

# COMP90057

## Advanced Theoretical Computer Science

Complexity

### Assignment Project Exam Help

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<https://powcoder.com>

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## This part of COMP90057: Complexity

Among problems that computers *can* solve, there are many that are **intractable**: they take an unreasonably long time to solve, or require an unreasonably large amount of space to solve.

Complexity: runtime bounds on a problem based on assessing the runtime of algorithms; space bounds; complexity classes of problems; methods to deal with intractable problems.

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## Preparatory reading/viewing

<http://cacm.acm.org/magazines/2009/9/>

[38904-the-status-of-the-p-versus-np-problem/fulltext](http://cacm.acm.org/magazines/2009/9/38904-the-status-of-the-p-versus-np-problem/fulltext)

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**Symposium on 50 Years of Complexity Theory: A Celebration of the Work  
of Stephen Cook** <https://powcoder.com>

<http://www.fields.utoronto.ca/activities/18-19/NP50>

<http://www.fields.utoronto.ca/video-archive/event/2774/2019>

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# Big-Oh Notation

In algorithms classes, you would have seen a definition of  $O()$  notation, something like this

$g = O(f)$  iff for all  $n > n_0 : g(n) < c \cdot f(n)$ ,

for some integers  $n_0$  and  $c$ . For example,

$$5n^3 + 2n^2 - 22n + 6 = O(n^3)$$

$$5n^3 + 2n^2 - 22n + 6 = O(n^4)$$

$$1 + 2 + \dots + n = O(n^2)$$

$$3n \log n = O(n \log n)$$

$$3n \log n = O(n^2)$$

You can also think of  $O(\cdot)$  as a set of functions, and thus write  $g \in O(f)$ . However, we'll stick to Sipser's notation convention.

# Polynomially Bounded Functions

To determine the time complexity of a *problem*, we examine the asymptotic behaviour of algorithms that solve that problem, that is, decide the underlying language. (We assume that the TM **halts on all inputs**.)

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We want to bound the running time of the algorithm/machine, asymptotically, as a function of input size.

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In particular, function  $t$  is **polynomially bounded** if  $t(n) = O(n^r)$  for some positive integer  $r$ .

# Textbook Questions

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Questions you can try at this point:

Exercise 7.1

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Exercise 7.2

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# Complexity Classes

Let  $M$  be a deterministic (single-tape) Turing machine.

The **time complexity** of  $M$  is the function  $t_M : \mathbb{N} \rightarrow \mathbb{N}$  defined by

$$t_M(n) = \max_{w \in \Sigma^*} \{ m \mid M(w) \text{ halts after exactly } m \text{ steps} \}$$

This focuses on the worst case input string for string-length  $n$ .

Let  $t : \mathbb{N} \rightarrow \mathbb{N}$  be a function. The complexity class (of languages) for function  $t$  is:

$$TIME(t(n)) = \left\{ L \mid \begin{array}{l} L \text{ is decided by some } M \\ \text{with } t_M(n) = O(t(n)) \end{array} \right\}$$

This focuses on the *best* of all machines that decide a language.

# Machine Model Affects Complexity

A one-tape Turing machine can decide the language

$$A = \{0^k 1^k \mid k \geq 0\}$$

in  $O(n \log n)$  time.

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The first phase checks whether the length of the input is even (and perhaps that it's of the form  $0^* 1^*$ ).

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The machine then repeatedly scans across its input, each time crossing off every second 0 and every second 1, checking after each scan that the number of non-crossed symbols remains even.

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Since this halves the number of 0s and 1s (integer division) in each scan, the time taken is  $O(n \log n)$  in the worst case.

The machine cannot decide  $A$  in linear time: indeed, a linear-time language would have to be **regular** (Problem 7.49).

# Model Affects Complexity

0000-01 - 1  
↑  
Turing

A two-tape deterministic machine has lower time complexity.

It can copy all 1s to its second tape and then match them against the 0s in linear time.

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The two kinds of Turing machine have the same computational power, but they have different complexity properties.

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Thus we lose some of the robustness of the Turing-machine model when we move from studying decidability/computability to (time) complexity.

# The Class $P$

Fortunately, it is possible to recover much of that robustness.

We need to think more broadly, considering all the polynomial-time deterministic deciders as one class.

*Definition:*  $P$  is the class of languages decidable by a **deterministic** Turing machine in polynomial time, viz.

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$$P = \bigcup_{k \in \mathbb{N}} \text{TIME}(n^k)$$

# Robustness of $P$

$$(x^4 + x^3)(x^{12} + 3x^2) \rightarrow x^{16}$$
$$((x^2 + 3x^2)^4 + (x^4 + 3x^2)^3) \rightarrow x^{48}, \dots$$

Recall that polynomial functions are closed under addition, multiplication, and composition.

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Class  $P$  is robust in the sense that all the deterministic computational models are polynomially invariant. Changing to some other deterministic machine model will not change the class  $P$ .

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Roughly,  $P$  corresponds to the class of problems that can be solved in a *reasonable* amount of time.



# Polynomial vs Exponential

Recall  $t(n)$  is *polynomially bounded* means it is  $O(n^r)$  for some positive integer  $r$ .

~~$2^{2^n} = (2^2)^n = 4^n \neq O(2^n)$~~   
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Here we take “exponential” to mean  $\Theta(2^{cn})$  for some constant  $c > 0$ .

There are functions, such as  $2^{\sqrt{n}}$  and  $n^{\log n}$  that grow faster than every polynomial function, but more slowly than every exponential function.

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Hence it is possible that  $P \subsetneq NP$ , and at the same time, that  $NP$ -complete problems can be solved in sub-exponential time.

We describe  $NP$  below.

$$\begin{aligned} 4^{4 \log_2 n} &= (2^2)^{4 \log_2 n} \\ &= 2^{8 \log_2 n} > (2^{\log_2 18})^8 \\ &= n^8 \end{aligned}$$

# Some Problems in P

6, 35

27, 24

The language

$$\text{size } (\langle x, y \rangle) \approx \log x + \log y.$$

$\text{RELPRIME} = \{\langle x, y \rangle \mid x \text{ and } y \text{ relatively prime}\}$

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is in P. Euclid's algorithm finds the greatest common divisor of  $x$  and  $y$ :  
accept iff  $\text{gcd}(x, y) = 1$ . What is the size of the input string?

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Likewise, the language

$$\text{size } (\langle n, \langle u, v \rangle, \langle u, v \rangle \rangle) \approx \log_2 n + m \log n$$

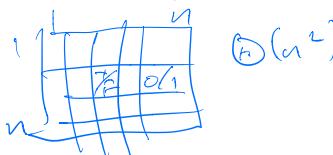
~~$\text{PATH} = \left\{ \langle G, s, t \rangle \mid G \text{ is a directed graph with a path from } s \text{ to } t \right\}$~~

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$\text{PATH} = \left\{ \langle G, s, t \rangle \mid \begin{array}{l} G \text{ is a directed graph} \\ \text{with a path from } s \text{ to } t \end{array} \right\}$

$\text{size } (G) = \log_2 n + m (2 \log n) \in O(m \log n)$

is in P. There are several deterministic algorithms for this problem, each running in polynomial time.



$\langle G, s, t \rangle \in \text{PATH}$ .

DFS = depth-first search

# Textbook Questions

Questions you can try at this point:

Exercise 7.3

Exercise 7.6

Exercise 7.8

Exercise 7.9

Exercise 7.10

Exercise 7.11

Problem 7.13

Problem 7.14

Problem 7.15

DPS running time =  
 $O(m+n)$ .

( $\Theta(m \log n)$ )

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 $n=15$   
 $\langle n, 2, e^{\pi}, \pi, \text{em} \rangle$

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$O(n^2)$

$m - O(\sqrt{n})$

**TRIANGLE** =  $\{ \langle a \rangle \mid a \text{ is undirected graph; } a \text{ contains a triangle} \}$ .

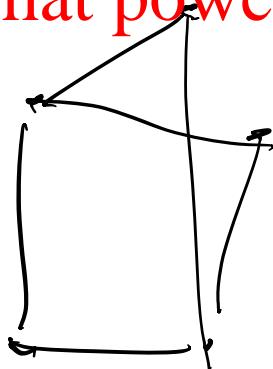
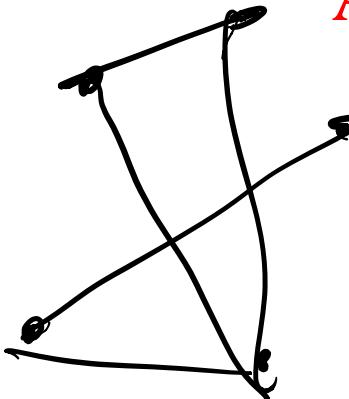
exist?  $\exists c$

$x$        $y$        $z$

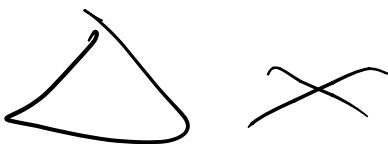
$$\{(x, y), (x, z), (y, z)\} \subseteq E.$$

$G = \langle V, E \rangle$  Assignment Project Exam Help  
 $\exists i, \forall n \in \{u_1, v_1, u_2, v_2, \dots, u_m, v_m\}$ .  
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vertex  
 1, 2, 3  
 1, 2, 4  
 1, 2, 5  
 ...  
 n-2, n-1, n.



$O(n^3 m)$   
 running time.

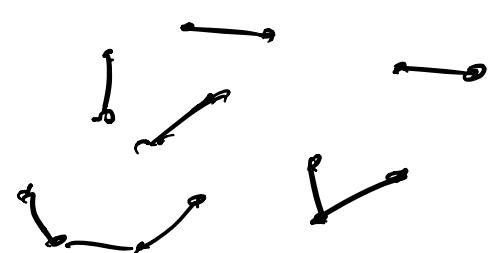
$e_1, e_2, e_3$



$e_1, e_2, e_4$



$e_{m-2}, e_{m-1}, e_m \rightarrow \Delta ?$



# Nondeterminism Makes a Difference?

The Hamiltonian path problem has underlying language  $HAMPATH =$

$$\left\{ \langle G, s, t \rangle \mid \begin{array}{l} G \text{ is a directed graph with a} \\ \text{Hamiltonian path from } s \text{ to } t \end{array} \right\},$$

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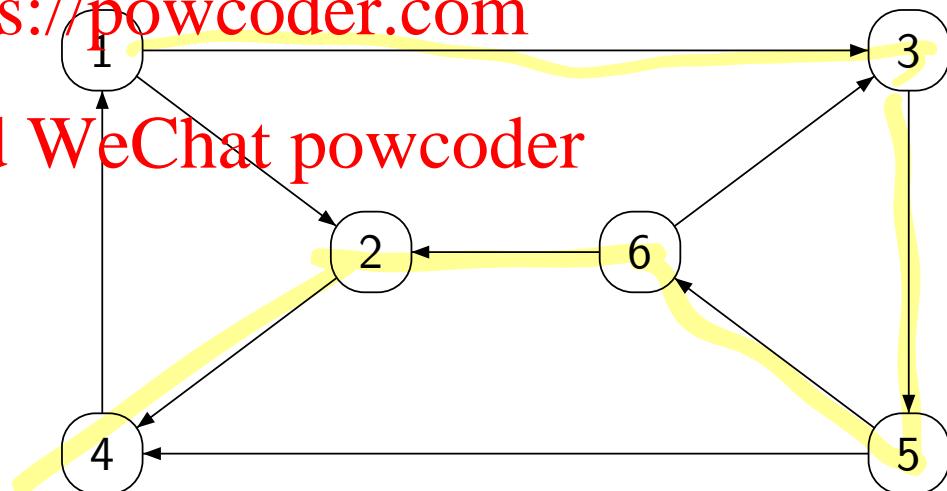
where a *Hamiltonian path* visits each node in  $G$  exactly once.

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Two instances:

Is there a Hamiltonian path from 1 to 4?

From 1 to 6?



# HAMPATH with Nondeterminism

Let  $G$  have  $m$  nodes. Represent  $G$  as, say,

$$x_1 \# y_1 \#\# \cdots \#\# x_n \# y_n \#\#\# m$$

We can build a multi-tape nondeterministic Turing machine  $N$  to solve the problem.

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One tape holds the graph (does not change).

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Machine  $N$  writes a guessed sequence of  $m$  integers, between 1 and  $m$ , on a different tape, each representing a node.

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$$\sqcup i_1 \sqcup i_2 \sqcup i_3 \sqcup \cdots \sqcup i_m$$

If there is a repeated node on the tape,  $N$  rejects. If  $i_1 \neq s$  or  $i_m \neq t$ ,  $N$  rejects. For each  $k$ , if  $(i_k, i_{k+1})$  is not an edge in  $G$ ,  $N$  rejects.

Otherwise,  $N$  accepts.

# Running-time Analysis

As written, to ease understanding of the process, these verification steps involve more than one tape.

We can simulate Assignment Project Exam Help  
the multi-tape machine on a single-tape machine via a polynomial “blowup” in speed. Anyway...

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The guess is done in polynomial (linear) time.

Checking for repetition and verifying the start and end nodes is done in polynomial time.

Verifying an edge (by looking up the representation) is also done in polynomial time.

# Do we need Nondeterminism?

Guessing in linear time is clearly very powerful.

To solve the same problem with a deterministic machine, it seems the best approach is exhaustive search through all possible orderings of  $m$  nodes.

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A deterministic machine can do this (on tape 2) in factorial time (there are  $m!$  sequences).

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Nobody has come up with a polynomial-time solution – if they did, then we would know that  $P = NP$  (see below).

# Polynomial Verifiability

*HAMPATH* has an important feature shared with many other fundamental computational problems: It is polynomially **verifiable**.

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On the one hand, **discovering** a Hamiltonian path seems difficult, and certainly we do not have a fast algorithm for it.

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On the other hand, if somebody claims that a certain path is a Hamiltonian path, **checking** their claim is easy: we can do that in polynomial time.

It is possible – and it seems likely – that **verifying** a Hamiltonian path is much easier than **determining its existence**.

# Polynomial Verifiability

A **verifier** for language  $A$  is an algorithm  $V$  (that is, a deterministic Turing machine) with **Assignment Project Exam Help**

$$A = \{ w \mid V \text{ accepts } \langle w, c \rangle \text{ for some string } c \}$$

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Graph  
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proposed path.

Language  $A$  is **polynomially verifiable** if it has a polynomial-time verifier: it runs in time that is polynomial in the length of  $w$ .

$$|c| = 2^{|w|} ?$$

# Polynomial Verifiability – complement

Notice that polynomial verifiability need not be closed under complement.

For example, consider ~~HAMPATH~~.

There is no obvious (polynomial-length) certificate for the ~~non-existence~~ of a path.  
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We don't know how to verify this non-existence without using the same exponential-time method that was needed for ~~determining~~ non-existence (without a certificate) in the first place.

# The Class $NP$

$NP$  is the class of languages that have polynomial-time verifiers.

**Theorem:** Language  $A$  is in  $NP$  iff  $A$  is decided by some nondeterministic polynomial-time Turing machine.

We can also phrase this as

$$NP = \bigcup_k NTIME(n^k)$$

$$NTIME(t(n)) = \left\{ L \middle| \begin{array}{l} L \text{ is decided by a nondeterministic} \\ \text{Turing machine in } O(t(n)) \text{ time} \end{array} \right\},$$

where the running time of an NTM is the *maximum* over all computation branches, given a particular input.

This definition is robust, in the sense that  $NP$  stays the same even when we change the machine to another nondeterministic model.

# P vs NP

In summary:

For  $P$ , membership can be **decided** quickly.

For  $NP$ , membership can be **verified** quickly

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Clearly  $P \subseteq NP$ . Is

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**P = NP?**

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Many computer scientists consider this the

**most important unanswered question**

in computer science.

# The Nature of an *NP* Problem

Recall the Hamiltonian path problem, *HAMPATH*:

*Given a directed graph  $G$ , and nodes  $s$  and  $t$ , is there a path, starting in  $s$ , ending in  $t$ , that visits each node exactly once?*

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This problem has the property of being **polynomially verifiable**.

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Discovering a Hamiltonian path seems difficult, indeed we do not have a fast algorithm for it.

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But if somebody claims to have a valid path, **checking** that claim is easy (in polynomial time): The path is a **certificate**.

It is possible that **verifying** a Hamiltonian path is much easier than determining its existence.

