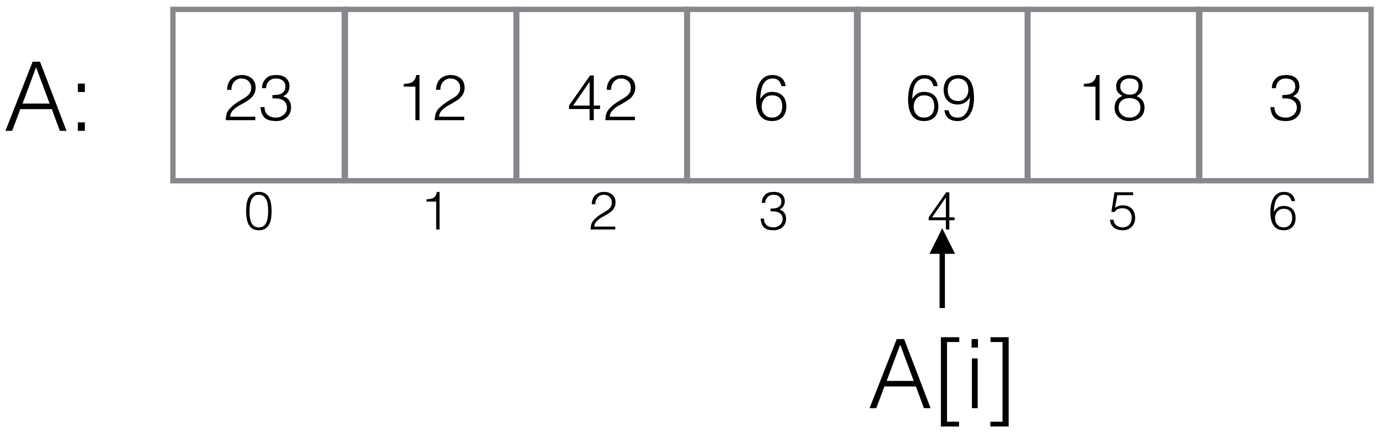

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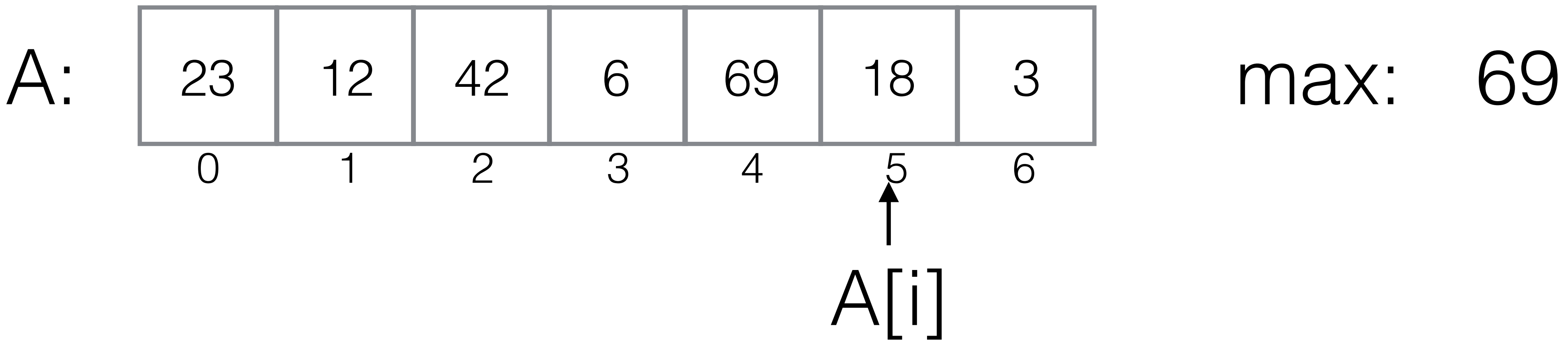

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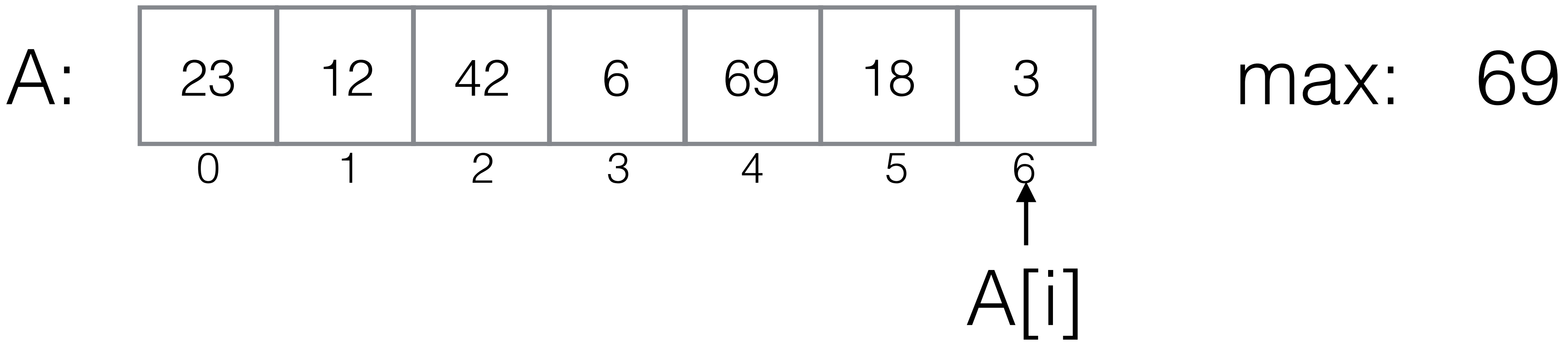

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Size of input, *n*: length of the array

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Size of input, *n*: length of the array

Basic operation: comparison “A[i] > max”

Example:
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(where *n* is length of the array)

Size of input, *n*: length of the array

Basic operation: comparison “A[i] > max”

Count the number of basic operations executed for an array of size *n:*

Example:
Finding Largest Element in an Array

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    for $i \leftarrow 1$ to $n - 1$ do
        if $A[i] > max$ then
            $max \leftarrow A[i]$
    return $max$
```

(where *n* is length of the array)

Size of input, *n*: length of the array

Basic operation: comparison “A[i] > max”

Count the number of basic operations executed for an array of size *n:*

1

Example:
Finding Largest Element in an Array

```
function MaxElement($A[\cdot], n$)
    $max \leftarrow A[0]$
    for $i \leftarrow 1$ to $n - 1$ do
        if $A[i] > max$ then
            $max \leftarrow A[i]$
    return $max$
```

(where n is length of the array)

Size of input, n: length of the array

Basic operation: comparison “A[i] > max”

Count the number of basic operations executed for an array of size n:

$$C(n) = \sum_{i=1}^{n-1} 1$$

Example:
Finding Largest Element in an Array

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function $\textsc{MaxElement}(A[\cdot], n)$
    $max \leftarrow A[0]$
    for $i \leftarrow 1$ to $n - 1$ do
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(where *n* is length of the array)

Size of input, *n*: length of the array

Basic operation: comparison “A[i] > max”

Count the number of basic operations executed for an array of size *n:*

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1)-1+1)$$

Example:
Finding Largest Element in an Array

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```
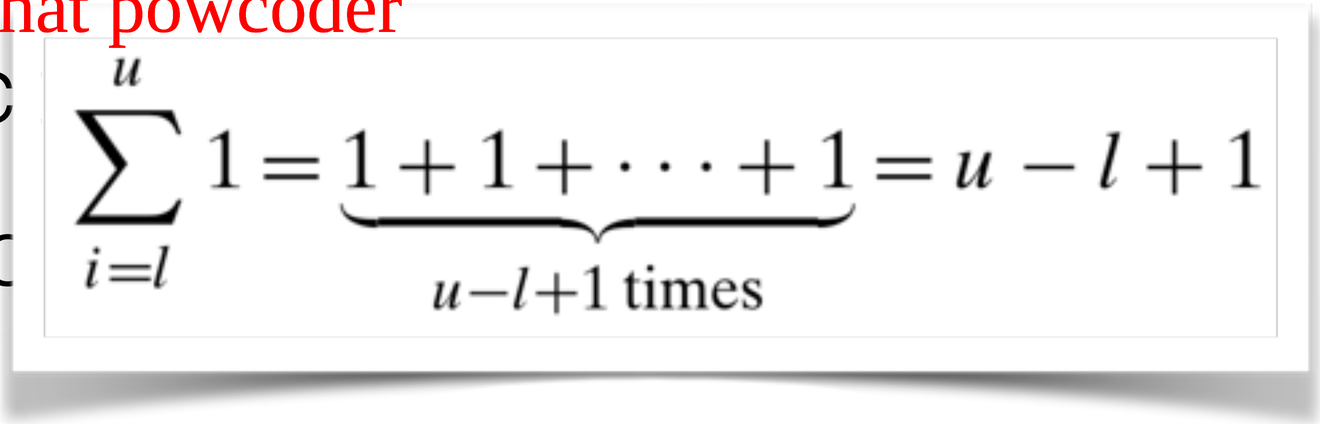

(where *n* is length of the array)

Size of input, *n*: length of the array

Basic operation: c



$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

Count the number of basic for an array of size *n:*

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1)$$

Example:
Finding Largest Element in an Array

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Count the number of basic

for an array of size *n:*

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1) = n - 1$$

Example:
Finding Largest Element in an Array

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(where *n* is length of the array)

Size of input, *n*: length of the array

Basic operation: c

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

Count the number of basic
for an array of size *n:*

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1) = n - 1 \in \Theta(n)$$

Example: Selection Sort

```
function SelSort($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ do
            if $A[j] < A[min]$ then
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$
```

23	12	42	6	69	18	3
0	1	2	3	4	5	6

Example: Selection Sort

```
function SelSort($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
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```

