COMP90038 Algorithms and Complexity

Lecture 11: Sorting with Divide-and-Conquer

(with thanks to Harald Søndergaard)

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Divide and Conquer

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- We earlier studied recursion as a powerful problem solving technique.

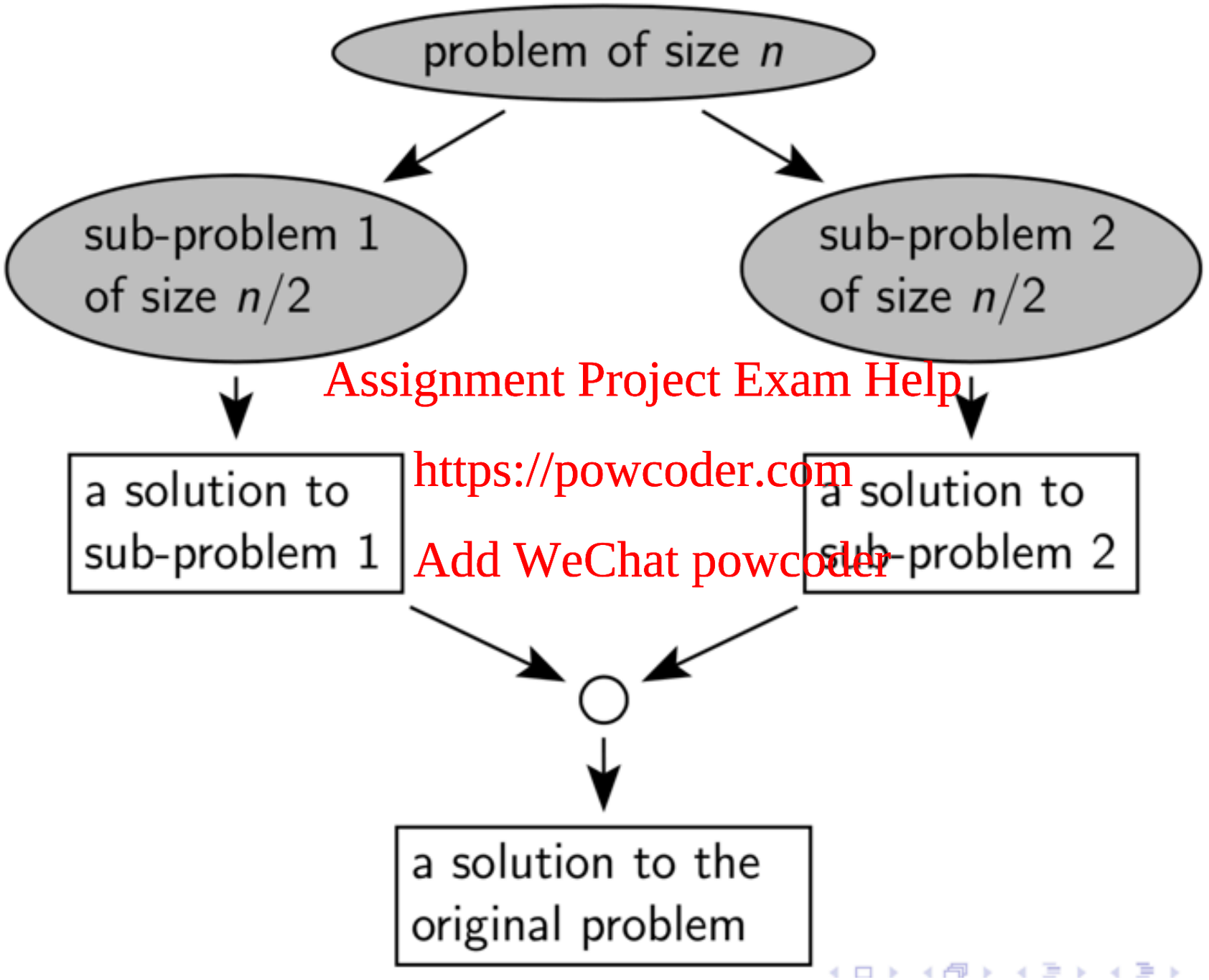
- The **divide-and-conquer** strategy tries to make the most of this idea:

1. Divide the given problem instance into smaller instances.

2. Solve the smaller instances recursively.
3. Combine the smaller solutions to solve the original instance.

- This works best when the smaller instances can be made to be of equal (or near-equal) size.

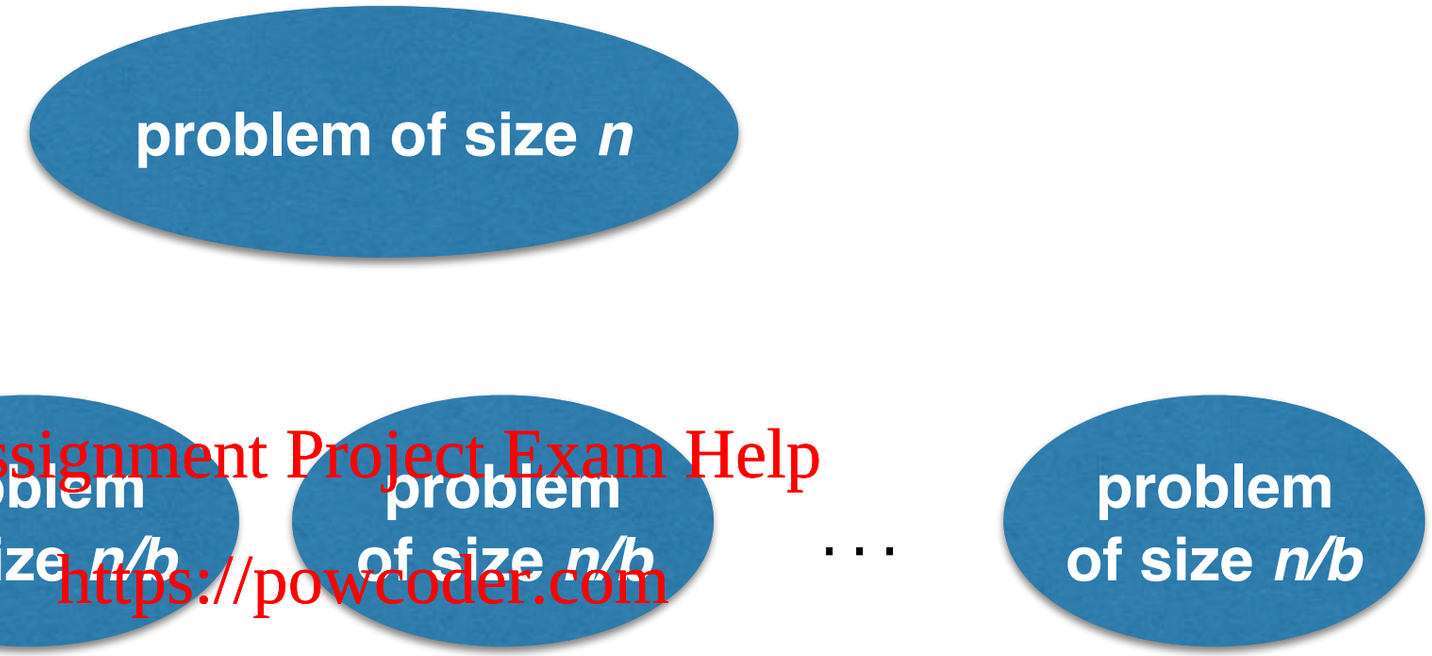
Split-Solve-and-Join Approach



Divide and Conquer Algorithms

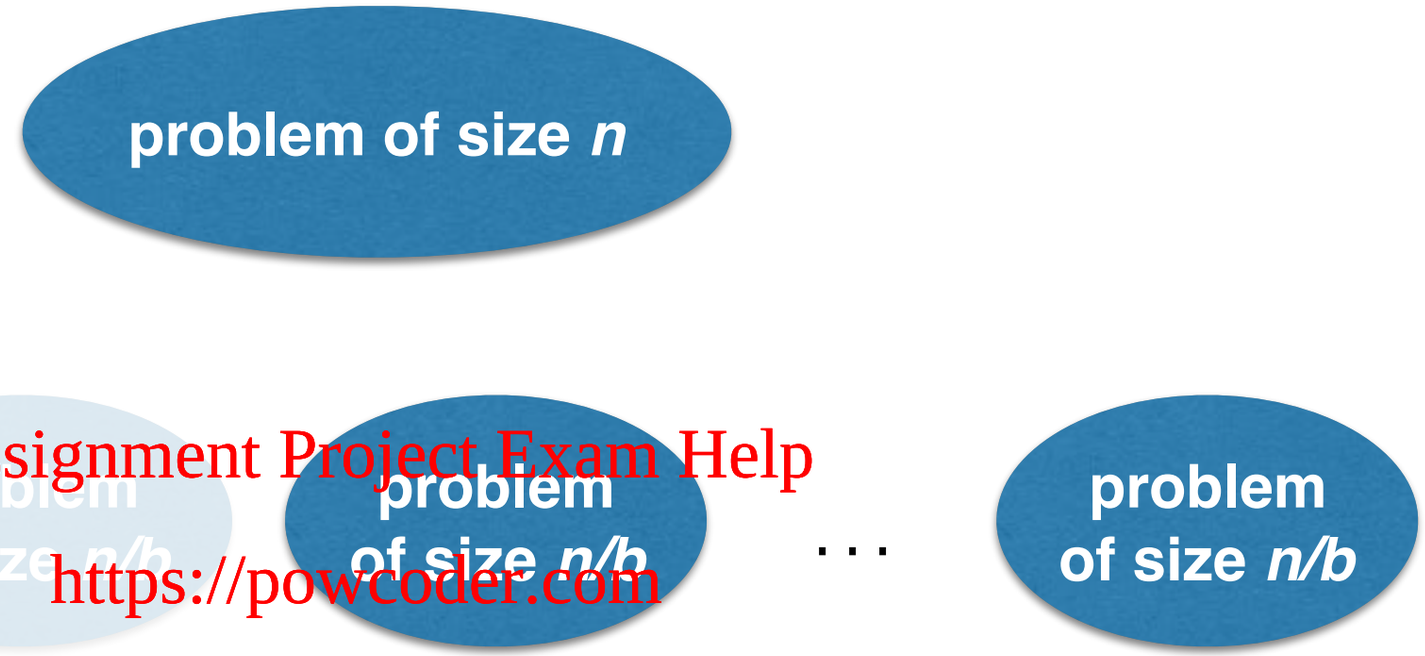
- We will discuss:
  - The Master Theorem
  - Mergesort
  - Quicksort
  - Tree traversal
  - Closest Pair revisited

Divide-and-Conqer General Case



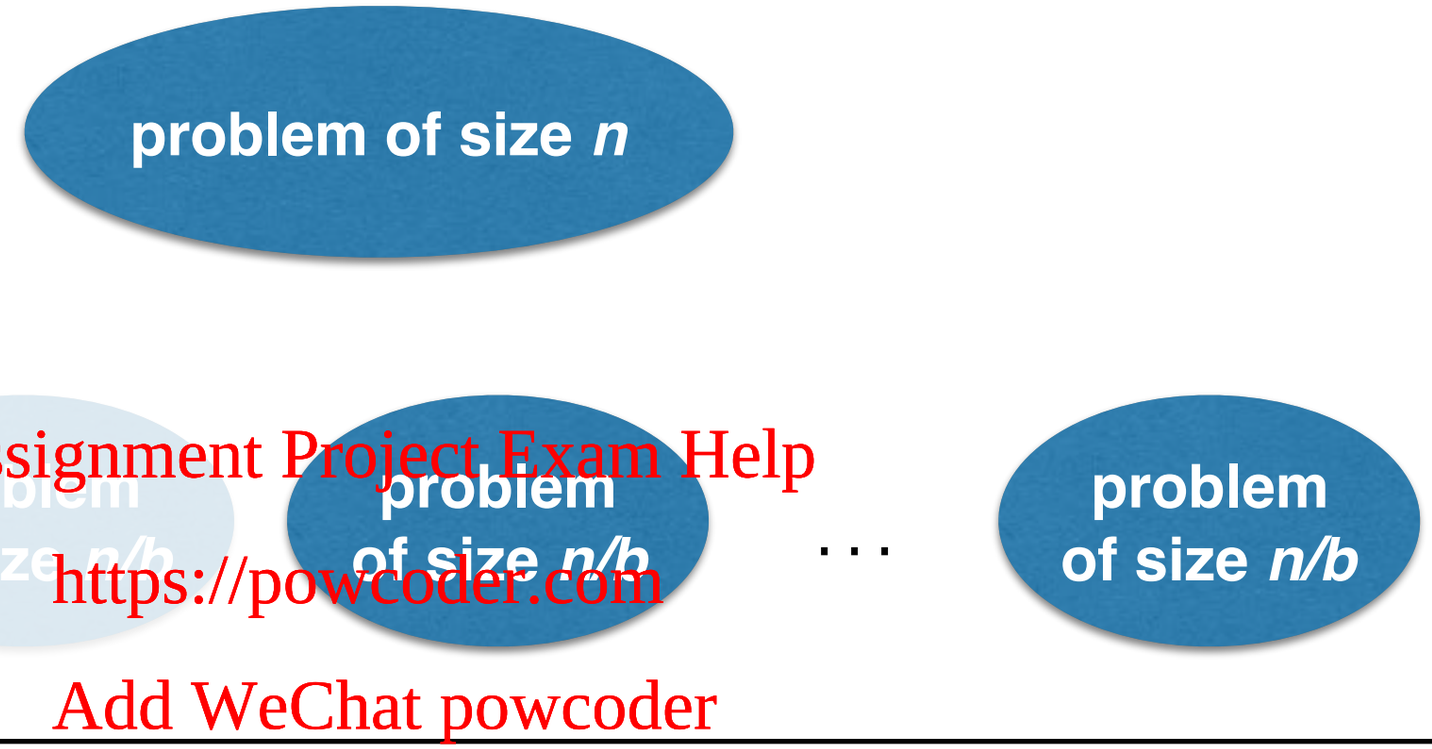
b sub-problems

Divide-and-Conqer General Case



only *a* sub-problems need to be solved

Divide-and-Conqer General Case



only *a* sub-problems need to be solved

Divide-and-Conquer Recurrences

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- What is the time required to solve a problem of size *n* by divide-and-conquer?

- For the general case, assume we split the problem into *b* instances (each of size *n/b*), of which *a* need to be solved:

$$T(n) = aT(n/b) + f(n)$$

where $f(n)$ expresses the time spent on dividing a problem into *b* sub-problems and combining the *a* results.

- (A very common case is $T(n) = 2T(n/2) + n$.)

- How to find closed forms for these recurrences?

The Master Theorem

- (A proof is in Levitin's Appendix B.)

- For integer constants $a \geq 1$ and $b > 1$, and function f with $f(n) \in \Theta(n^d)$, $d \geq 0$, the recurrence

$$T(n) = aT(n/b) + f(n)$$

(with T(1) = c) has solutions, and

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

- Note that we also allow a to be greater than b.

Master Theorem: Example 1

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$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

Master Theorem: Example 1

THE UNIVERSITY OF MELBOURNE

$T(n) = 2T(n/2) + n$

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So, by the Master Theorem, $T(n) \in \Theta(n \log n)$

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THE UNIVERSITY OF MELBOURNE

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Master Theorem: Example 1

THE UNIVERSITY OF MELBOURNE

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

$T(n) = 2(2T(n/4) + (n/2)) + n$

$1 \times n$

Master Theorem: Example 1

THE UNIVERSITY OF MELBOURNE

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

$T(n) = 4T(n/4) + 2(n/2) + n$

Master Theorem: Example 1

THE UNIVERSITY OF MELBOURNE

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

$T(n) = 4(2T(n/8) + n/4) + 2(n/2) + n$

Master Theorem: Example 1

THE UNIVERSITY OF MELBOURNE

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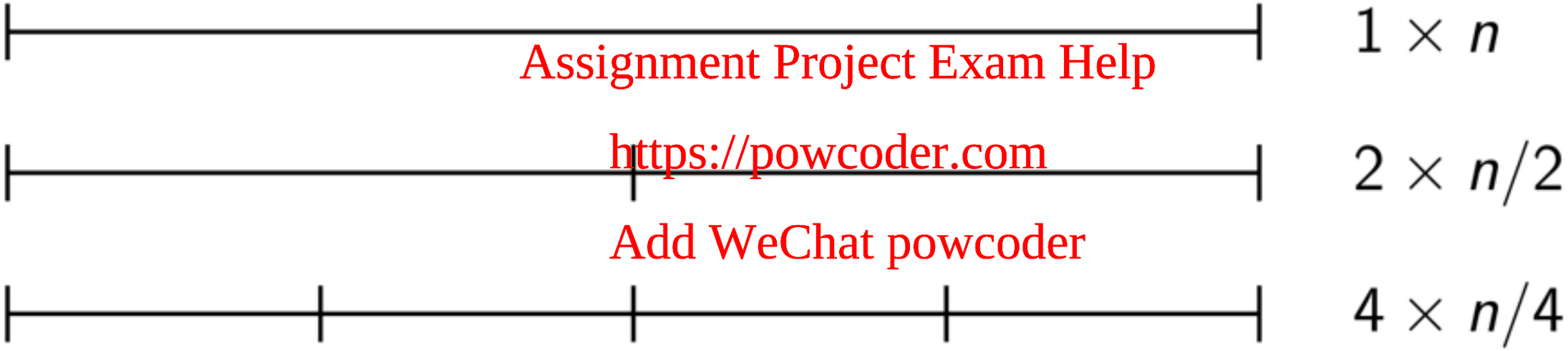
$T(n) = 8T(n/8) + 4(n/4) + 2(n/2) + n$



Master Theorem: Example 1

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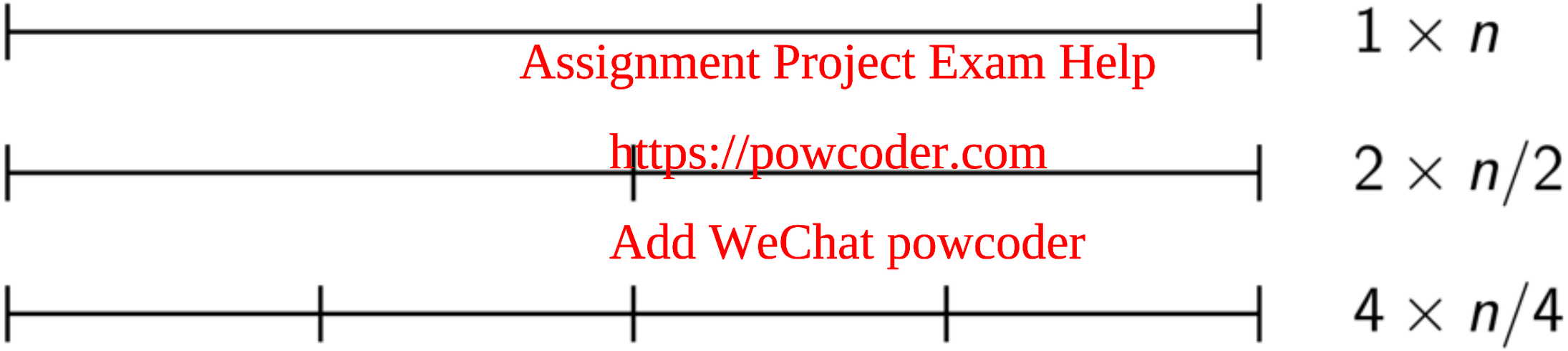
$T(n) = 2T(n/2) + n$ $\quad$ a = 2, b = 2, d = 1

$T(n) = 8(2T(n/16) + n/8) + 4(n/4) + 2(n/2) + n$



Master Theorem: Example 1

THE UNIVERSITY OF MELBOURNE

$T(n) = 2T(n/2) + n$ $\quad$ a = 2, b = 2, d = 1

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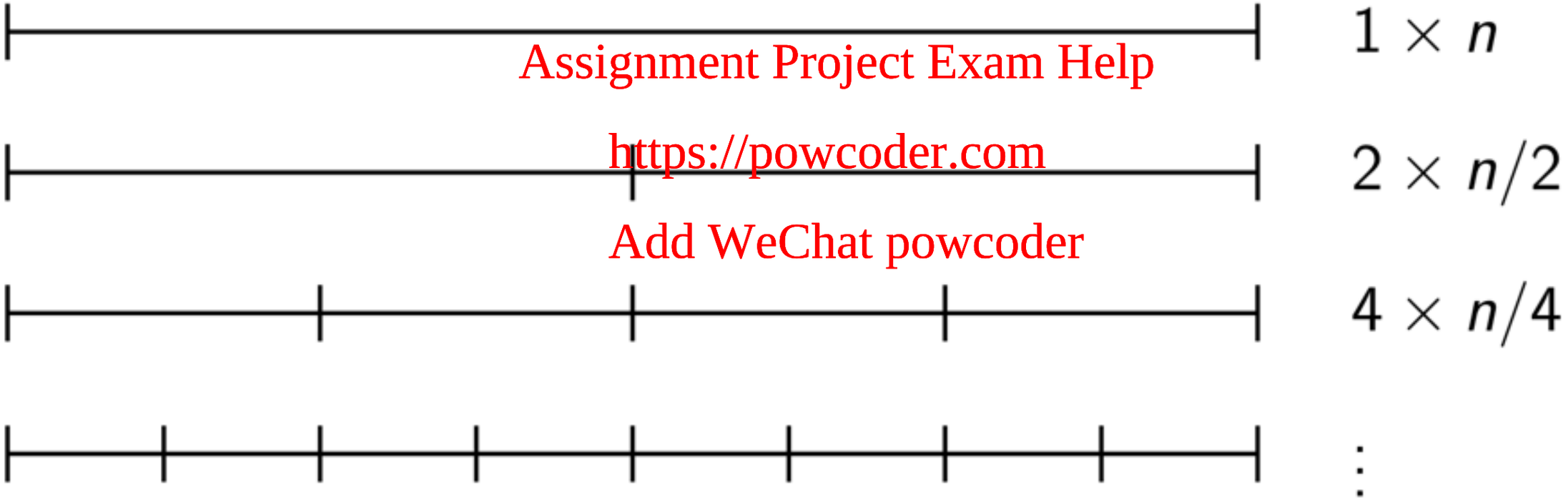

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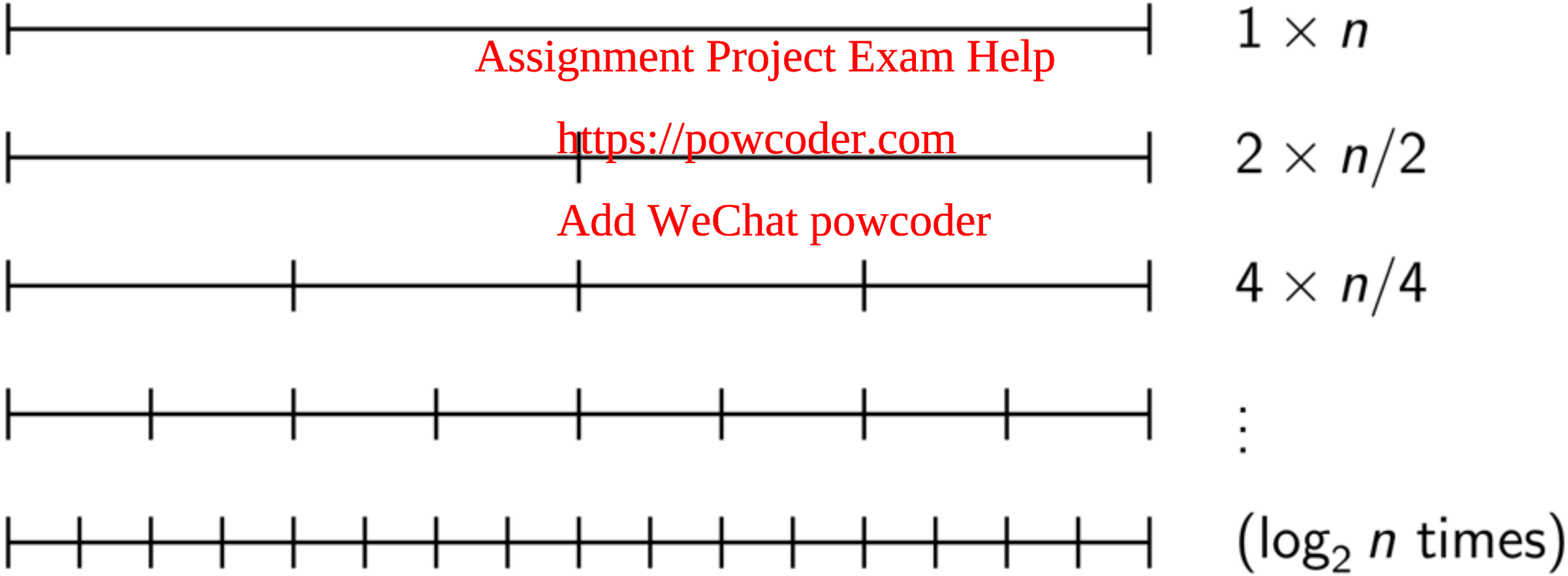