# Some Problems in $NP$

A clean  $NP$  problem is

$$SUBSET-SUM = \{\langle S, t \rangle \mid \sum_{x \in A} x = t \text{ for some } A \subseteq S\}$$

(There are several variants; here we consider  $A$  and  $S$  to be multisets that allow repetition of elements.)

Example:  $S = \{2, 2, 5, 7, 8\}$ . Is there a subset with sum  $t = 17$ ?  $t = 6$ ?

No polynomial-time decider is known, but the description of  $A$  itself provides a succinct certificate.

2, 2, 5, 7, 8.

A nondeterministic TM decides  $SUBSET-SUM$  in polynomial time:

- ① Guess a subset  $A \subseteq S$ , nondeterministically.
- ② Accept if  $\sum_{x \in A} x = t$ , otherwise reject.

# Textbook Questions

Questions you can attempt at this stage:

Exercise 7.7

Exercise 7.12

Problem 7.16

Problem 7.19

A  
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<https://powcoder.com>  
is a group.  
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# Closure under Complement?

Recall that HAMPATH has no obvious certificate of the non-existence of a path.

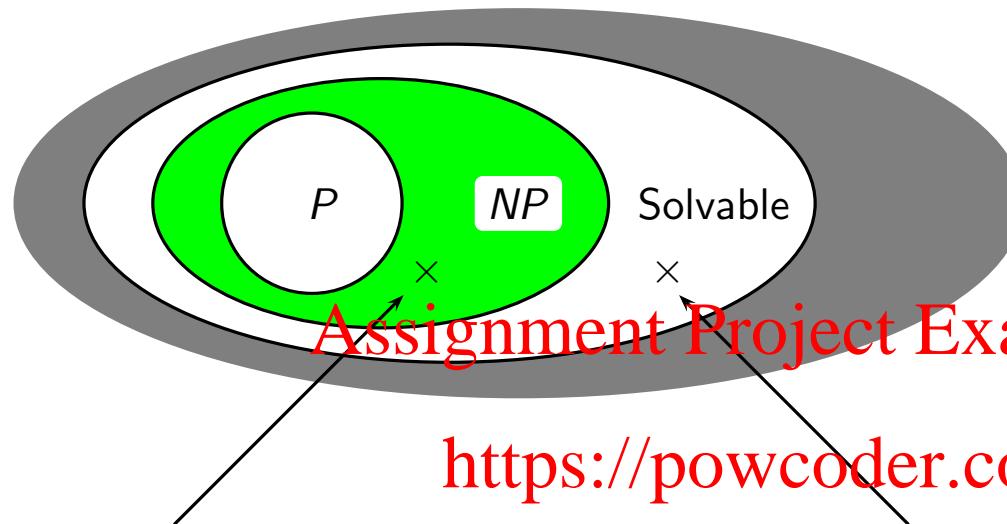
Similarly SUBSET-SUM may not be in  $NP$ .

Namely, how could we quickly verify that a subset of  $S$  sums to  $t$ ?

$$\langle S, t \rangle \dashv A \subseteq S \quad \sum_{a \in A} a = t.$$

$$\text{SUBSET-SUM} \neq \{ \langle S, t \rangle \mid \nexists A \subseteq S \text{ such that } \sum_{a \in A} a = t \}.$$

# Venn diagram



Problem which **provably** can be solved by a NDTM in polynomial time, though **not** by a deterministic TM.

**Does one exist?**

deciding whether two regexs represent different languages. Four allowed operators: union, concatenation, Kleene star, and squaring (two copies)

Solvable problem that **provably** cannot be solved in polynomial time by a NDTM (“nondeterministically intractable”).  
**EXPSPACE-complete problems**, e.g.,...

# Polynomial-Time Reducibility

Via the concept of reducibility, we can classify problems by *hardness*.

Crucial point: when we talk about reducing problem  $A$  to problem  $B$  with regard to time complexity, the (mechanistic) reduction must take place in polynomial time.

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$A$  is polynomial-time (mapping) reducible to  $B$ ,  $A \leq_P B$ , iff there is some polynomial-time (deterministic) computable function  $f : \Sigma^* \rightarrow \Sigma^*$ , such that for all  $w$ ,

$$w \in A \text{ iff } f(w) \in B.$$

We say that  $f$  is a polynomial-time reduction from  $A$  to  $B$ .

# Reducibility and Language Classification

Since the class of polynomials is closed under composition, we have:

$$A \leq_P A \quad A \leq_P B, B \leq_P C \Rightarrow A \leq_P C.$$

**Theorem:** The relation  $\leq_P$  is a pre-order: reflexive and transitive (and not anti-symmetric. Can have  $A \leq_P B$  and  $B \leq_P A$ , with  $A \neq B$ ).

However, we can define an equivalence relation  $\equiv_P$  on the set of all languages by saying  $A \equiv_P B$  iff  $A \leq_P B \leq_P A$ .

The collection of equivalence classes forms a partial order.

**Theorem:** If  $A \leq_P B$  and  $B \in P$  then  $A \in P$ .

Can you prove this?

$$A \leq_P B$$
  
$$B \subseteq C$$
  
$$f$$
  
$$g$$

$$w \xrightarrow{f(w)} g(f(w))$$
  
$$\text{poly}_1(w)$$
  
$$\uparrow$$
  
$$\text{poly}_2(f(w))$$

$$|f(w)| \leq_P \text{poly}_1(w),$$
  
$$\text{poly}_2(\text{poly}_1(w)).$$

# $NP$ -Completeness

$B$  is  **$NP$ -hard** iff every  $A \in NP$  is reducible to  $B$  in polynomial time.

$B$  is  **$NP$ -complete** iff

- ①  $B \in NP$ , and
- ②  $B$  is  $NP$ -hard.

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We sometimes denote the class of  $NP$ -complete languages by  $NPC$ .

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The  $NP$ -complete languages are interesting because:

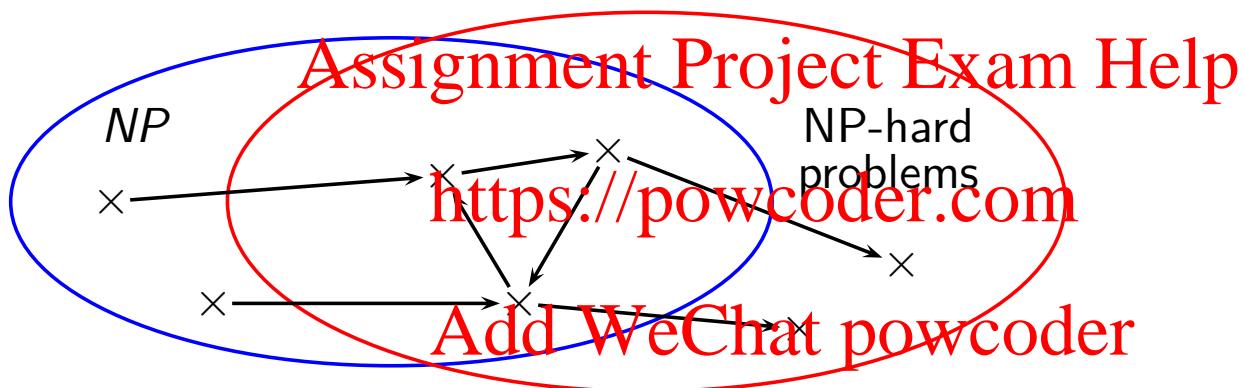
**Theorem:** If  $B \in NPC$  and  $B \in P$  then  $P = NP$ .

**Theorem:** If  $B \in NP\text{-hard}$  and  $B \leq_P C$  then  $C \in NP\text{-hard}$ .

(If also  $C \in NP$ , then  $C \in NPC$ .)

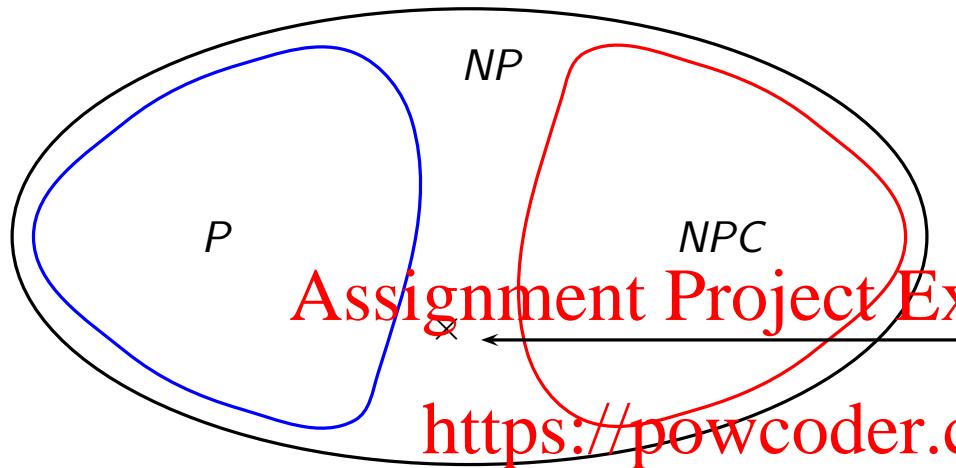
# NPC Diagrammatically

*NP*-complete problems are the *NP*-hard problems in *NP*.



Arrows indicate polynomial-time reduction (a transitive relation).

# The Fabled “Intermediate” Class NPI



If  $P \neq NP$  then we have this picture.

Moreover, if  $P \neq NP$ , this gap is known to be inhabited (this is Karp's Theorem).

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Let  $NPI = NP \setminus (P \cup NPC)$ .

Difficult problems that in spite of many attempts have not been proven  $NP$ -complete, such as **graph isomorphism**, are sometimes suggested as candidate members of  $NPI$ .

<http://www.quantamagazine.org/20151214-graph-isomorphism-algorithm/>

# Textbook Questions

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Questions to attempt:

Problem 7.18

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Problem 7.20

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# Proving *NP*-Completeness

The plan for demonstrating *NP*-completeness:

- The *ancestor* of all *NPC* problems: *SAT*.
  - ① Show  $SAT \in NP$ .
  - ② Show that *SAT* is *NP-hard*.
- For each subsequent problem *B*:
  - ① Show  $B \in NP$ .
  - ② Show *B* is *NP-hard* by describing a polytime reduction from some other known *NP-hard* problem to *B*.

Cook followed exactly this path in his 1971 paper.

However, his proof relied on a weaker concept of (Turing) reducibility.

# Propositional Logic

To describe *SAT*, we introduce some basics of propositional logic.

Let the Boolean values be 0 and 1 and let  $V$  be a set of Boolean variables.

A **truth assignment** is a (total) function from  $V$  to  $\{0, 1\}$ .

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Well-formed propositional formulas include 0, 1, and  $x$  for all  $x \in V$ .

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Moreover, if  $\varphi$  and  $\psi$  are well-formed formulas, so are  $\neg\varphi$  (negation),  $\varphi \wedge \psi$  (conjunction), and  $\varphi \vee \psi$  (disjunction)—and we can add more connectives if we want.

# Propositional Logic

Truth tables for  $\neg$ ,  $\wedge$ , and  $\vee$  determine truth value of formula  $\varphi$  Ans OR

For  $x \in V$ ,  $x$  and  $\neg x$  are **literals**.

$x$	$\neg x$	$\neg x$	$\neg x \wedge y$	$\neg x \vee y$
0	1	1	0	0
1	0	0	0	1

A **clause** is a disjunction of literals (using  $\vee$ ).

A formula is in **conjunctive normal form** (CNF) if it is a conjunction ( $\wedge$ ) of clauses. Every formula has an equivalent formula in CNF – though converting one to the other might require more than polynomial time.

( $\ell_1 \vee \ell_2 \vee \ell_3 \vee \dots$ ) **Add WeChat powcoder** ( $\ell_1 \wedge \ell_2 \wedge \dots$ )

Formula  $\varphi$  is **satisfiable** if there is **some** truth assignment to the variables such that the truth value of formula  $\varphi$  is 1.

Language  $SAT = \{\varphi \mid \varphi \text{ is in CNF and is satisfiable}\}$ .  $(x \vee \neg y \vee z) \wedge (\neg x \vee y \vee z)$

If  $\varphi$  has  $n$  (distinct) variables, then there are  $2^n$  relevant truth assignments.

# Example formulas in CNF

\phi

$$x_1 = 1 \quad x_3 = 1$$

$$\varphi = (x_1 \vee \cancel{x_2} \vee \cancel{x_3}) \wedge (\cancel{x_1} \vee x_2 \vee x_3)$$

$$\psi = (x_1 \vee x_2) \wedge (\cancel{x_1} \vee \cancel{x_2}) \wedge (x_1 \vee \cancel{x_2}) \wedge (\cancel{x_1} \vee x_2)$$

Which of these formulas is satisfiable?

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\psi

# $SAT$ is in $NP$

To show that  $SAT$  is in  $NP$ , we build a two-tape nondeterministic TM over alphabet  $\{0, 1, \wedge, \vee, \#, \square, \leftarrow, \oplus, \ominus\}$ . Let  $\bar{i}$  denote a string of  $i$  1s and encode

$x_i$  by  $\bar{i} \oplus$

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For example,  $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3)$  yields the input string

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1#11#111\square1⊕∨11⊖∧1⊖∨111⊕

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variables formula

The NTM will check that this is a valid  $SAT$  instance.

Then it will guess-and-check as usual, using tape 2 to store the guessed truth assignment.

Why are these labels (allowed to be) in unary?

# SAT is in NP

$$x_1 \in 0 \quad x_2 \in 0 \quad x_3 \in 1$$

Tape 2:  $1 \leftarrow 0 \# 11 \leftarrow 0 \# 111 \leftarrow 1 \sqcup$

Tape 1:  $1 \# 11 \# 111 \square 1 \oplus \vee 11 \ominus \wedge 1 \ominus \vee 111 \oplus$

The latter checking happens in  $\Theta(n^2)$  time:

- Scan to the formula on tape 1.
- While there is another clause:
  - For each literal
    - If the truth value **differs** from the tape-1 value (i.e.,  $0 \neq \oplus$ ,  $1 \neq \ominus$ ), scan for next  $\vee$ : if  $\wedge$  (or  $\sqcup$ ) is found before  $\vee$ , **reject**
    - Else (the variable's value **agrees**) break out of this for loop
  - Scan just past next  $\wedge$  to the start of the next clause
  - If no  $\wedge$  is found (we have checked every clause) **accept**.

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Question to attempt:

Exercise 7.5

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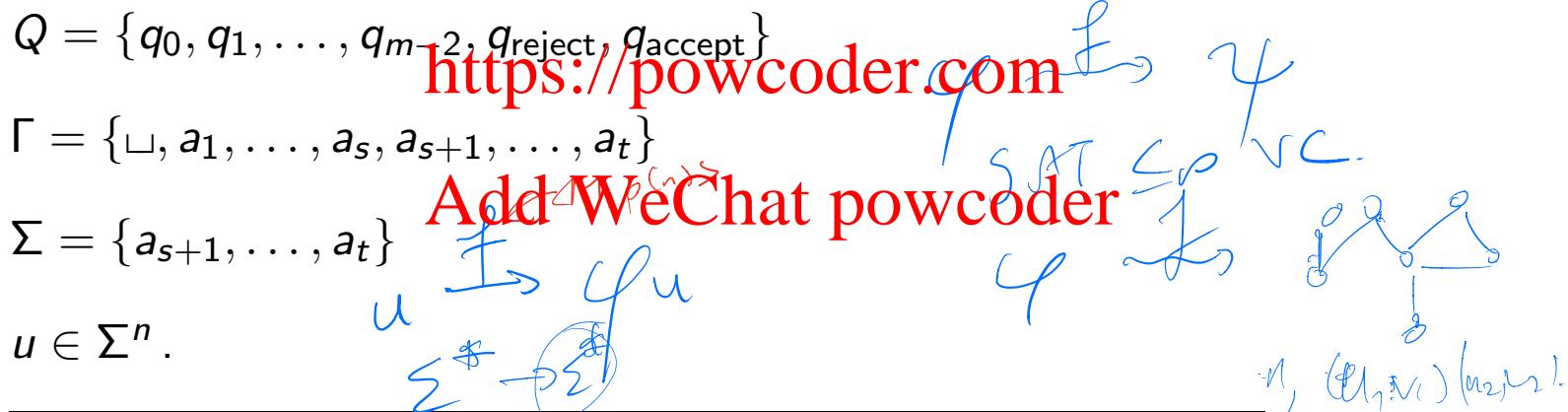
# Modelling NTM Computation

The important part is to show that  $SAT$  is  $NP$ -hard.