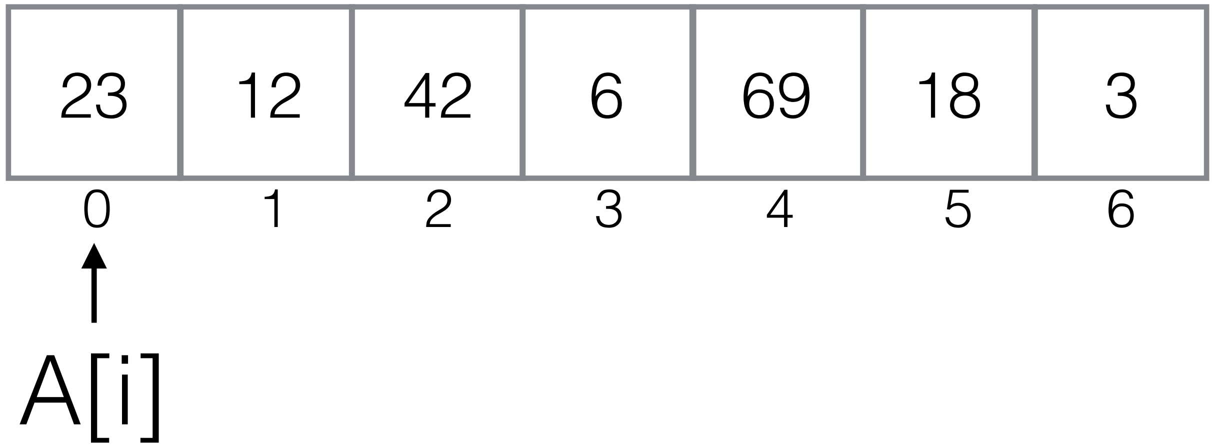

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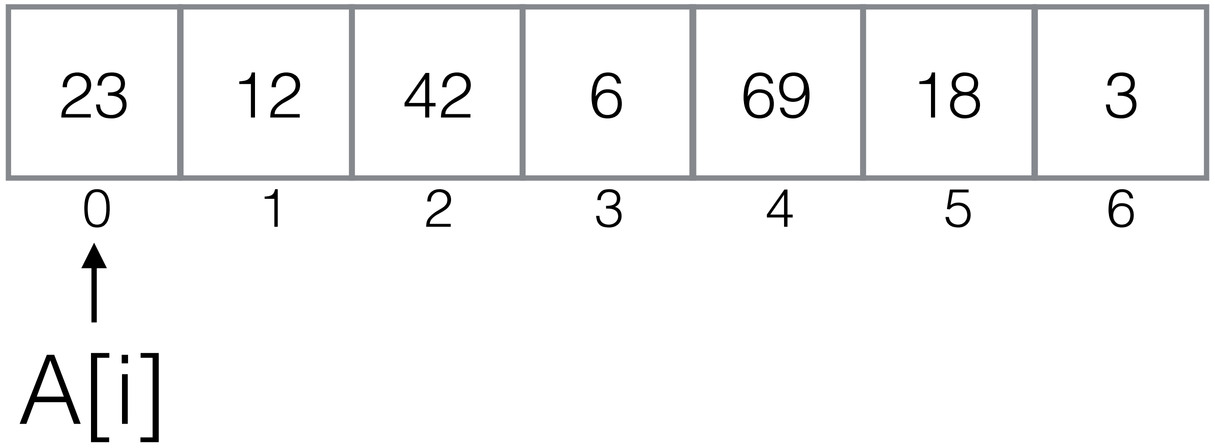

23	12	42	6	69	18	3
0	1	2	3	4	5	6

min: 0

Example: Selection Sort

```
function SELSORT($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ do
            if $A[j] < A[min]$ then
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        swap $A[i]$ and $A[min]$
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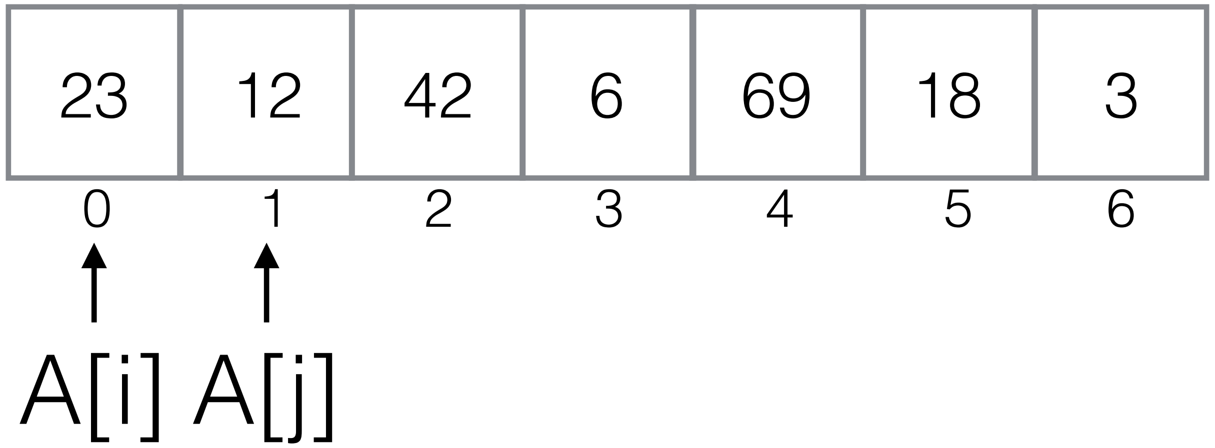

23	12	42	6	69	18	3
0	1	2	3	4	5	6

min: 0

Example: Selection Sort

```
function $\textsc{SelSort}(A[\cdot], n)$
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        swap $A[i]$ and $A[min]$
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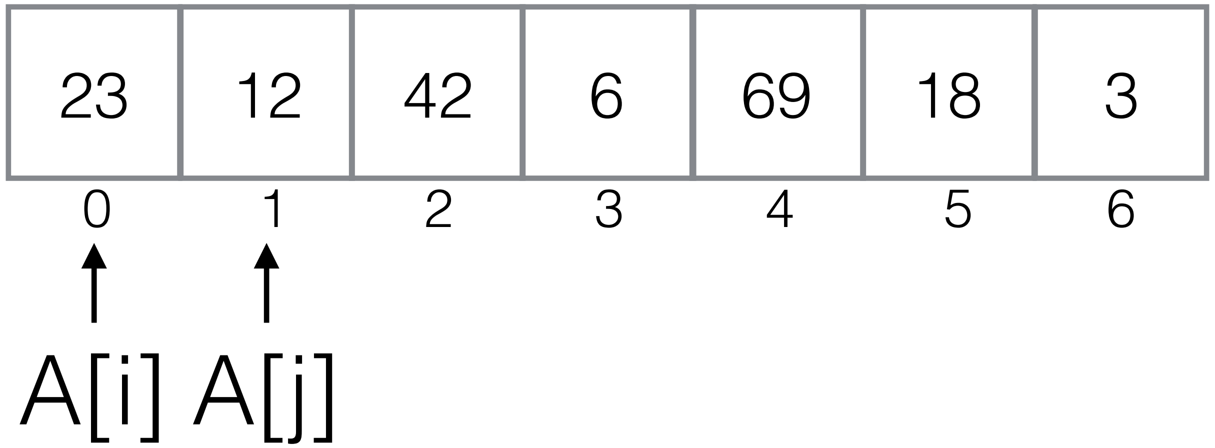

23	12	42	6	69	18	3
0	1	2	3	4	5	6

min: 1

Example: Selection Sort

```
function SelSort($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
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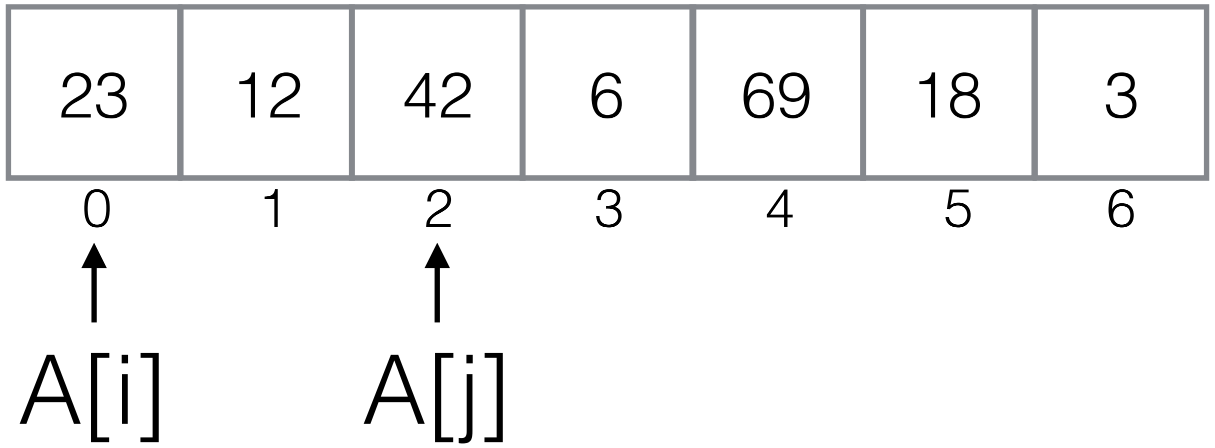

23	12	42	6	69	18	3
0	1	2	3	4	5	6

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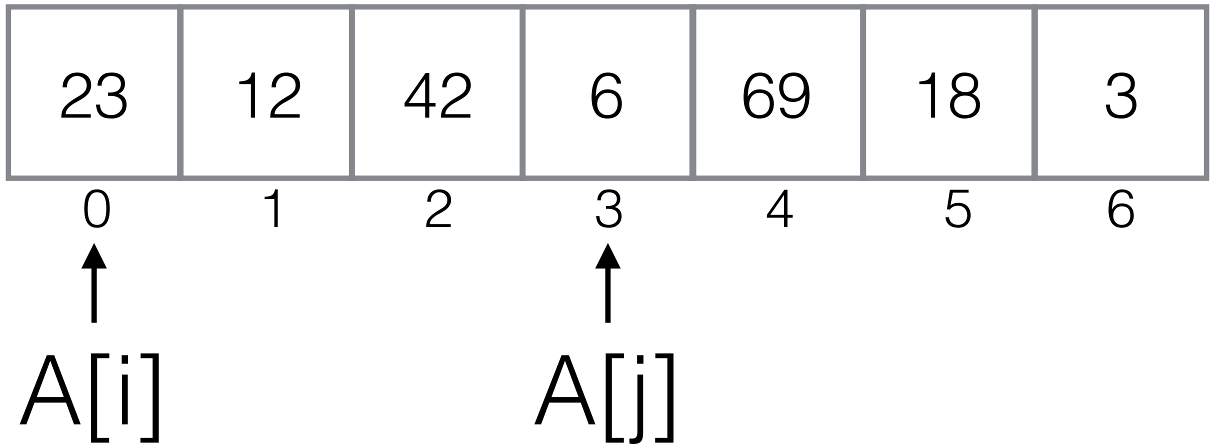

23	12	42	6	69	18	3
0	1	2	3	4	5	6

min: 1

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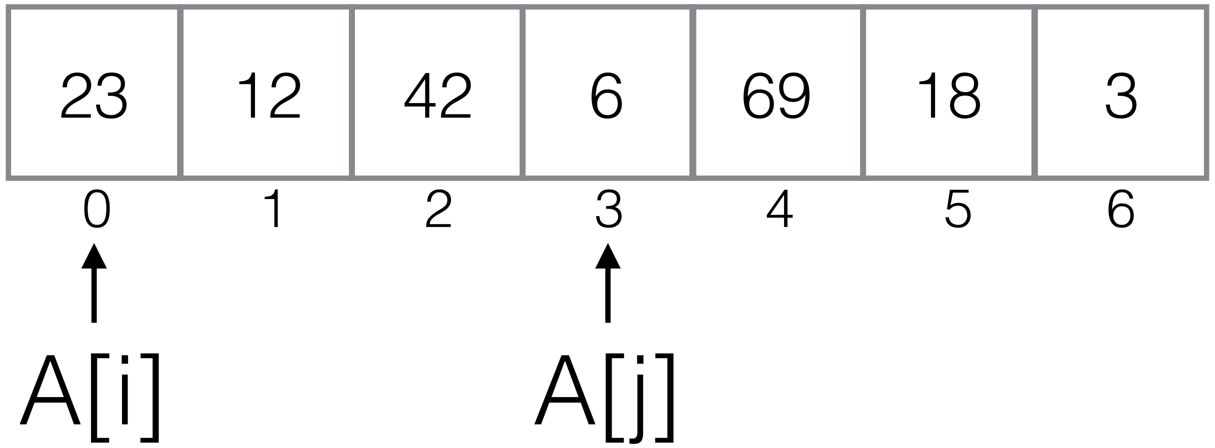

23	12	42	6	69	18	3
0	1	2	3	4	5	6

min: 3

Example: Selection Sort