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THE UNIVERSITY OF MELBOURNE

$T(n) = 2T(n/2) + n$ $\quad$ a = 2, b = 2, d = 1

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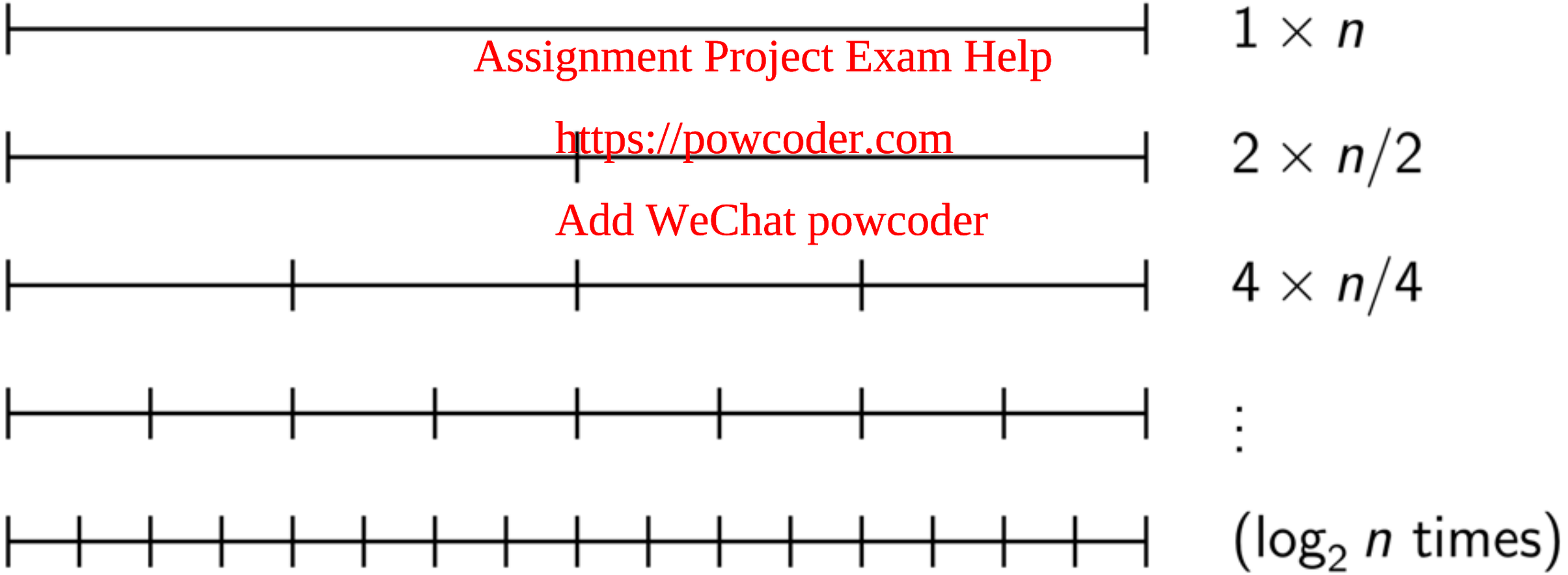

Master Theorem: Example 2

THE UNIVERSITY OF MELBOURNE

$T(n) = 4T(n/4) + n$

a = 4, b = 4, d = 1

$a = b^d$

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

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Master Theorem: Example 2

THE UNIVERSITY OF MELBOURNE

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

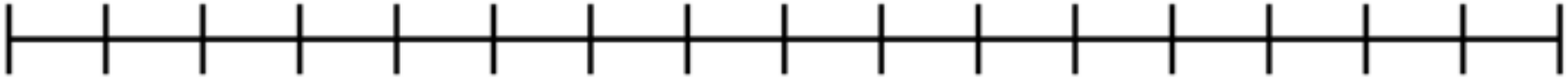
$T(n) = 16(4T(n/64) + n/16)\ 4(n/4) + n$

Master Theorem: Example 2

THE UNIVERSITY OF MELBOURNE

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

$T(n) = 64T(n/64) + 16(n/16) + 4(n/4) + n$



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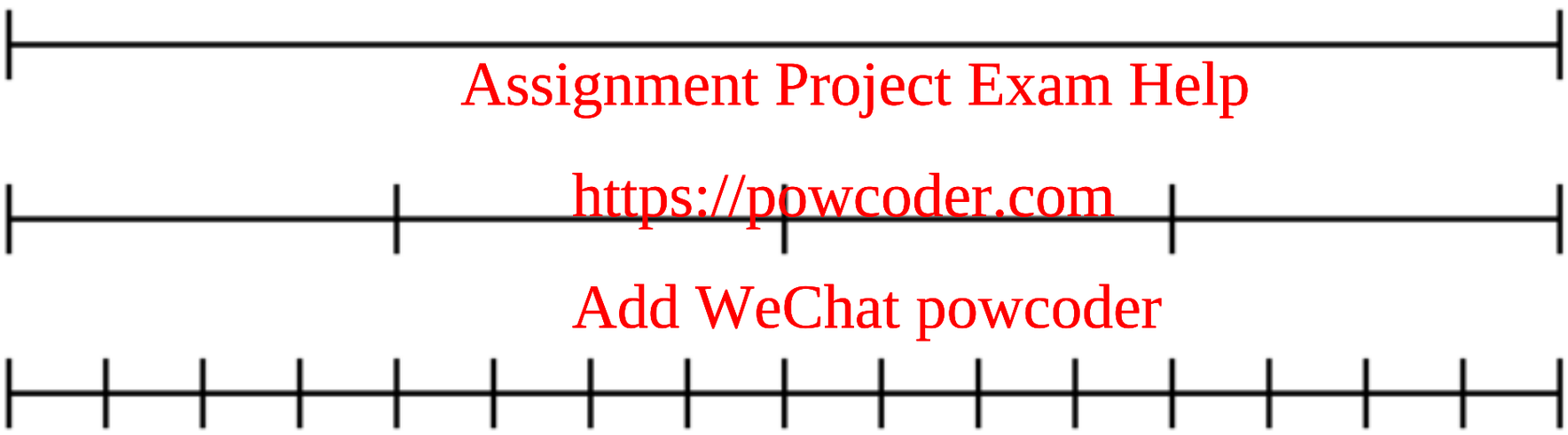

($\log_4$ n times)

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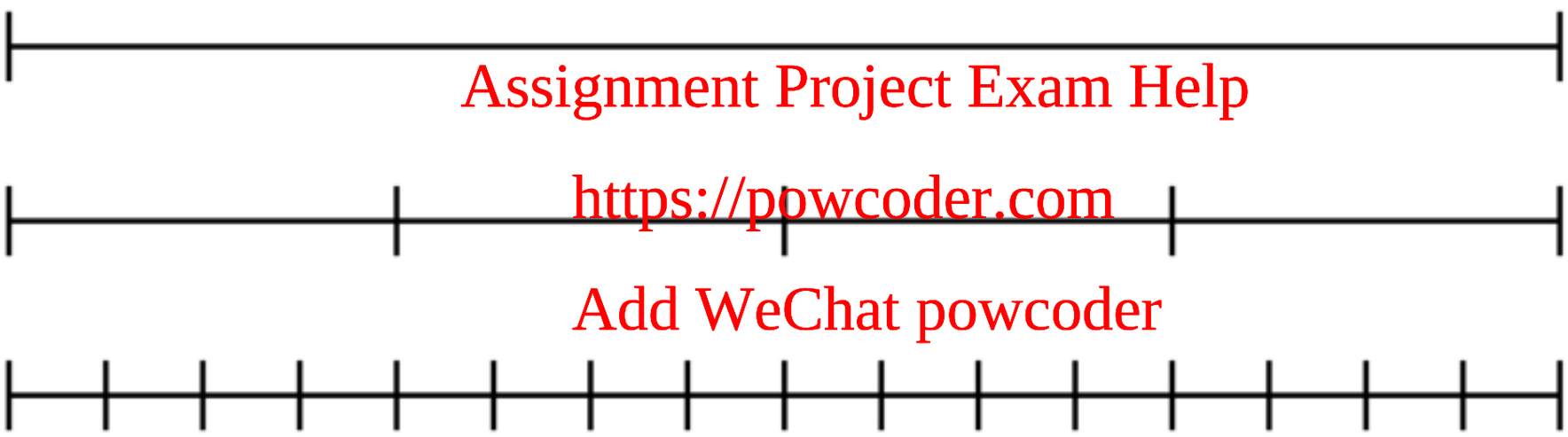

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$T(n) \in \Theta(n \log n)$

Master Theorem: Example 3

THE UNIVERSITY OF MELBOURNE

$T(n) = T(n/2) + n$ a = 1, b = 2, d = 1

Master Theorem: Example 3

THE UNIVERSITY OF MELBOURNE

$T(n) = T(n/2) + n$

a = 1, b = 2, d = 1

$a < b^d$

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

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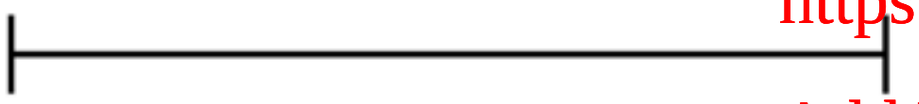

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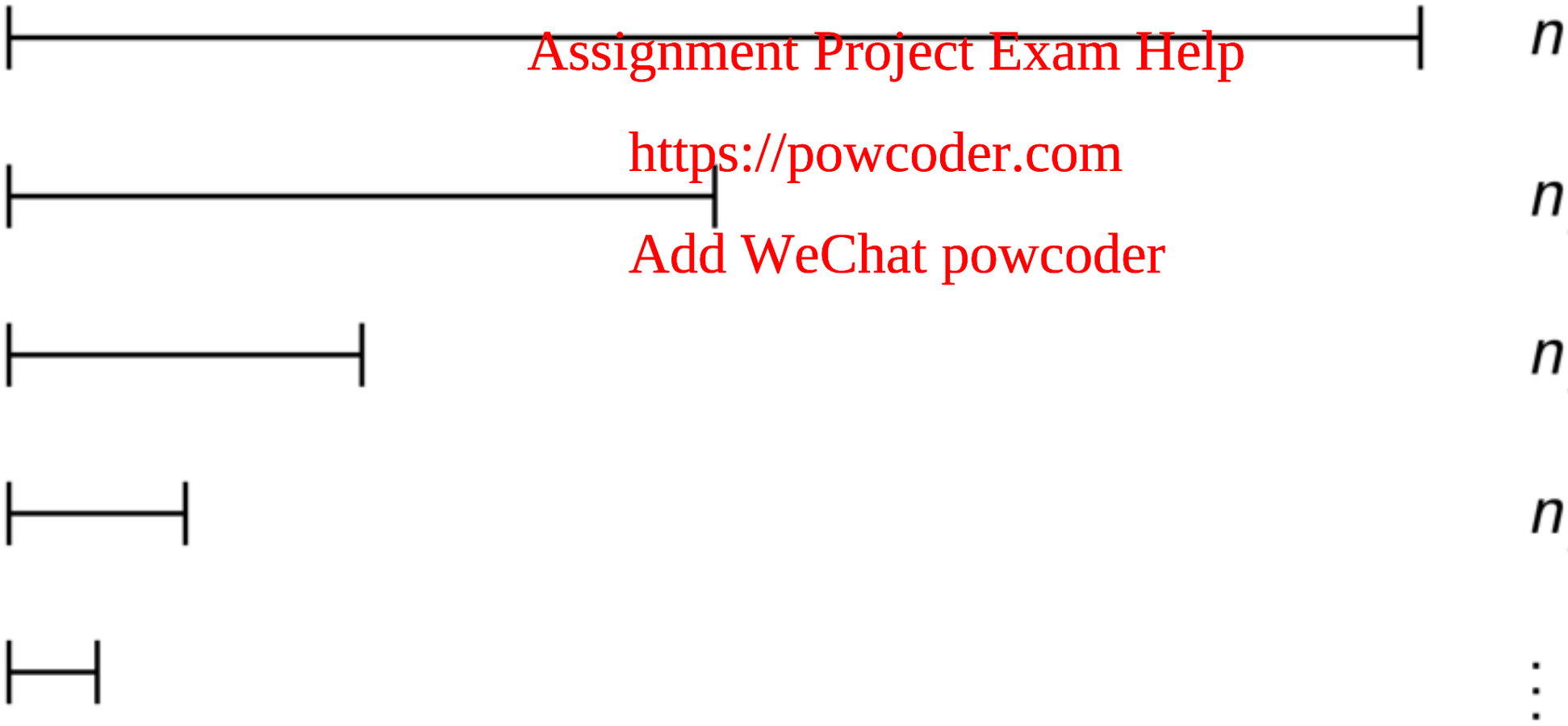

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Master Theorem: Example 4

THE UNIVERSITY OF MELBOURNE

$T(n) = 2T(n/2) + n^2$ a = 2, b = 2, d = 2

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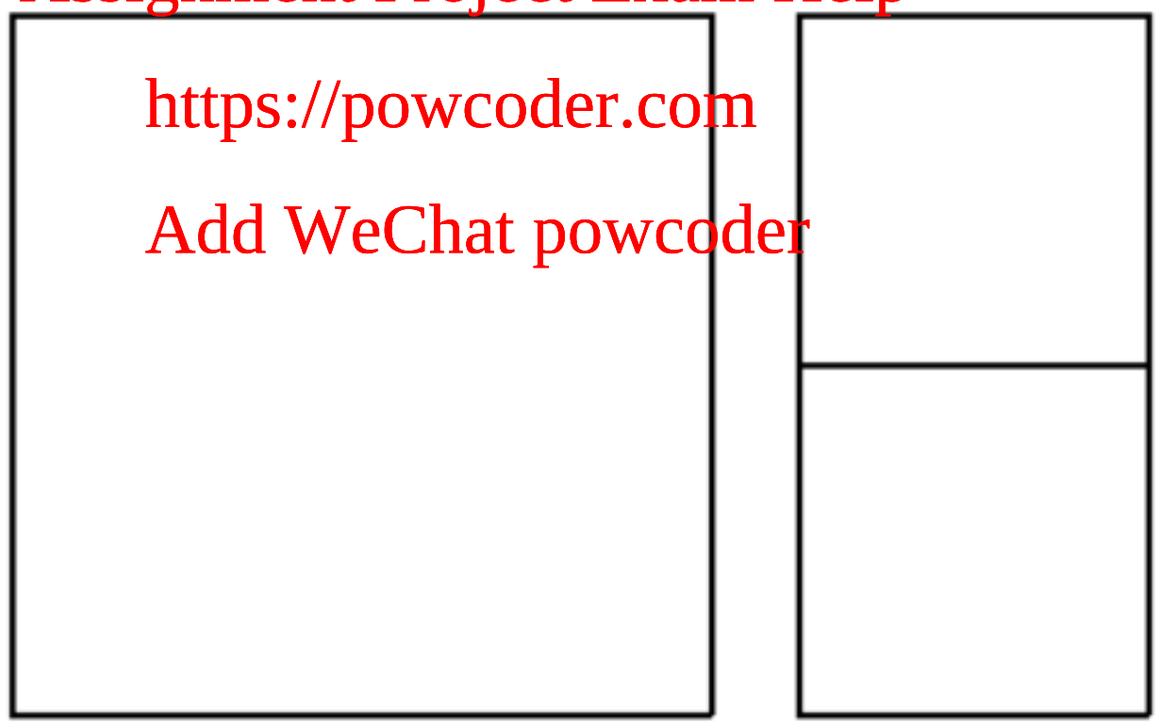

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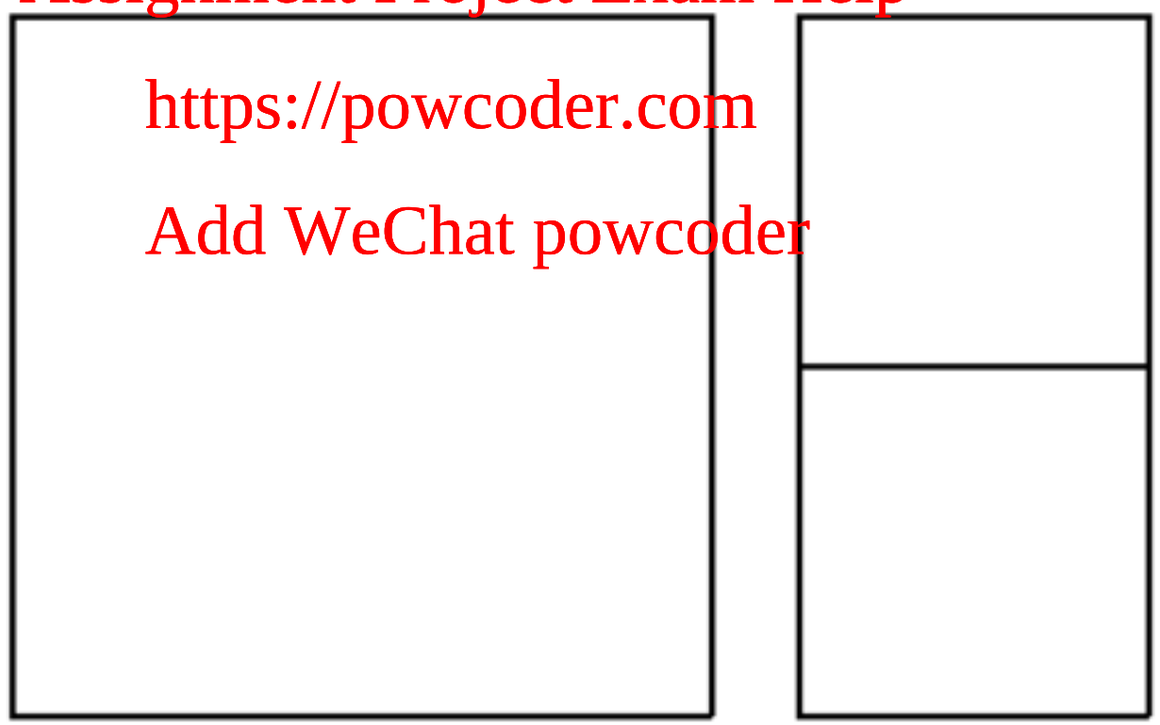

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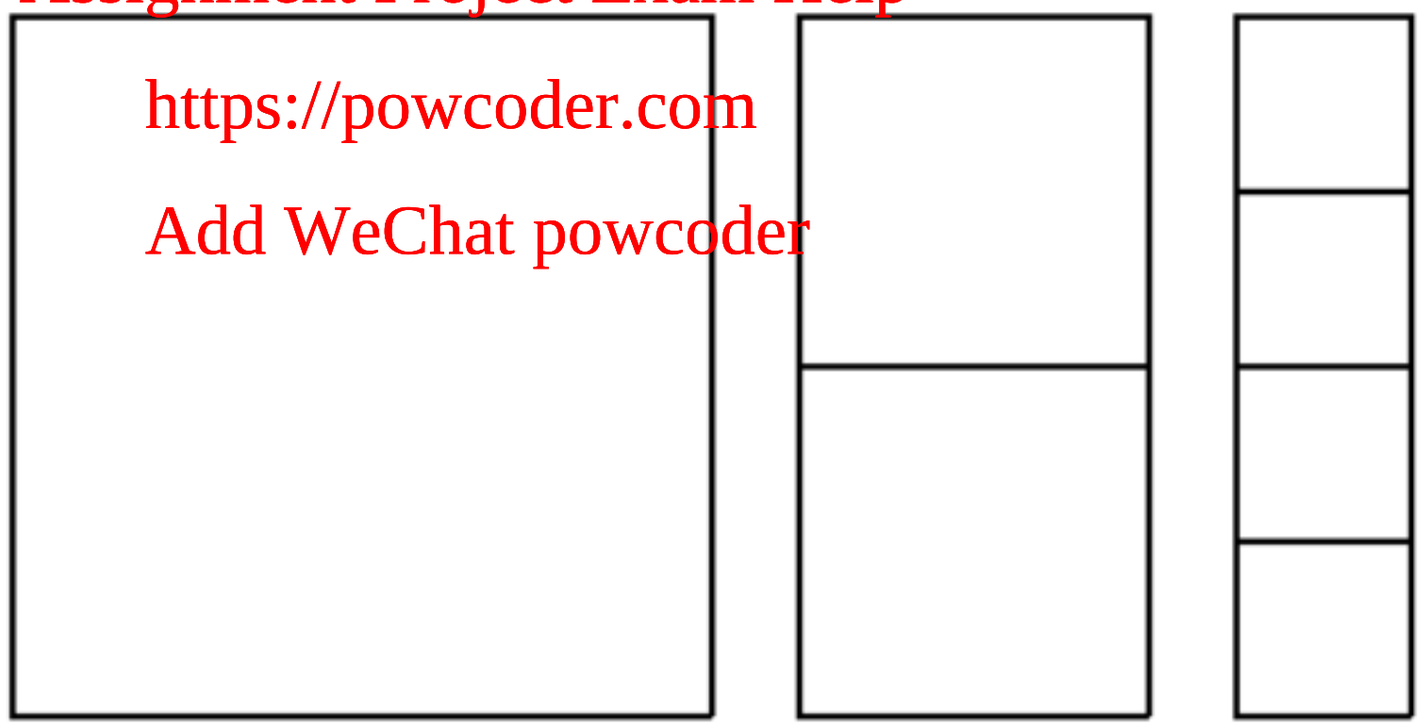

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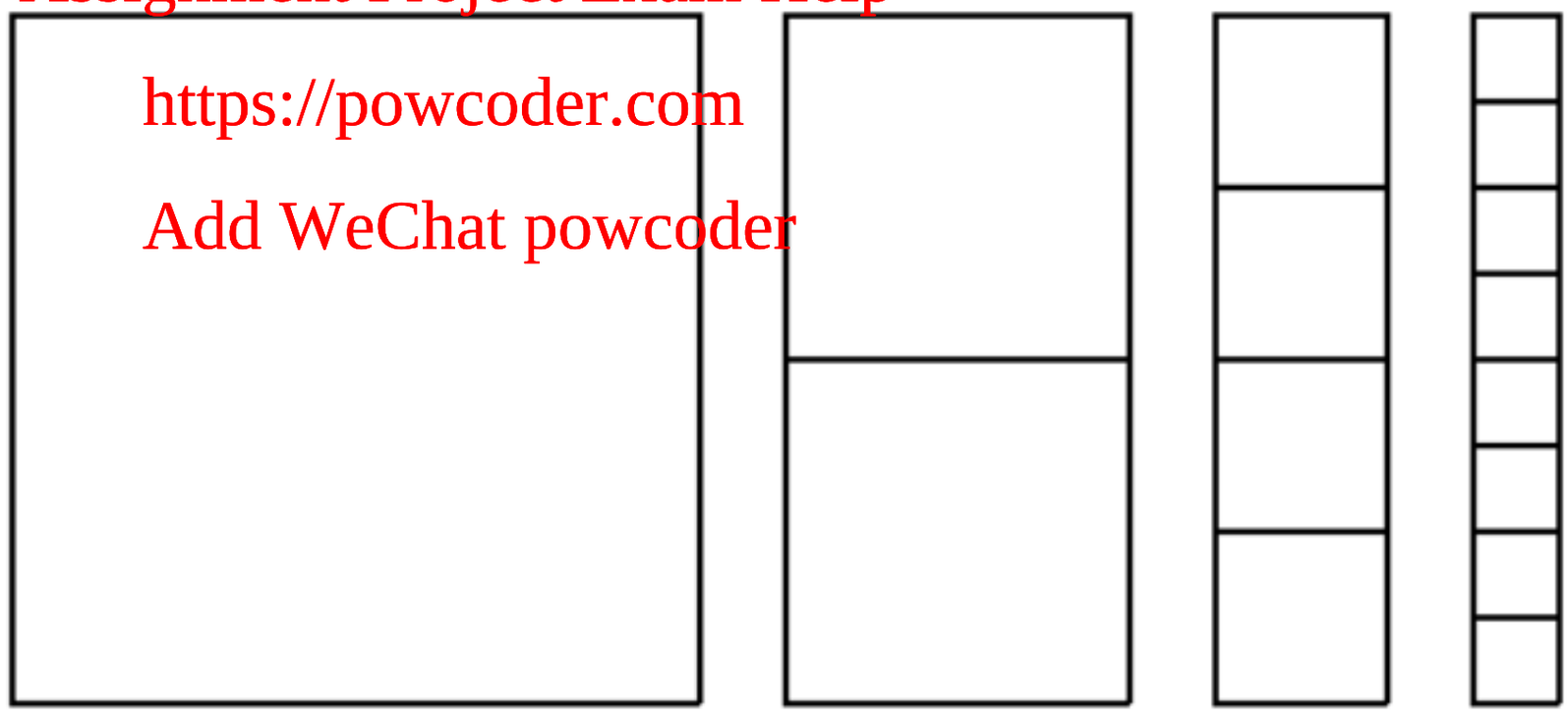

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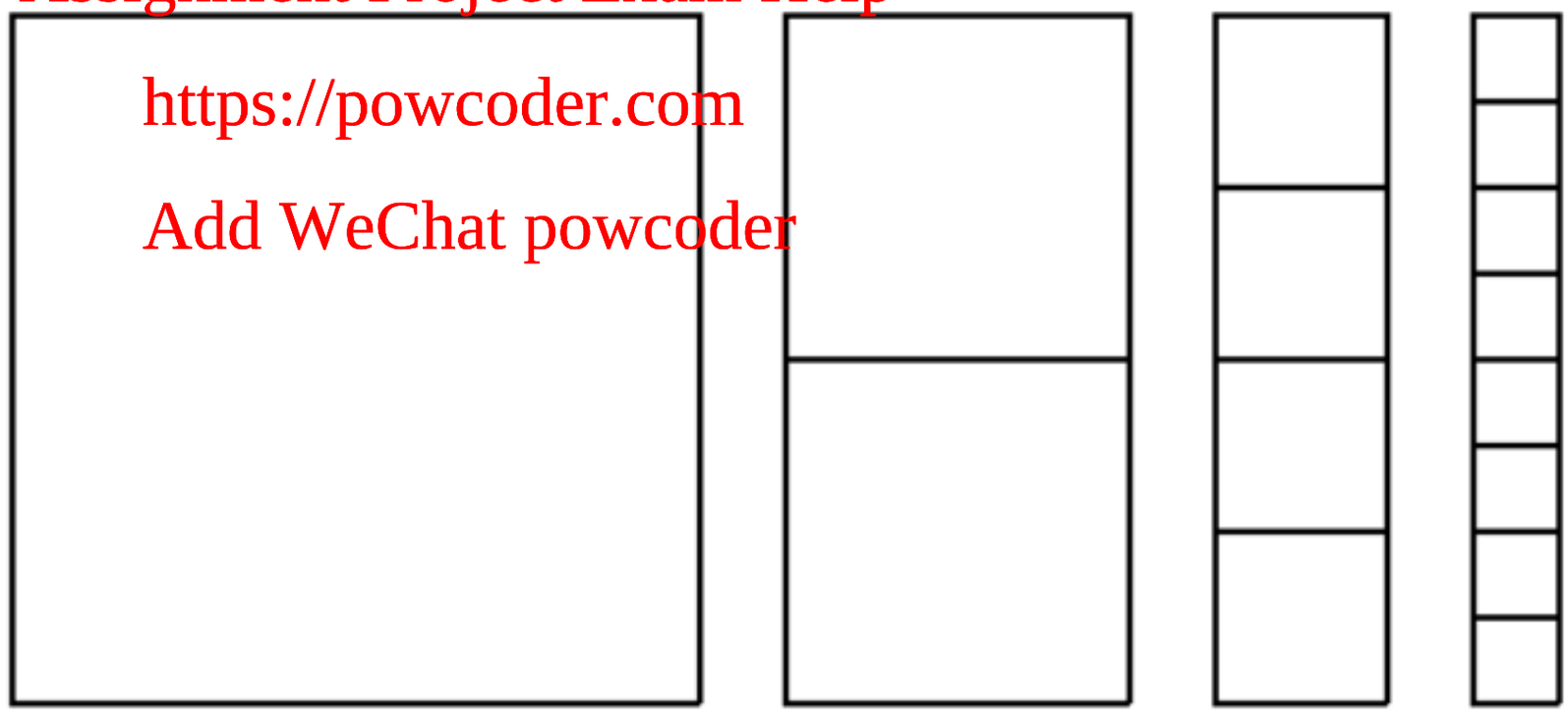
$T(n) \in \Theta(n^2)$