Let  $M$  be a nondeterministic TM, whose running time is bounded by polynomial  $p$  (that is, given input of size  $n$ , on **every** computation branch  $M$  halts within  $p(n)$  steps.)

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$$Q = \{q_0, q_1, \dots, q_{m-2}, q_{\text{reject}}, q_{\text{accept}}\}$$



We construct a formula  $\varphi_u$  that mimics  $M$ 's operation on  $u$ .

Formula  $\varphi_u$  and its construction depend on  $p(n)$ .

$A \leq_p B$        $A = \{0, 00, 0000, 010, -\}$

$f(u) f: \Sigma^* \rightarrow \Sigma^*$        $B = \{1, 11, 111, 101, -\}$

$u \in A$  if and only if  $f(u) \in B$ .

$$2^n n^2$$

1 2 1  
2 4 4  
3 8 9  
4 16 16  
5 32 25

$A \in NP$

$A \leq_p SAT$

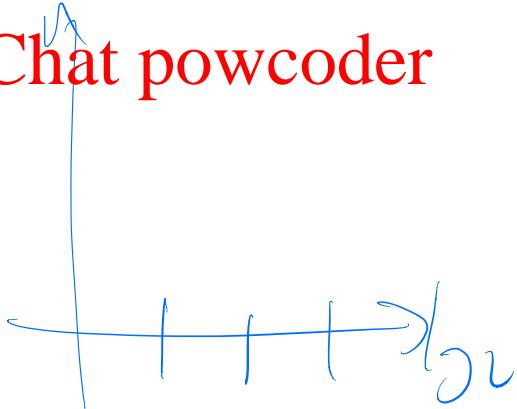
$u \in A \iff \varphi_u \in SAT$ .

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<https://powcoder.com>

6 64 36  
7 128 49  
10 1024 100  
20 10<sup>6</sup> 400  
30 10<sup>9</sup> 900

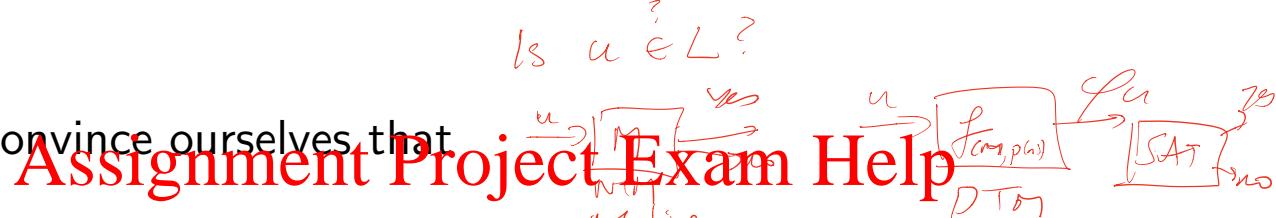
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# What To Look Out For in the Proof

The idea is to create a formula  $\varphi_u$  that it is satisfiable iff the NTM  $M$  accepts  $u$ .

We need to convince ourselves that



- $\varphi_u$  emerges as a formula in CNF;
- the size of  $\varphi_u$  is polynomial in the size of  $u$  (in particular, the process to create  $\varphi_u$  runs in polynomial time);
- if  $\varphi_u$  is satisfiable, then  $M$  accepts  $u$ ;
- if  $M$  accepts  $u$ , then  $\varphi_u$  is satisfiable.

In short,  $\varphi_u$  will speak the truth, the whole truth, and nothing but the truth, about  $M$ 's computation history on input  $u$ .

# Modelling NTM Computation: Variables Used

The number of propositional variables in the formula is **huge**, but **polynomial** in  $n$ , the size of  $u$ .

Machine  $M$ :  $m + 1$  states,  $t + 1$  symbols, and at most  $p(n)$  time steps.

Hence at most  $p(n)$  tape cells are touched. **Why??**

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The variables have the following format, with the suggested interpretations:

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$Q_{ik}$	$0 \leq i \leq m$ $0 \leq k \leq p(n)$	$M$ is in state $q_i$ at time $k$
$P_{jk}$	$0 \leq j \leq p(n)$ $0 \leq k \leq p(n)$	$M$ is scanning position $j$ at time $k$
$S_{jrk}$	$0 \leq j \leq p(n)$ $0 \leq r \leq t$ $0 \leq k \leq p(n)$	Tape position $j$ contains symbol $a_r$ at time $k$

# Modelling NTM Computation: Starting the NTM

Capturing **initial configuration** (with input string  $u = a_{r_1} \cdots a_{r_n}$ ): Take the conjunction of

$Q_{00}$   
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 $\wedge$   $S_{000}$   
 $\wedge$   $S_{1r_10}$   
 $\wedge$   $S_{2r_20}$   
  
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 $\vdots$   
 $S_{nr_n0}$   
 $S_{(n+1)00}$   
 $\vdots$   
 $\wedge$   $S_{p(n)00}$

# Modelling NTM Computation: Invariants

Generate these formulas for each time-step  $k \in [0, p(n)]$ :

To capture **state invariants**

(Remember,  $a \Rightarrow b \equiv \neg a \vee b$ )

$(0 \leq i < i' \leq m)$ :

$$a \Rightarrow b$$

$$\neg b \Rightarrow \neg a$$

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$\bigwedge_{i=0}^m Q_{ik}$  At all times,  $M$  is in some state,

$Q_{ik} \Rightarrow \neg Q_{i'k}$  and not in two different states

$\neg Q_{ik} \vee \neg Q_{i'k}$

To capture **tape-head invariants** ( $0 \leq j < j' \leq p(n)$ ):

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$\bigwedge_{j=0}^{p(n)} P_{jk}$  At all times, the tape head is somewhere,

$P_{jk} \Rightarrow \neg P_{j'k}$  and not in two different positions

To capture **symbol invariants**, for  $j \in [0, p(n)]$  ( $0 \leq r < r' \leq t$ ):

$\bigwedge_{r=0}^t S_{jrk}$  At all times, each cell contains some symbol,

$S_{jrk} \Rightarrow \neg S_{jr'k}$  and no cell contains two different symbols

# Modelling NTM Computation: Tape Consistency

Symbols **not** under the tape head are **not** changed.

$$(S_{jrk} \wedge R_{jk}) \Rightarrow S_{j(r+1)}$$

(for all  $j, k \in [0, p(n)]$ ,  $r \in [0, t]$ ).  
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# Modelling NTM Computation: NTM Transitions

Capturing the (possible) transition  $(q_i, a_r) \mapsto (q_{i'}, a_{r'}, d)$ : (here  $i$  could equal  $i'$  and  $r$  could equal  $r'$ )

(this allows for nondeterminism)

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$$(Q_{ik} \wedge P_{jk} \wedge S_{jrk}) \Rightarrow Q_{i'(k+1)}$$

$$(Q_{ik} \wedge P_{jk} \wedge S_{jrk}) \Rightarrow S_{i'(k+1)}$$

$$(Q_{ik} \wedge P_{jk} \wedge S_{jrk}) \Rightarrow P_{j'(k+1)}$$

where

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$$j' = \begin{cases} j + 1 & \text{if } d = R \\ j \ominus 1 & \text{if } d = L \end{cases}$$

Again, these clauses are added for each time-step  $k$ .

Here,  $\ominus$  denotes cut-off subtraction ( $0 \ominus 1 = 0$ ).

# Modelling NTM Computation: Nondeterministic Transitions

Since this is a nondeterministic TM, there may be multiple transitions from a given state  $q_i$  with a given symbol  $a_r$ , under the tape head. Only one of the several possible transitions occurs, since the machine cannot be in multiple states at the same time.

The set of clauses specifying a possible transition need to be added for each of the **possible** transitions from a given state and symbol pair. To model this, we add the clauses from the previous slide for each possible transition, separated by disjunctions ( $\vee$ ).

To transform this formula into Conjunctive Normal Form, we apply the distributive law  $P \vee (Q \wedge R) \leftrightarrow (P \vee Q) \wedge (P \vee R)$ . This increases the size of the formula only locally, overall a constant-factor increase in size. The reduction is still **polynomial** in the size of the input,  $u$ .

# Modelling NTM Computation: Padding

To capture **inactivity after halting**, when the machine is in state  $q_{\text{accept}}$  or  $q_{\text{reject}}$ , i.e.,  $i \in [m-1, m]$ , for  $j, k \in [0, p(n)]$ ,  $r \in [0, t]$ :

$$\begin{aligned} (Q_{ik} \wedge P_{ik} \wedge S_{jrk}) &\Rightarrow Q_{i(k+1)} \\ (Q_{ik} \wedge P_{jk} \wedge S_{jrk}) &\Rightarrow P_{j(k+1)} \\ (Q_{ik} \wedge P_{jk} \wedge S_{jrk}) &\Rightarrow S_{jr(k+1)} \end{aligned}$$

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Recall that  $p(n)$  is the *longest* nondeterministic computation branch.

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To capture acceptance:

$\text{https://powcoder.com}^{Q_{mp(n)}}$

(That is, we are in state  $q_{\text{accept}}$  at time  $p(n)$ .)

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# Modelling NTMs

That's it!

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The formula we have constructed is in CNF.

The construction ensures that <https://powcoder.com>

- if  $u \in L(M)$  then  $\varphi_u$  has a satisfying assignment, and
- if  $u \notin L(M)$  then  $\varphi_u$  does not have a satisfying assignment.

# *NP*-Hard vs Exponential

Many interesting problems have been shown to be *NP*-hard, without anybody being able to show that they are in *NP*.

Such a problem may well turn out to require exponential time, even if  $P = NP$ ; indeed it may be undecidable.

<https://powcoder.com>

It is possible that some problem may require exponential time, and yet it is not *NP*-hard. Nothing in our definitions rules this out.

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Finally, it is possible that some *NP*-hard problem does not require exponential time. (For example, we may have  $P = NP$  and the problem we consider is *P*-complete.)

# *SAT* is *NP*-Complete

[Read §7.4 and §7.5 of the Sipser text.]

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We established that every problem  $A \in NP$  can be reduced to *SAT* in polynomial time. <https://powcoder.com>

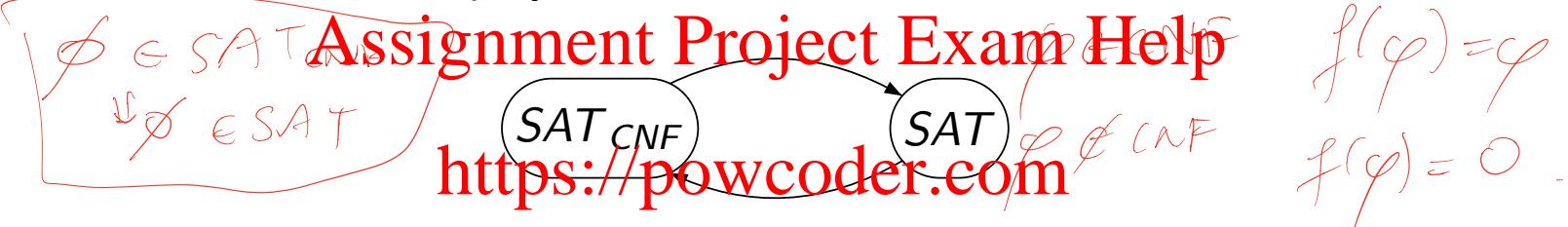
Our construction reduced every *NP* problem to  $SAT_{CNF}$ , the satisfiability problem for formulas in Conjunctive Normal Form.

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# *SAT* is *NP*-Complete

But then *SAT* (for arbitrary propositional formulas) is also *NP*-complete:

- ① It is in *NP* (as for the CNF case) **Actually, let's check this!**, and
- ② There is a trivial poly-time reduction of  $SAT_{CNF}$  to  $SAT$ .



The reduction from right to left is covered by the general reduction we did in the last lecture.

(A direct reduction is not trivial, by the way: We know how to turn an arbitrary formula into CNF, but that technique can cause an exponential blow-up! However, in the reduction we don't have to preserve equivalence in the formula. We need **only** show that the  $SAT$  formula is satisfiable if and only if the  $SAT_{CNF}$  formula is satisfiable.)

# 3SAT

A propositional formula is in **3-CNF** if it is in CNF and every clause has exactly three literals (no two of which involve the same variable). An example is

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 $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$

*3SAT* is the language of satisfiable 3-CNF formulas.

Let us show that it is *NP*-complete

~~3SAT~~ ~~CNF~~  
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First, *3SAT* is in *NP*, as the machine that solves  $SAT_{CNF}$  in polynomial time also works for *3SAT*.

So it suffices to show  $SAT_{CNF} \leq_P 3SAT$ .

# 3SAT

The reduction is as follows. Given an instance of  $SAT_{CNF}$ :

$$C_1 \wedge C_2 \wedge \cdots \wedge C_k$$

we turn each clause  $C$  into a conjunction of 3-literal clauses as follows:

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- If  $C$  is a single literal  $\ell$ , produce four clauses using two fresh variables  $x$  and  $y$ : <https://powcoder.com>

$$(\ell \vee x \vee y) \wedge (\ell \vee \neg x \vee y) \wedge (\ell \vee x \vee \neg y) \wedge (\ell \vee \neg x \vee \neg y)$$

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- If  $C$  is  $(\ell_1 \vee \ell_2)$ , produce two clauses using a fresh variable  $x$ :

$$(\ell_1 \vee \ell_2 \vee x) \wedge (\ell_1 \vee \ell_2 \vee \neg x)$$

- If  $C$  has three literals, leave it as it is.

Clearly satisfiability is preserved for these clauses.

If  $C$  has  $n > 3$  literals, say

$$(\ell_1 \vee \ell_2 \vee \ell_3 \vee \cdots \vee \ell_n),$$

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produce  $n - 2$  clauses using  $n - 3$  fresh variables  $x_3, \dots, x_{n-1}$ :

<https://powcoder.com>

$$\begin{aligned} & (\ell_1 \vee \ell_2 \vee x_3) \\ & \wedge (\neg x_3 \vee \ell_3 \vee x_4) \\ & \vdots \\ & \wedge (\neg x_{n-2} \vee \ell_{n-2} \vee x_{n-1}) \\ & \wedge (\neg x_{n-1} \vee \ell_{n-1} \vee \ell_n) \end{aligned}$$

Why does this preserve satisfiability?

⇒ Let  $t$  be a truth assignment that satisfies

*Well*

$$(\ell_1 \vee \ell_2 \vee \ell_3 \vee \cdots \vee \ell_n)$$

and let  $\ell_j$  be the ‘first’ literal that is set to true. We satisfy

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$$\wedge (\neg x_3 \vee \ell_3 \vee x_4)$$

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$$\wedge (\neg x_{n-2} \vee \ell_{n-2} \vee x_{n-1})$$

$$\wedge (\neg x_{n-1} \vee \ell_{n-1} \vee \ell_n)$$

$$(\neg x_{j+1} \vee (\ell_{j+1} \vee x_j))$$

$$(\neg x_j \vee \ell_j \vee x_{j+1})$$

$$(x_{j+1} \vee \ell_{j+1} \vee x_{j+2})$$

by extending  $t$  so that each of  $x_3, \dots, x_j$  is true, while the remaining fresh variables,  $x_{j+1}, \dots, x_{n-1}$  are false.

Let  $t'$  be a truth assignment that satisfies

$$\begin{aligned} & (\ell_1 \vee \ell_2 \vee x_3) \\ \wedge \quad & (\neg x_3 \vee \ell_3 \vee x_4) \end{aligned}$$

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$$\wedge \quad (\neg x_{n-2} \vee \ell_{n-2} \vee x_{n-1})$$

$$\wedge \quad (\neg x_{n-1} \vee \ell_{n-1} \vee \ell_n)$$

Then  $t'$  also satisfies  $C = (\ell_1 \vee \ell_2 \vee \ell_3 \vee \dots \vee \ell_n)$ .  
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Proof by contradiction: assume that it doesn't. Then every  $\ell_j$  is false.

So  $x_3$  is true, hence  $x_4$  is, and so on, including  $x_{n-1}$ .

But then  $(\neg x_{n-1} \vee \ell_{n-1} \vee \ell_n)$  is false.

We have a contradiction; so  $C$  is satisfiable.

Sipser's proof of the *NP*-completeness of 3SAT (Corollary 7.42) is rather different (as is his proof of the *NP*-completeness of SAT).

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It is instructive to read both.

From Sipser's proof, it is clear that he considers something like  $x \vee x \vee x$  an acceptable 3-CNF formula. We choose to rule that out.