```
function SelSort($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
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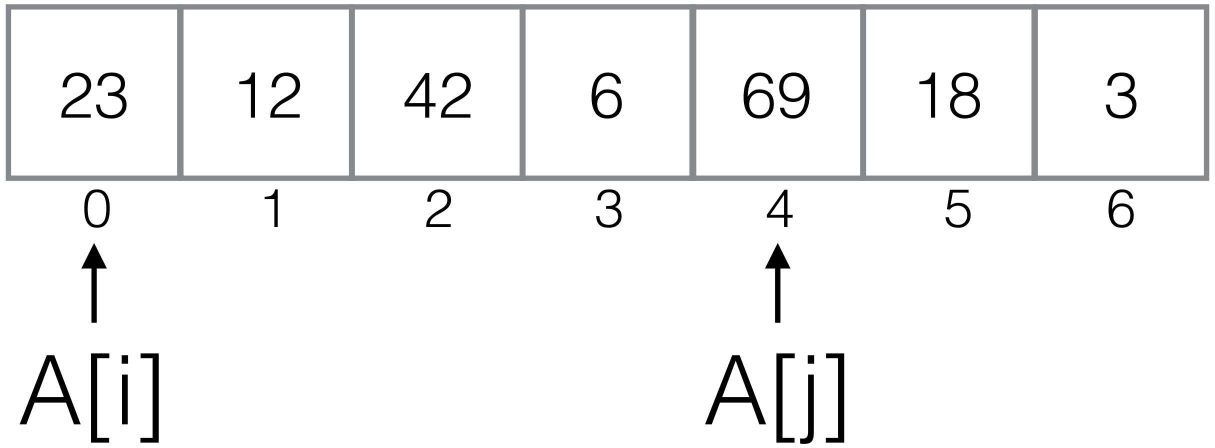

23	12	42	6	69	18	3
0	1	2	3	4	5	6

min: 3

Example: Selection Sort

```
function SELSORT($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
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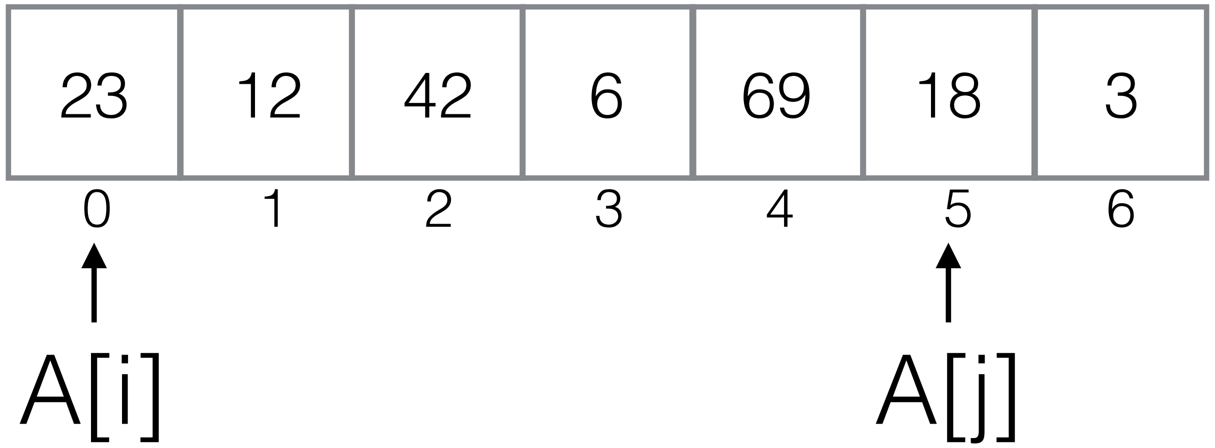

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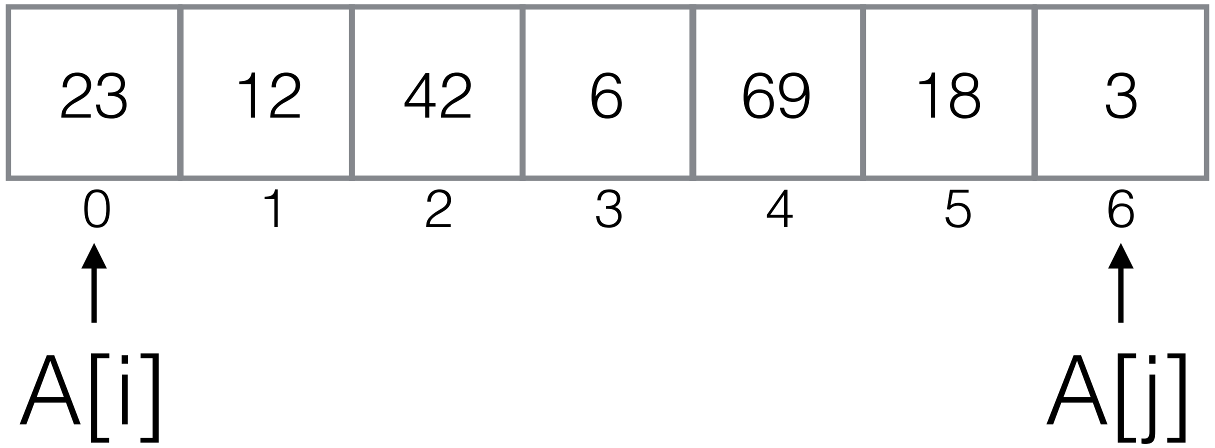

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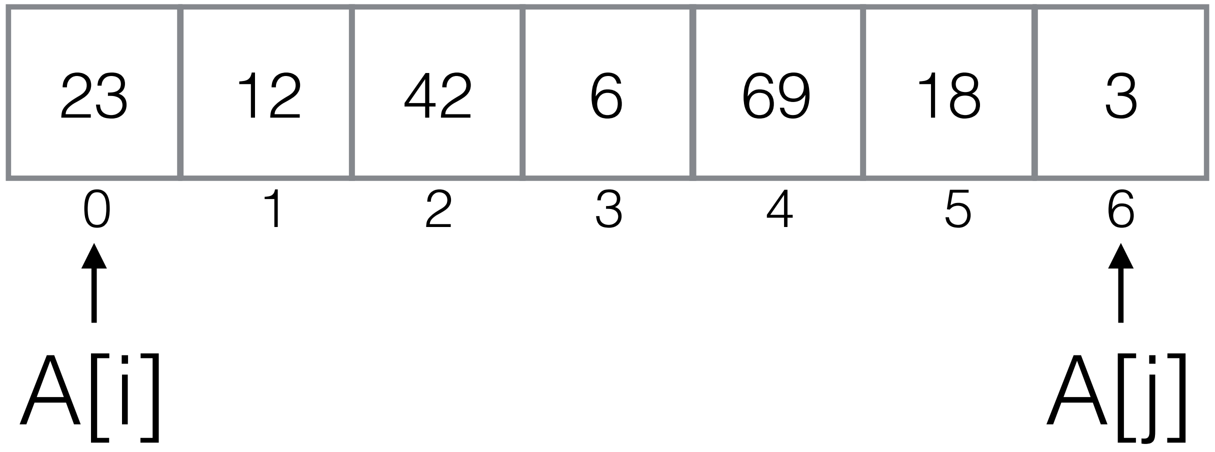

23	12	42	6	69	18	3
0	1	2	3	4	5	6

min: 6

Example: Selection Sort