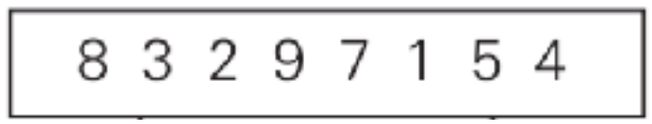
Mergesort

- Perhaps the most obvious application of divide-and-conquer:

- To sort an array (or a list), cut it into two halves, sort each half, and merge the two results.


```
procedure MergeSort(A[], n)                                  ▷ Sort A[0]..A[n − 1]
    if n > 1 then
        for i ← 0 to ⌊n/2⌋ − 1 do                            ▷ Copy left half of A to B
            B[i] ← A[i]
        for i ← 0 to ⌈n/2⌉ − 1 do                            ▷ Copy right half of A to C
            C[i] ← A[⌊n/2⌋ + i]
        MergeSort(B, ⌊n/2⌋)                                  ▷ Sort B
        MergeSort(C, ⌈n/2⌉)                                  ▷ Sort C
        Merge(B, ⌊n/2⌋, C, ⌈n/2⌉, A)                        ▷ Merge B and C into A
```

Mergesort



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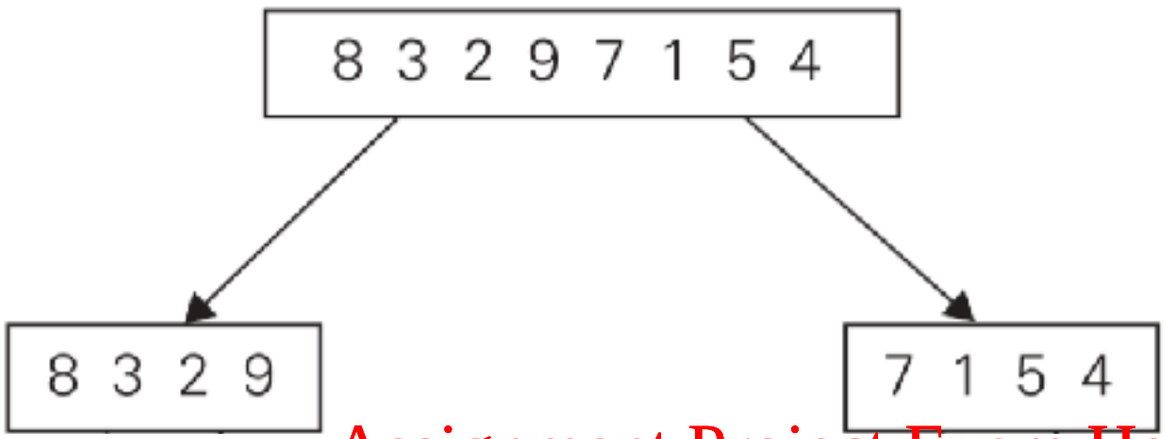

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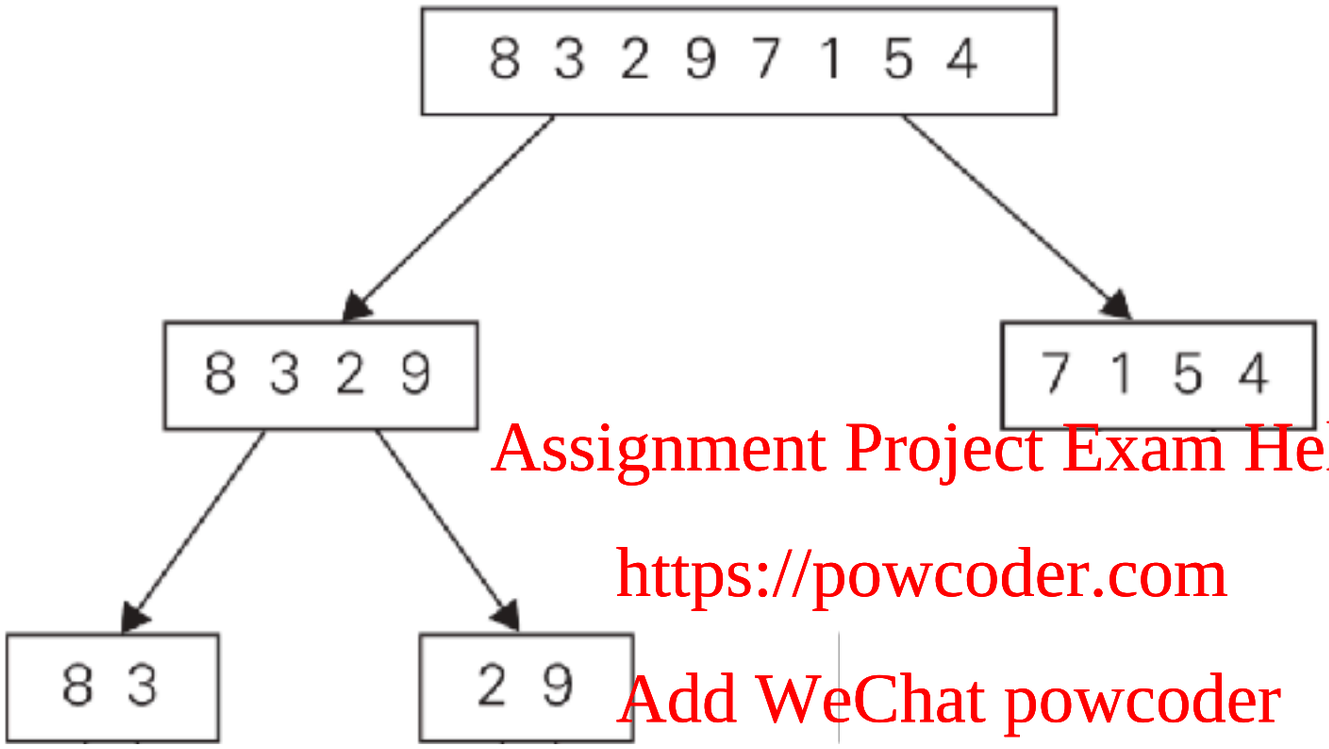

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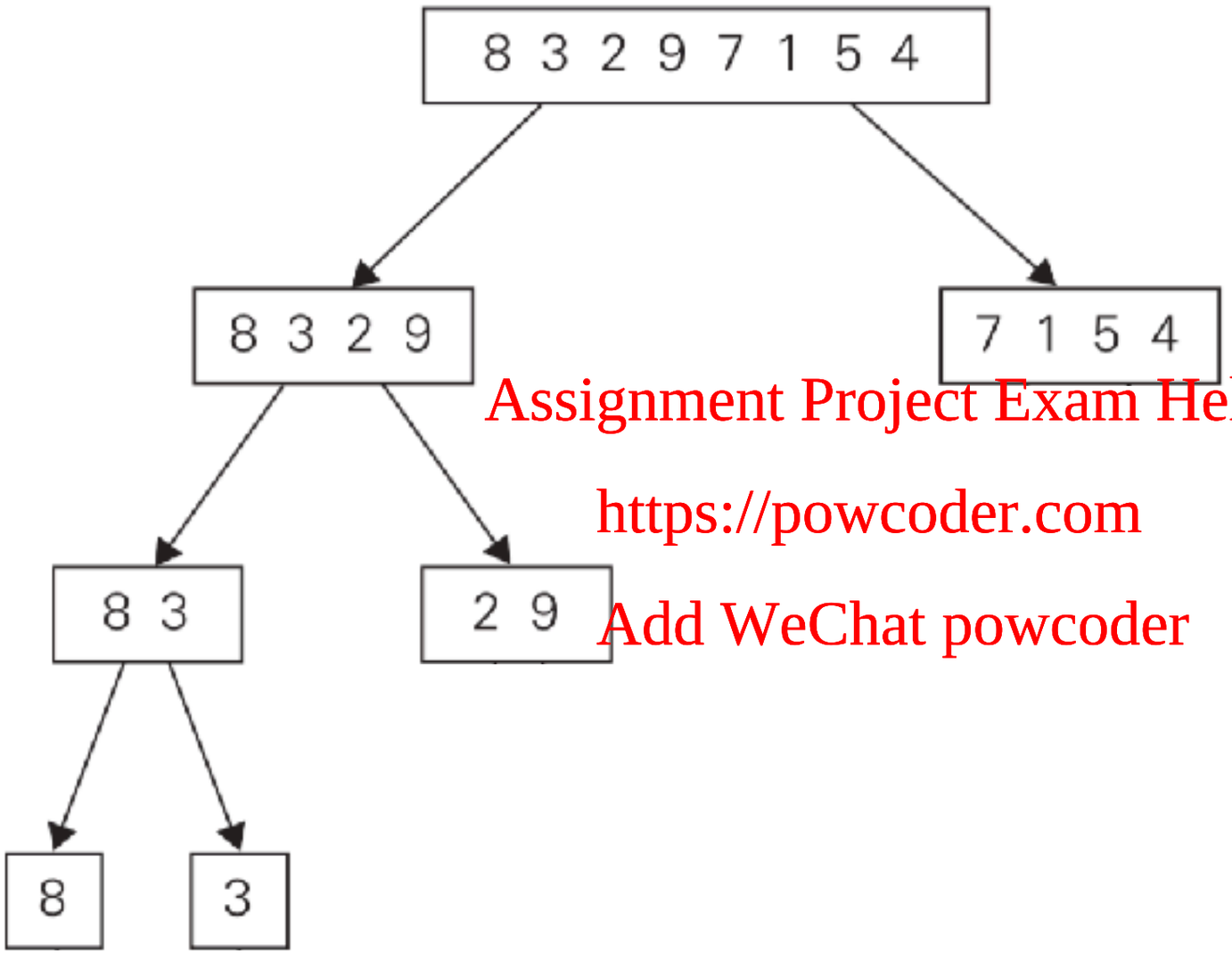

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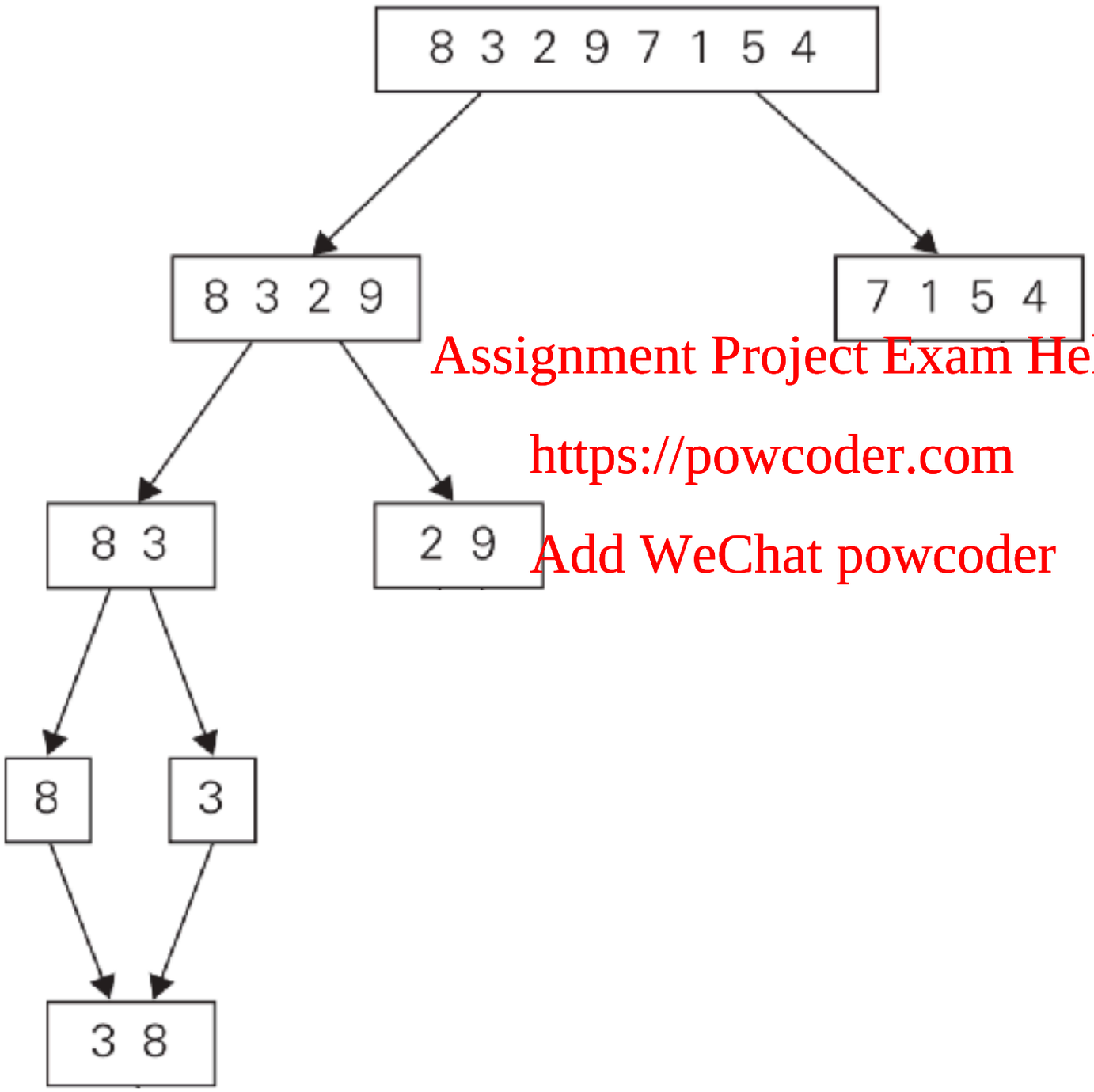

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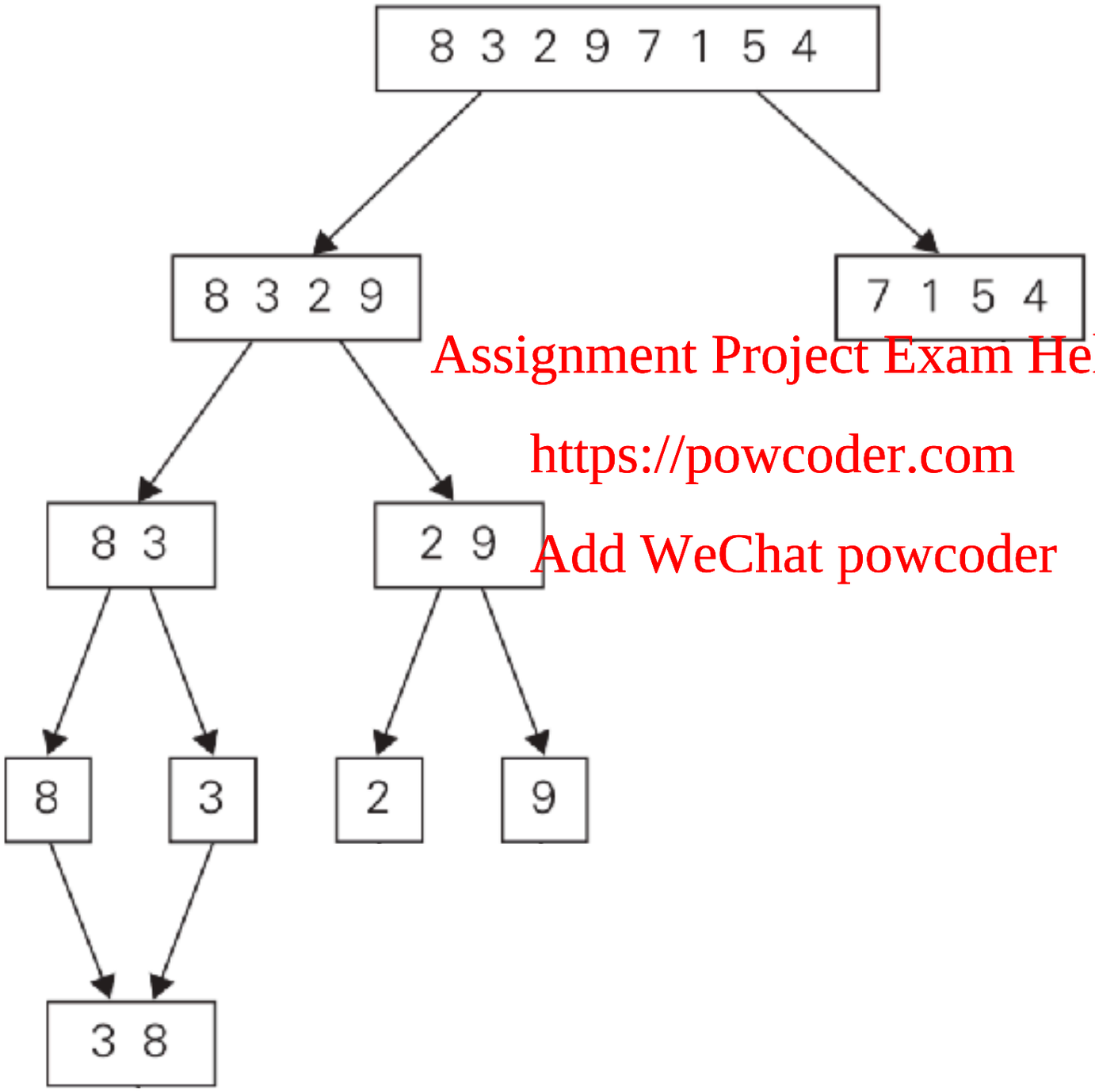

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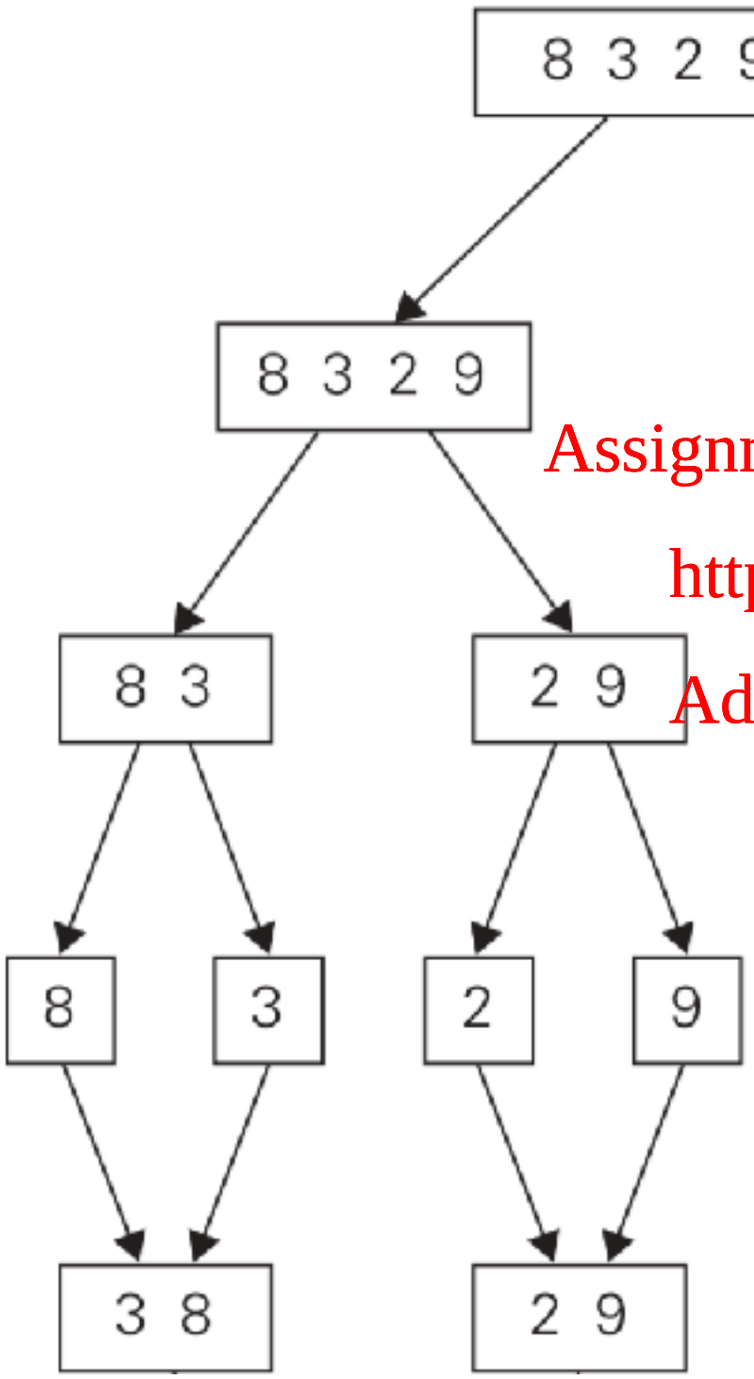

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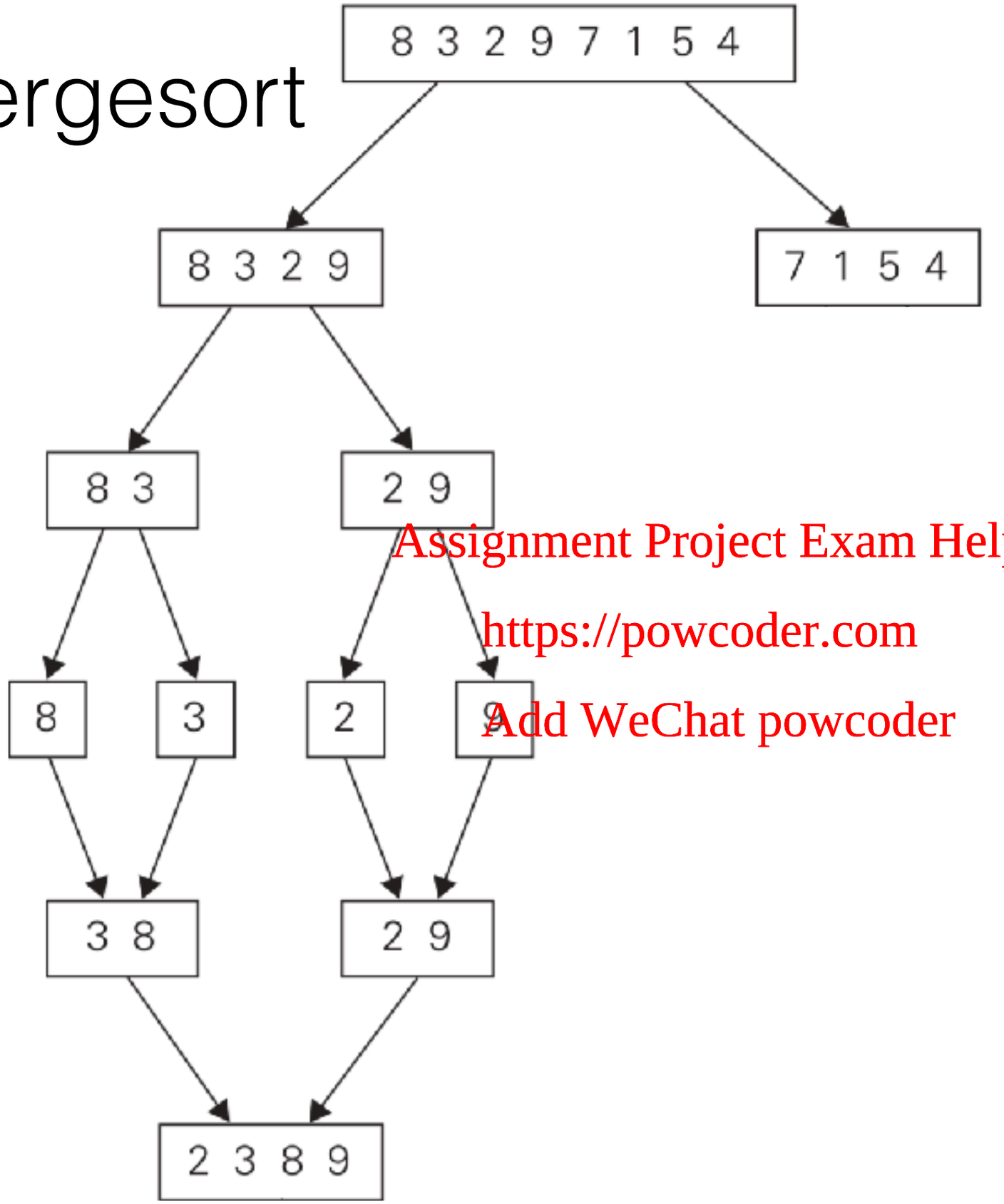