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Either way can be justified, but our assumption that the three literals in a 3-CNF clause involve different variables makes it more obvious that later proofs of *NP*-hardness, which are based on reductions from 3SAT, actually work.

# Textbook Questions

Questions you can try at this point:

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Problem 7.22

Problem 7.24

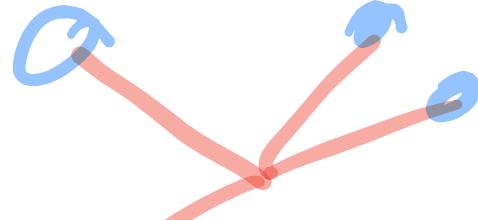
Problem 7.25

Problem 7.26

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# The Vertex Cover Problem



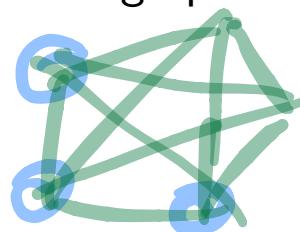
Let  $G = (V, E)$  be an undirected graph.

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 $S \subseteq V$  is a **vertex cover** of  $G$  iff for each edge  $(v_1, v_2) \in E$ , at least one of  $v_1, v_2$  is in  $S$ .

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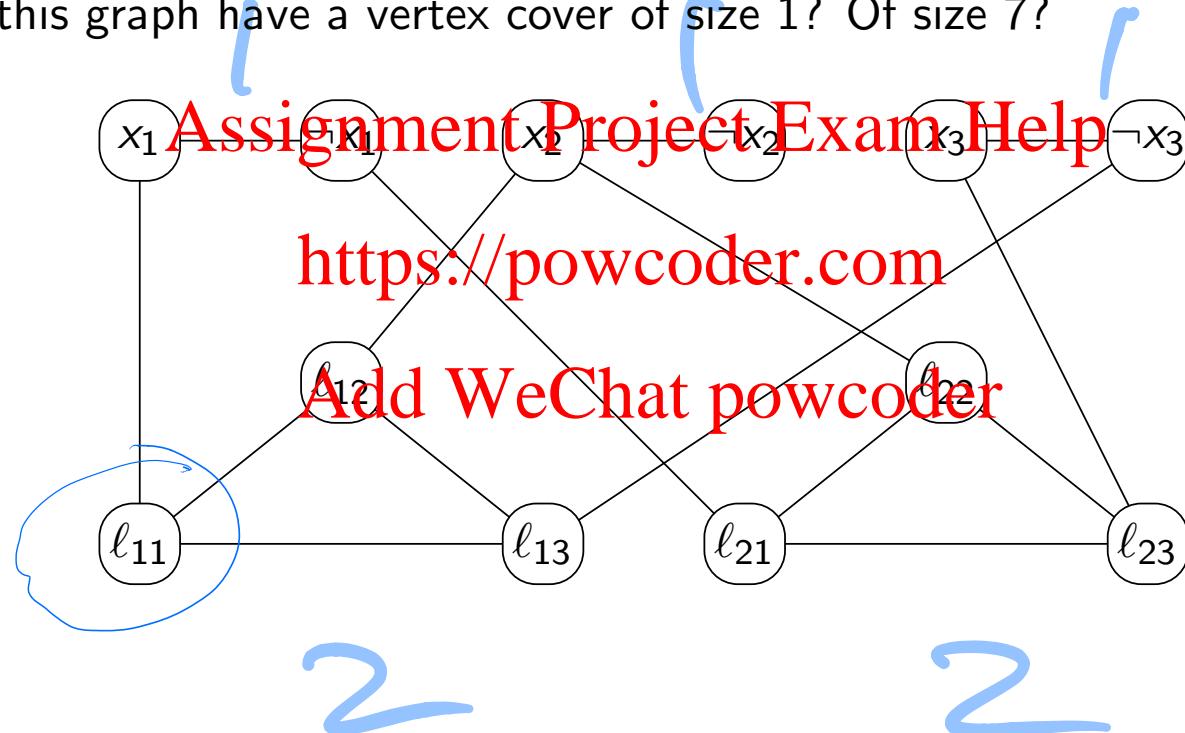
*VERTEX-COVER* =

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 $\{ \langle G, k \rangle \mid G \text{ is a graph that has a vertex cover of size } k \}$



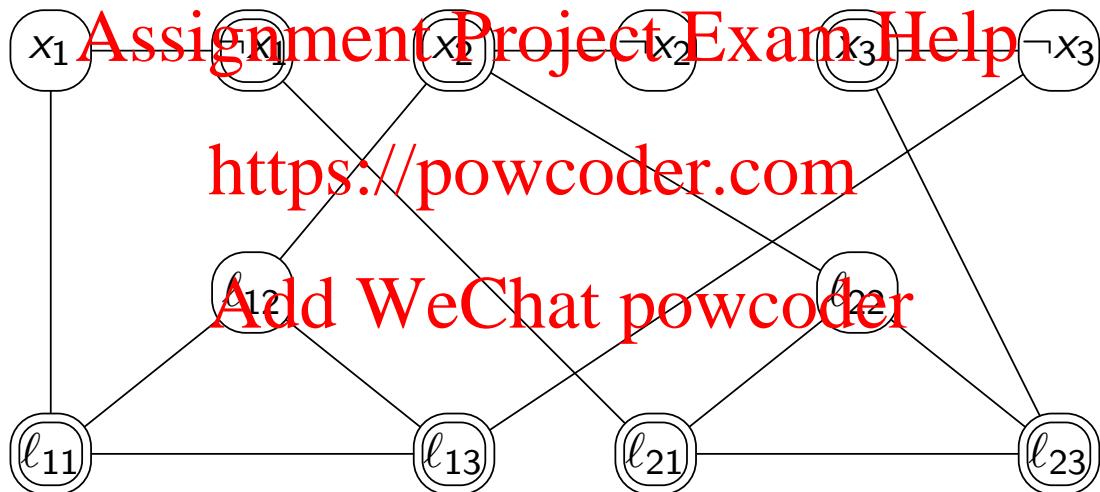
# Vertex Cover Example

Does this graph have a vertex cover of size 1? Of size 7?



# Vertex Cover Example

Does this graph have a vertex cover of size 1? Of size 7?



# Vertex Cover is in NP

We show that  $VERTEX-COVER =$

$$\{ \langle G, k \rangle \mid G \text{ is a graph that has a vertex cover of size } k \}$$

is  $NP$ -complete.

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The problem (language) is in  $NP$  because a nondeterministic TM can guess a set of  $k$  nodes  $S$  and check in polynomial time whether they “cover”  $E$ .

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All it needs is a linear scan through  $E$ , and for each edge, a linear scan of the guess  $S$ .

$$G = \langle n, (u_1, v_1), (u_2, v_2), \dots, (u_m, v_m) \rangle$$
$$S = \{ s_1, s_2, \dots, s_k \}$$

# Reducing 3SAT to Vertex Cover

For a given 3-CNF formula  $\varphi$  we construct  $\langle G, k \rangle$  such that  $G$  has a vertex cover of size  $k$  iff  $\varphi$  is satisfiable.

Let  $\varphi = (\ell_{11} \vee \ell_{12} \vee \ell_{13}) \wedge \dots \wedge (\ell_{m1} \vee \ell_{m2} \vee \ell_{m3})$  have  $n$  variables  $x_1, \dots, x_n$ .

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We produce  $3m + 2n$  nodes, one per **literal**, and two per **variable**:

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 $\{\ell_{kj} \mid 1 \leq k \leq m, 1 \leq j \leq 3\}$

$\{x_i, \neg x_i \mid 1 \leq i \leq n\}$   
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and  $6m + n$  edges:

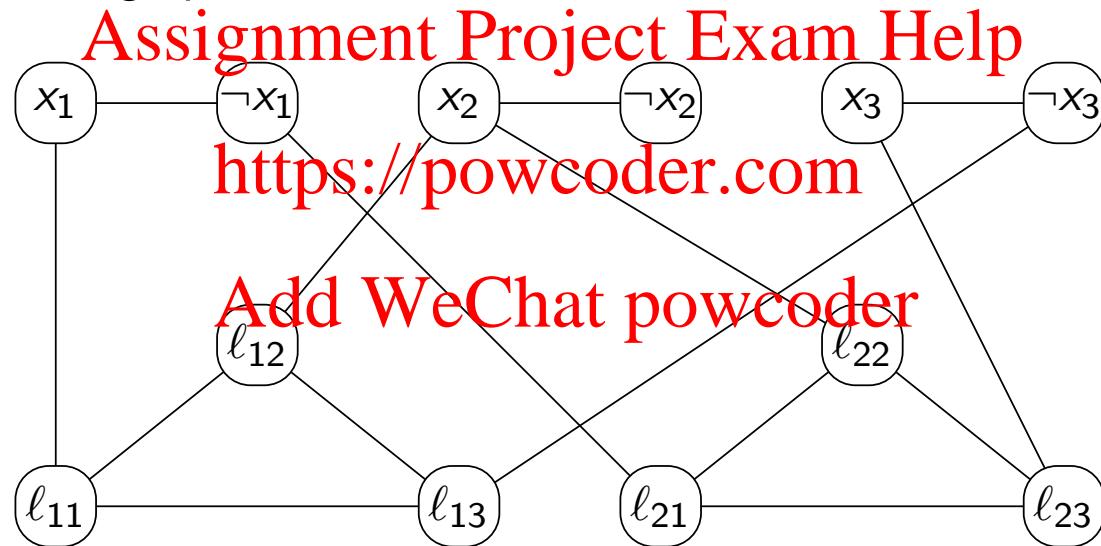
$$\begin{aligned} T_i &= \{(x_i, \neg x_i)\} \\ C_k &= \{(\ell_{k1}, \ell_{k2}), (\ell_{k2}, \ell_{k3}), (\ell_{k3}, \ell_{k1})\} \\ L_k &= \{(\ell_{k1}, v_{k1}), (\ell_{k2}, v_{k2}), (\ell_{k3}, v_{k3})\} \end{aligned}$$

where  $v_{kj}$  is  $x_i$  or  $\neg x_i$ , the same as  $\ell_{kj}$ .

# 3SAT to Vertex Cover

The generated instance is  $\langle G, k = 2m + n \rangle$ .

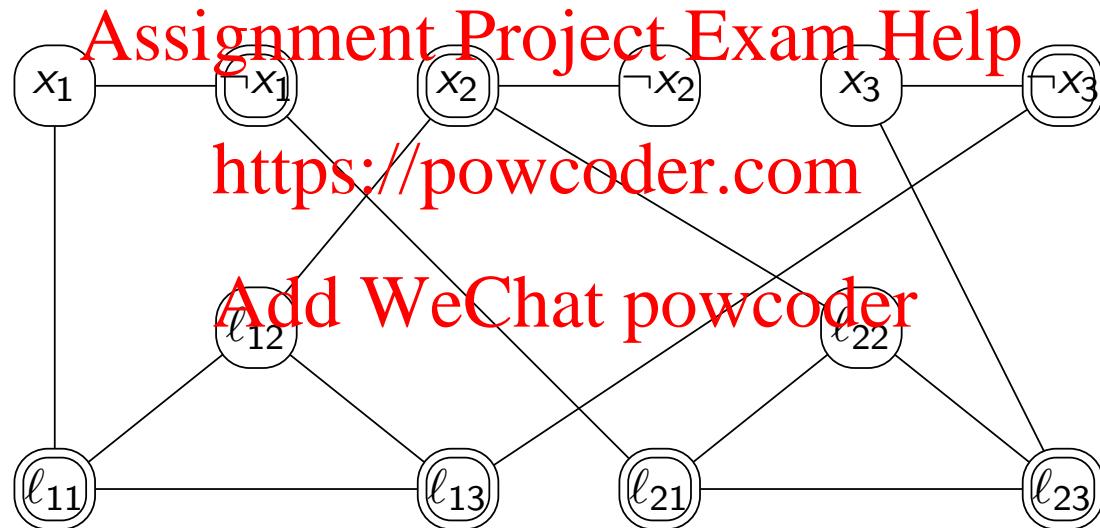
For example,  $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$  is translated to the problem of whether this graph has a cover of size  $2 \cdot 2 + 3 = 7$ :



By this construction, every vertex cover contains **at least**  $2m + n$  nodes.

# 3SAT to Vertex Cover Example

$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$  is translated to the problem of whether this graph has a cover of size  $2 \cdot 2 + 3 = 7$ :



# 3SAT to Vertex Cover

Clearly, the graph is constructed in polynomial time.

We need to show that  $G$  has a vertex cover of size  $2m + n$  iff  $\varphi$  is satisfiable.

<https://powcoder.com>

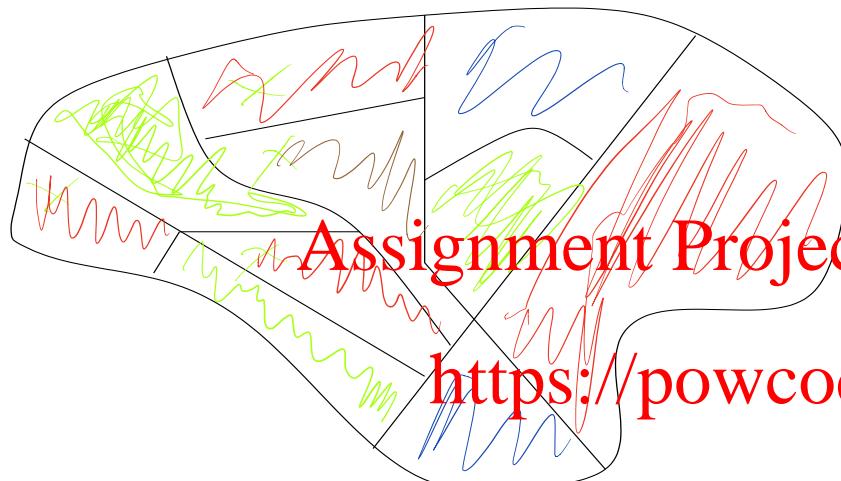
⇒ To be a cover of size  $2m + n$ ,  $S$  includes exactly one “variable gadget” node and exactly two “clause gadget” nodes. This defines a truth assignment  $t$ , as it includes exactly one of  $x_i$  and  $\neg x_i$ . Moreover, this  $t$  makes each clause true: The  $\ell_{kj}$  that is not in the cover  $S$  is set to true.

# 3SAT to Vertex Cover

Let  $t$  satisfy  $\psi$ . That gives us  $m$  nodes (for each  $i$ , either  $x_i$  or  $\neg x_i$ ) in the cover.

Now for each clause  $C_k$ , there must be some  $\ell_{kj}$  that is true with assignment  $t$ . Add to the cover  $S$  the two nodes corresponding to the two other literals in  $C_k$ . The result is a cover of the required size,  $n + 2m$ .

# Map and Graph Colouring



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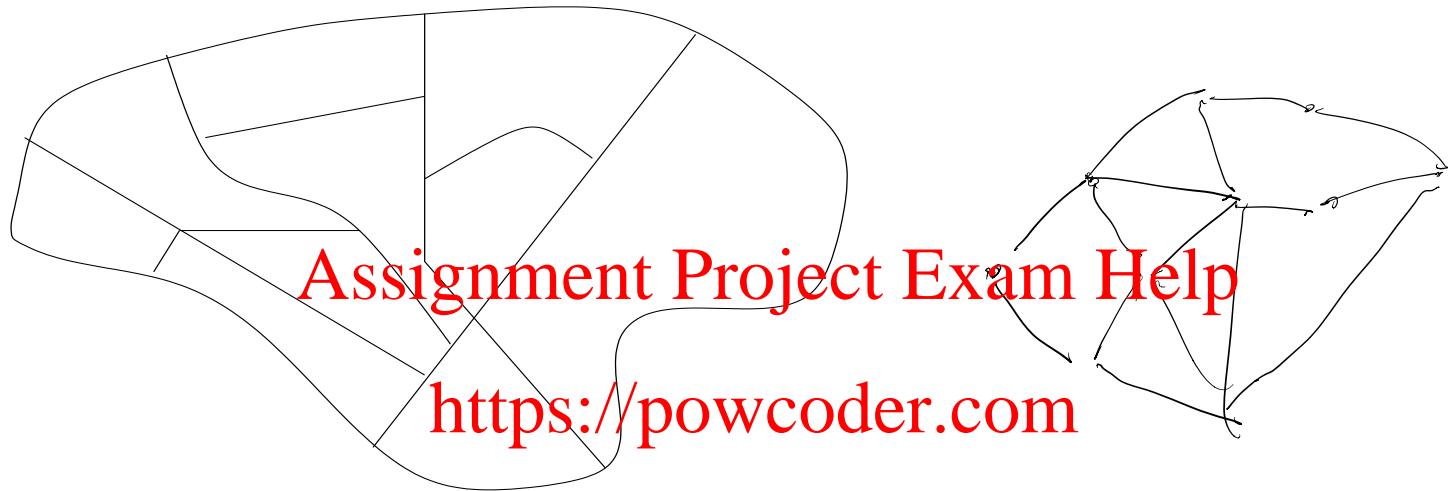
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How few colours suffice to colour this map?