```
function SelSort($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
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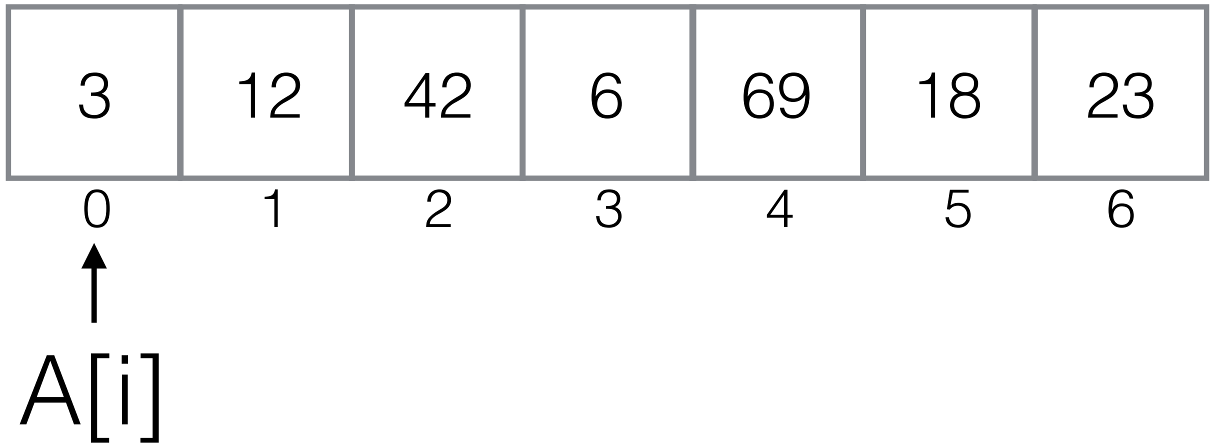