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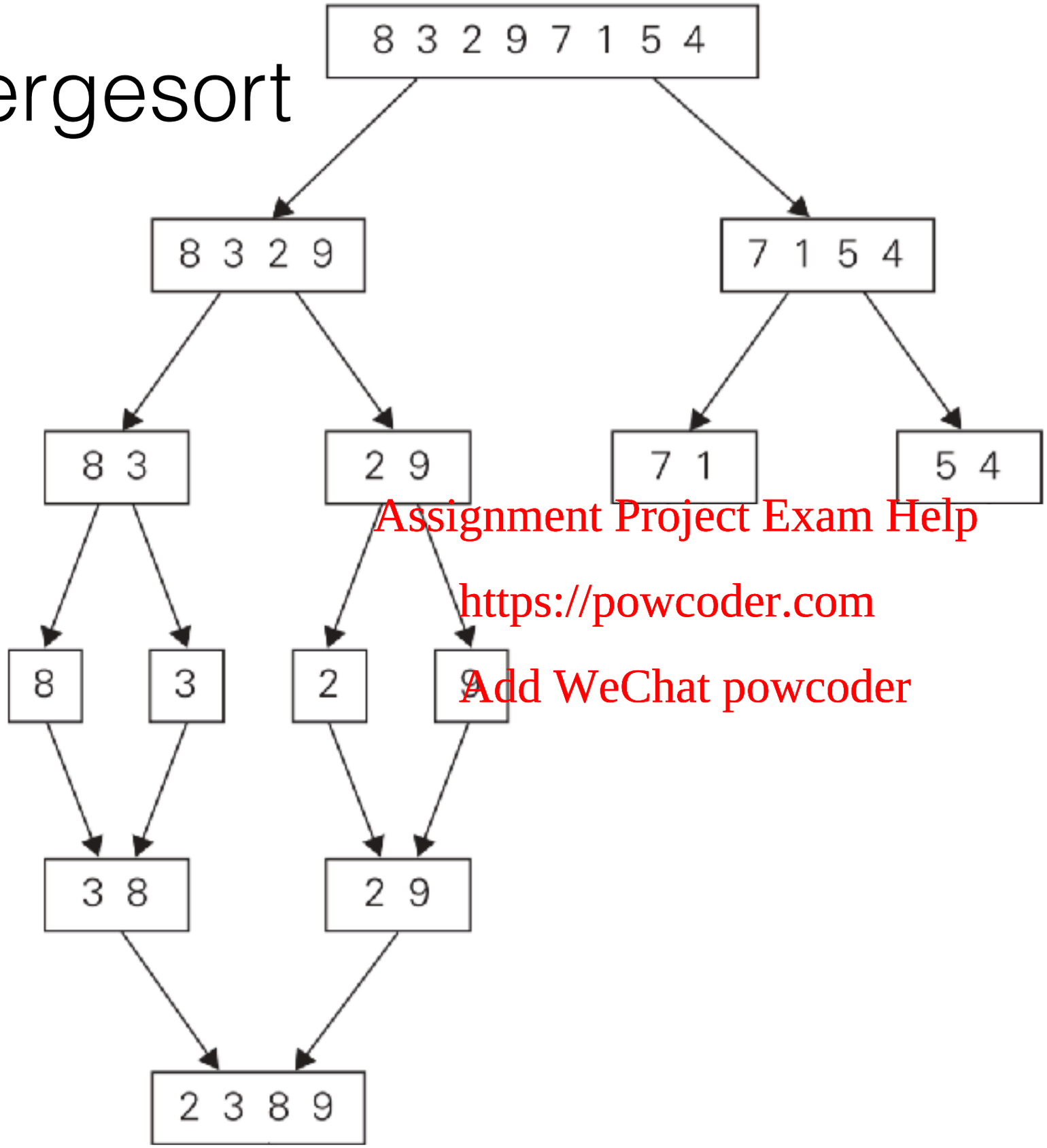

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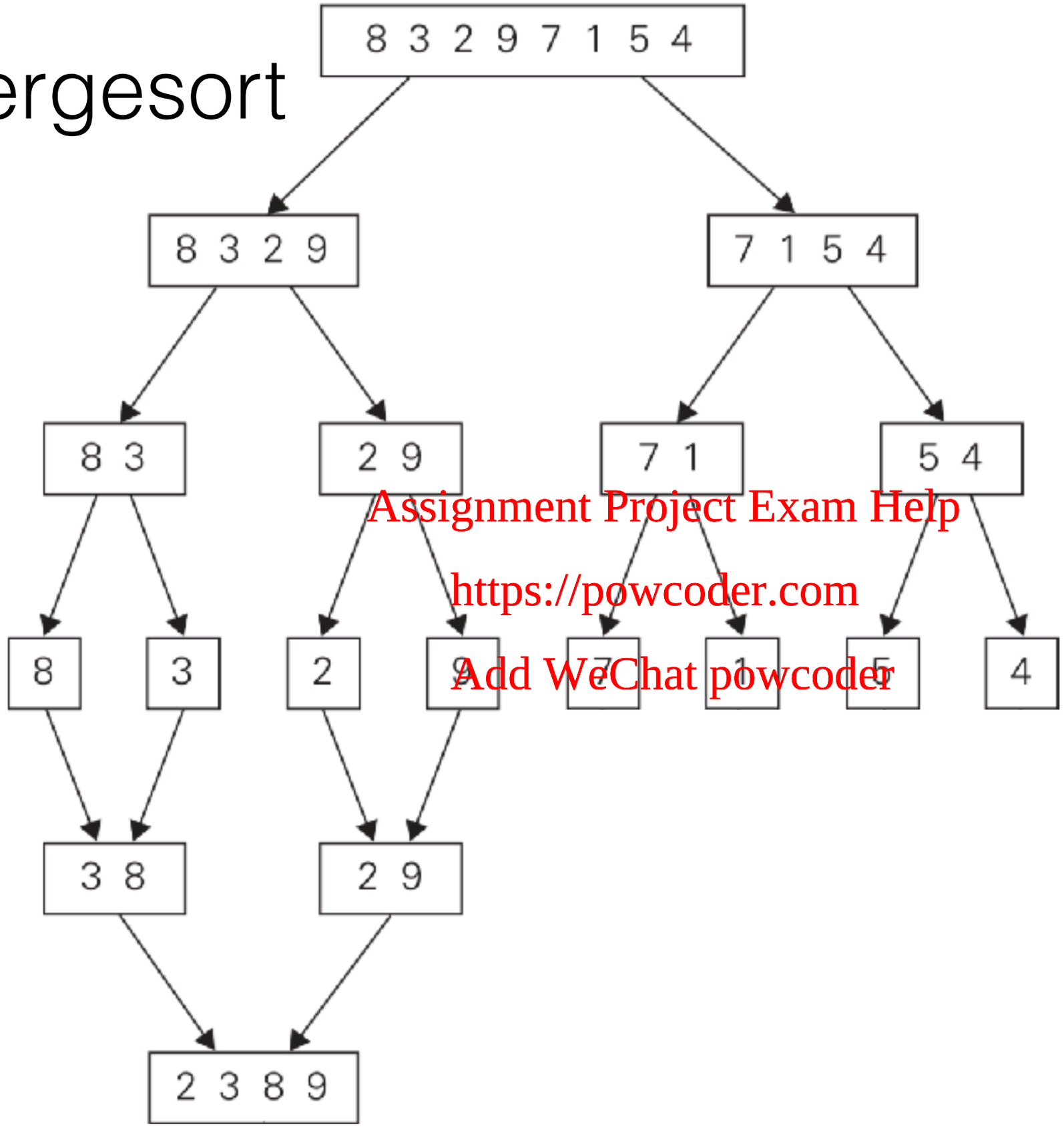

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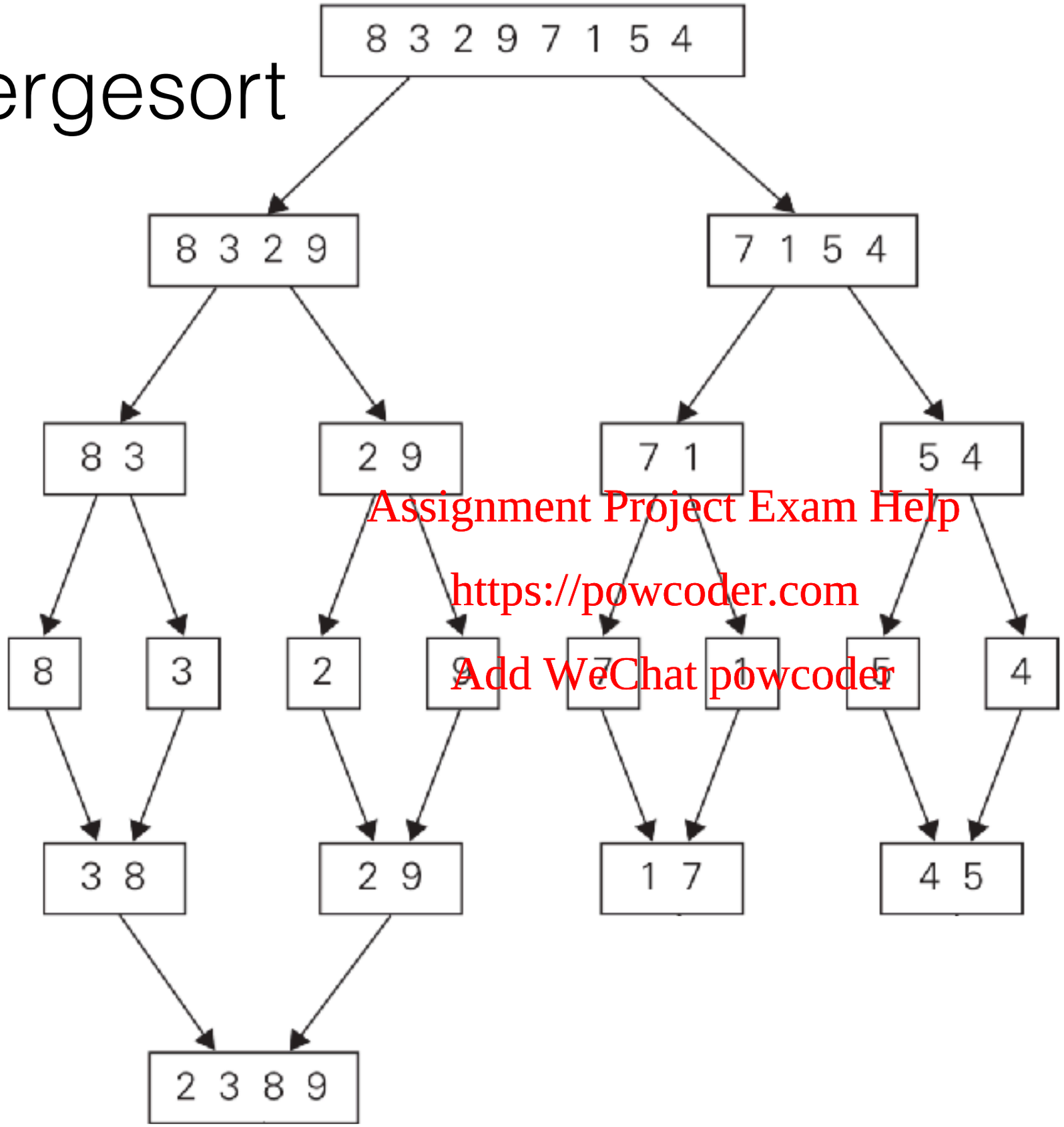
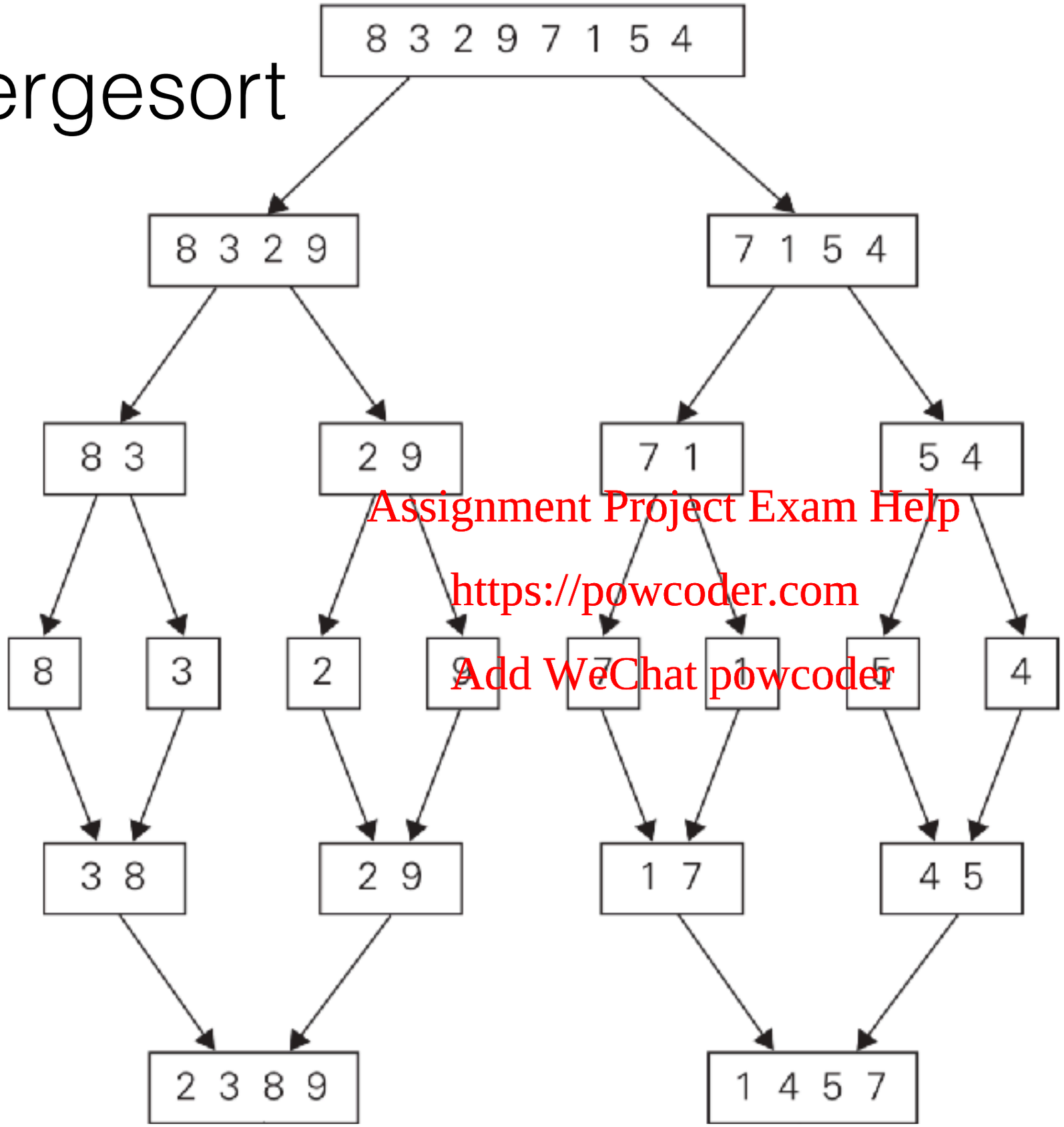


Mergesort



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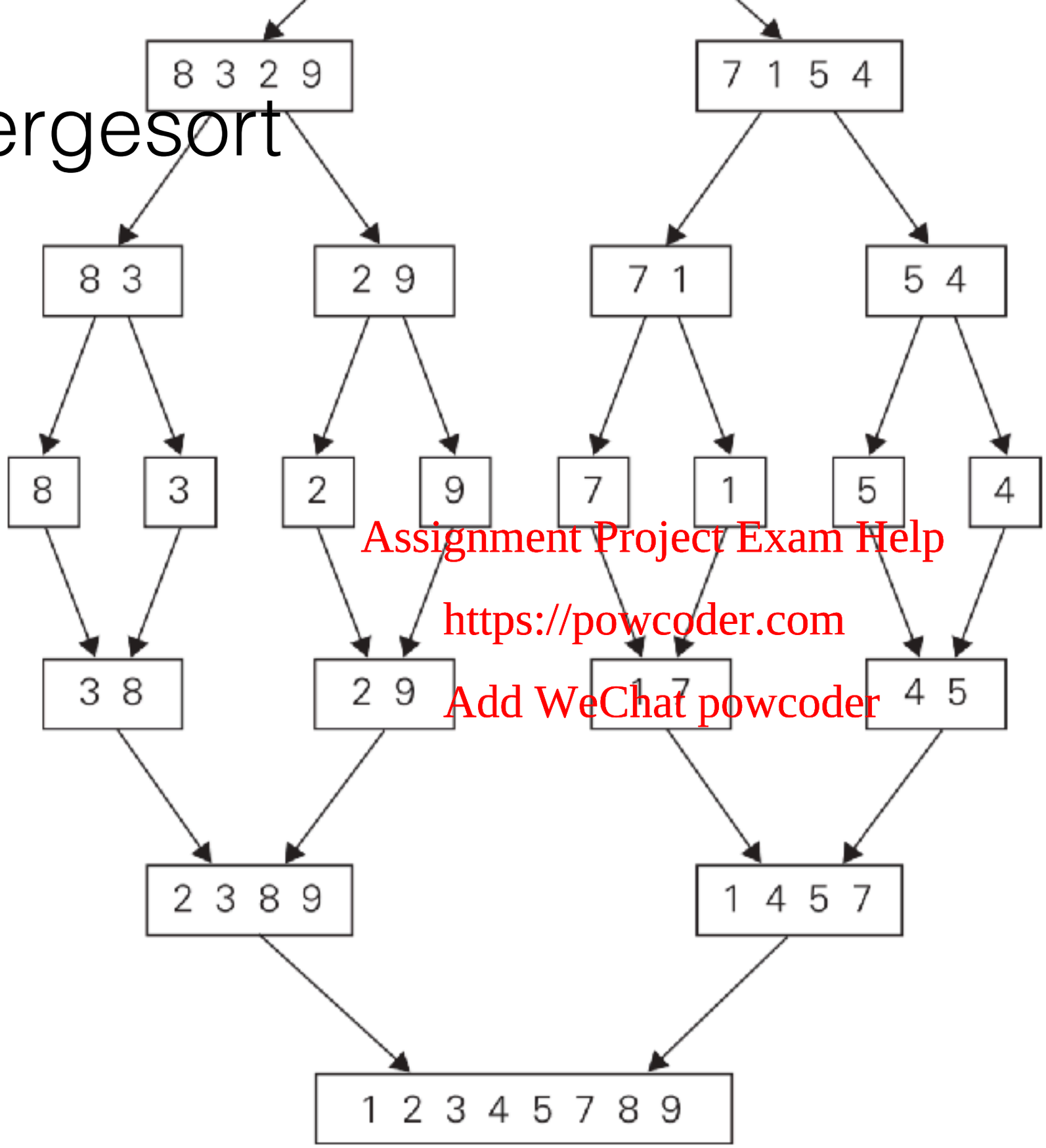
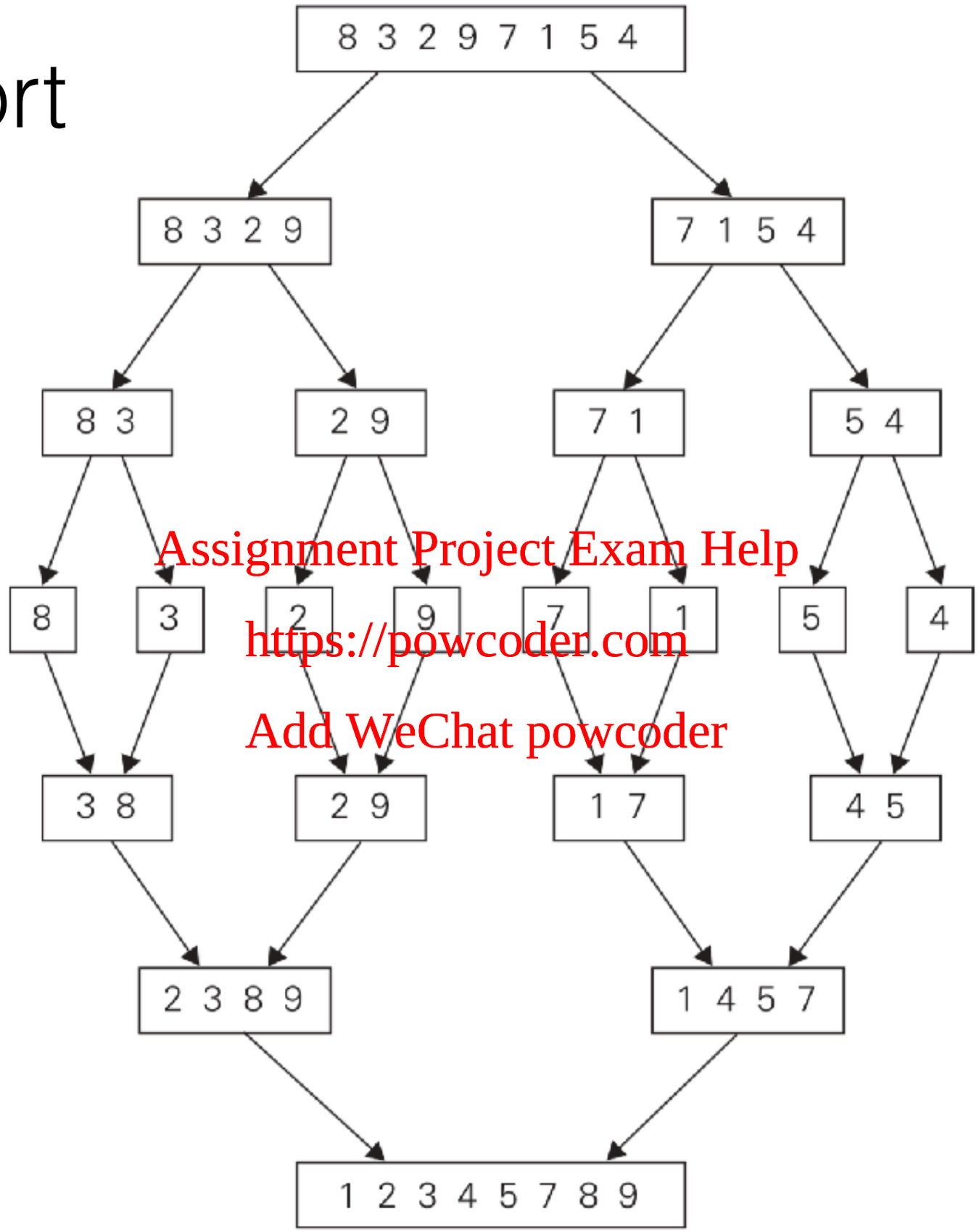


Mergesort



Mergesort: Merging Arrays

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procedure MERGE($B[\cdot], p, C[\cdot], q, A[\cdot]$)

$i \leftarrow 0; j \leftarrow 0; k \leftarrow 0$

while $i < p$ and $j < q$ **do**

if $B[i] \leq C[j]$ **then**

$A[k] \leftarrow B[i]$

$i \leftarrow i + 1$

else

$A[k] \leftarrow C[j]$

$j \leftarrow j + 1$

$k \leftarrow k + 1$

if $i = p$ **then**

copy $C[j]..C[q-1]$ to $A[k]..A[p+q-1]$ ▷ (a for loop)

else

copy $B[i]..B[p-1]$ to $A[k]..A[p+q-1]$ ▷ (a for loop)

Mergesort: Merging Arrays

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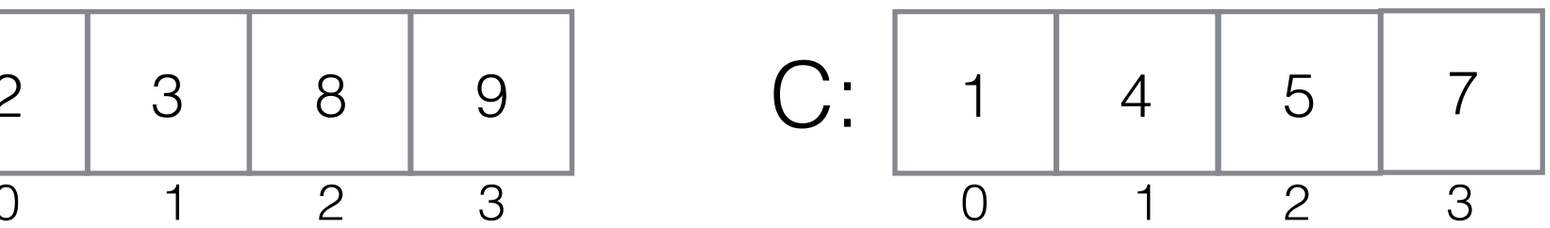

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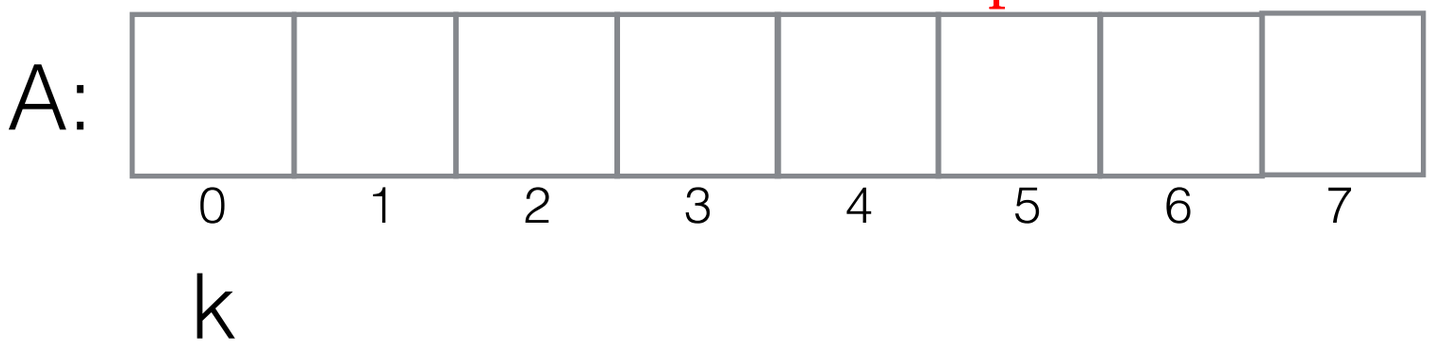
q: 4



Mergesort: Merging Arrays

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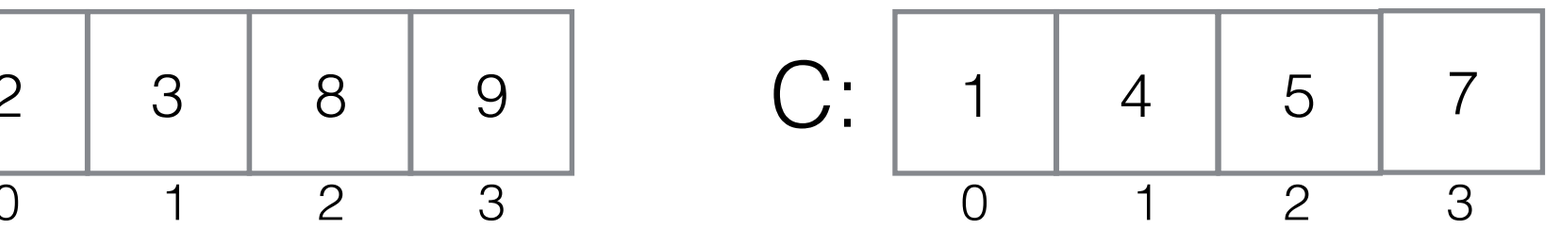