No two countries that share a border can have the same colour.

# Map and Graph Colouring



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Four colours suffice in the plane! (Appel and Haken, 1976)

Tractability of  $n$ -colouring:

- 1-colouring is trivial.
- 2-colouring is easy.
- 3-colouring is NP-complete.
- 4-colouring is trivial.

# *NP* and *co-NP*

With **deterministic** machines, reversing the yes/no outcomes decides the complementary problem.

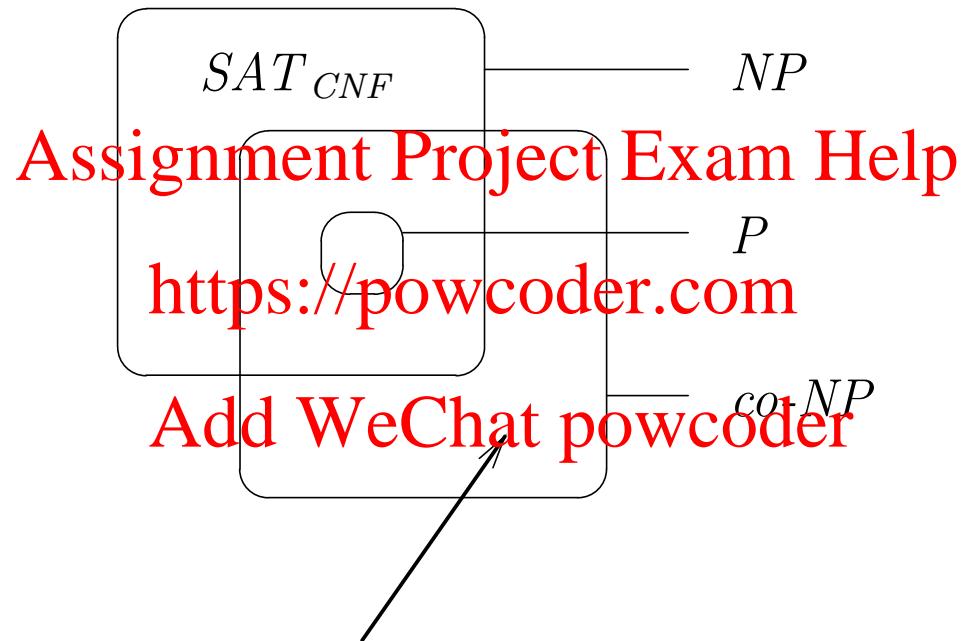
Using a nondeterministic machine to guess-and-check it is not clear that this property should hold!

For example, how does one guess-and-check that a propositional formula is unsatisfiable?

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It is not known whether *NP* is closed under complement. (Of course, if *P* were equal to *NP* then it would be.)

The class of languages *co-NP* is those that are complements of the *NP* languages.



# *NP* and *co-NP* Example

The dual of  $SAT_{DNF}$ , that is, the *satisfiability* problem for propositional formulas in DNF, is in *co-NP*.

**Satisfiability:**  $\varphi$  is true for some assignment of variables.

**Validity:**  $\varphi$  is true for all assignments of variables.

$$\begin{array}{c} ((\vee \vee) \wedge (\sim \vee)) \sim \sim \sim \\ ((x_1 \bar{y} \wedge z) \vee (\sim \sim) \vee (\sim \wedge \sim)) \\ T F T \end{array}$$

# *INDSET* is *NP*-Complete

Given an undirected graph  $G = (V, E)$ , an **independent (vertex) set** is a set  $S \subseteq V$  satisfying  $S^2 \cap E = \emptyset$  (no edges in induced subgraph of  $S$ ).

Decision problem: Does a graph  $G$  have an independent set of size  $k$ ?

$$INDSET = \left\{ \langle G, k \rangle \mid \begin{array}{l} G \text{ is an undirected graph with} \\ \text{an independent set of size } k \end{array} \right\}$$

Let us show that *INDSET* is *NP*-complete.

# INDSET is NP-Complete

$v_1, v_2, v_3, \dots, v_m$        $S_1, S_2, \dots, S_n$ .

*INDSET* is in *NP*, as an independent vertex set provides a succinct certificate.

$$(x_1 \vee y \vee z) \quad x_1 = F, y = T, z = F$$

To show *NP*-hardness, reduce *3SAT* to *INDSET*.

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Each size- $m$  clause  $c$  (from the *3SAT* instance) is mapped to a subgraph of  $2^m - 1$  nodes: each node corresponds to a (partial) truth assignment that satisfies  $c$ .

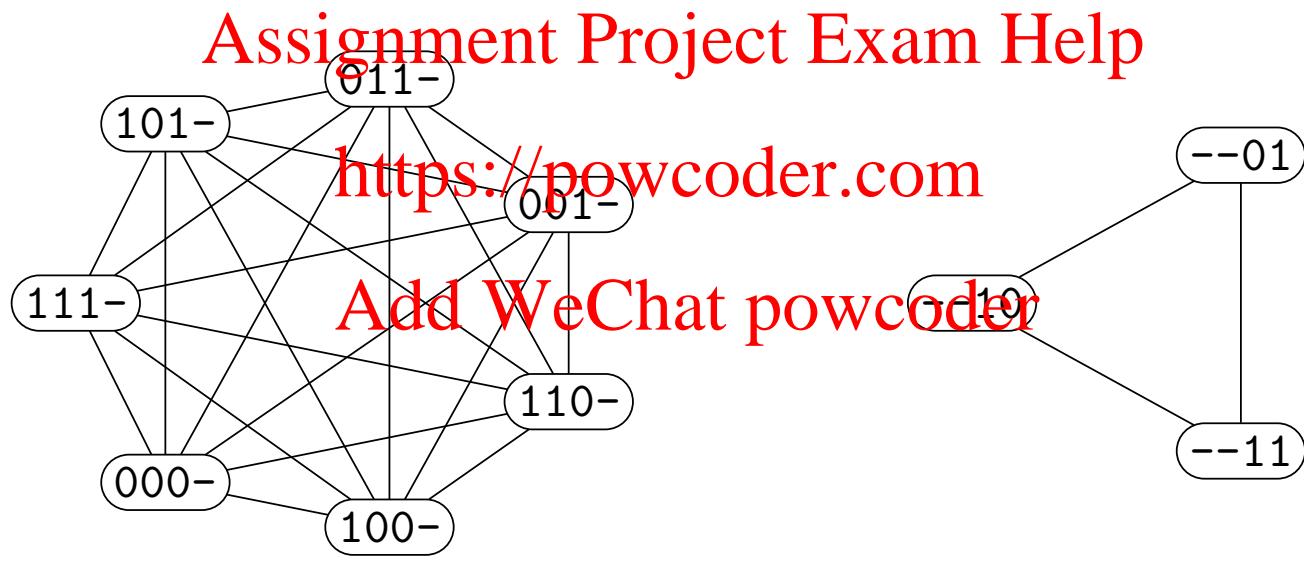
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There is an **edge** between every pair of nodes in which the two associated truth assignments are somehow **incompatible**.

That is, for some variable  $v$  in both partial truth assignments, one node assigns  $v$  true, while the other node assigns  $v$  false.

# *3SAT* to *INDSET* Reduction: Example

For  $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_3 \vee x_4)$ , form two subgraphs. Clearly, each subgraph is (internally) complete.

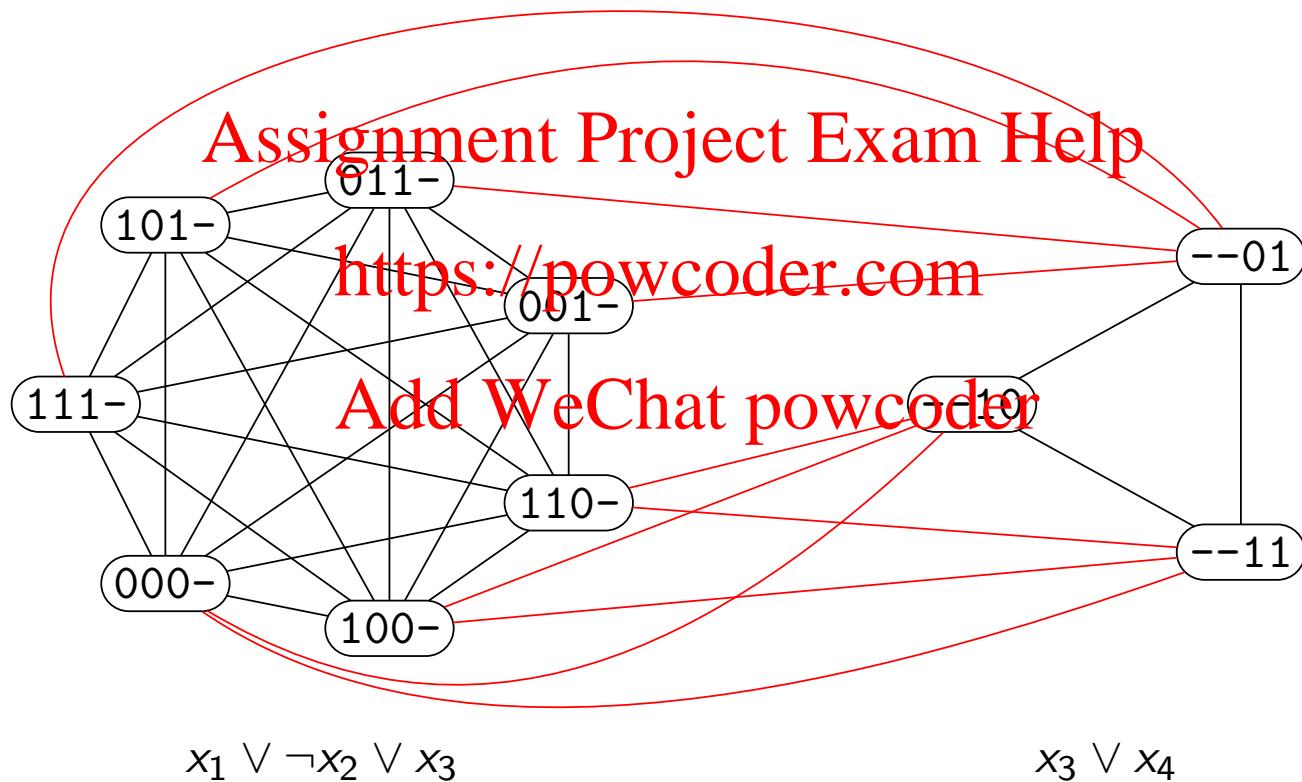


$$x_1 \vee \neg x_2 \vee x_3$$

$$x_3 \vee x_4$$

# *3SAT* to *INDSET* Reduction: Example

Now add remaining links, between incompatible assignments:



# *3SAT* to *INDSET* Reduction

The reduction takes a  $k$ -clause formula  $\varphi$  and produces the *INDSET*-instance  $\langle G, k \rangle$ .

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Since, in particular, each clause has at most 3 literals, the computation of graph  $G$  is polynomial in the size of  $\varphi$ .

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Since each of the  $k$  subgraphs is complete, an independent set in  $G$  has at most  $k$  nodes.

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Indeed, an independent set of size  $k$  must correspond to a satisfying assignment for  $\varphi$ .

# Textbook Question

## Assignment Project Exam Help

Question you can try at this point:

Problem 7.23

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# Reducing 3SAT to SUBSET-SUM

We already saw that

$$\text{SUBSET-SUM} = \{\langle S, t \rangle \mid \sum_{x \in A} x = t \text{ for some } A \subseteq S\}$$

is in  $NP$ .

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Let us reduce 3SAT to SUBSET-SUM.

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Say  $\varphi$  has  $n$  variables and  $k$  clauses.

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We construct a set of  $2n + 2k$  numbers and a "target" sum

$$t = \underbrace{1 1 1 \cdots 1}_{n} \underbrace{3 3 3 \cdots 3}_{k}$$

The numbers in the set are chosen in a clever way, which we demonstrate by example.

# $3SAT$ to $SUBSET-SUM$

$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$  yields the numbers shown in the table.

Numbers are written in decimal, or at least base 6, notation.

This set of  $2n + 2k$  numbers can be produced in polynomial time. (As shown here, it's a multiset, but we can easily add extra columns to make it a set.)

Each “clause column” (upper right part of table) must have one, two, or three 1s (since derived from 3-CNF formula).

	1	2	3	$c_1$	$c_2$
$x_1$	1	0	0	1	0
$\neg x_1$	1	0	0	0	1
$x_2$	0	1	0	0	1
$\neg x_2$	0	1	0	1	0
$x_3$	0	0	1	1	1
$\neg x_3$	0	0	1	0	0
$c_1$	0	0	0	1	0
	0	0	0	1	0
$c_2$	0	0	0	0	1
	0	0	0	0	1
$t$	1	1	1	3	3

# $3SAT$ to $SUBSET-SUM$

Zero 1s in a “clause column” in a subset corresponds to the clause not being satisfied; anything else means the clause is satisfied.

The clause rows (bottom part of the table) are simply fillers that allow a clause **column** with one or more 1s to sum to 3.

Clearly, a subset summing to  $t$  corresponds to a satisfying truth assignment for the formula.

	1	2	3	$c_1$	$c_2$
$x_1$	1	0	0	1	0
$\neg x_1$	1	0	0	0	1
$x_2$	0	1	0	0	1
$\neg x_2$	0	1	0	1	0
$x_3$	0	0	1	1	1
$\neg x_3$	0	0	1	0	0
$c_1$	0	0	0	1	0
	0	0	0	1	0
$c_2$	0	0	0	0	1
	0	0	0	0	1
$t$	1	1	1	3	3

# Pseudo-Polynomial Algorithms

It is tempting to suggest that *SUBSET-SUM* can be solved in polynomial time by a table-filling method.

Example: Given  $S = \{2, 5, 8, 9\}$  and target sum 13, we fill in a  $13 \times 4$  table, as shown here.

<https://powcoder.com>

The  $i$ th column gives the sums that could be constructed using the first  $i$  elements of the set.

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A sparse representation, growing a list, and stopping as soon as we find 13, would be more practical, for example: [0], [0,2], [0,2,5,7], [0,2,5,7,8,10,13,15].

$\log_2 k$

$\langle t, a_1, a_2, \dots, a_n \rangle$

Sum	2	5	8	9
0	T	T	T	T
1	F	F		F
2	T	T	T	
3	F	F	F	F
4	F	F	F	F
5	F	T	T	
6	F	F	T	
7	F	T	T	
8				
9				
13	F	F		

# Pseudo-Polynomial Algorithms

However, we require that numerical methods involve “reasonable” representation of numbers, such as decimal (or even binary).

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The table above is polynomial in the size  $n$  of the set. However, it is not polynomial in the ~~size~~ of the description of the target  $t$ , which would only be  $O(\log t)$ . Unless the integers in the input have unary representation, the table is in fact ~~exponential~~ in the size of input.

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Nevertheless, intractability of *SUBSET-SUM* depends strongly on the fact that large numbers are allowed. For sets of “small” numbers, the problem is tractable.

# Textbook Question

Assignment Project Exam Help

Question you can try at this point:

Problem 7.17

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# Hamiltonian Path is *NP*-Complete

Recall the Hamiltonian path problem  $HAMPATH =$

$\{(S, g, t) \mid \begin{array}{l} G \text{ is a directed graph with a} \\ \text{Hamiltonian path from } s \text{ to } t \end{array}\}$

where a Hamiltonian path visits each node in  $G$  exactly once.

$HAMPATH$  is in *NP* since a Hamiltonian path can be (guessed and) checked in polynomial time.