3	12	42	6	69	18	23
0	1	2	3	4	5	6

min: 6

Example: Selection Sort

```
function SelSort($A[\cdot]$, $n$)
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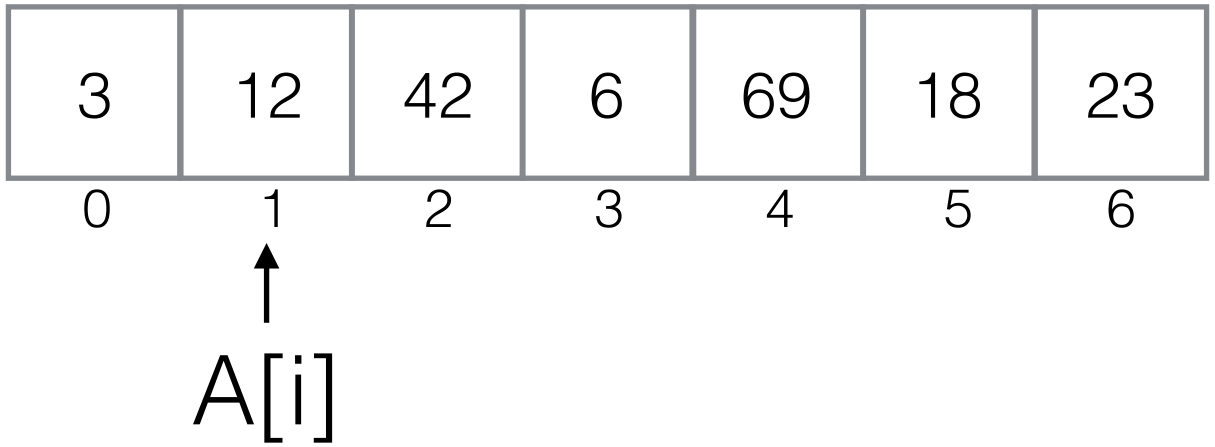

3	12	42	6	69	18	23
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min: 6

Example: Selection Sort

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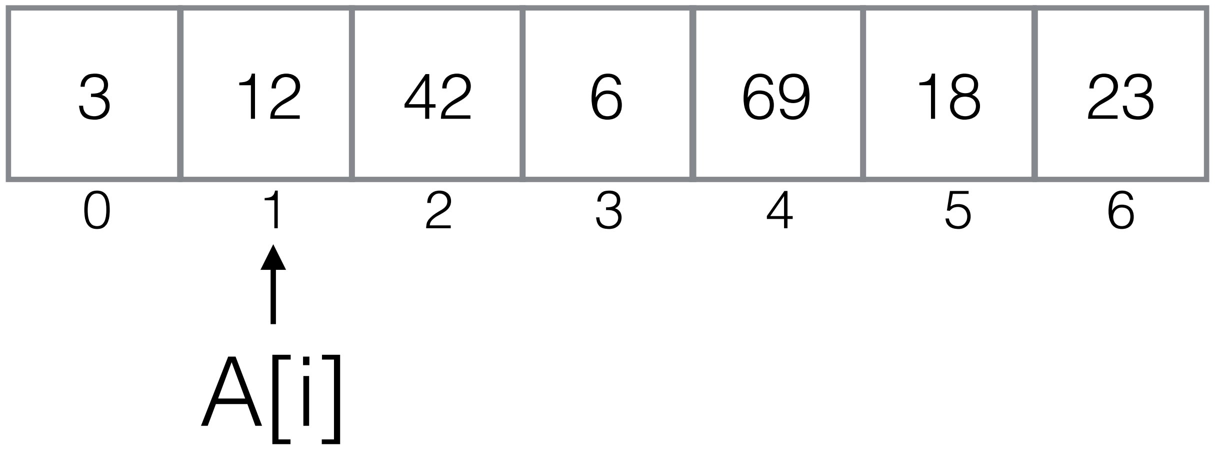

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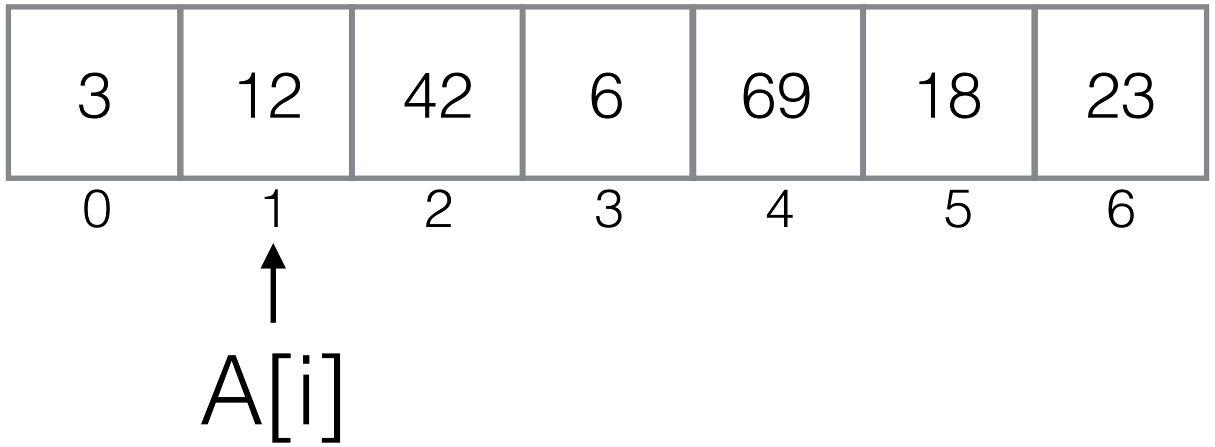

3	12	42	6	69	18	23
0	1	2	3	4	5	6

min: 1

Example: Selection Sort

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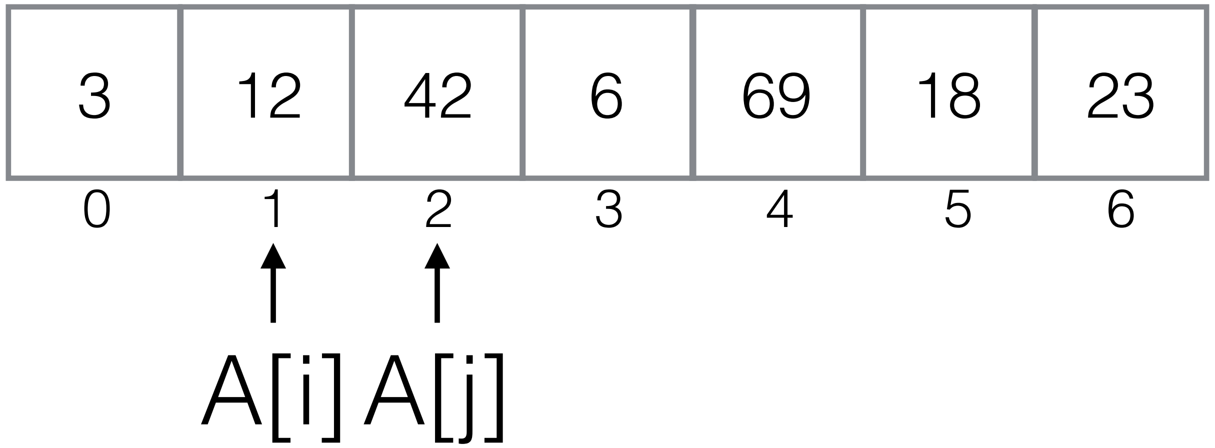

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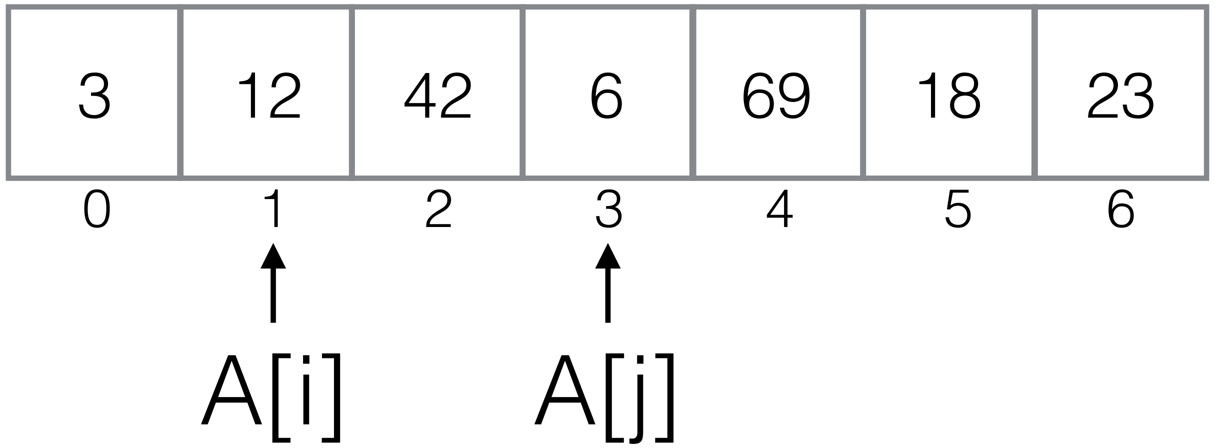

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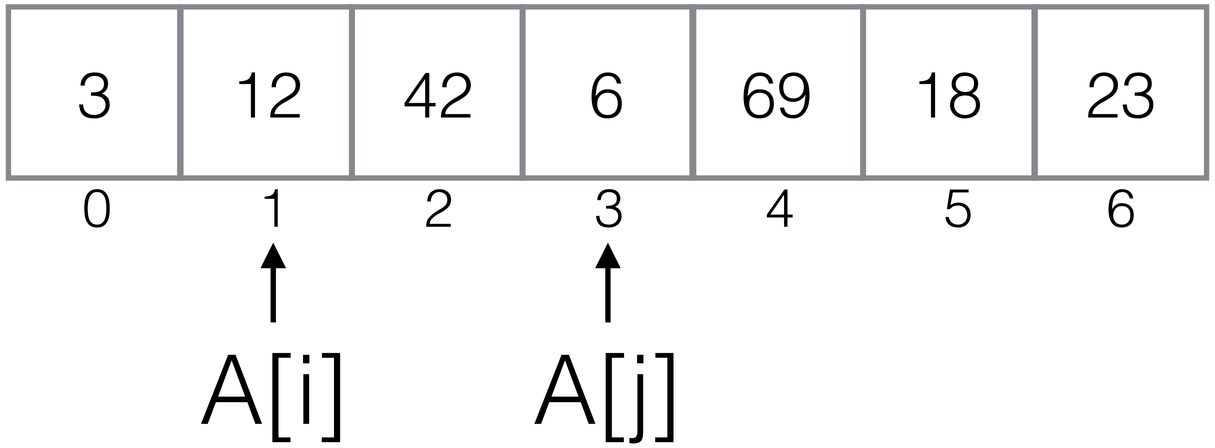

3	12	42	6	69	18	23
0	1	2	3	4	5	6

min: 3

Example: Selection Sort

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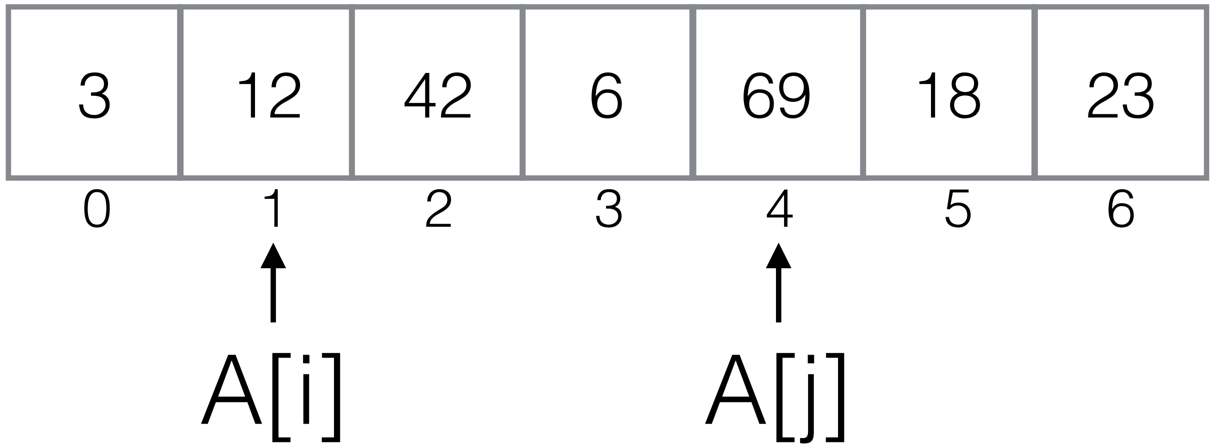

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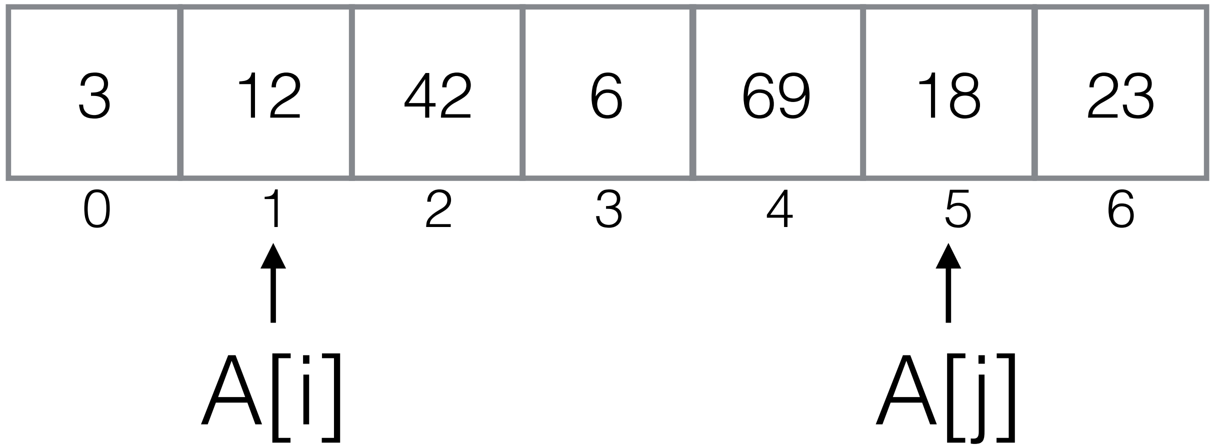

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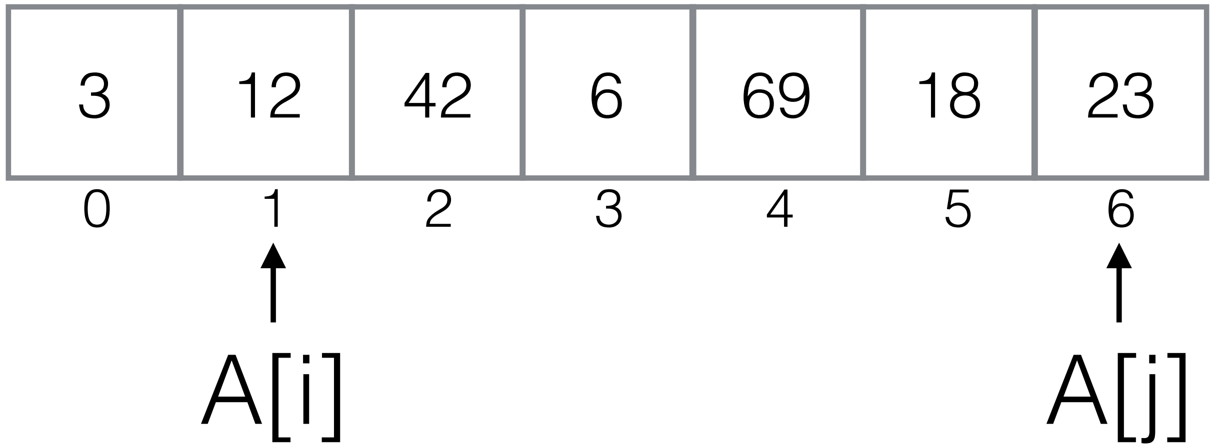

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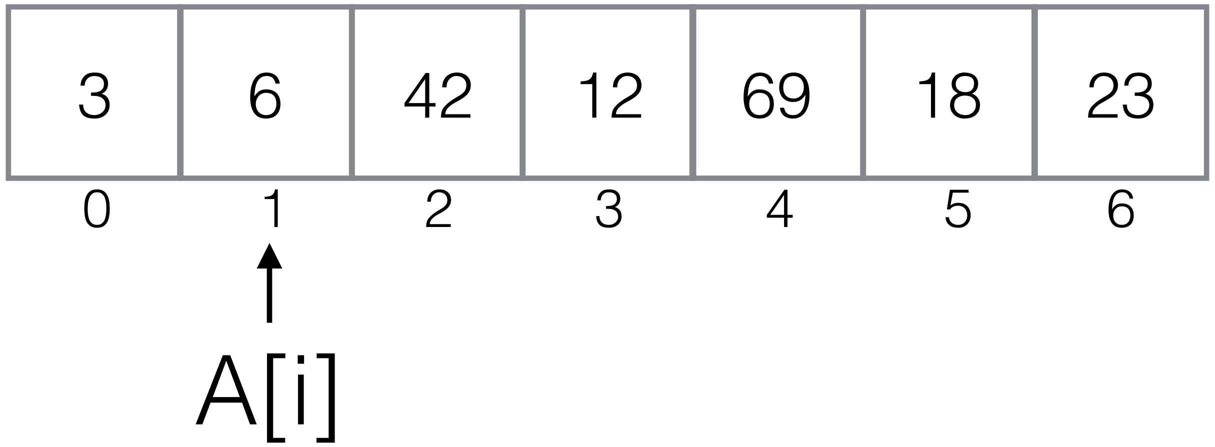

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0	1	2	3	4	5	6

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Example: Selection Sort

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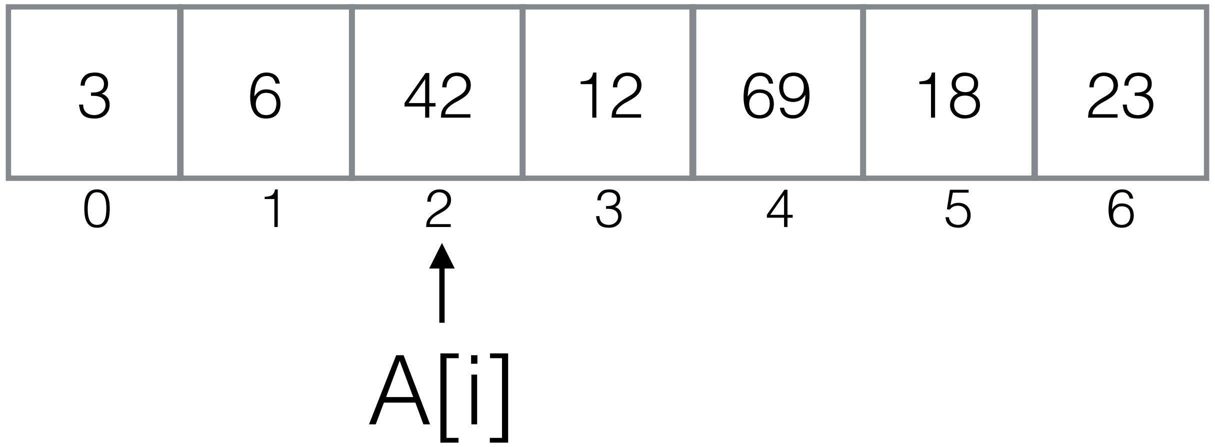
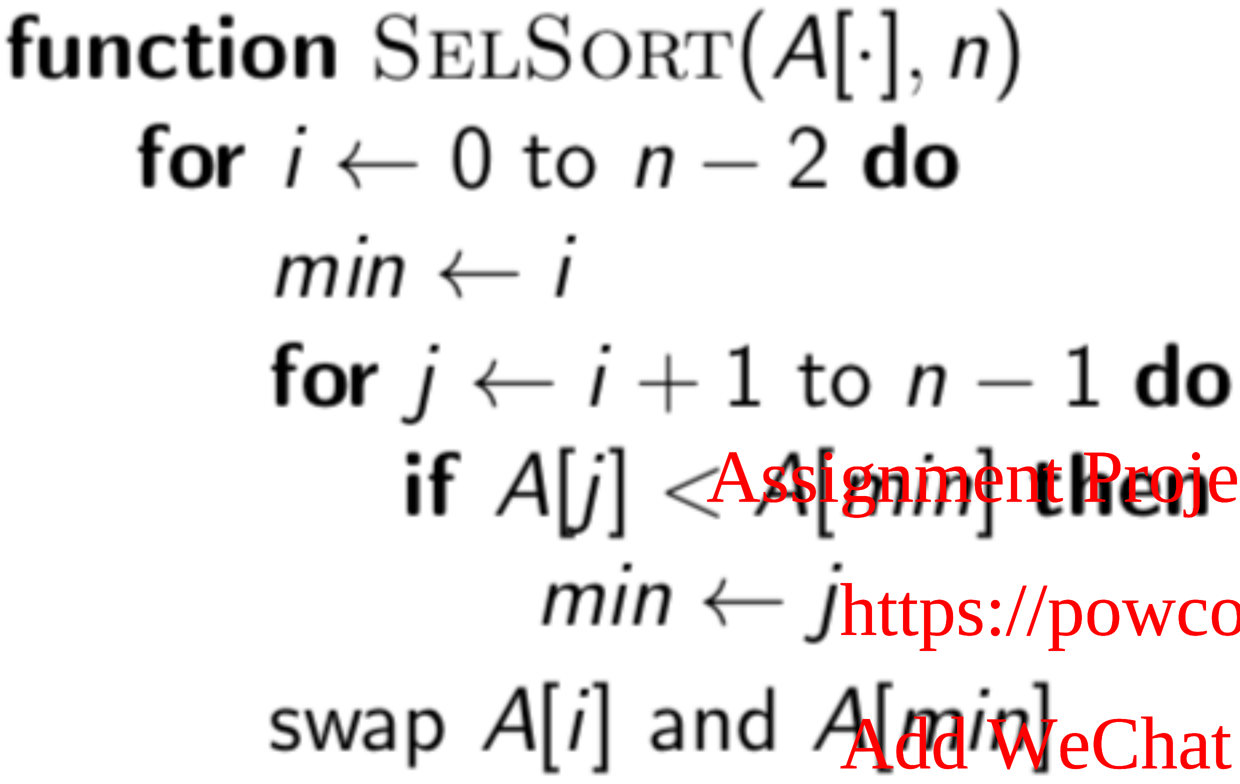


3	6	42	12	69	18	23
0	1	2	3	4	5	6

min: 3

Example: Selection Sort



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Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

Example: Selection Sort

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Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

C(n) =

Example: Selection Sort

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function SELSORT($A[\cdot]$, $n$)
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        for $j \leftarrow i+1$ to $n-1$ do
            if $A[j] < A[min]$ then
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$
```