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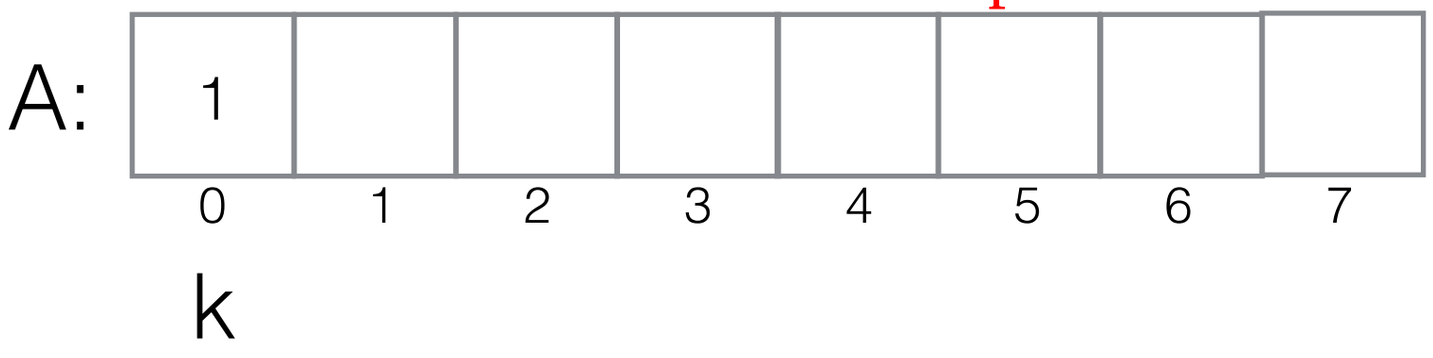

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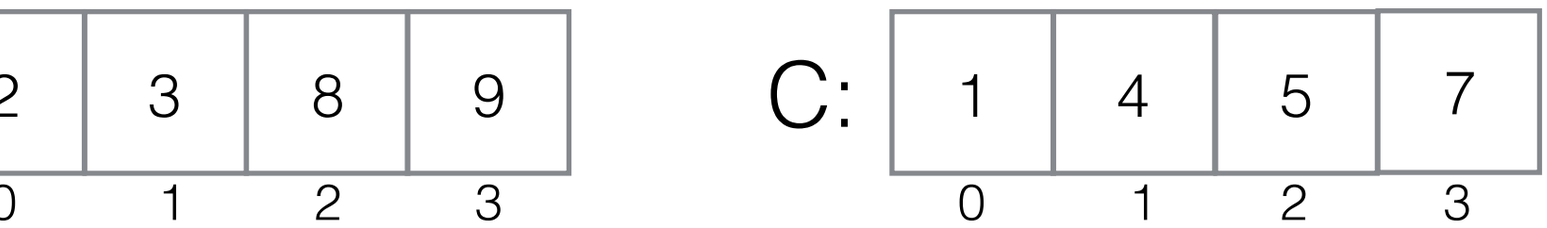

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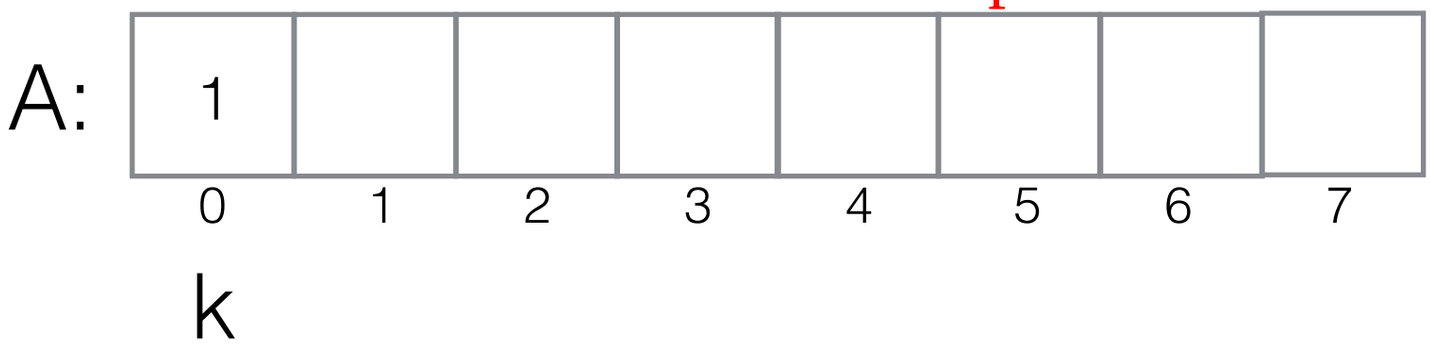

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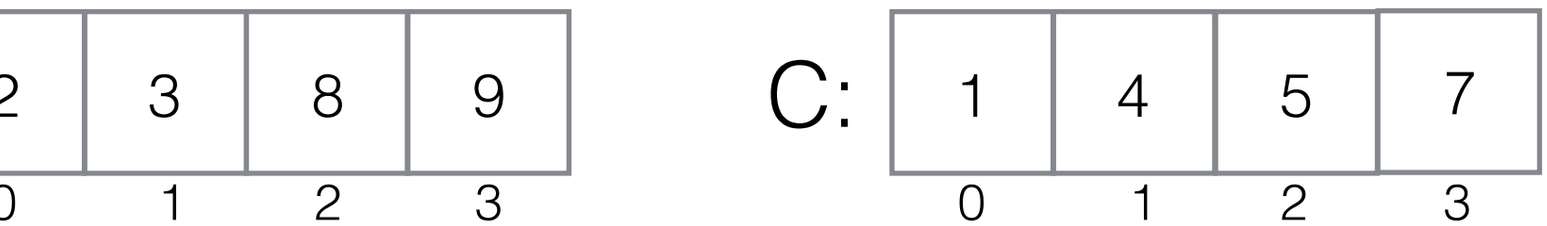

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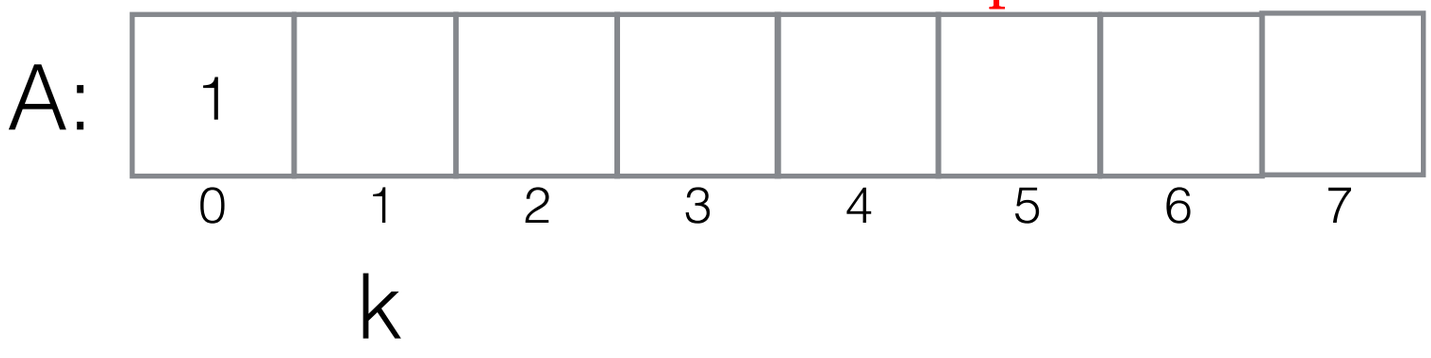

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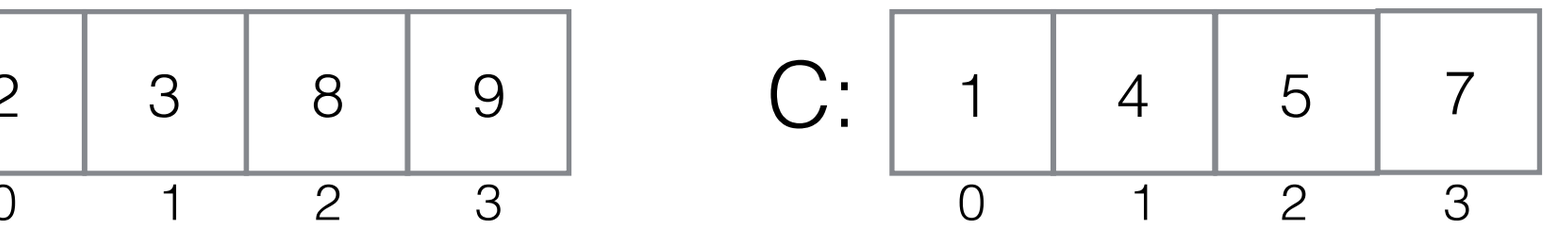

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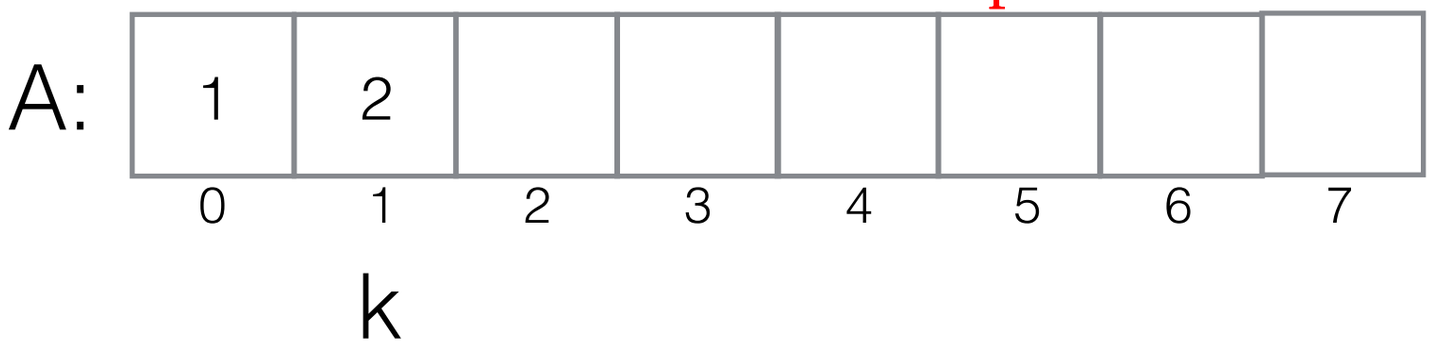

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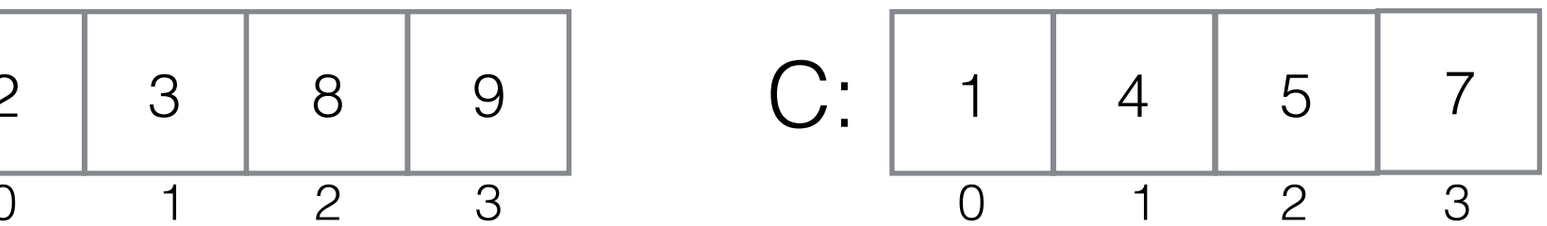

p: 4

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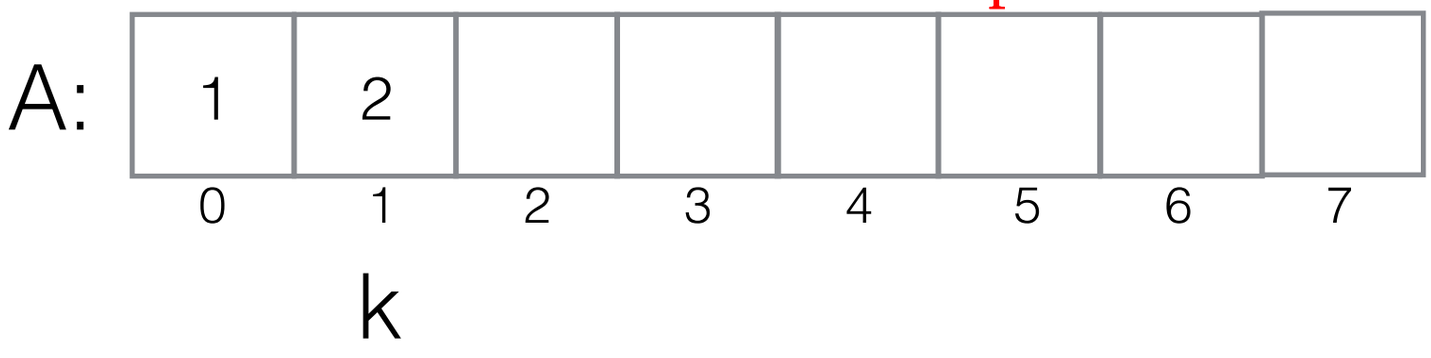

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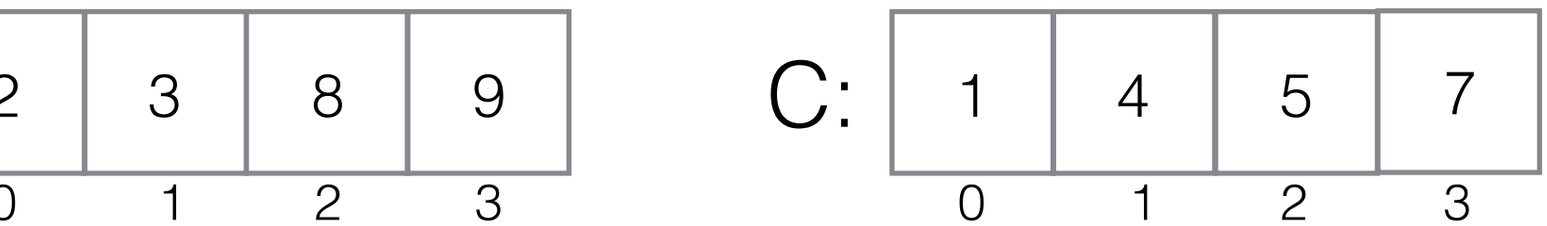

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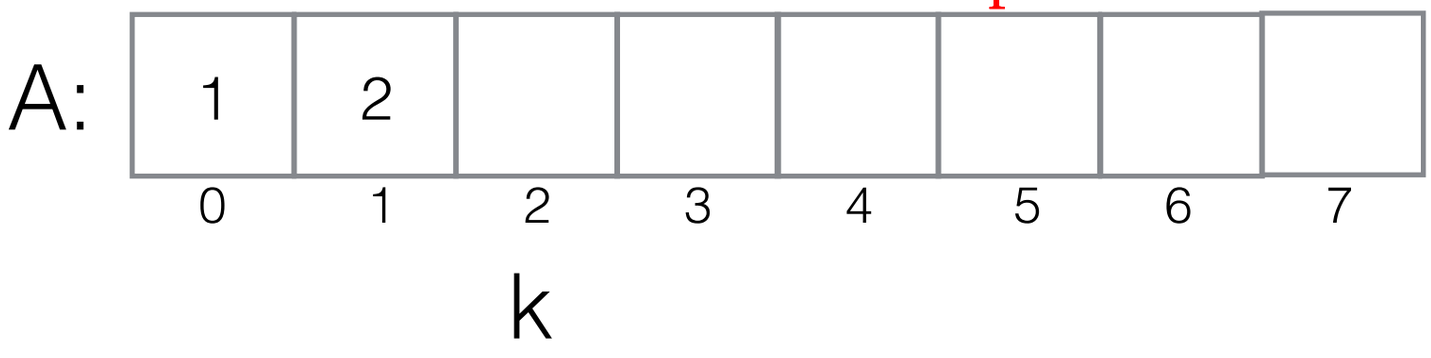

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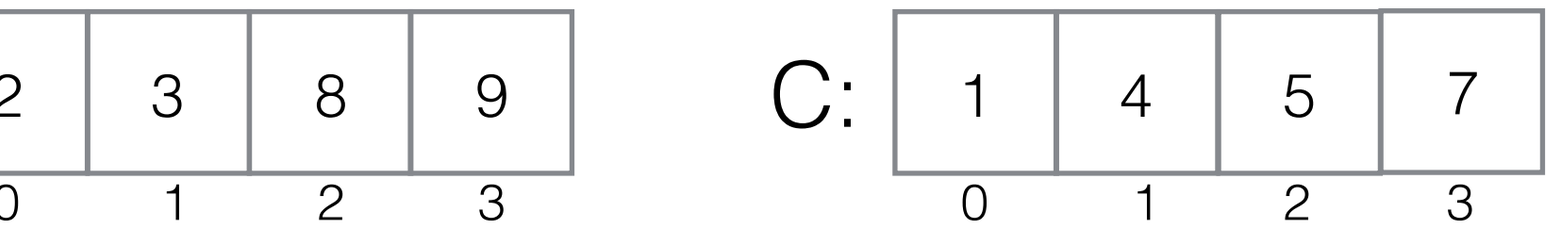

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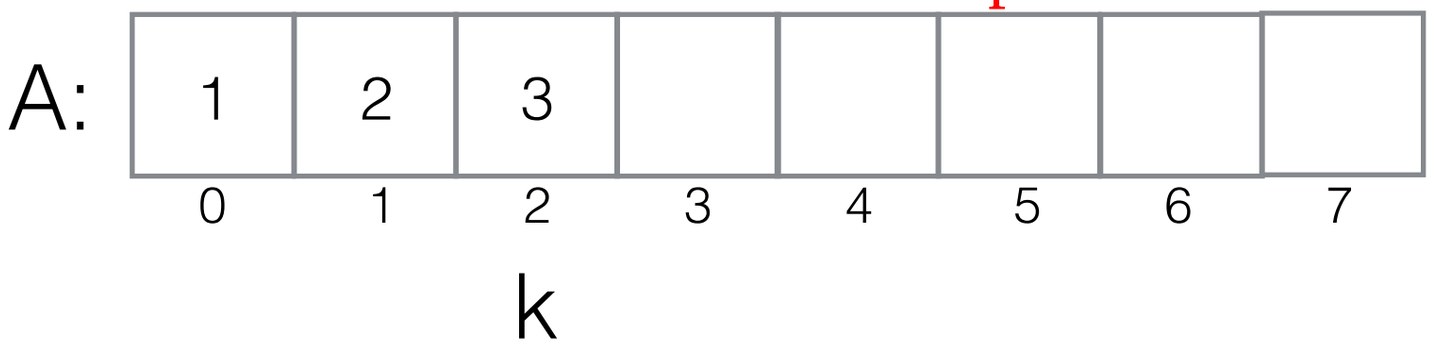

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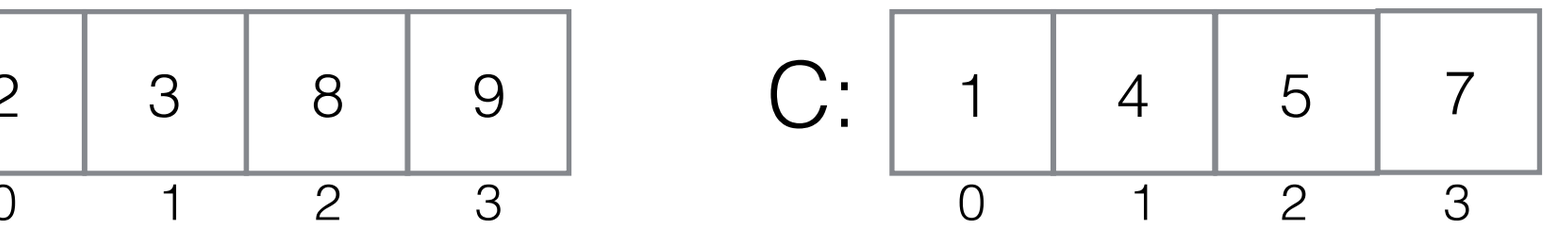

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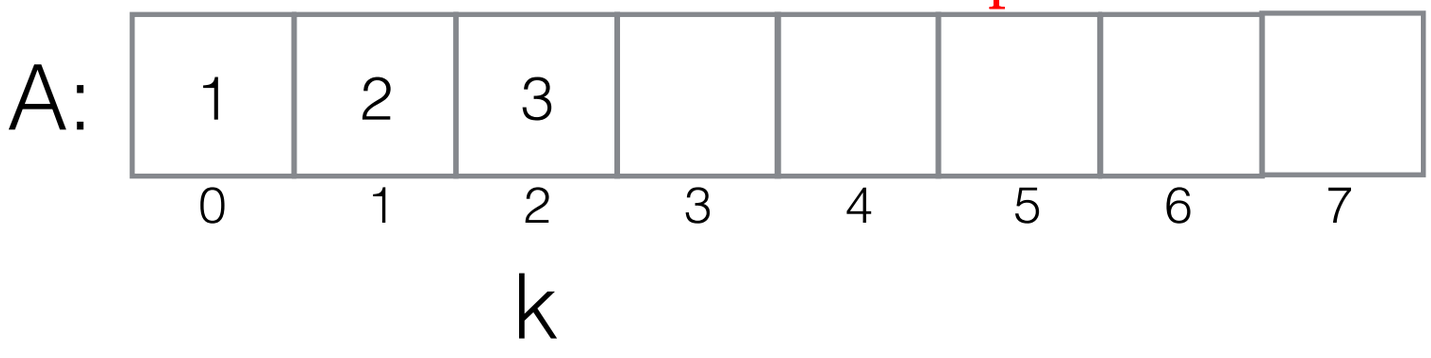

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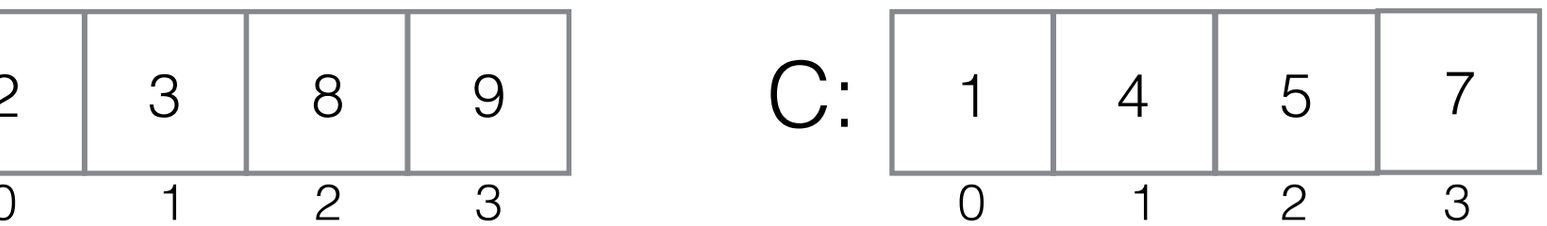

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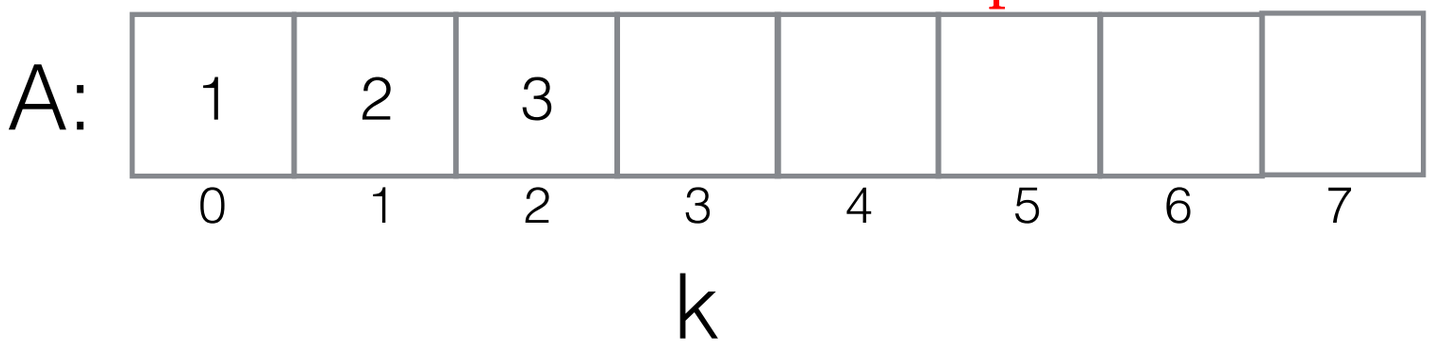

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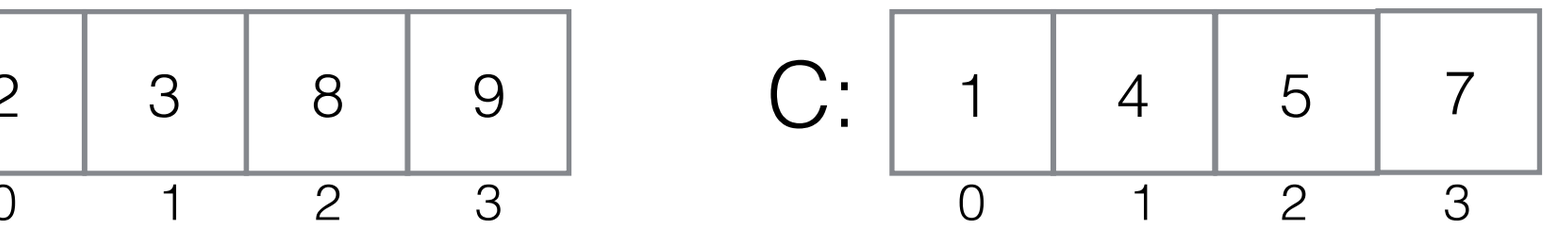

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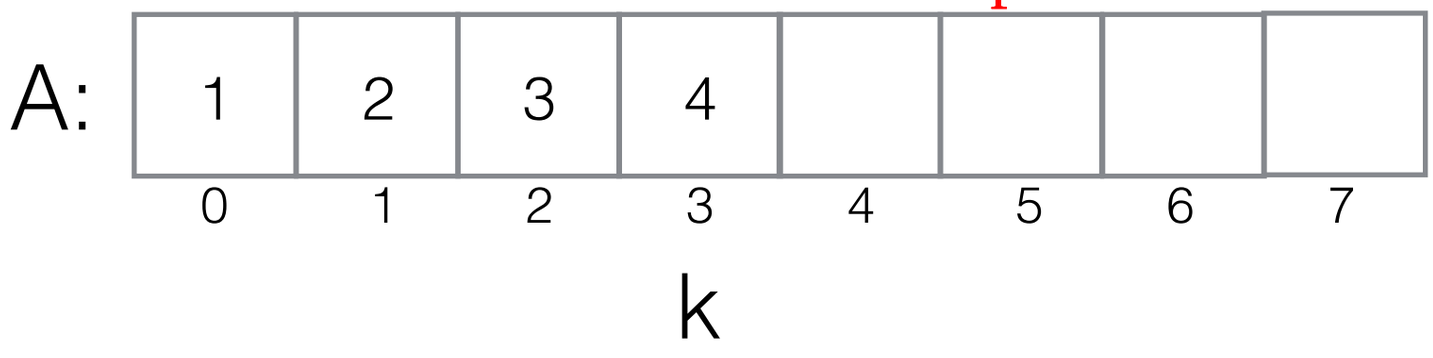