By reduction from  $3SAT$ , we now establish that  $HAMPATH$  is *NP-complete*

# Reducing 2SAT to HAMPATH

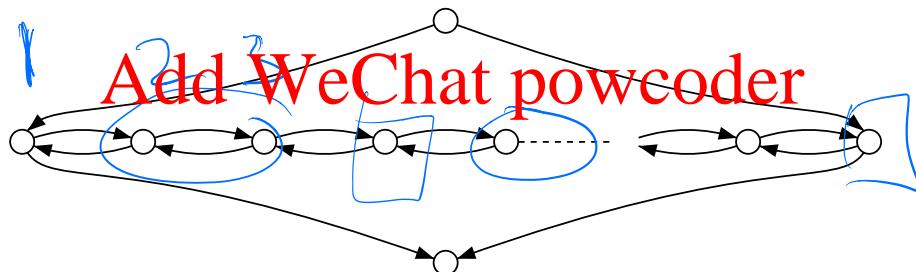
Assume we have a 3-CNF formula  $\varphi$  with  $k$  clauses.

We need to construct (in polynomial time) a graph  $G$  in which a Hamiltonian path from  $s$  to  $t$  exists iff  $\varphi$  is satisfiable.

$$\varphi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \dots \wedge (a_k \vee b_k \vee c_k)$$

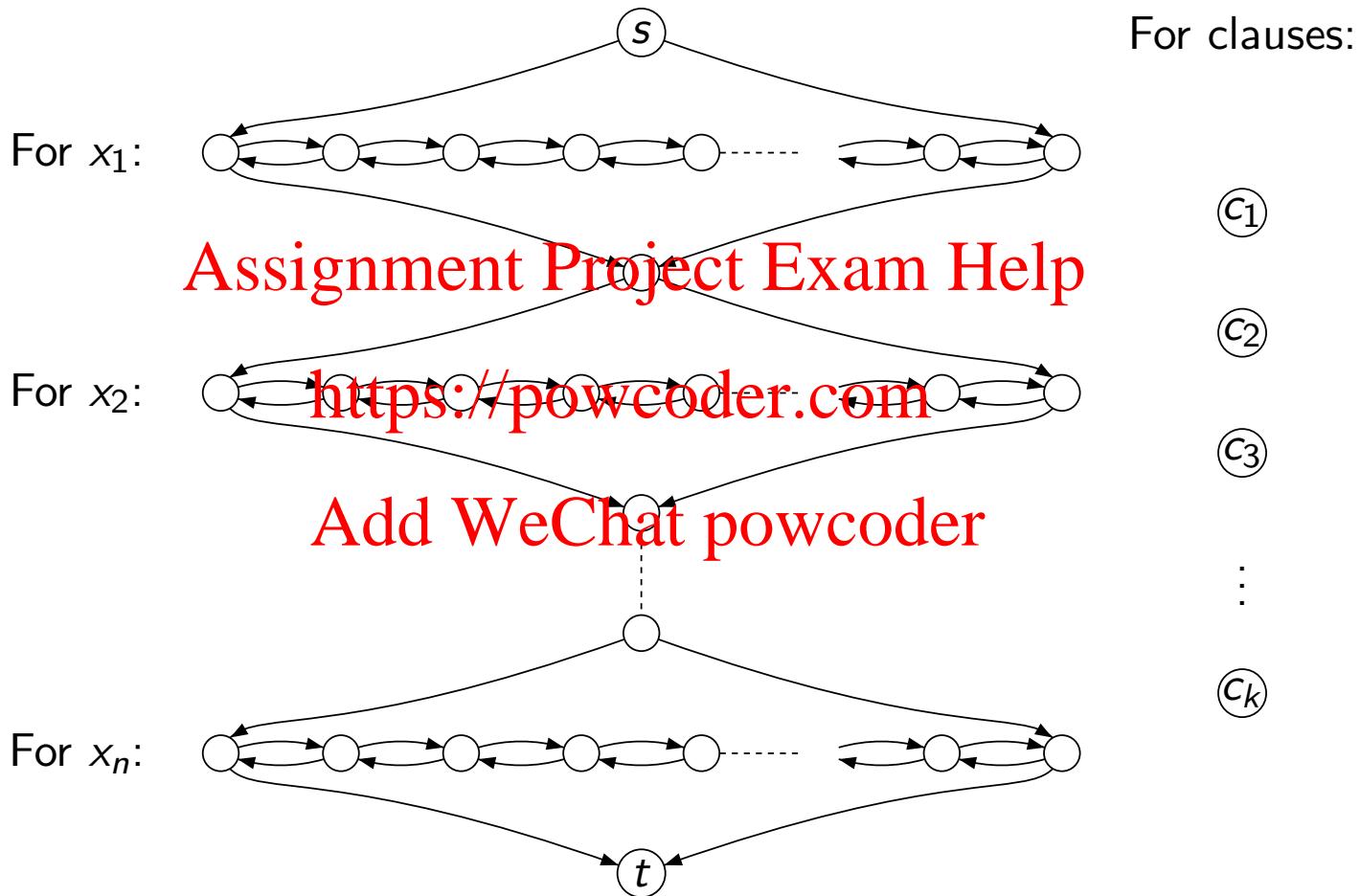
For each variable  $x_i$  in  $\varphi$ , we produce a “diamond” gadget like this:

<https://powcoder.com>



Apart from the nodes on the perimeter, which are shared between gadgets, there are  $3k + 1$  nodes, two adjacent nodes per clause, with a separator node either side of the pair. We also construct  $k$  separate nodes, one per clause.

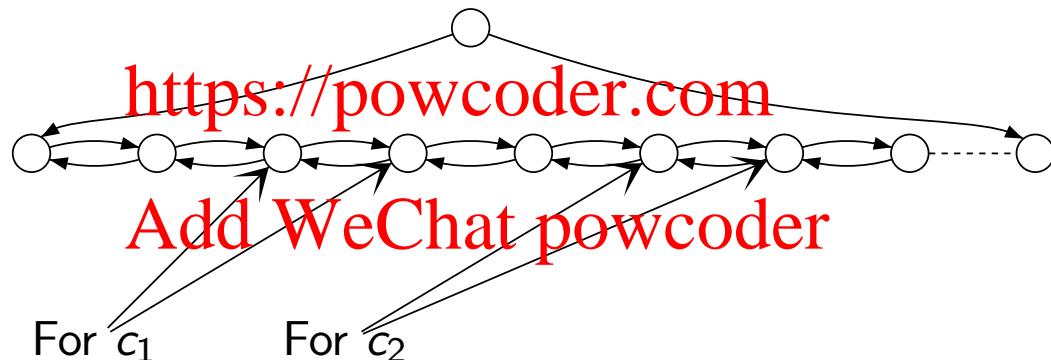
# Reducing SAT to HAMPATH



# Reducing *SAT* to *HAMPATH*

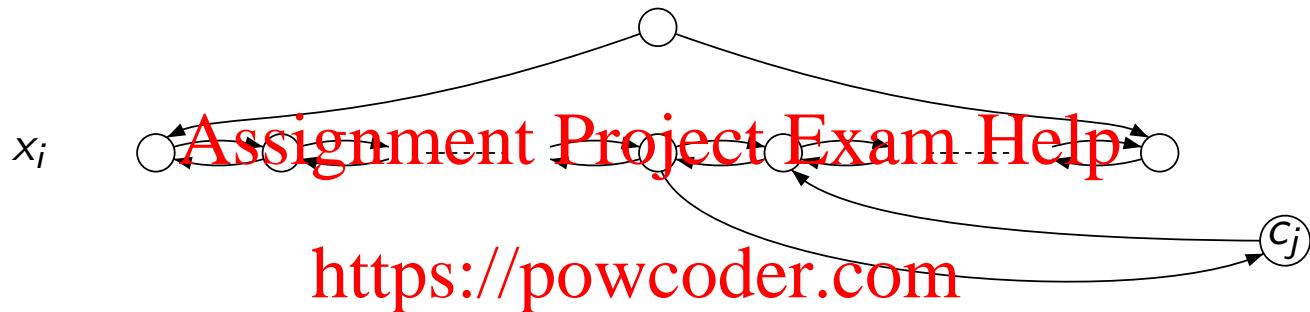
The chain of  $3k + 1$  nodes comprises one **pair** per clause, plus separator nodes on either side:

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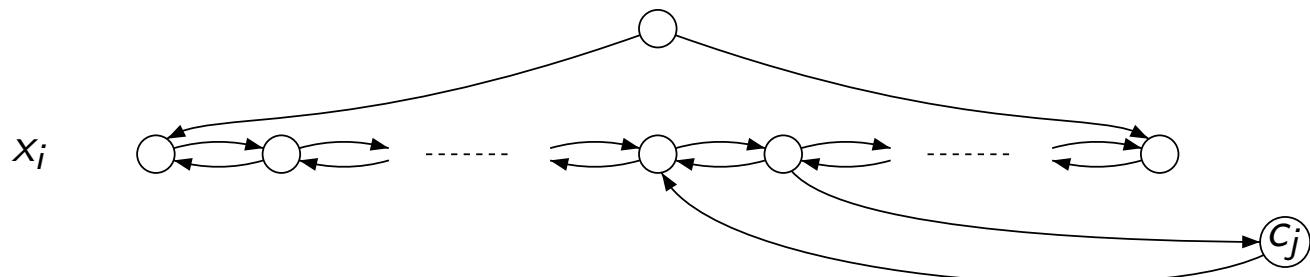


# Reducing $SAT$ to $HAMPATH$

Consider  $x_i$ 's chain. If  $x_i$  occurs in  $c_j$ , we add two edges:



If  $\neg x_i$  occurs in  $c_j$ , the linking edges are reversed:

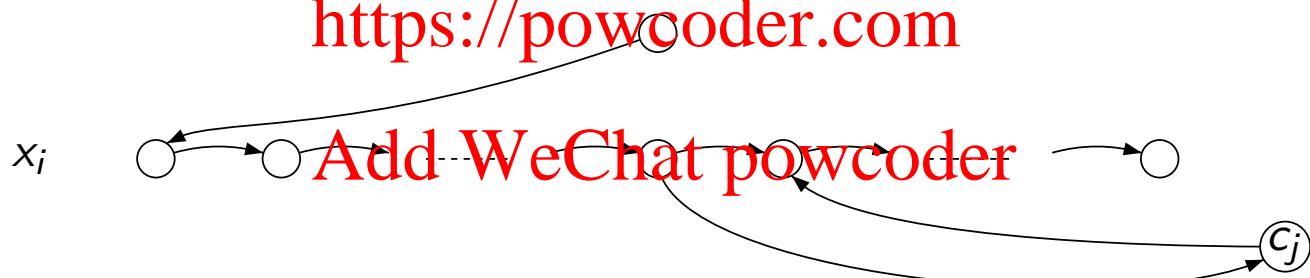


# Reducing $SAT$ to $HAMPATH$

The idea is that a traversal down through the sequence of diamonds corresponds to a truth assignment to the variables.

Walking left-to-right through the  $i^{\text{th}}$  chain corresponds to setting  $x_i$  to true.

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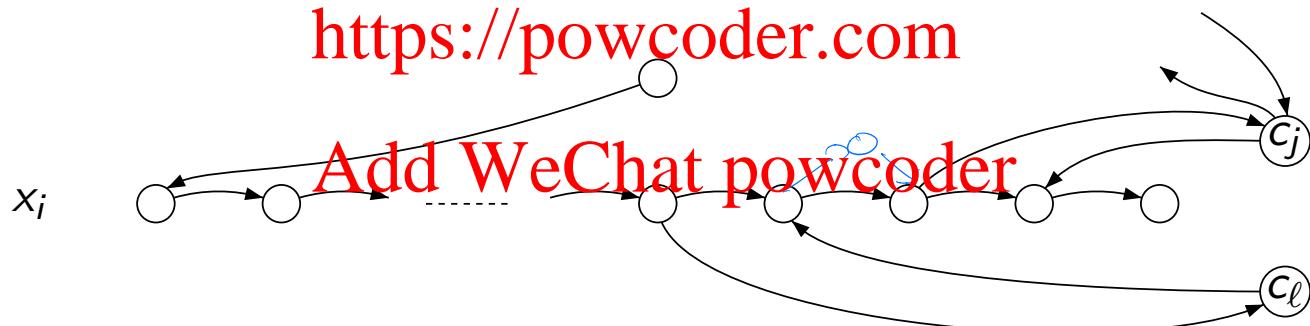
On the other hand, walking right-to-left through the  $i^{\text{th}}$  chain corresponds to setting  $\neg x_i$  to true.

# Reducing *SAT* to *HAMPATH*

Given a satisfying truth assignment, there is clearly a Hamiltonian path through the graph.

All clause nodes  $c_j$  can be reached, but going from a particular chain node to  $c_j$  is optional.

<https://powcoder.com>



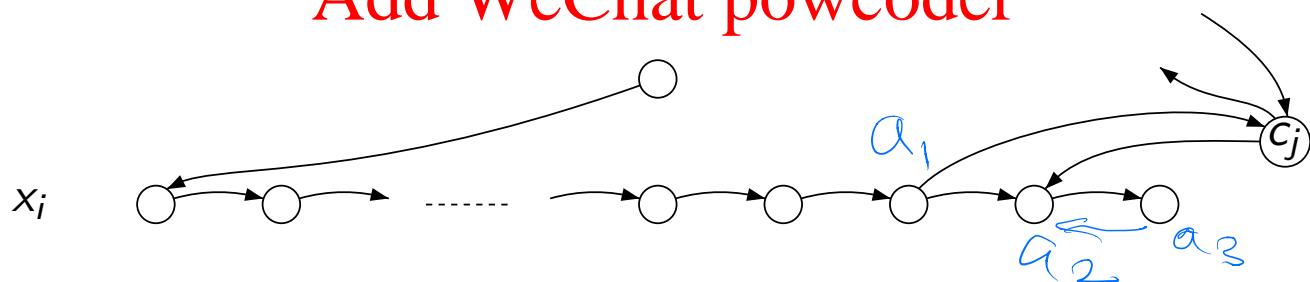
A single  $x_i$  may make several different clauses true.

# Reducing SAT to HAMPATH

Conversely, every Hamiltonian path through the graph from  $s$  to  $t$  determines a satisfying truth assignment.

If a path takes us from a chain node to a clause node, we must return to the neighbouring (pair) node in the chain. Suppose we don't: we exit from  $a_1$  to clause node  $c_j$ . Then the next node in the chain from  $a_1, a_2$ , is cut off. Depending on its status,  $a_2$ 's only possible entry points are  $a_1, c_j$  or the next node in the chain,  $a_3$ . Nodes  $a_1, c_j$  aren't available as entry points, as they lead elsewhere; node  $a_3$  is needed as the exit point.

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So the graph traversal must correspond to a satisfying truth assignment.

The graph is produced ('computed') in polynomial time.

# Textbook Questions

Questions you can try at this point:

Problem 7.21 **Assignment Project Exam Help**

Problem 7.27

Problem 7.28

Problem 7.29

Problem 7.30

Problem 7.31

Problem 7.37

Problem 7.43

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# Decision vs Witness Production or Search

Our definitions of (mapping) reduction and of *NP* apply to **decision problems**—they are phrased in terms of language membership.

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But often we are interested in more than a yes/no answer—we want a **witness**.

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Is that substantially harder than getting a yes/no?

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In the context of **optimization problems**, we ask not “is there a route of length no more than  $k$ ?”, but “what is the shortest route?”

Is this **optimization** problem substantially harder than getting a yes/no?

# Decision vs Witness Production

It turns out that, if  $P = NP$  then we can produce witnesses for  $L \in NP$  with only a polynomial-time overhead.

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The next slide exemplifies this, for *SAT*: check that the overhead indeed is polynomial.