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1$$

Example: Selection Sort

```
function SelSort($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ do
            if $A[j] < A[min]$ then
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$
```

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n - 1 - i)$$

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n-1-i) = \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i$$

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
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Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n-1-i) = \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2}$$

Example: Selection Sort

```
function SelSort($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ do
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ do
            if $A[j] < A[min]$ then
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$
```

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n - 1 - i) = \sum_{i=0}^{n-2} (n - 1) - \sum_{i=0}^{n-2} i$$

$$= (n - 1)^2 - \frac{(n - 2)(n - 1)}{2} = \frac{n(n - 1)}{2} \in \Theta(n^2)$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
        for $j \leftarrow 0$ to $n - 1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n - 1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)    ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
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            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 0

$$\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix} \qquad \begin{bmatrix} & \\ & \end{bmatrix}$$

A B C

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)    ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
        for $j \leftarrow 0$ to $n - 1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n - 1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 0

$$\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & \\ & \end{bmatrix}$$

A B C

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)    ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
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            for $k \leftarrow 0$ to $n - 1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 0

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 0 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
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                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 0

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 40 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
        for $j \leftarrow 0$ to $n - 1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n - 1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 0

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 40 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MATRIXMULT($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 0

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$)    ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
        for $j \leftarrow 0$ to $n - 1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n - 1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \quad \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 0 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$    ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
        for $j \leftarrow 0$ to $n - 1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n - 1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 1

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 0 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 1

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 10 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 1

k: 1

$$\underset{A}{\begin{bmatrix}5 & 7\\ 3 & 4\end{bmatrix}} \underset{B}{\begin{bmatrix}8 & 2\\ 9 & 6\end{bmatrix}} \qquad \underset{C}{\begin{bmatrix}103 & 10\\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
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            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

i: 0

j: 1

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 52 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
        for $j \leftarrow 0$ to $n - 1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n - 1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

Basic operation: multiplication A[i,k] * B[k,j]

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
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            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) =$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
        for $j \leftarrow 0$ to $n - 1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n - 1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \qquad 1$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ do
        for $j \leftarrow 0$ to $n-1$ do
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ do
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return $C$
```

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