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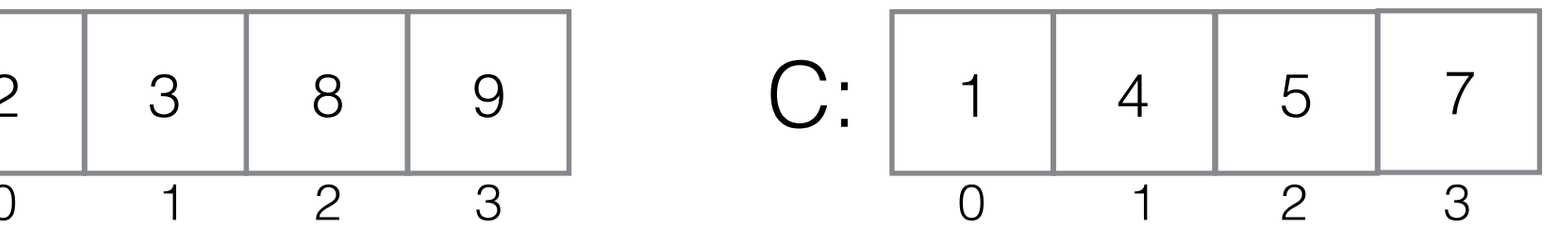

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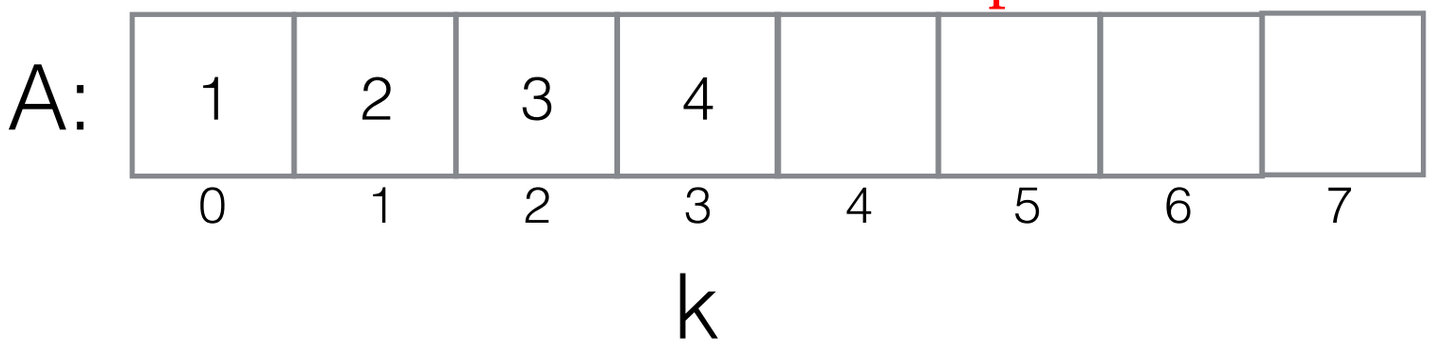

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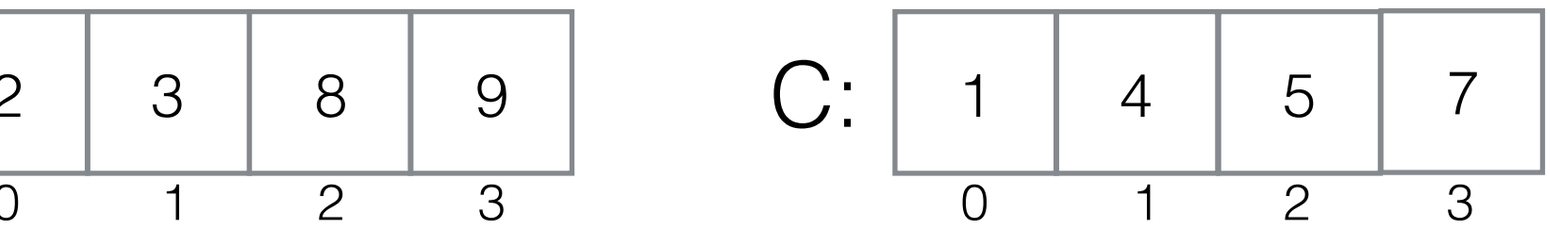

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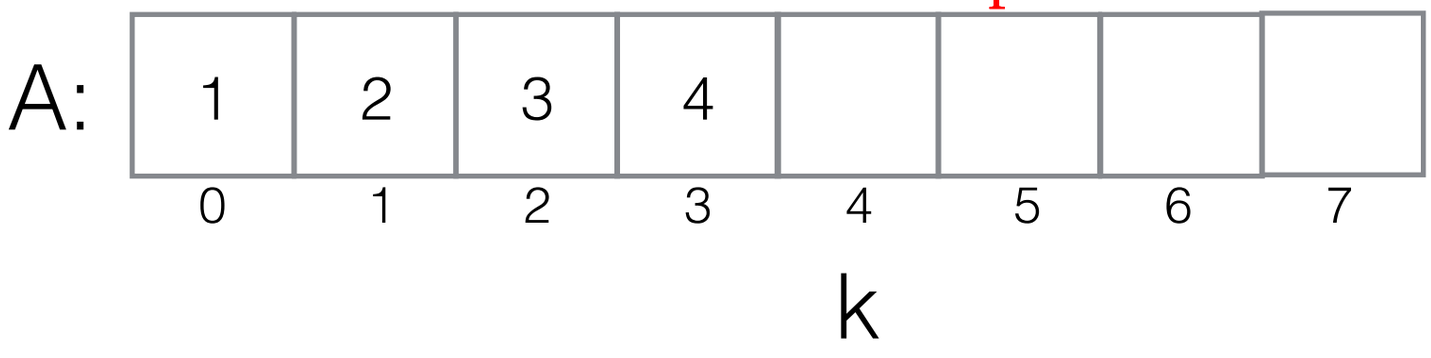

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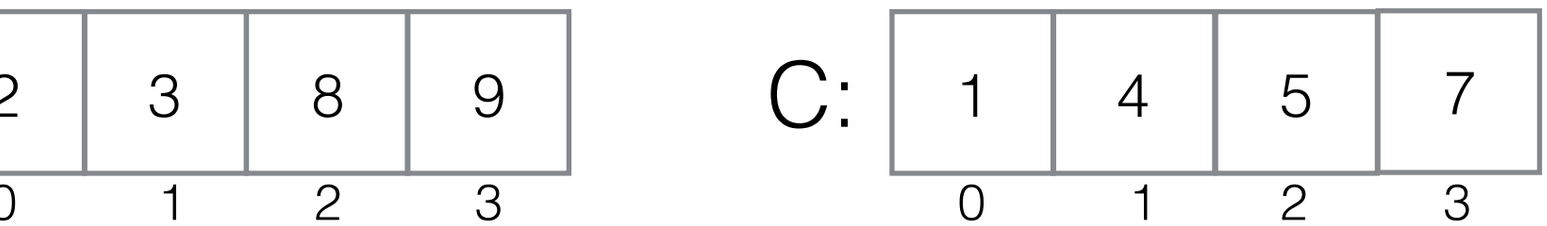

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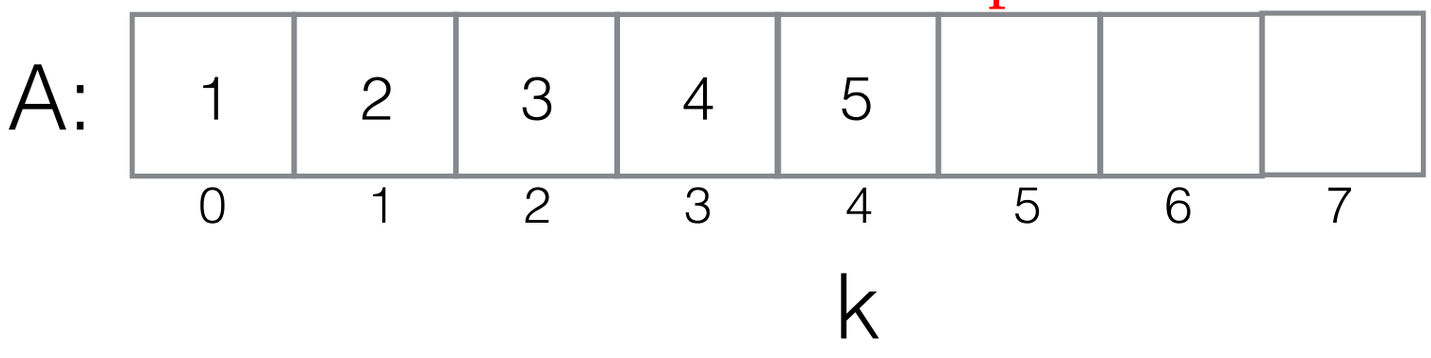

Mergesort: Merging Arrays

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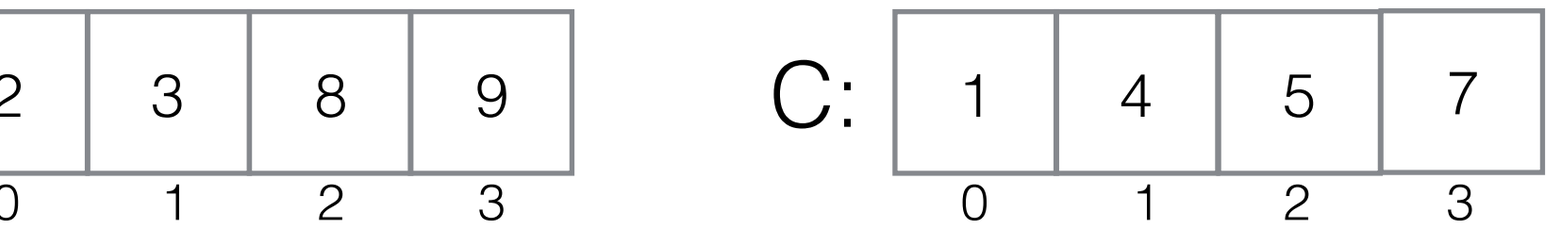

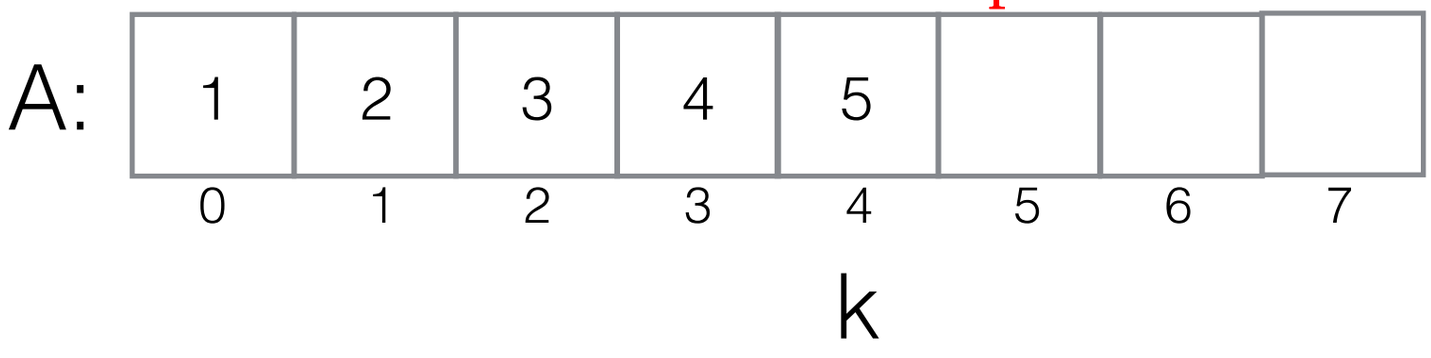

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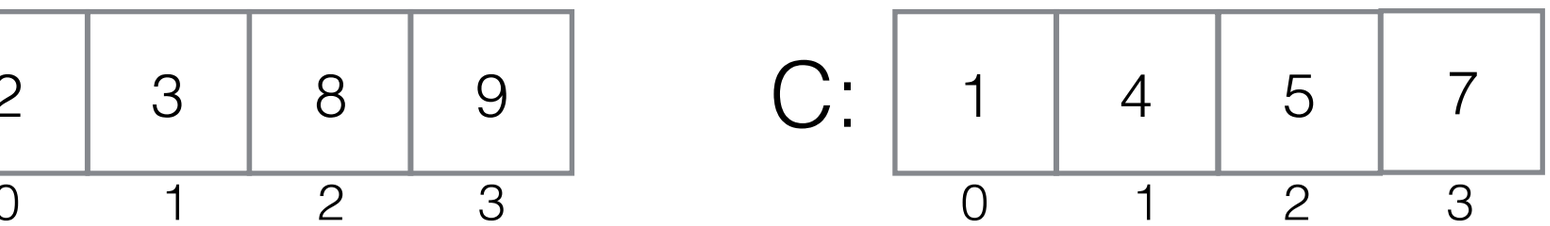

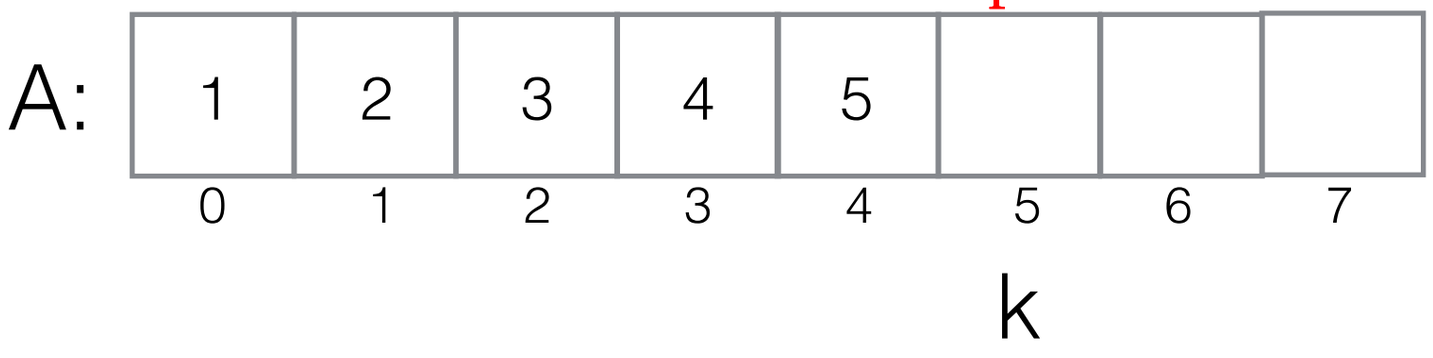

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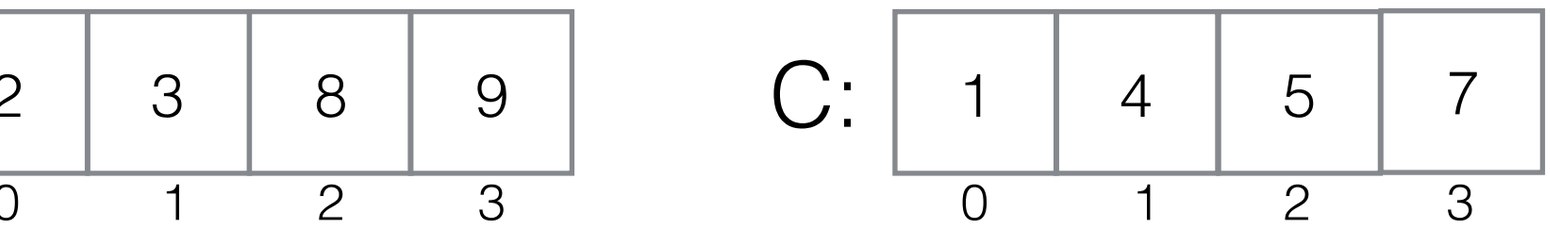

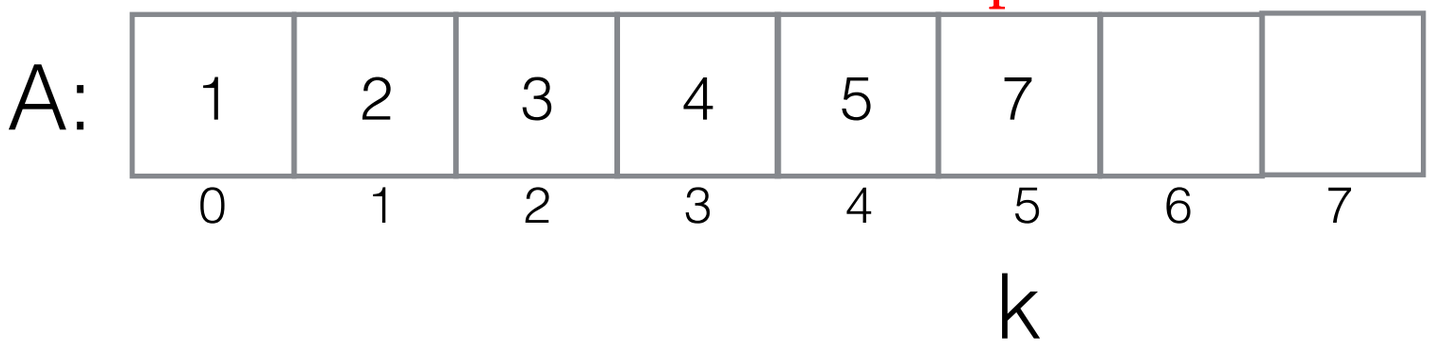

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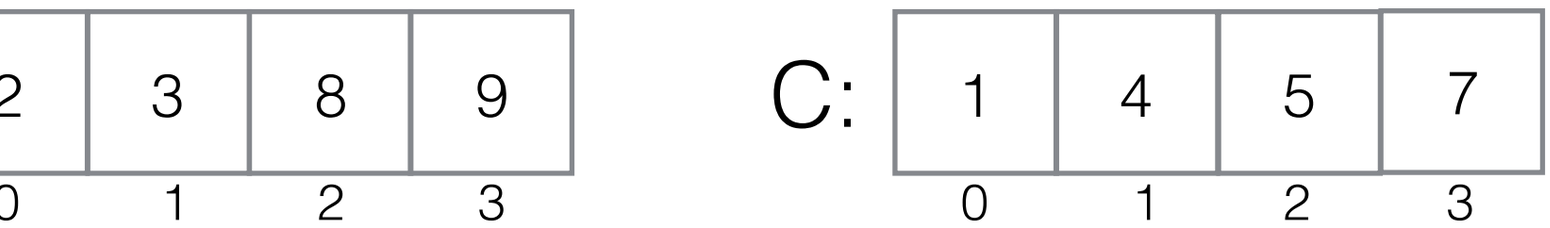

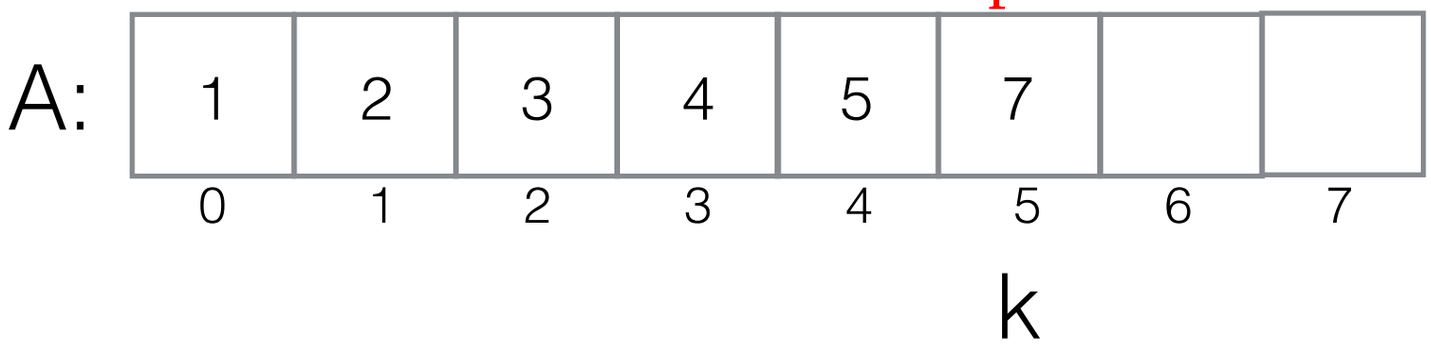

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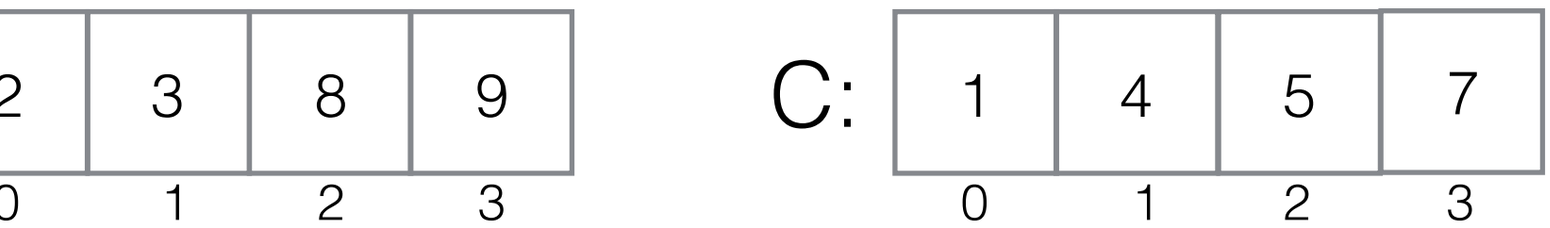

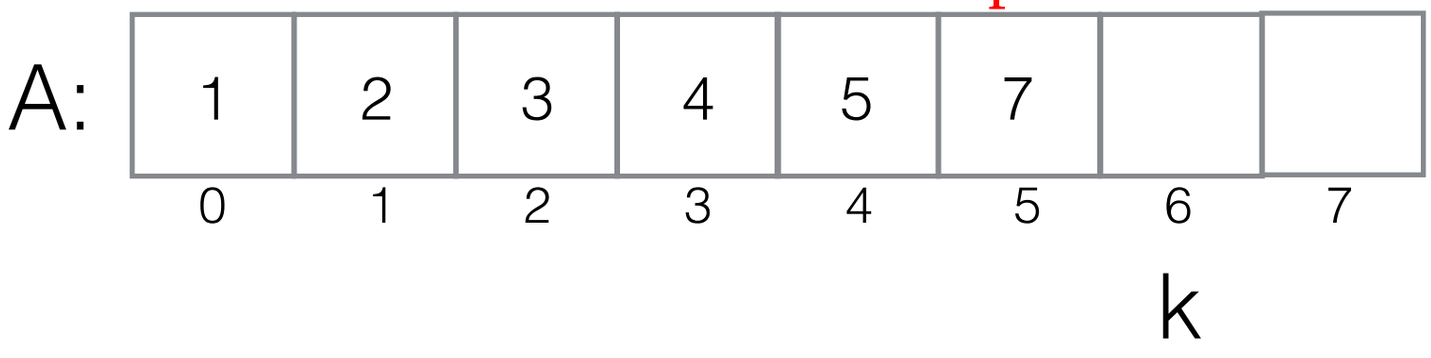

Mergesort: Merging Arrays

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procedure MERGE($B[\cdot], p, C[\cdot], q, A[\cdot]$)

$i \leftarrow 0; j \leftarrow 0; k \leftarrow 0$

while $i < p$ and $j < q$ **do**

if $B[i] \leq C[j]$ **then**

$A[k] \leftarrow B[i]$

$i \leftarrow i + 1$

else

$A[k] \leftarrow C[j]$

$j \leftarrow j + 1$

$k \leftarrow k + 1$

if $i = p$ **then**

copy $C[j]..C[q-1]$ to $A[k]..A[p+q-1]$ ▷ (a for loop)

else

copy $B[i]..B[p-1]$ to $A[k]..A[p+q-1]$ ▷ (a for loop)

Mergesort: Merging Arrays

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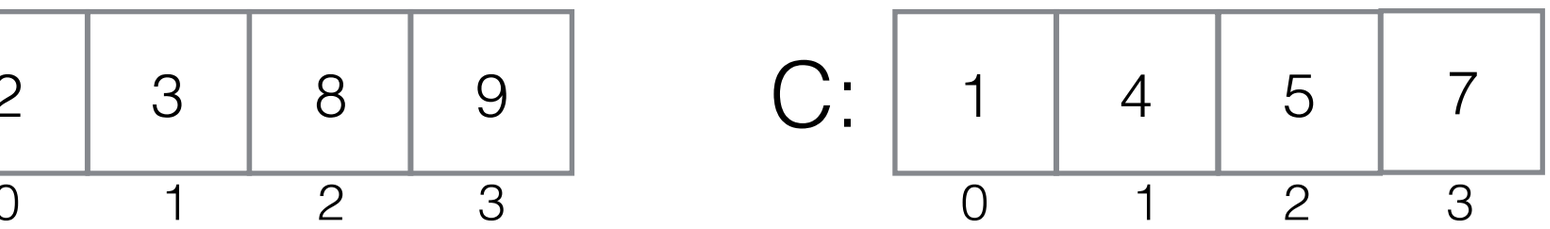