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# Decision vs Search

If  $M$  were a polytime decider for  $SAT$ , formula  $\varphi$  has  $n$  variables, and if  $M(\varphi) = \text{'yes'}$ , then a satisfying assignment is found thus:

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```
t := []
For i ∈ [1, n]:
    ti0 := [xi ↦ 0]
    ti1 := [xi ↦ 1]
    If M(ti0(φ))
        φ := ti0(φ)
        t := ti0 ∘ t
    else
        φ := ti1(φ)
        t := ti1 ∘ t
Output t
```

# Decision vs Search

Again, if  $P = NP$  then the typical “optimization” problem is not substantially harder than its decision version.

For example, given a graph  $G$  with  $n$  nodes, we can find the size of its smallest vertex cover by invoking the decider for  $VERTEX-COVER$  up to  $\log_2 n$  times.

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Similarly the decision problem

$$MINVC = \left\{ \langle G, k \rangle \middle| \begin{array}{l} G \text{ is a graph whose minimal} \\ \text{vertex cover has size } k \end{array} \right\}$$

can be decided by running the deterministic decider for  $VERTEX-COVER$  on the two instances  $\langle G, k - 1 \rangle$  and  $\langle G, k \rangle$ .

Unclear whether  $MINVC$  is in  $NP$ , in  $coNP$ , (or neither).

# Textbook Question

## Assignment Project Exam Help

Question you can try at this point:

Problem 7.46

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## Review of this block

We saw how to reduce *SAT* to several different *NP-hard* problems, both directly and indirectly (via other *NP-hard* problems).

We explored a little of the difference between a decision problem and an “answer” problem.

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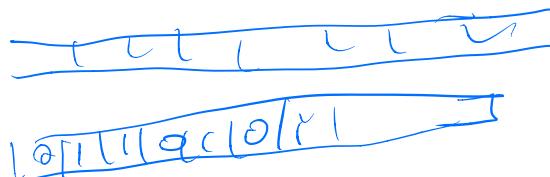
## What's next

Randomised algorithms: probabilistic TMs

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Space complexity: look at the beginning of Chapter 8 in the Sipser text.

# Probabilistic Turing Machines



input



Consider a Turing machine that has the ability to make random choices as it computes.

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A mental model: The machine has access to an infinite tape of random bits, initialised freshly for each run of the machine.

Each nondeterministic step is a coin toss step, with two legal next moves, of equal probability. (There is no loss of generality in considering just two moves.)

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# What Does Acceptance Mean?

A deterministic Turing machine either accepts or rejects its input, with the **same outcome** every time the machine is run on the same input.

A nondeterministic machine shares this property, though only one accepting branch is needed to accept the input.

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In contrast, a probabilistic Turing machine accepts its input with some **probability**: unless this probability is 0 or 1, the result can vary between runs.

For example, it may take strings in  $\{0, 1\}^*$  as input, and accept strings beginning with 0 with probability  $2^{-n}$ , where  $n$  is the length of the string.

# Probability of Acceptance

Let  $M$  be a probabilistic Turing machine (PTM) and  $w$  be an input string.

When running  $M$  on  $w$ , each computation branch  $b$  has probability  $Pr[b] = 2^{-k}$ , where  $k$  is the number of coin-tosses on branch  $b$ .

The probability that  $M$  accepts  $w$  is

$$Pr[M \text{ accepts } w] = \sum_{\text{accepting } b} Pr[b]$$

$$Pr[M \text{ rejects } w] = 1 - Pr[M \text{ accepts } w].$$

# Recognition with Error Probability

Let  $0 \leq \varepsilon < \frac{1}{2}$ .

$M$  recognises  $L$  with error probability  $\varepsilon$  iff

- ① If  $w \notin L$  then  $\Pr[M \text{ rejects } w] \geq 1 - \varepsilon$
- ② If  $w \in L$  then  $\Pr[M \text{ accepts } w] \geq 1 - \varepsilon$ .

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The error probability may depend on the length  $n$  of input, as happened with the example PTM above.

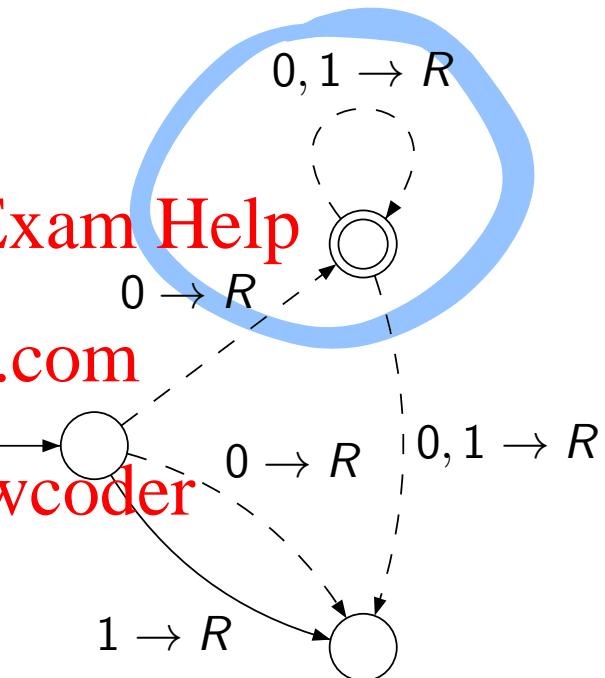
# An Example of a PTM

Here is a PTM for the language mentioned before.

The machine accepts each binary string starting with 0 with probability  $2^{-n}$ ,

where  $n$  is the length of the string; every other string is rejected.

A dashed edge indicates a transition step that has probability 1/2.



# The Class BPP

Language  $L$  is in BPP iff  $L$  is decided ~~in polynomial time~~ by some PTM  $M$ , with an error probability of  $1/3$ .

Note that  $M$  must run in polynomial time, regardless of the outcome of coin-tosses along the way.

That is, the probabilistic computation ‘tree’ must have a height (across all branches) which is polynomially bounded.

# The Amplification Lemma

The choice of  $1/3$  in the definition of BPP is rather arbitrary. It's simply a value strictly between  $0$  and  $1/2$ , indeed the "simplest" fraction in that range.

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This follows from the [amplification lemma](#): with a polynomial-time effort, an error probability can be made exponentially small.

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Sipser gives a proof, and we shall see an example shortly.

# The Class RP

Language  $L$  is in RP iff  $L$  is recognised in polynomial time by some PTM  $M$ , with a one-sided error probability, as follows:

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- ① If  $w \notin L$  then  $\Pr[M \text{ rejects } w] = 1$ .
- ② If  $w \in L$  then  $\Pr[M \text{ accepts } w] \geq \frac{1}{2}$ .

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RP stands for “randomised polynomial”.

For such machines, a ‘yes’/‘accept’ answer can be trusted.

# “Almost Certainly” vs “Certainly”

Again, for  $L \in RP$ , when deciding membership of  $L$ , repeated runs of  $M$  on the same input will give independent, equally reliable, results.

With an error probability of  $1/3$ , say, the likelihood of getting a wrong ‘no’ answer from 100 consecutive trials is  $3^{-100}$ .

Success is so close to certainty that, for our purposes, it is. Hardware failure has a much higher likelihood.

# An Example RP Problem

$\{ \langle G \rangle \mid G \text{ undirected \& contains a triangle} \}$ .

Input: An undirected graph  $G = (V, E)$ .

Output: Yes if  $G$  contains a triangle.

Method:

Repeat the following  $k$  times:

Select an edge  $(x, y) \in E$  uniformly at random.

Select a node  $z \in V$  uniformly at random.

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Output 'yes' if  $(x, z) \in E$  and  $(y, z) \in E$ , and exit.

Output 'no'.

Clearly the first of the two conditions for RP membership is satisfied: If  $G$  contains no triangle then we can only get a 'no'.

# An Example RP Problem

As for the second condition, let  $n = |V|$  and  $m = |E|$ . If  $G$  has a triangle, the probability that it is selected in a particular iteration is  $3/(m(n - 2))$ .

For  $k$  iterations, the probability that the triangle will be found is

$$\frac{3}{m(n-2)} \left(1 - \frac{3}{m(n-2)}\right)^k$$

For small  $x$ ,  $(1 - x)^k \approx e^{-kx}$ .

If we pick  $k = 1/x$ ,  $1 - e^{-kx} \approx 0.63$ .

Thus with  $k = m(n - 2)/3$ , the probability that a graph with a triangle is accepted is well over  $1/2$ .

# An Example RP Problem

The complexity of the algorithm is polynomial.

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In particular,  $k = O(mn)$ , so the number of rounds is at most quadratic in the input size (assuming  $n = \Omega(m)$ )

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The running time for each round is linear in the input size.

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Hence the problem is in *RP*.

# And with Amplification . . .

The randomized method we described has one-sided error: it never gives a false positive result.

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However, it could reject input that should be accepted.

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But the probability if this happening can be made as small as we want.

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Running the triangle algorithm 30 times, the likelihood of a false negative is less than one in a billion.

# Where RP Lives

**Theorem:**  $P \subseteq RP \subseteq NP$ .

The first inclusion follows immediately from the definitions.

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For the second, a nondeterministic TM  $N$  can simulate a probabilistic TM  $M$  (with the same time bound) for language  $L$ .

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Whenever  $M$  uses a random bit,  $N$  chooses nondeterministically between the two values.

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If  $w$  is in  $L$ , then  $M$  has a probability of at least  $1/2$  of accepting  $w$ , so there must be **some** sequence of choices for  $N$  that leads to acceptance.

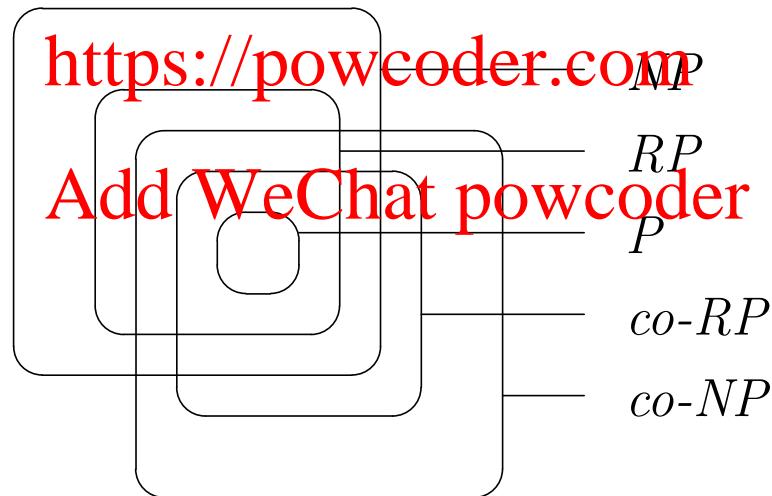
If  $w$  is not in  $L$ , no such sequence exists, so  $N$  will reject.

# The Class co-RP

*RP* is one-sided, so it has a natural dual class.

A co-*RP* language has a PTM  $M$  that accepts members with probability 1, and rejects non-members with probability  $\geq 1/2$ .

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# An Example co-RP Problem

An algebraic circuit is like a Boolean circuit, except inputs are integers and the “gates” perform addition, subtraction, and multiplication.

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The polynomial identity testing problem asks whether two algebraic circuits compute the same function (a polynomial).

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This is an example of a problem for which a fast randomized algorithm exists.

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Unlike the case of the triangle problem discussed above, no efficient deterministic algorithm is currently known for polynomial identity testing.

# Open Problems

Many interesting questions remain open.

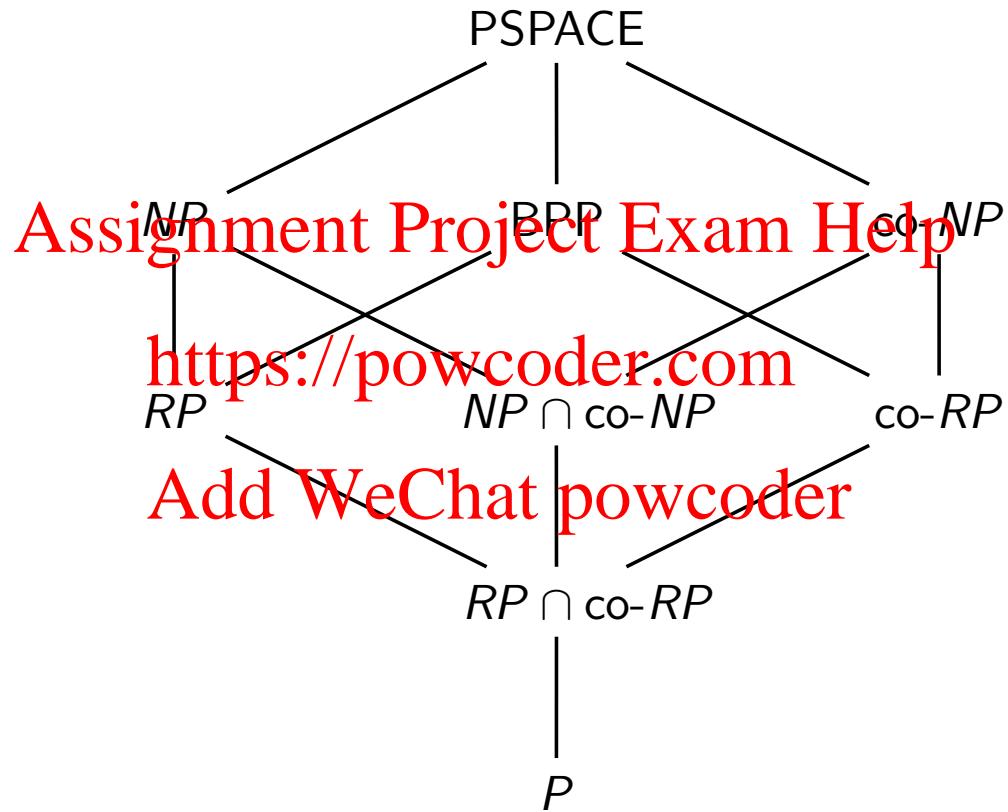
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For example, is  $RP = \text{co-}RP$ ?

Is  $RP \subseteq NP \cap \text{co-}NP$ ? ( $\text{if } RP = \text{co-}RP \text{ the answer is 'yes'}$ .)

Is  $BPP \subseteq NP$ ? Add WeChat powcoder

Is  $BPP = P$ ? (Believed to be yes)

# Where Some Randomized Classes Fit



# Summarising RP and co-RP

$L \in RP \cap \text{co-}RP$  means:

(RP) There is a PTM  $M$  running in time  $p(n)$  such that

$w \in L \Rightarrow \Pr[M \text{ accepts } w] = 1$

- $w \in L \Rightarrow \Pr[M \text{ accepts } w] \geq 1/2$

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(co-RP) There is a PTM  $M'$  running in time  $p'(n)$  such that

- $w \notin L \Rightarrow \Pr[M' \text{ accepts } w] = 1$
- $w \notin L \Rightarrow \Pr[M' \text{ rejects } w] \geq 1/2$

In the first case, a ‘yes’ can be trusted.

In the second case, a ‘no’ can be trusted.

# Understanding RP $\cap$ co-RP

We can run  $M$  and  $M'$  on input  $w$  in time  $p(n) + p'(n)$ .

There are three possible outcomes:

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$M$ says	$M'$ says	We can conclude
yes	yes	$x \in L$
no	no	$x \notin L$
no	yes	?

*yes      no      ?*

The combined PTM is **sound** (we can trust its answer, if it gives one), and it will answer ‘?’ with probability less than  $1/2$ .

# Monte Carlo and Las Vegas Methods

The randomized methods discussed so far are Monte Carlo algorithms.

These run in (guaranteed) polynomial time and err with some probability less than  $1/2$ .