```
function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, $n$)        ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n - 1$ do
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    return $C$
```

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

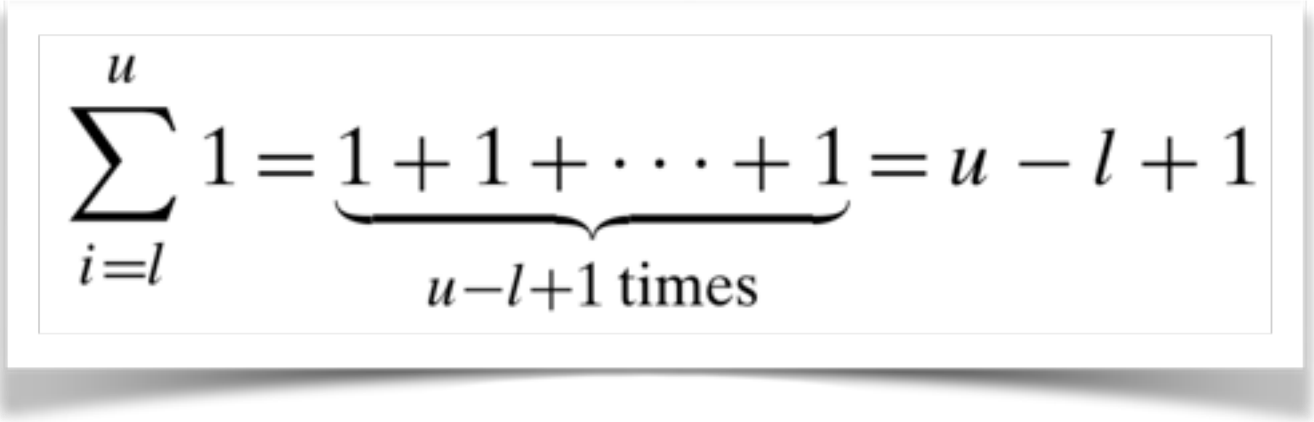
$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

Example: Matrix Multiplication

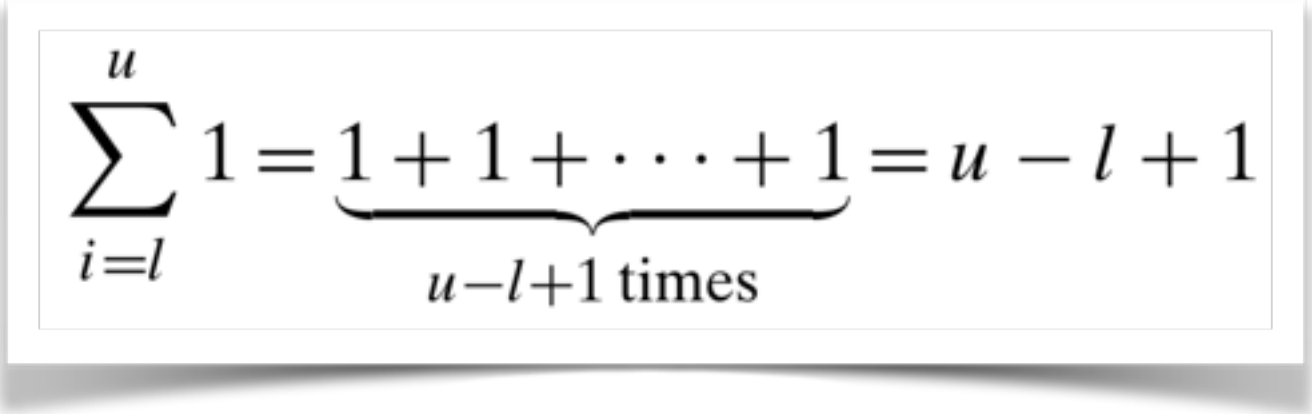


$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1)$$

Example: Matrix Multiplication



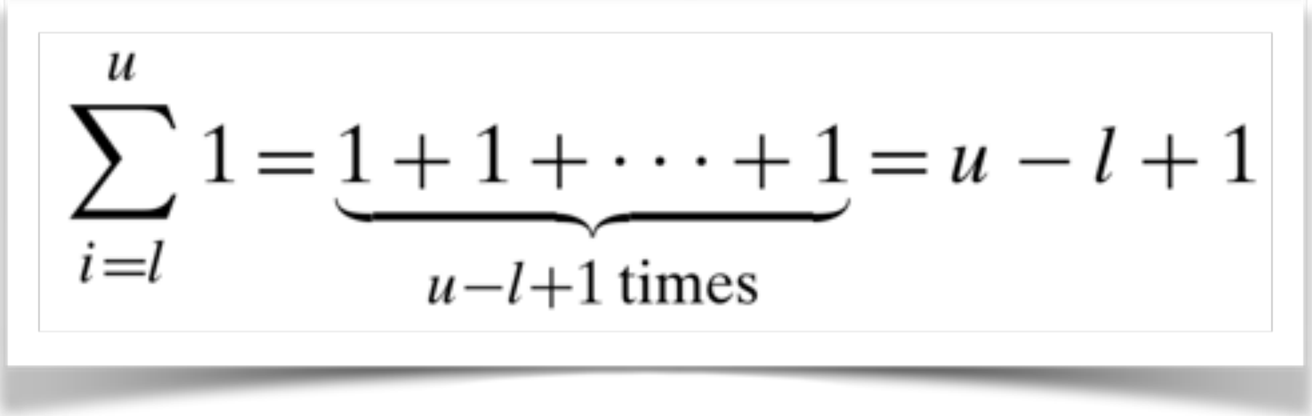
$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n$$

Example: Matrix Multiplication

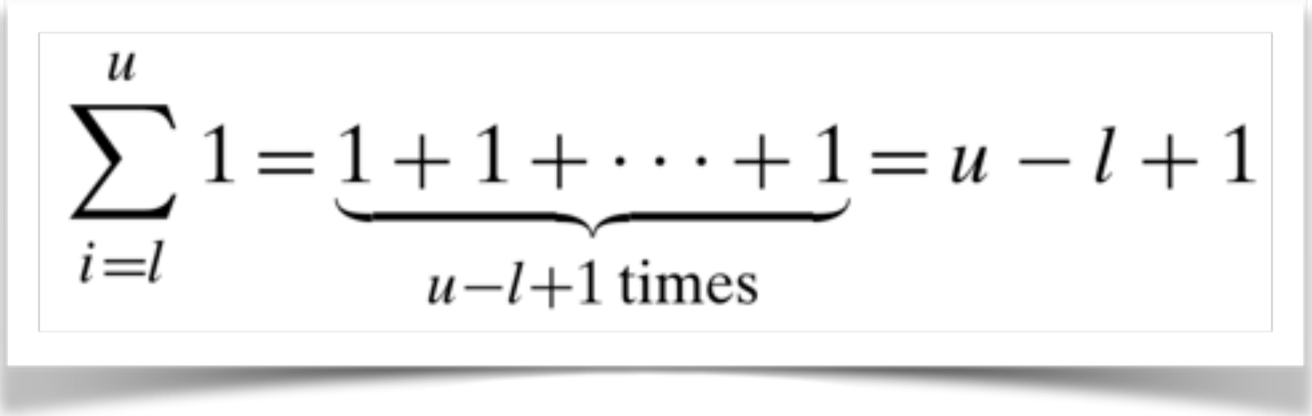
$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$



$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n \; = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

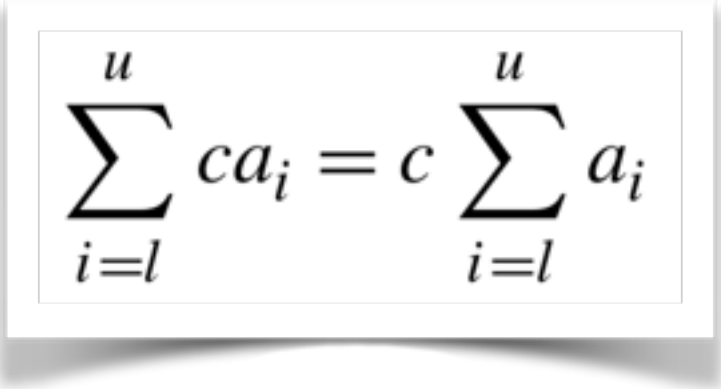
Example: Matrix Multiplication



$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n \ = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$



$$\sum_{i=l}^{u} ca_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n \; = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right)$$

$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n \ = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right) \ = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right) = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

$$= n^3$$

Example: Matrix Multiplication

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1)-0+1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n \; = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right) \quad = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} ca_i = c \sum_{i=l}^{u} a_i$$

$$= n^3$$

$$\in \Theta(n^3)$$

Analysing Recursive Algorithms

```
function $F(n)$
    if $n = 0$ then return $1$
    else return $F(n - 1) \cdot n$
```

Analysing Recursive Algorithms

```
function F(n)
    if $n = 0$ then return 1
    else return $F(n-1) \cdot n$
```

$F(5)$

Analysing Recursive Algorithms

```
function $F(n)$
    if $n = 0$ then return 1
    else return $F(n-1) \cdot n$
```

$$F(5) = F(4) \cdot 5$$

Analysing Recursive Algorithms

```
function $\mathrm{F}(n)$
    if $n = 0$ then return 1
    else return $\mathrm{F}(n-1) \cdot n$
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

```
function $F(n)$
    if $n = 0$ then return 1
    else return $F(n-1) \cdot n$
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

```
function F(n)
    if $n = 0$ then return 1
    else return $F(n-1) \cdot n$
```

$$\begin{aligned} F(5) &= F(4) \cdot 5 \\ &= (F(3) \cdot 4) \cdot 5 \\ &= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\ &= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \end{aligned}$$

Analysing Recursive Algorithms

```
function $F(n)$
    if $n = 0$ then return 1
    else return $F(n - 1) \cdot n$
```

$$\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}$$

Analysing Recursive Algorithms

```
function F(n)
    if $n = 0$ then return 1
    else return $F(n-1) \cdot n$
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

```
function F($n$)
    if $n = 0$ then return 1
    else return F$(n - 1) \cdot n$
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 5!
\end{aligned}
$$

Analysing Recursive Algorithms

```
function $F(n)$
    if $n = 0$ then return $1$
    else return $F(n-1) \cdot n$
```

Analysing Recursive Algorithms

```
function F(n)
    if $n = 0$ then return 1
    else return $F(n - 1) \cdot n$
```

Basic operation: multiplication

Analysing Recursive Algorithms

```
function F(n)
    if $n = 0$ then return 1
    else return $F(n-1) \cdot n$
```

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

Analysing Recursive Algorithms

```
function F(n)
    if n = 0 then return 1
    else return F(n − 1) · n
```

Basic operation:
multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) =$$

Analysing Recursive Algorithms

```
function F(n)
    if $n = 0$ then return 1
    else return $F(n-1) \cdot n$
```

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$

Analysing Recursive Algorithms

```
function F(n)
    if $n = 0$ then return 1
    else return $F(n - 1) \cdot n$
```

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$

$$M(n) =$$

Analysing Recursive Algorithms

```
function F($n$)
    if $n = 0$ then return 1
    else return F($n - 1$) $\cdot$ $n$
```

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$
\begin{aligned}
M(0) &= \quad 0 \\
M(n) &= \qquad\qquad\qquad 1
\end{aligned}
$$

Analysing Recursive Algorithms

```
function F(n)
    if n = 0 then return 1
    else return F(n − 1) · n
```

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$

$$M(n) = \qquad\qquad + 1$$

Analysing Recursive Algorithms

```
function F(n)
    if n = 0 then return 1
    else return F(n − 1) · n
```

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$
$$M(n) = \quad M(n-1) + 1$$

Analysing Recursive Algorithms

```
function $\mathrm{F}(n)$
    if $n = 0$ then return 1
    else return $\mathrm{F}(n-1) \cdot n$
```

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$
\begin{aligned}
M(0) &= 0 \\
M(n) &= M(n-1) + 1
\end{aligned}
$$

Need to express M(n) in **closed form** (i.e. non-recursively)

Analysing Recursive Algorithms

```
function F(n)
    if $n = 0$ then return 1
    else return $F(n - 1) \cdot n$
```

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$
\begin{aligned}
M(0) &= \quad 0 \\
M(n) &= \quad M(n-1) + 1
\end{aligned}
$$

Need to express M(n) in **closed form** (i.e. non-recursively)

Try: **“telescoping”** aka **“backward substitution”**

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ? $M(n-1) = M((n-1)-1) + 1$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) =$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2) + 1) + 1$$

Telescoping (aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2) + 1) + 1$$

$$= M(n-2) + 2$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2) + 1) + 1$$

$$= M(n-2) + 2$$

$$= (M(n-3) + 1) + 2$$

Telescoping (aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1}$$

$$\boxed{M(0) = 0}$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2) + 1) + 1$$
$$= M(n-2) + 2$$
$$= (M(n-3) + 1) + 2$$
$$= M(n-3) + 3$$

Telescoping
(aka Backward Substitution)

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$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\ldots \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2)+1)+1$$

$$= M(n-2) + 2$$

$$= (M(n-3)+1)+2$$
$$= M(n-3) + 3$$
$$\ldots$$
$$= M(n-n) + n$$

Telescoping
(aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2) + 1) + 1 \\ &= M(n-2) + 2 \\ &= (M(n-3) + 1) + 2 \\ &= M(n-3) + 3 \\ &\dots \\ &= M(n-n) + n \\ &= M(0) + n \end{aligned}$$

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\ldots \\ &= M(n-n)+n \\ &= M(0)+n \\ &= n \end{aligned}$$

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2) + 1) + 1 \\ &= M(n-2) + 2 \\ &= (M(n-3) + 1) + 2 \\ &= M(n-3) + 3 \\ &\ldots \\ &= M(n-n) + n \\ &= M(0) + n \\ &= n \end{aligned}$$

Closed form:

Telescoping (aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$\begin{aligned}
M(n) &= (M(n-2)+1)+1 \\
&= M(n-2)+2 \\
&= (M(n-3)+1)+2 \\
&= M(n-3)+3 \\
&\ldots \\
&= M(n-n)+n \\
&= M(0)+n \\
&= n
\end{aligned}$$

Closed form:

$$\boxed{M(n) = n}$$

Telescoping (aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1}$$

$$\boxed{M(0) = 0}$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2)+1)+1$$

$$= M(n-2) + 2$$

$$= (M(n-3)+1)+2$$
$$= M(n-3) + 3$$
$$\ldots$$
$$= M(n-n) + n$$
$$= M(0) + n$$
$$= n$$

Closed form:

$$\boxed{M(n) = n}$$

Complexity:

Telescoping (aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2) + 1) + 1 \\ &= M(n-2) + 2 \\ &= (M(n-3) + 1) + 2 \\ &= M(n-3) + 3 \\ &\dots \\ &= M(n-n) + n \\ &= M(0) + n \\ &= n \end{aligned}$$

Closed form:

$$\boxed{M(n) = n}$$

Complexity:

$$\boxed{M(n) \in \Theta(n)}$$

Example:
Binary Search in Sorted Array