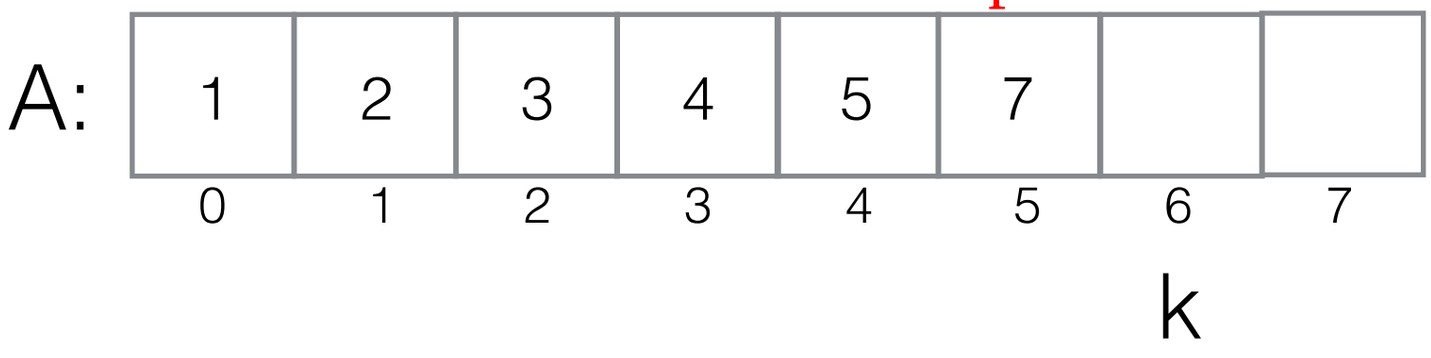

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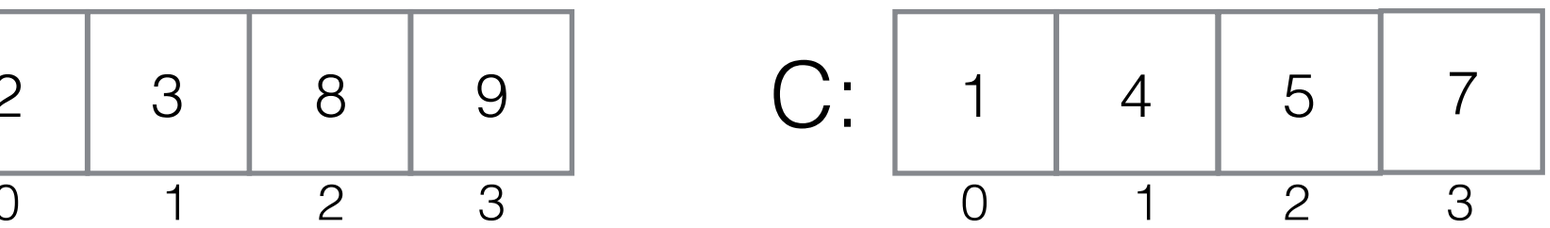

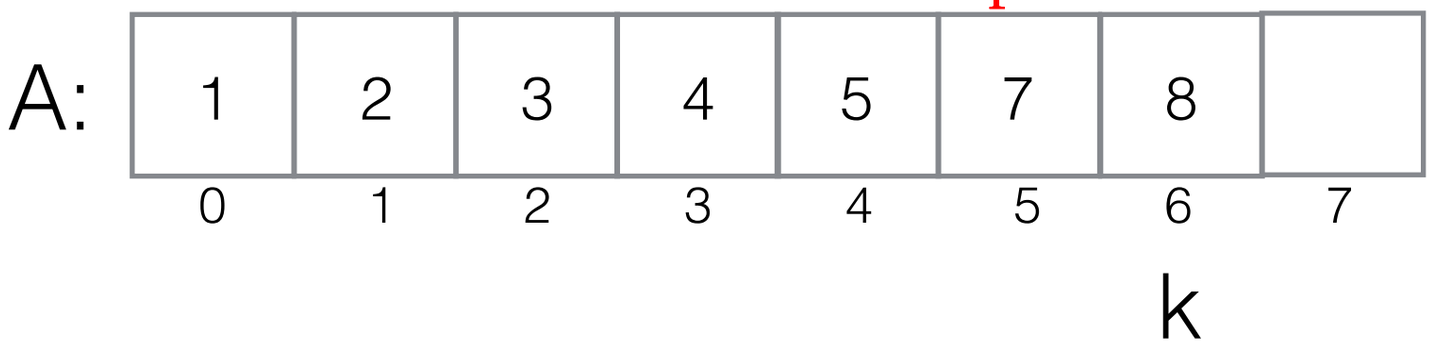

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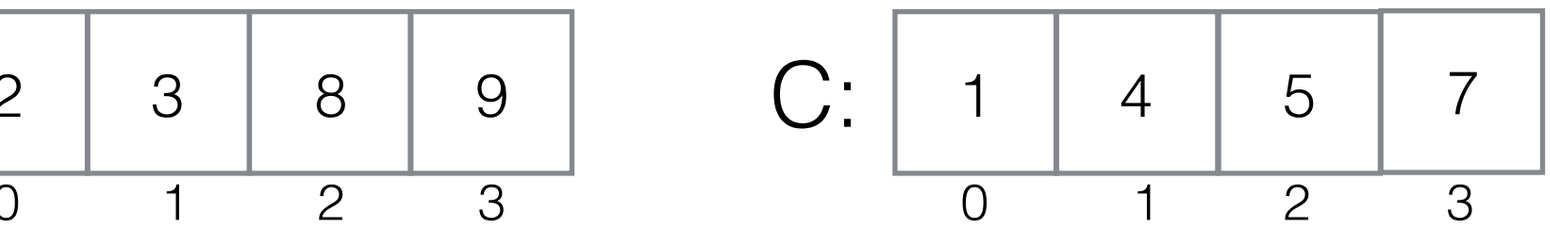

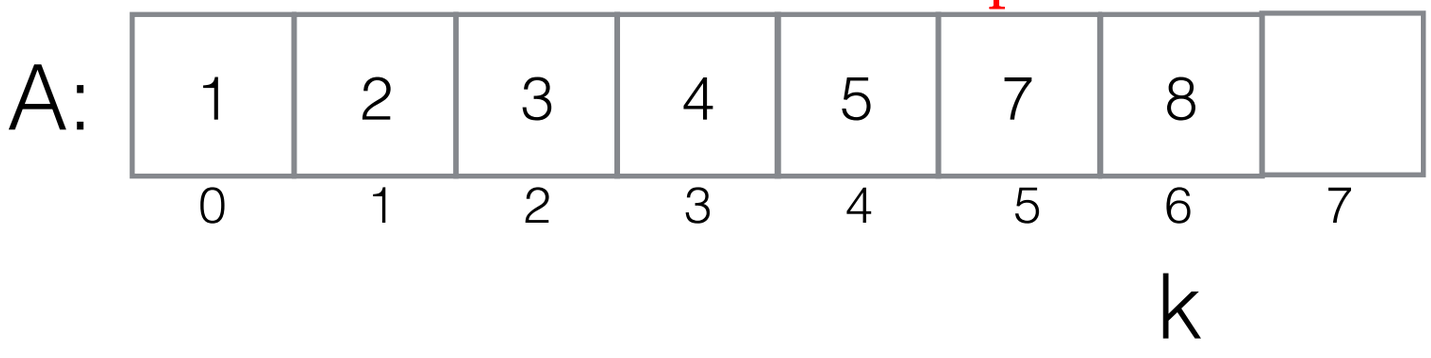

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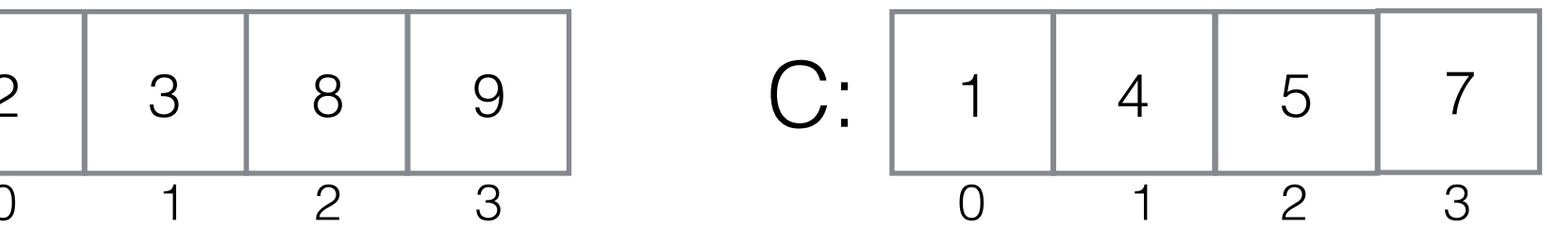
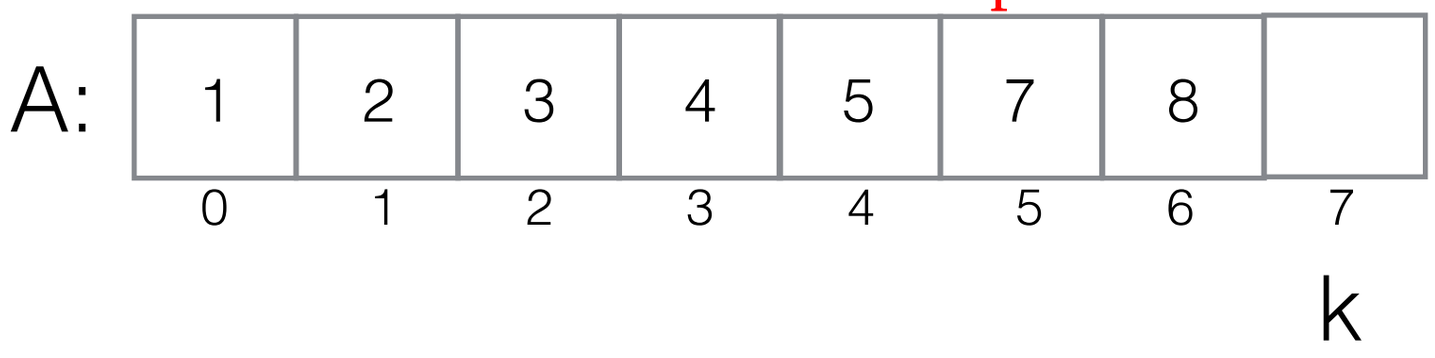


i

j

p: 4

q: 4



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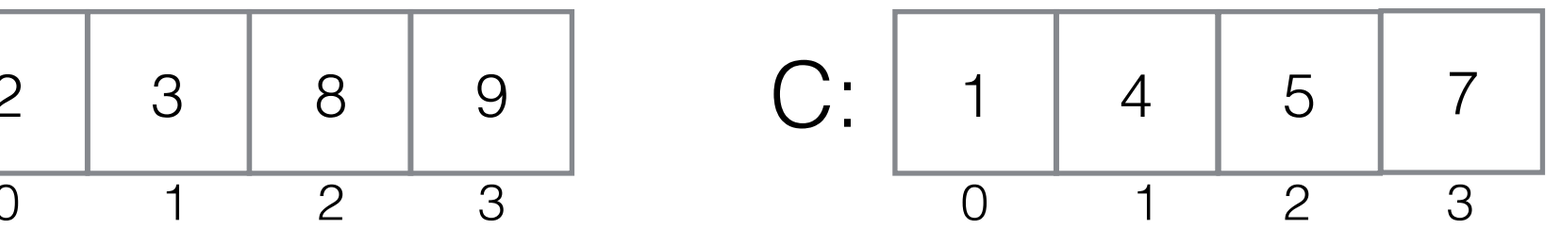

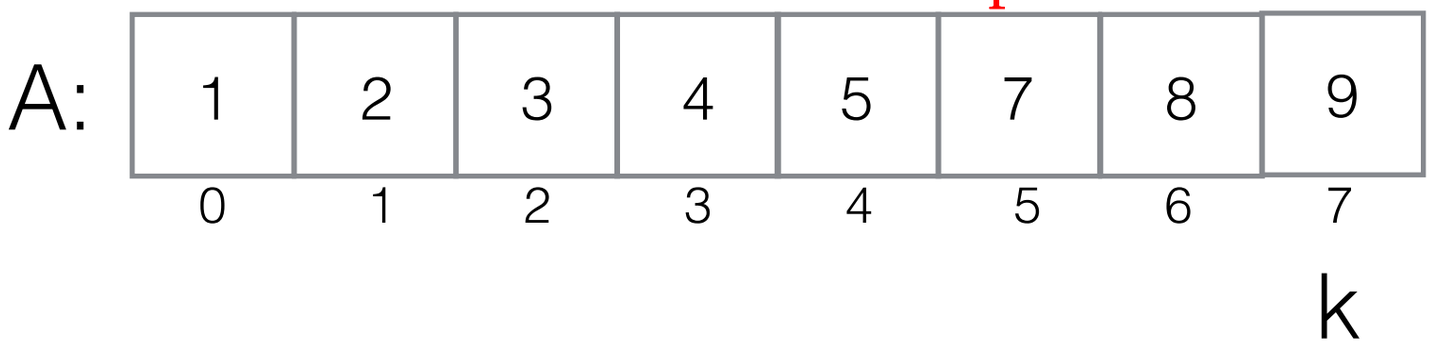

Mergesort: Analysis

- How many comparisons will MERGE need to make in the worst case, when given arrays of size ⌊n/2⌋ and ⌈n/2⌉?

- If the largest and second-largest elements are in different arrays, then n – 1 comparisons. Hence the cost equation for Mergesort is

$$C(n) = \begin{cases} 0 & \text{if } n < 2 \\ 2C(n/2) + n - 1 & \text{otherwise} \end{cases}$$

- By the Master Theorem, C(n) ∈ Θ(n log n).

Mergesort: Properties

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- For large n, the number of comparisons made tends to be around 75% of the worst-case scenario.

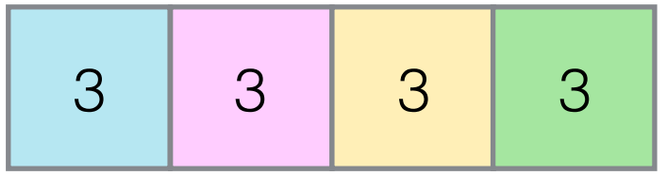
- Is mergesort stable? ?

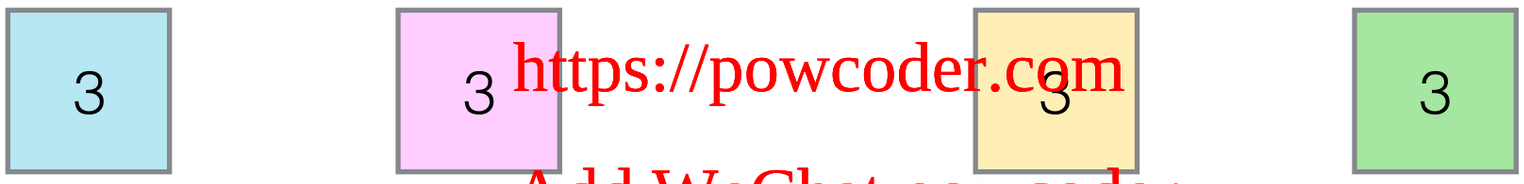
- Is mergesort in-place? no

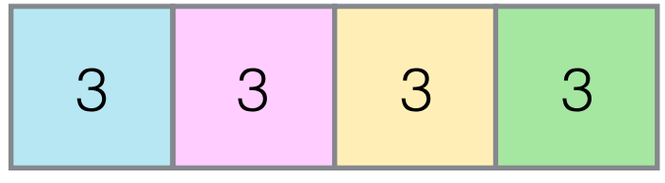
- If comparisons are fast, mergesort ranks between quicksort and heapsort (covered next week) for time, assuming random data.

- Mergesort is the method of choice for linked lists and for very large collections of data.

Mergesort: Stability







Mergesort: Properties

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- For large n, the number of comparisons made tends to be around 75% of the worst-case scenario.

- Is mergesort stable? yes

- Is mergesort in-place? no

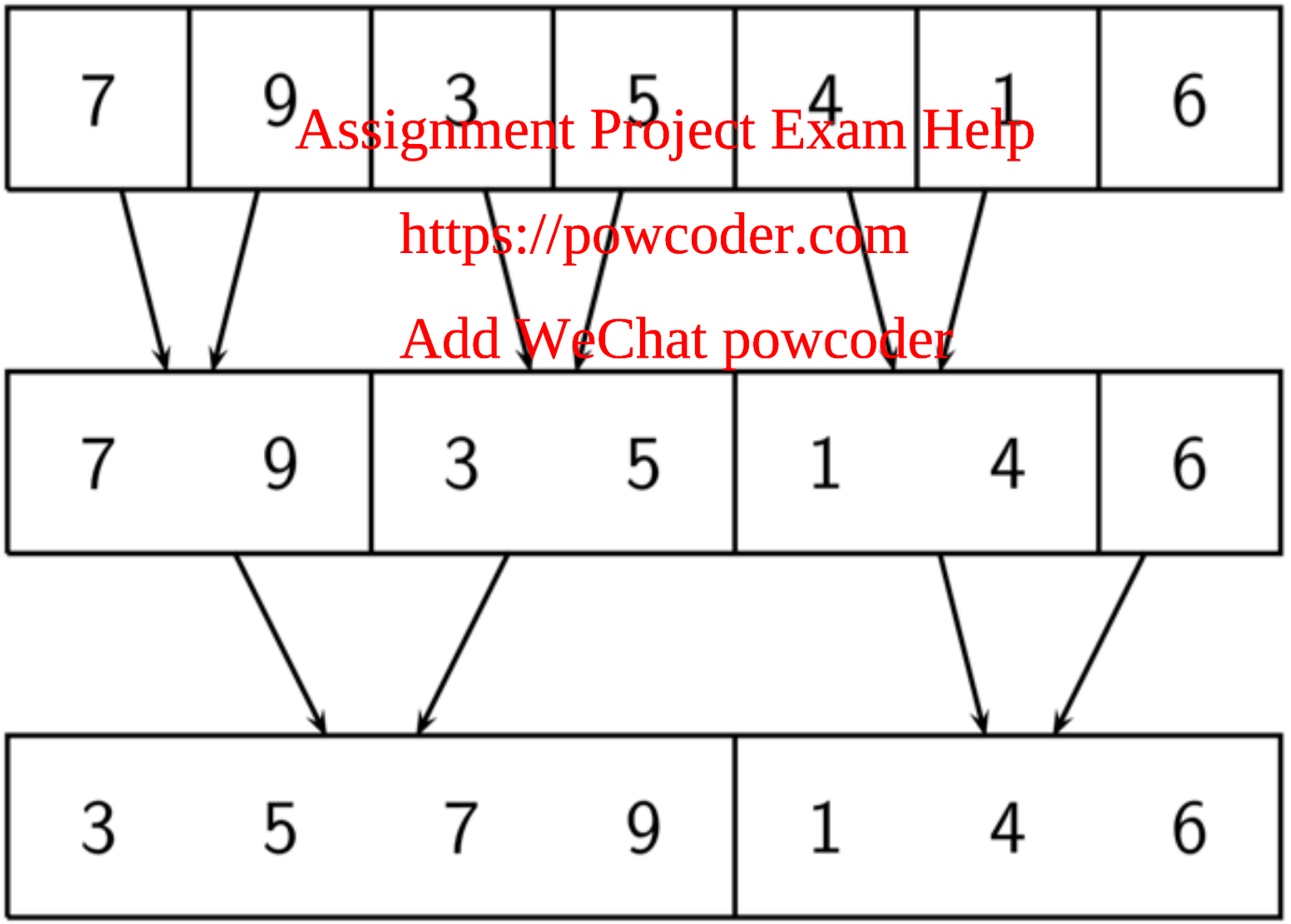
- If comparisons are fast, mergesort ranks between quicksort and heapsort (covered next week) for time, assuming random data.

- Mergesort is the method of choice for linked lists and for very large collections of data.

Bottom-Up Mergesort

- An alternative way of doing mergesort:

- Generate **runs** of length 2, then of length 4, and so on:



Quicksort

- Quicksort takes a divide-and-conquer approach that is different to mergesort's.

- It uses the **partitioning** idea from QuickSelect, picking a pivot element, and partitioning the array around that, so as to obtain this situation:

$$\underbrace{A[0] \ldots A[s-1]}_{\text{all are } \leq A[s]} \quad A[s] \quad \underbrace{A[s+1] \ldots A[n-1]}_{\text{all are } \geq A[s]}$$

- The element A[s] will be in its final position (it is the (s + 1)th smallest element).

- All that then needs to be done is to sort the segment to the left, recursively, as well as the segment to the right.

Quicksort

- Very short and elegant:

procedure QUICKSORT($A[\cdot], lo, hi$)
  if $lo < hi$ **then**
    $s \leftarrow$ PARTITION(A, lo, hi)
    QUICKSORT($A, lo, s - 1$)
    QUICKSORT($A, s + 1, hi$)

- Initial call: Quicksort(A, 0, n – 1).

Quicksort: Example

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procedure QUICKSORT($A[\cdot], lo, hi$)
  if $lo < hi$ **then**
    $s \leftarrow$ PARTITION(A, lo, hi)
    QUICKSORT($A, lo, s-1$)
    QUICKSORT($A, s+1, hi$)

A:

9	23	8	41	22	3	37
0	1	2	3	4	5	6

Quicksort: Example

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procedure QUICKSORT($A[\cdot], lo, hi$)
  if $lo < hi$ **then**
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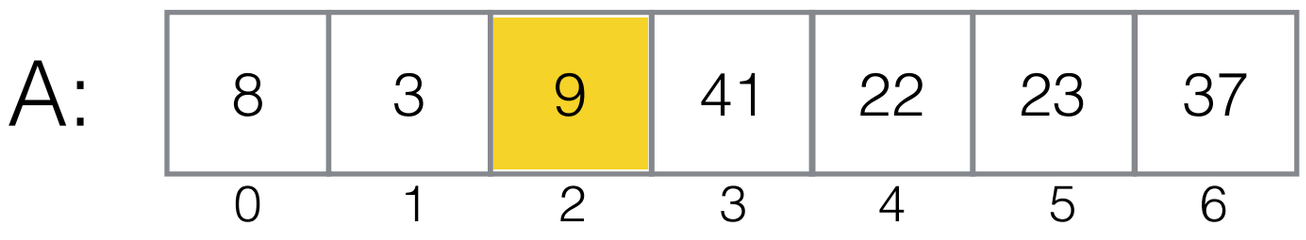

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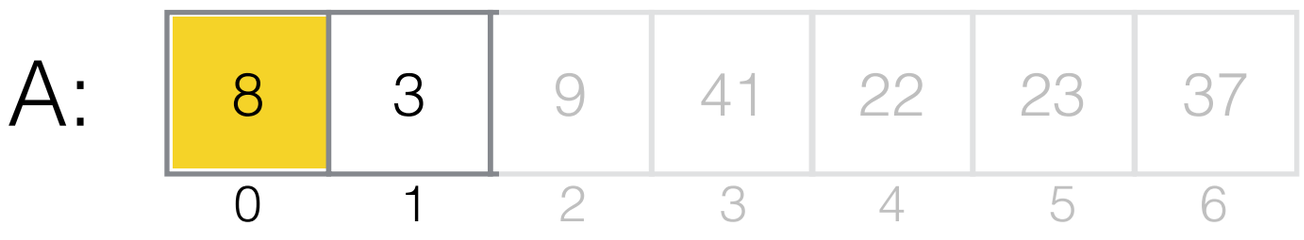

Quicksort: Example

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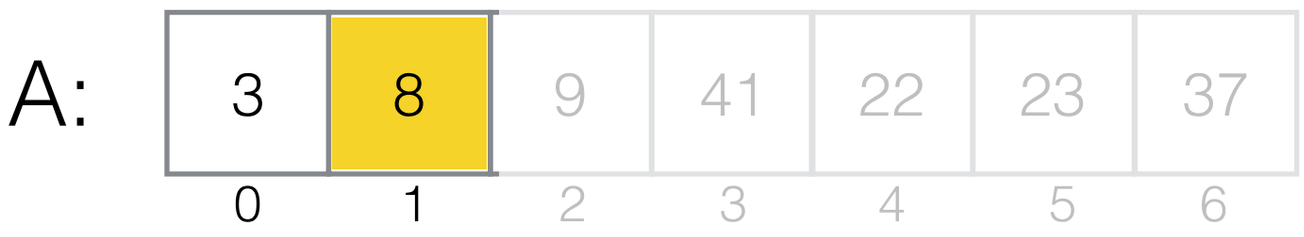

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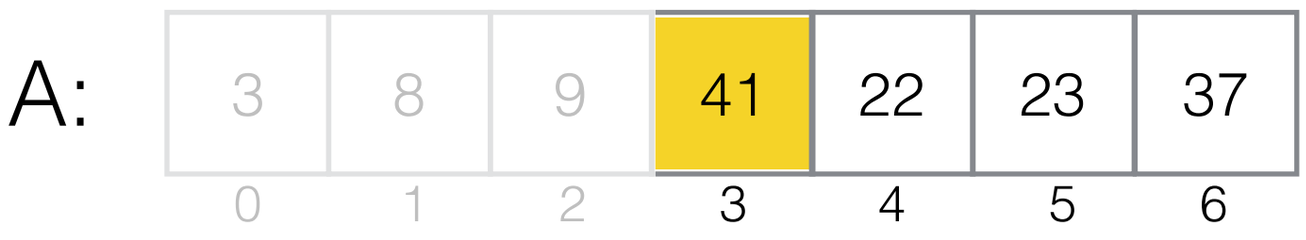

Quicksort: Example

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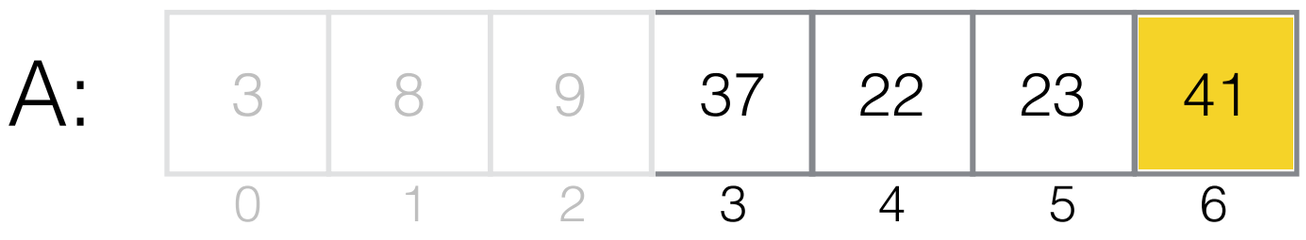

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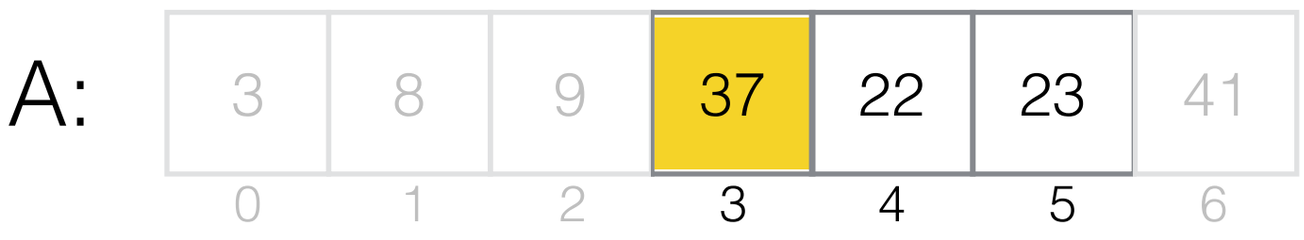