```
function $\textsc{BinSearch}(A[\cdot], lo, hi, key)$
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return $\textsc{BinSearch}(A, lo, mid - 1, key)$
        else return $\textsc{BinSearch}(A, mid + 1, hi, key)$
```

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,0,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 0

A:

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,0,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot]$, $lo$, $hi$, $key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A$, $lo$, $mid - 1$, $key$)
        else return BinSearch($A$, $mid + 1$, $hi$, $key$)
```

lo: 0

hi: 6

A:

4	9	13	22	41	83	96
0	1	2	3	4	5	6

BinSearch(A,0,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 0

hi: 6

key: 41

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,0,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 0

hi: 6

key: 41

mid: 3

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,0,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 0

hi: 6

key: 41

mid: 3

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,0,6,41)

BinSearch(A,4,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,4,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
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    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

A:

4	9	13	22	41	83	96
0	1	2	3	4	5	6

BinSearch(A,4,6,41)

Example:
Binary Search in Sorted Array

```
function BINSEARCH($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BINSEARCH($A, lo, mid - 1, key$)
        else return BINSEARCH($A, mid + 1, hi, key$)
```

lo: 4

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,4,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
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    if $A[mid] = key$ then return $mid$
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            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 4

hi: 6

A:

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,4,6,41)

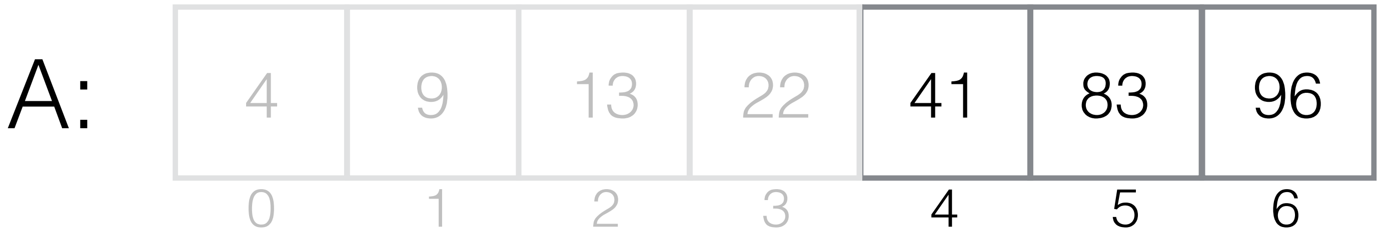
Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 4

hi: 6

key: 41



A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,4,6,41)

Example:
Binary Search in Sorted Array

```
function $\textsc{BinSearch}(A[\cdot], lo, hi, key)$
    if $lo > hi$ then return $-1$
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    else
        if $A[mid] > key$ then
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        else return $\textsc{BinSearch}(A, mid + 1, hi, key)$
```

lo: 4

hi: 6

key: 41

mid: 5

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,4,6,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 4

hi: 6

key: 41

mid: 5

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,4,6,41)

BinSearch(A,4,4,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

A:

4	9	13	22	41	83	96
0	1	2	3	4	5	6

BinSearch(A,4,4,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

A:

4	9	13	22	**41**	83	96
0	1	2	3	**4**	5	6

BinSearch(A,4,4,41)

Example:
Binary Search in Sorted Array

lo: 4

```
function BINSEARCH($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BINSEARCH($A, lo, mid - 1, key$)
        else return BINSEARCH($A, mid + 1, hi, key$)
```

A:

4	9	13	22	41	83	96
0	1	2	3	4	5	6

BinSearch(A,4,4,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 4

hi: 4

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,4,4,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 4

hi: 4

key: 41

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,4,4,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot]$, $lo$, $hi$, $key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A$, $lo$, $mid - 1$, $key$)
        else return BinSearch($A$, $mid + 1$, $hi$, $key$)
```

lo: 4

hi: 4

key: 41

mid: 4

A:

4	9	13	22	41	83	96
0	1	2	3	4	5	6

BinSearch(A,4,4,41)

Example:
Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

lo: 4

hi: 4

key: 41

mid: 4

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

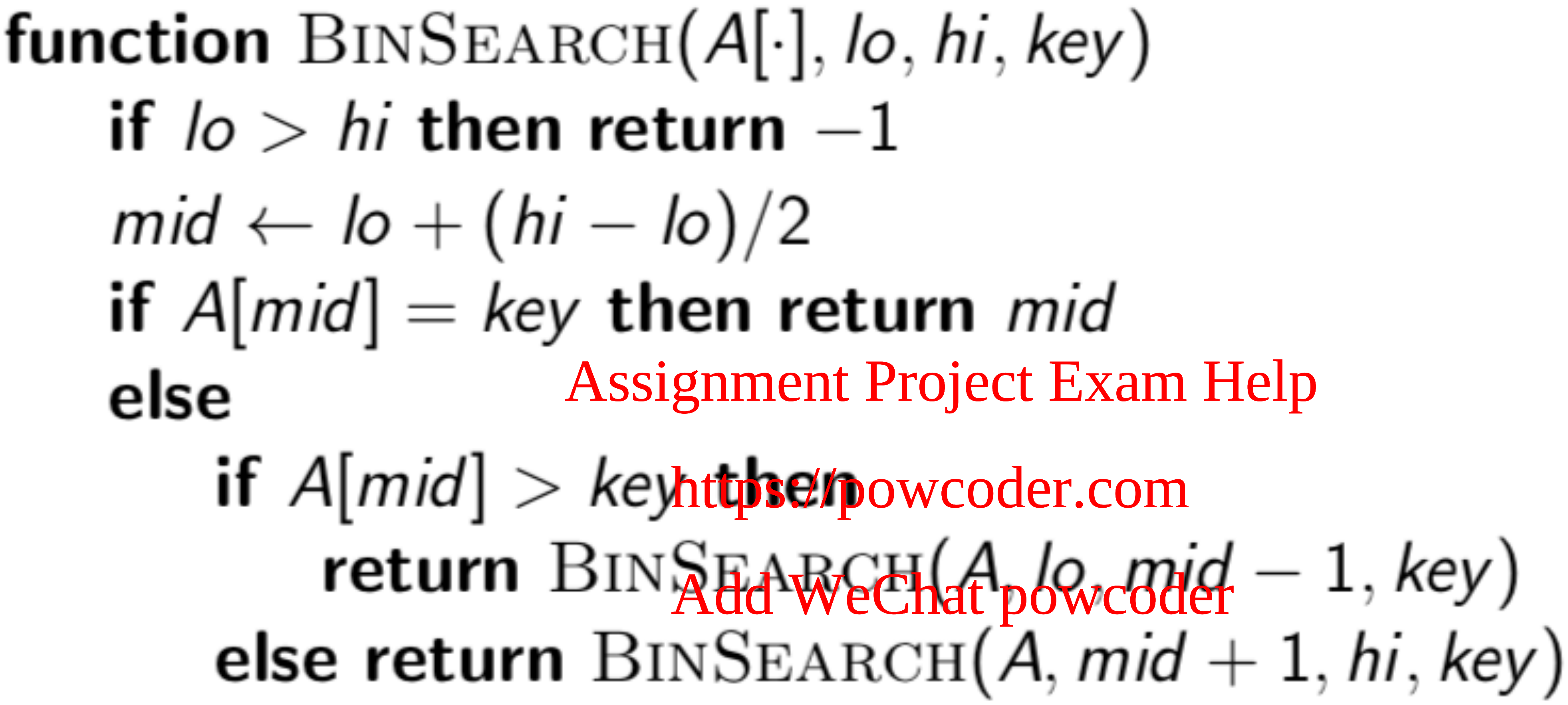
BinSearch(A,4,4,41)

returns 4

Example: Binary Search in Sorted Array

```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
    $mid \leftarrow lo + (hi - lo)/2$
    if $A[mid] = key$ then return $mid$
    else
        if $A[mid] > key$ then
            return BinSearch($A, lo, mid - 1, key$)
        else return BinSearch($A, mid + 1, hi, key$)
```

Example:
Binary Search in Sorted Array



```
function BinSearch($A[\cdot], lo, hi, key$)
    if $lo > hi$ then return $-1$
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$$C(n) \in \Theta(\log n)$$

Logarithmic Functions Have Same Rate of Growth

In *O, Ω, Θ,* expressions we can just write “log” for any logarithmic function no matter what the base is

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Back to Euclid

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function EUCLID($m, n$)
    while $n \neq 0$ do
        $r \leftarrow m \bmod n$
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Why? After two iterations, m becomes m mod n; also

$$1 < n < m \implies m \bmod n < m/2$$

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(for nested loops: count number of times outer loop is executed, multiply by cost of inner loop)

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See also Cormen's Appendix A or Levitin's Appendix A.

Levitin's Appendix B is a tutorial on recurrence relations.

The Road Ahead

- You'll get much more familiar with asymptotic analysis as we use it on algorithms we meet in this course.

- Next week we begin our study of algorithms by looking at **brute force** approaches