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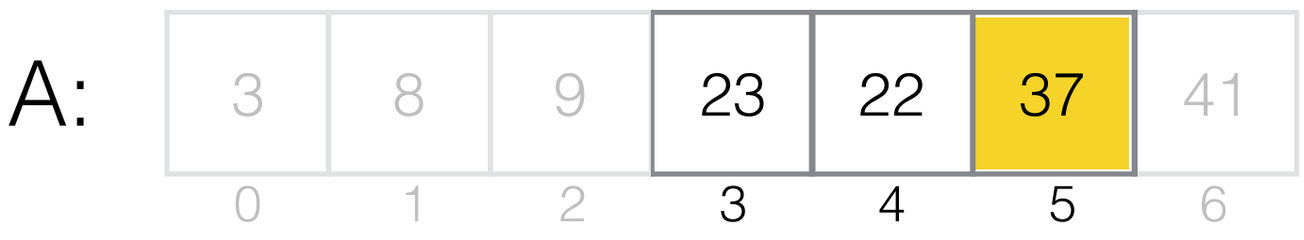

Quicksort: Example

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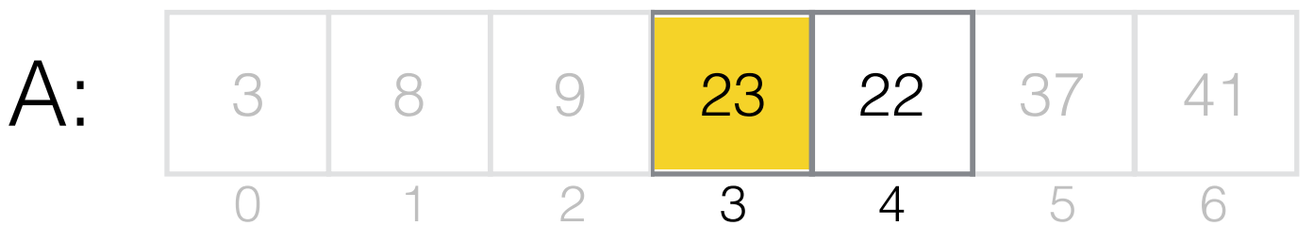
```
procedure QUICKSORT(A[·], lo, hi)
    if lo < hi then
        s ← PARTITION(A, lo, hi)
        QUICKSORT(A, lo, s − 1)
        QUICKSORT(A, s + 1, hi)
```

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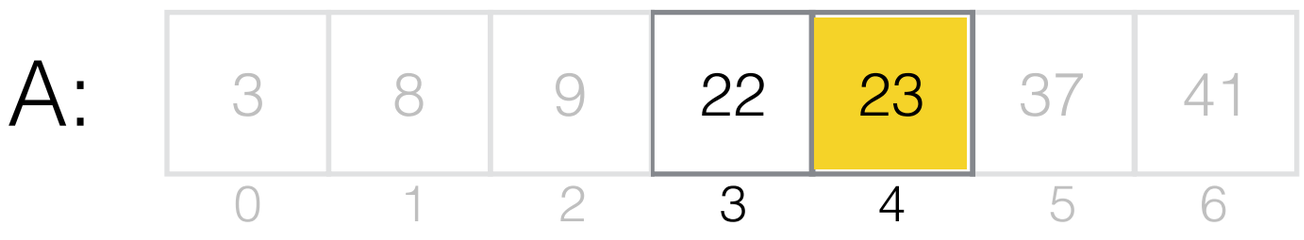

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A:

3	8	9	22	23	37	41
0	1	2	3	4	5	6

Hoare Partitioning

- The standard way of doing partitioning in Quicksort

```
function PARTITION(A[·], lo, hi)
    p ← A[lo]; i ← lo; j ← hi
    repeat
        while i < hi and A[i] ≤ p do i ← i + 1
        while j ≥ lo and A[j] > p do j ← j − 1
        swap(A[i], A[j])
    until i ≥ j
    swap(A[i], A[j])                    ▷ Undo the last swap
    swap(A[lo], A[j])                   ▷ Bring pivot to its correct position
    return j
```


Hoare Partitioning

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```
function PARTITION(A[·], lo, hi)
    p ← A[lo]; i ← lo; j ← hi
    repeat
        while i < hi and A[i] ≤ p do i ← i + 1
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    return j
```


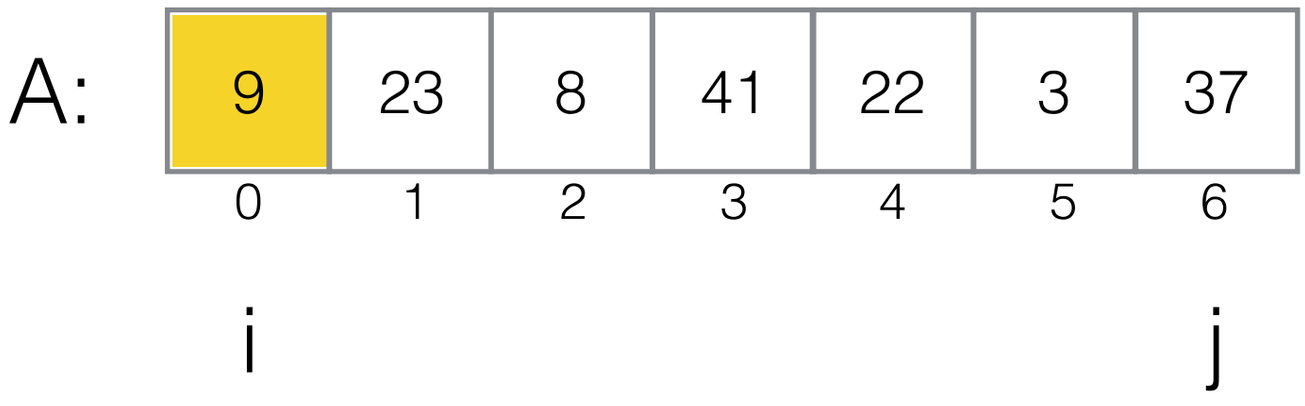

Hoare Partitioning

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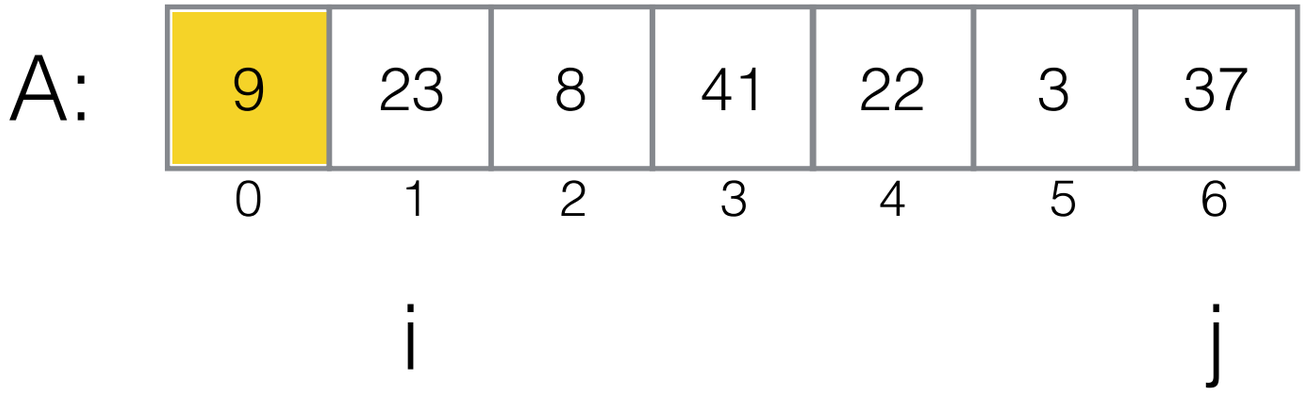

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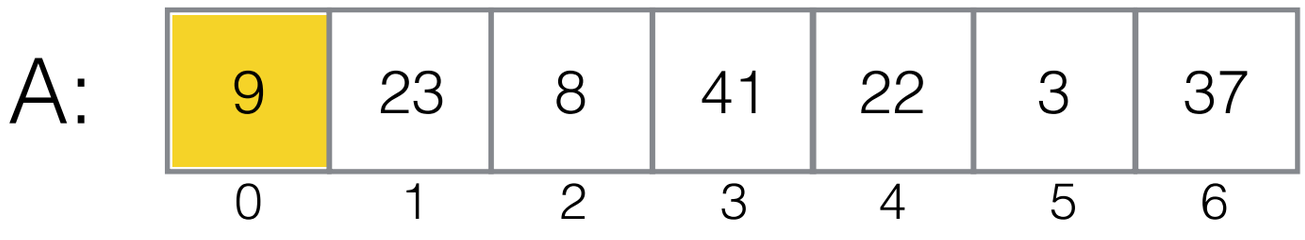

p: 9

Hoare Partitioning

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    p ← A[lo]; i ← lo; j ← hi
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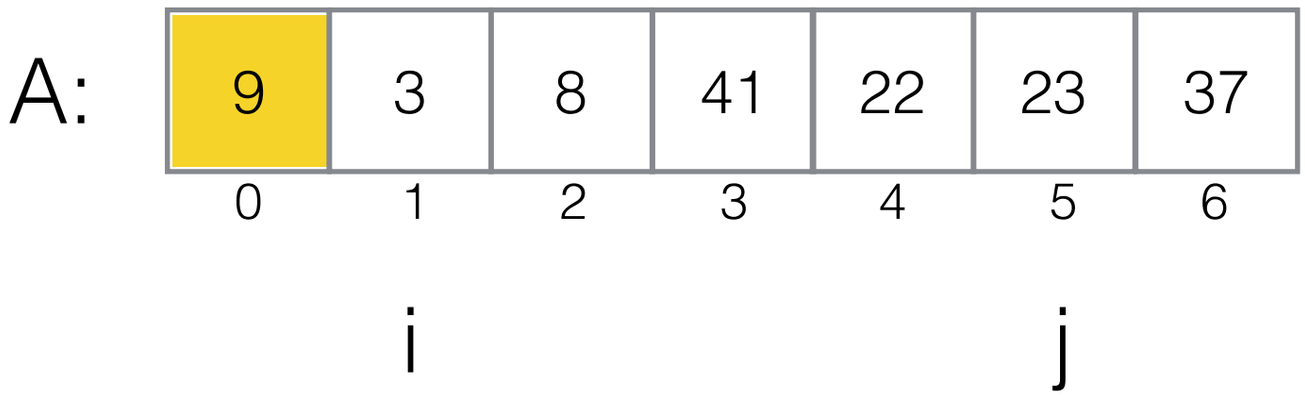

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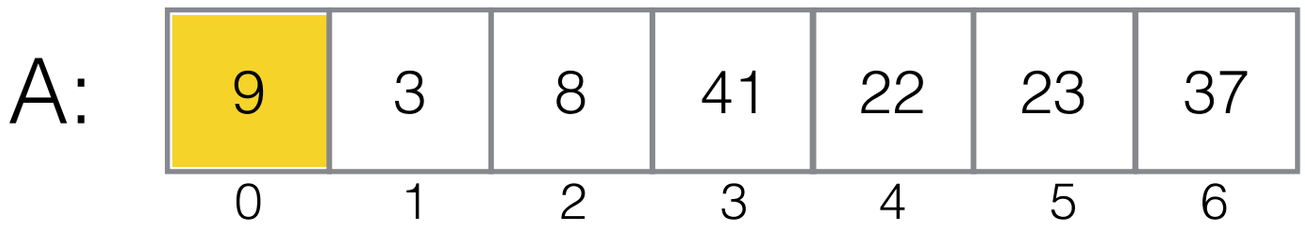

p: 9

Hoare Partitioning

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  $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi$
  repeat
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    while $j \geq lo$ and $A[j] > p$ **do** $j \leftarrow j - 1$
    $swap(A[i], A[j])$
  until $i \geq j$
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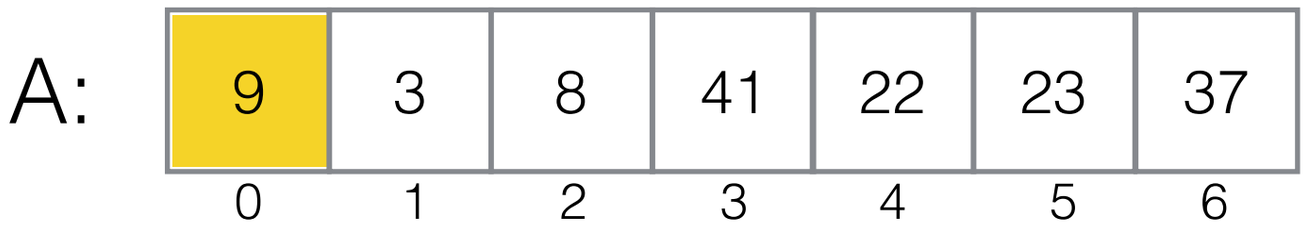

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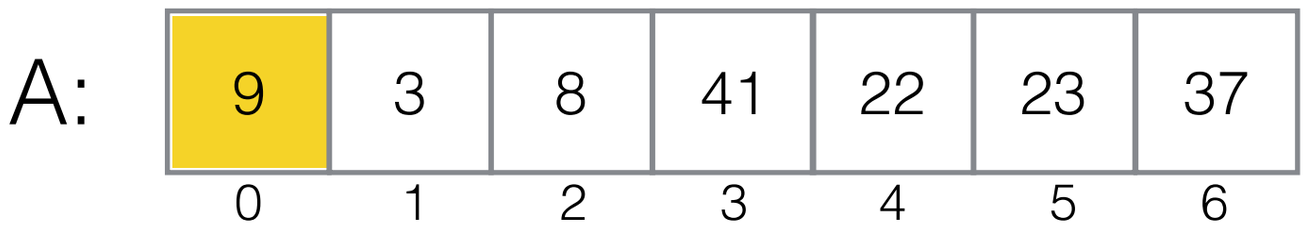

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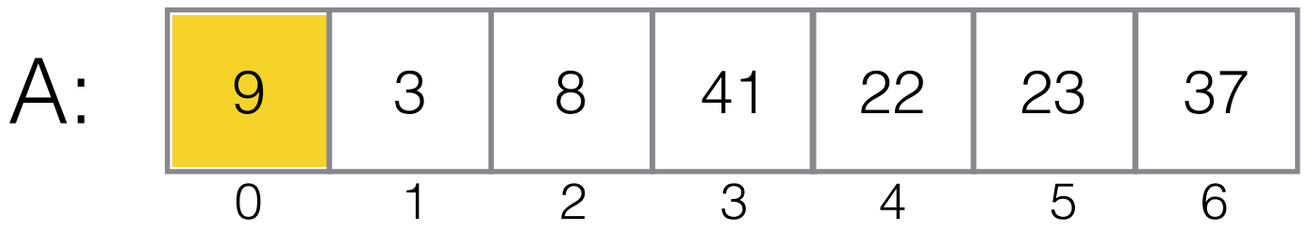

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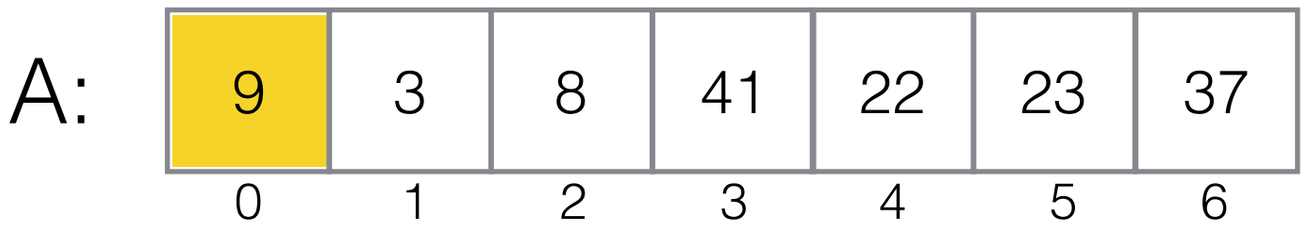

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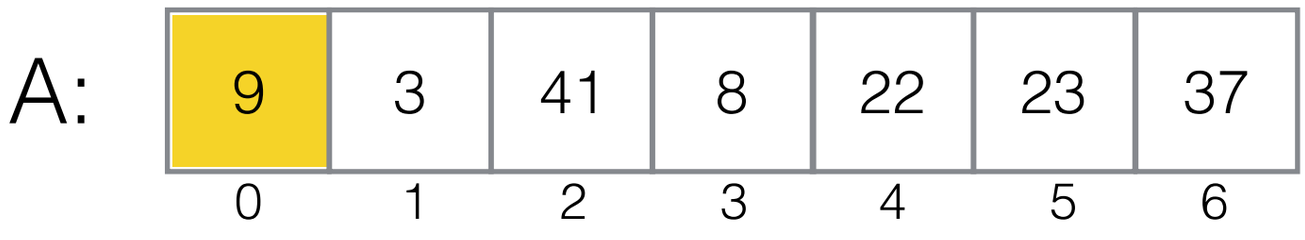

p: 9

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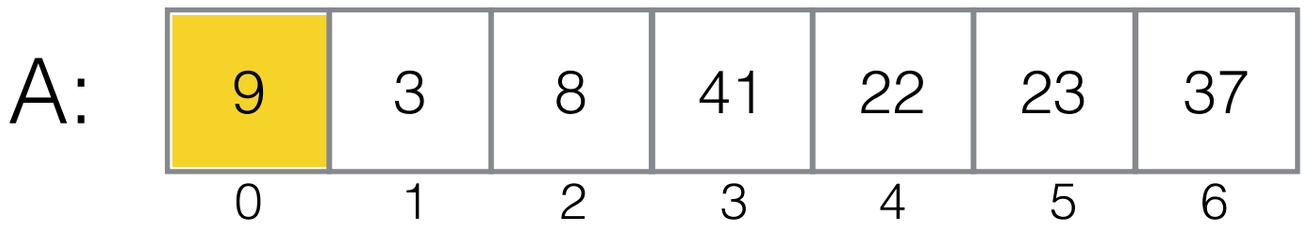

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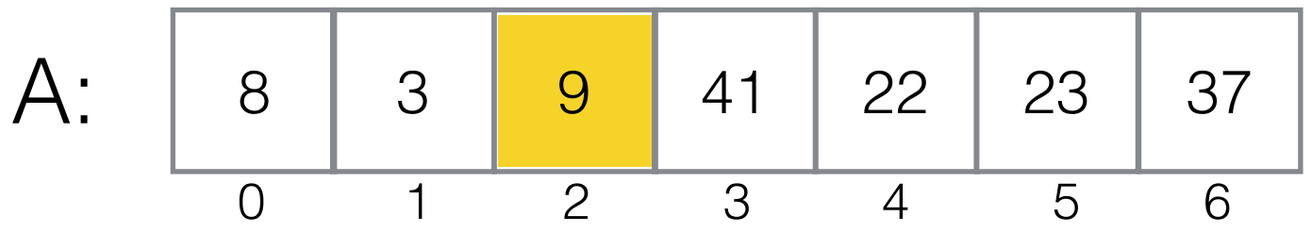

Quicksort Analysis: Best Case Analysis

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- The best case happens when the pivot is the median; that results in two sub-tasks of equal size.

$$C_{best}(n) = \begin{cases} 0 & \text{if } n < 2 \\ 2C_{best}(n/2) + n & \text{otherwise} \end{cases}$$

The 'n' is for the n key comparisons performed by Partition.

- By the Master Theorem, $C_{best}(n) \in \Theta(n \log n)$, just as for mergesort, so quicksort's best case is (asymptotically) no better than mergesort's worst case.

Quicksort Worst Case

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A:

Quicksort Analysis: Worst Case Analysis

- The worst case happens if the array is already sorted.

- In that case, we don't really have divide-and-conquer, because each recursive call deals with a problem size that has only been decremented by 1:

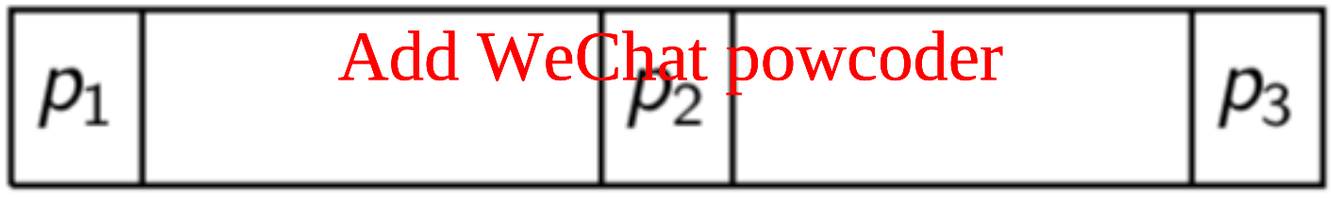
$$C_{worst}(n) = \begin{cases} 0 & \text{if } n < 2 \\ C_{worst}(n-1) + n & \text{otherwise} \end{cases}$$

- That is, $C_{worst}(n) = n + (n - 1) + \cdots + 3 + 2 \in \Theta(n^2)$.

Quicksort Improvements: Median-of-Three

- It would be better if the pivot was chosen randomly.

- A cheap and useful approximation to this is to take the median of three candidates, A[lo], A[hi], and A[⌊(lo + hi)/2⌋].



- Reorganise the three elements so that p_1 is the median, and p_3 is the largest of the three.

- Now run quicksort as before.

Quicksort Improvements: Median-of-Three

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- In fact, with median-of-three, we can have a much faster version than before, simplifying tests in the innermost loops:

function PARTITION($A[\cdot], lo, hi$)
  $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi + 1$
  repeat
    ~~**while** $i < hi$ and $A[i] < p$ **do** $i \leftarrow i + 1$~~
    repeat $i \leftarrow i + 1$ **until** $A[i] \geq p$
    ~~**while** $j \geq lo$ and $A[j] > p$ **do** $j \leftarrow j - 1$~~
    repeat $j \leftarrow j - 1$ **until** $A[j] \leq p$
    $swap(A[i], A[j])$
  until $i \geq j$
  $swap(A[i], A[j])$
  $swap(A[lo], A[j])$
  return j

Quicksort Improvements: Early Cut-Off

- A second useful improvement is to stop quicksort early and switch to insertion sort. This is easily implemented:

procedure SORT($A[\cdot], n$)
    QUICKALMOSTSORT($A, 0, n-1$)
    INSERTIONSORT(A, n)

procedure QUICKALMOSTSORT($A[\cdot], lo, hi$)
    if $lo + 10 < hi$ **then**
        $s \leftarrow$ PARTITION(A, lo, hi)
        QUICKALMOSTSORT($A, lo, s-1$)
        QUICKALMOSTSORT($A, s+1, hi$)

Quicksort Properties

- With these (and other) improvements, quicksort is considered the best available sorting method for arrays of random data.

- A major reason for its speed is the very tight inner loop in PARTITION.

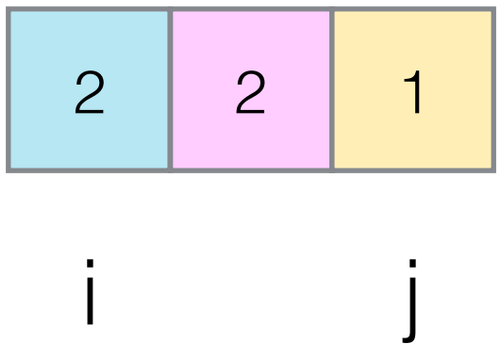
- Although mergesort has a better performance guarantee, quicksort is faster on average.

- In the best case, we get $\lceil \log_2 n \rceil$ recursive levels. It can be shown that on random data, the expected number is $2 \log_e n \approx 1.38 \log_2 n$. So quicksort's average behaviour is very close to the best-case behaviour.

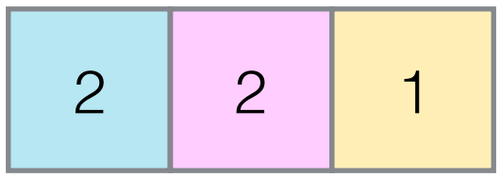
- Is quicksort stable? ?

- Is it in-place? yes

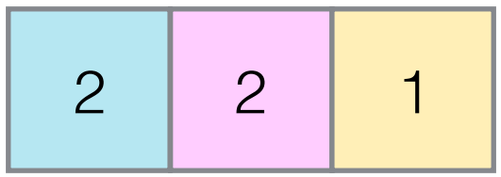
Quicksort Stability



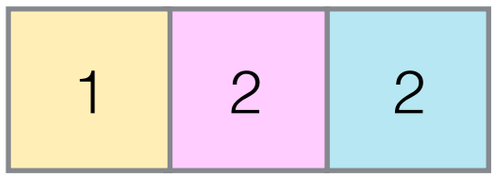
Quicksort Stability



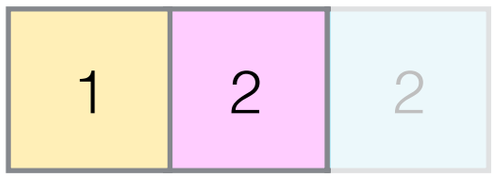
Quicksort Stability



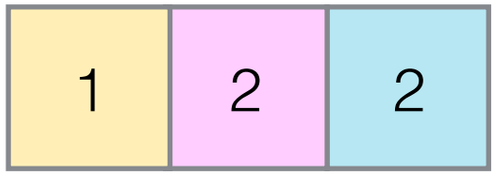
Quicksort Stability



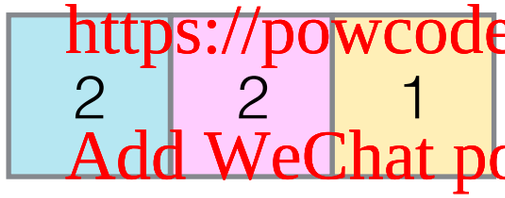
Quicksort Stability



Quicksort Stability



This is where we finished



This is where we started

Not stable

Quicksort Properties

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- With these (and other) improvements, quicksort is considered the best available sorting method for arrays of random data.

- A major reason for its speed is the very tight inner loop in PARTITION.

- Although mergesort has a better performance guarantee, quicksort is faster on average.

- In the best case, we get $\lceil \log_2 n \rceil$ recursive levels. It can be shown that on random data, the expected number is $2 \log_e n \approx 1.38 \log_2 n$. So quicksort's average behaviour is very close to the best-case behaviour.

- Is quicksort stable? no

- Is it in-place? yes

Next up

- Tree traversal methods, plus we apply the divide-and-conquer technique to the closest-pair problem.