RP and co-RP contain problems that have ~~the~~ Monte Carlo algorithms.

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A Las Vegas algorithm also uses randomization, and it never errs.

However, it does not have the polynomial-time guarantee for every computation branch. Instead, it runs in expected polynomial time.

# The Class ZPP

The Z in ZPP stands for “zero error”.

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ZPP corresponds to the Las Vegas type of algorithms, which always give the correct answer, but have a random execution time (even on fixed input), the expectation of which is polynomial.

In the following several slides, we prove the interesting result that  $ZPP = RP \cap \text{co-}RP$ .

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$$\text{ZPP} \subseteq RP \cap \text{co-}RP$$

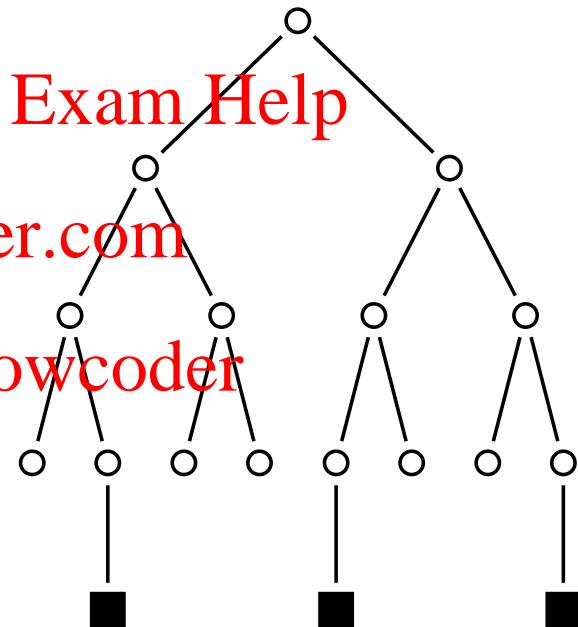
Let  $M$  be a PTM for a problem in ZPP and consider  $M$ 's computation tree for input  $w$ , of size  $n$ . Let  $q(n)$  be the expected (polynomial) running time.

At a given polynomial depth,  $p(n)$ , we

may have subtrees of super-polynomial depth (shown in black).

Let  $M'$  be the PTM which works like  $M$ , except (if it hasn't stopped already) it stops after  $p(n)$  steps and answers '?'.

We show that  $p$  can be chosen so that the probability  $\pi$  of  $M'$  answering '?' can be brought as close to 0 as we want.



$$\text{ZPP} \subseteq RP \cap \text{co-}RP$$

Let  $s(n)$  be the average length of the super-polynomial paths (so  $s(n)$  is super-polynomial). Let  $\pi$  be the probability that  $M'$  says '?'.

$$X \geq 0$$

$$\Pr[X \geq a \cdot E[X]] \leq \frac{1}{a}.$$

$$Y = 0 \cdot \Pr[X < a \cdot E[X]] + 1 \cdot \Pr[X \geq a \cdot E[X]]$$

$$\Pr[Y \geq a \cdot E[X]] \leq \frac{1}{a} \cdot \Pr[X \geq a \cdot E[X]]$$

$$\Pr[Y \geq a \cdot E[X]] \leq \Pr[X \geq a \cdot E[X]]$$

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The expected running time of  $M$  is  $(1 - \pi) \cdot p(n) + \pi \cdot s(n)$ , which we know is  $q(n)$ . Isolating  $\pi$ , we have

$$\pi = \frac{q(n) - p(n)}{s(n) - p(n)}$$

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$$\Pr[t(n) \geq r(n)] \leq \frac{1}{r(n)}.$$

Since the denominator is super-polynomial, we can choose  $p(n)$  so that  $\pi$  is arbitrarily small.

$$E[t(n)] = q(n)$$

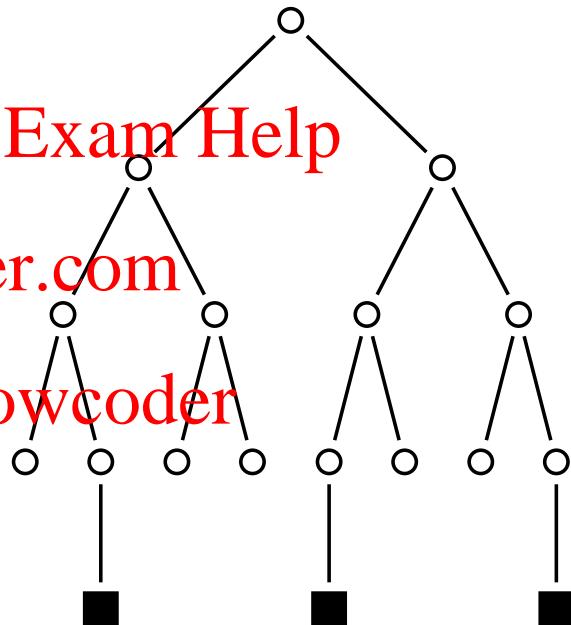
$$\Pr[t(n) \geq nq(n)] \leq \frac{1}{n}.$$

$$\text{ZPP} \subseteq RP \cap \text{co-}RP$$

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But this means that  $M'$  satisfies the conditions for membership of  $RP \cap \text{co-}RP$ .

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$$RP \cap \text{co-}RP \subseteq \text{ZPP}$$

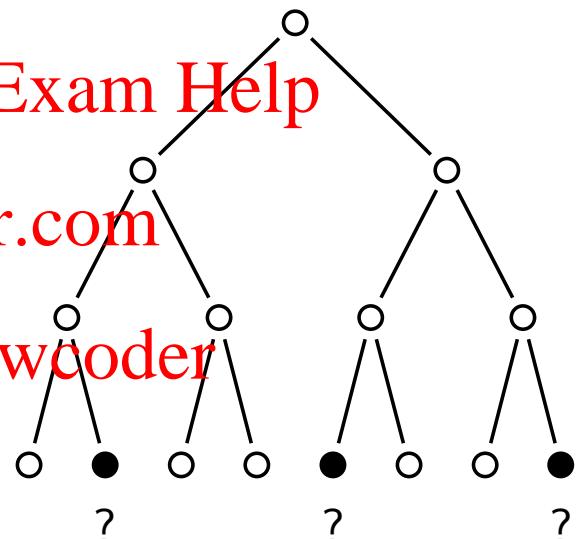
Now let  $M$  be a machine for a problem in  $RP \cap \text{co-}RP$ . Let  $1/k$  be an upper bound on the probability that  $M$  says ‘?’ ( $k > 1$ ).

Machine  $M$  has polynomial running time  $r(n)$ .

Since  $RP \subseteq NP$ , we can always solve the problem brute-force style, without randomization, in time at most  $k^{q(n)}$ , for some polynomial  $q(n)$ .

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2<sup>poly</sup>      k<sup>poly</sup>



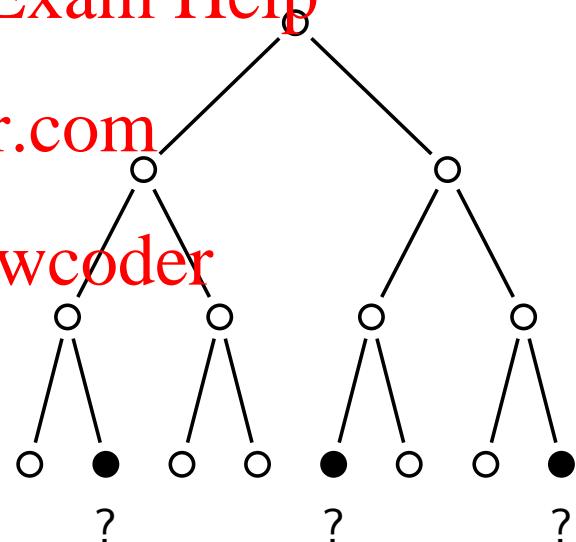
To build a corresponding machine  $M'$  that satisfies the conditions for membership of ZPP, we just run  $M$  up to  $q(n)$  times.

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As soon as  $M$  provides a yes or no answer, machine  $M'$  returns this answer and halts.

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If none of the  $q(n)$  runs of  $M$  provides an answer, then  $M'$  solves the instance the slow way: deterministically brute force.



$$RP \cap \text{co-}RP \subseteq \text{ZPP}$$

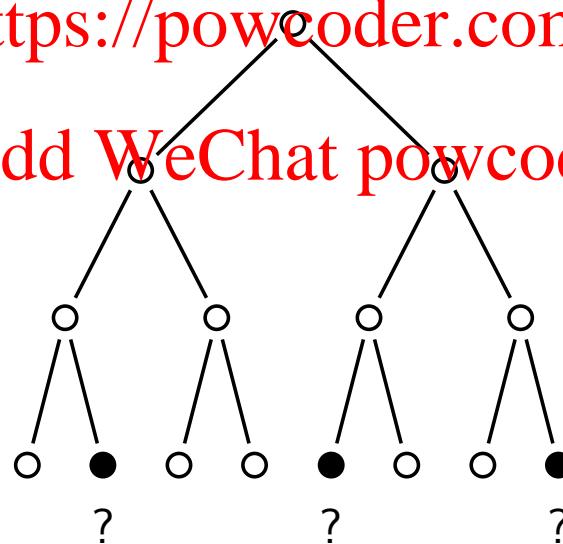
The expected running time of  $M'$  is at most

$$r(n) \cdot q(n) + \underbrace{k^{-q(n)}}_{Pr['?']} \cdot k^{q(n)} = r(n)q(n) + 1,$$

which is polynomial in  $n$ .

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# Textbook Questions

Questions you can try at this point:  
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Problem 10.11

Problem 10.15

Problem 10.19

Problem 10.22

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# Space Complexity

Just like time, ~~Assignment Project Exam Help~~ is a relevant resource to study.

The **space complexity** of a Turing machine is the number of distinct tape positions it visits, as a function of the size of input.

This leads to a complexity measure that is robust

# Notation

It is common to prefix the terms “TIME” and “SPACE” with “N” for “nondeterministic”, and “LOG”, “P”, “EXP”, etc., for complexity:

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LOGTIME	NLOGTIME
LOGSPACE	NLOGSPACE
PTIME	NPTIME
PSPACE	NPSPACE
EXPTIME	NEXPTIME
EXPSPACE	NEXPSPACE
⋮	

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# *SPACE* and *NSPACE*

We can define space complexity classes analogous to *TIME* and *NTIME*:

$$SPACE(f(n)) = \left\{ L \mid \begin{array}{l} L \text{ is decided by some deterministic} \\ \text{Turing machine in } O(f(n)) \text{ space} \end{array} \right\}$$

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and

$$NSPACE(f(n)) = \left\{ L \mid \begin{array}{l} L \text{ is decided by some nondeterministic} \\ \text{Turing machine in } O(f(n)) \text{ space} \end{array} \right\}$$

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In analogy with *P* and *NP*,

$$\text{PSPACE} = \bigcup_k SPACE(n^k)$$

$$\text{NPSPACE} = \bigcup_k NSPACE(n^k)$$

# SAT is in PSPACE

$\boxed{F_1 \ F_2 \ F_3 \ F_4}$

$x_0$

$x_1$

$\curvearrowright$

SAT can be solved in linear space thus:

We solve SAT in exponential time by generating all possible truth assignments, and evaluating the formula on each of them.

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Apart from space for the formula, we only need space for **one** truth assignment at a time, the one currently being evaluated.

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Overwriting each truth assignment with the next, in the same space, means that space consumption is linear in the size of input.

# Polynomial Space

Clearly  $P \subseteq PSPACE$ : If a Turing machine makes only a polynomial number of moves, then it cannot visit a super-polynomial number of cells.

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On the other hand, a polynomial-space-bounded TM can only make an *exponential* number of moves before it must repeat a configuration.

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For a machine with  $s$  states,  $\gamma$  symbols in the tape alphabet, at most  $f(n)$  tape cells visited, the number of configurations is  $s f(n) \gamma^{f(n)}$ . A machine that halts cannot repeat a configuration!

From this, it follows that  $PSPACE \subseteq EXPTIME$ , which is  $\bigcup_k TIME(2^{n^k})$ .

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Question you can try at this point:

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Exercise 8.4

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# Savitch's Theorem

Perhaps surprisingly,  $\text{PSPACE} = \text{NPSPACE}$ .

To demonstrate this, we show that a deterministic TM can simulate an NTM in a fairly small amount of space.

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**Theorem:** For a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  with  $f(n) \geq n$ ,

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 $\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n))$

# Node Reachability in Sub-Linear Space

A central tool for proving space-complexity results.

Path Existence Assignment Project Exam Help

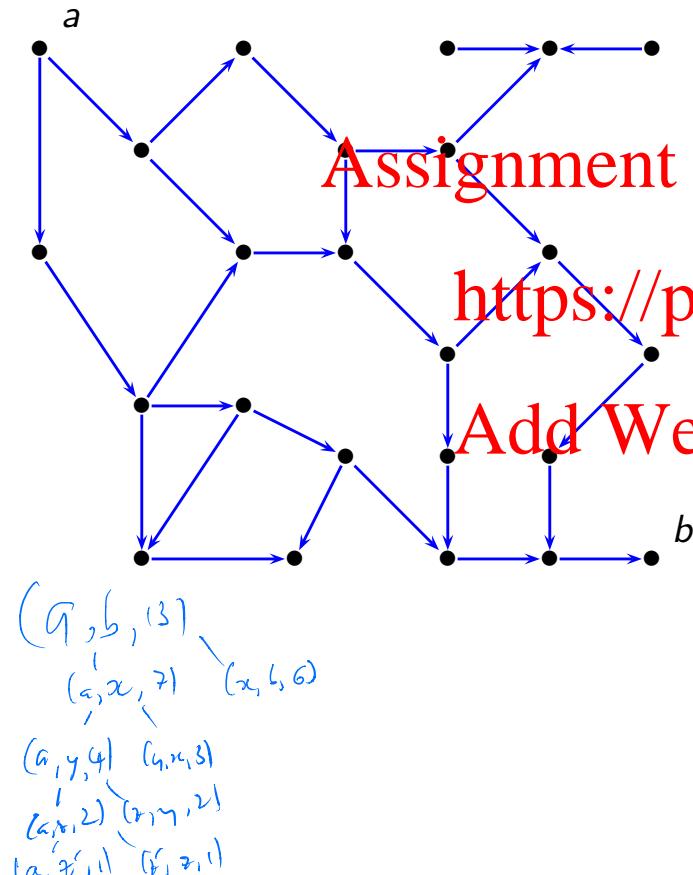
Given a graph  $G$ , nodes  $a$  and  $b$ , and a number  $n$ , we can ask “is node  $b$  reachable from node  $a$  via a path that is at most  $n$  steps long?”

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We can decide this in  $O(n \log n)$  space and exponential time.

We apply this to configurations of a Turing machine: the combination of tape contents, head location, and state.

# Node Reachability Example



In this directed graph, we can go from  $a$  to  $b$  in 13 steps if and only if:

There exists some node  $x$  for which we can:

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- 1 Go from  $a$  to  $x$  in 7 steps
  - 2 Go from  $x$  to  $b$  in 6 steps.

So we test this for all nodes  $x$ .

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The runtime stack will never be deeper than  $\log n$  (and each frame holds little more than the numerical path-length bound).

Remember, we can reuse the space when we ‘pop’ the stack frame.

# Savitch's Theorem

This insight leads us to the following

**Theorem:** For a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  with  $f(n) \geq \log n$ ,

$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n))$$

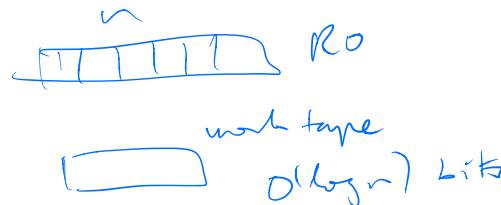
Suppose  $N$  is an NTM working in space  $f(n)$ .

We construct a deterministic TM  $M$  to simulate  $N$ .

Machine  $M$  uses a recursive procedure **canyield**, which takes configurations  $c_1$  and  $c_2$  together with a number  $t$  and outputs *accept* iff  $c_2$  can be obtained from  $c_1$  in  $t$  steps or fewer, according to the rules of  $N$ .

# How Many Configurations?

Assuming



$s$  states in  $Q$

$\gamma$  symbols in  $\Gamma$

$f(n)$  cells touched on tape

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there at most  $s \cdot f(n) \cdot \gamma^{f(n)}$  different configurations.

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That is, for suitable constant  $d$ , no more than  $2^{d \cdot f(n)}$  configurations. (We can achieve this by treating the presence of the tape head as an extra symbol.)

(To achieve the  $f(n) \geq \log n$  condition, we treat the input as read only, so that the number of configurations is  $n 2^{O(f(n))}$ : provided  $f(n) \geq \log n$ , this is  $2^{O(f(n))}$ . Without this, we require  $f(n) \geq n$ , just to account for the input.)

# Savitch's Theorem

canyield( $c_1, c_2, t$ ) procedure

- ① If  $t = 1$ , accept if  $c_1 = c_2$  or  $c_1$  yields  $c_2$  in one step, otherwise reject.
- ② For each configuration  $c$ ,
- ③ Run canyield( $c_1, c, \lceil \frac{t}{2} \rceil$ )
- ④ Run canyield( $c, c_2, t - \lceil \frac{t}{2} \rceil$ )
- ⑤ If both accept then accept, otherwise reject.

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Now we can define  $M$ , which simulates  $N$ .

To have a single accept configuration,  $M$  can alter  $N$  slightly so that  $N$  clears its tape and rewinds upon accepting.

# Savitch's Theorem

Let  $c_s$  be the start configuration of  $N$  on  $w$ , and let  $c_a$  be the accept configuration.

$M$  simply runs  $\text{canyield}(c_s, c_a, 2^{df(n)})$  for some suitable constant  $d$ .

Each level of recursion uses  $O(f(n))$  additional space, and the maximum depth of recursion is  $O(\log(2^{df(n)}))$ , which is  $O(f(n))$ , for  $f(n) \geq \log n$ .

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Therefore the overall machine  $M$  simulates  $N$  in  $O(f^2(n))$  space.

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Therefore for any function  $f : \mathbb{N} \rightarrow \mathbb{N}$  with  $f(n) \geq \log n$ ,

$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n))$$

(We do not know  $f(n)$ , but can run `canyield` with increasing values to determine this.)

# Complexity Class Relations

Since  $NP \subseteq \text{NPSPACE} = \text{PSPACE}$ , we have

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$P \subseteq NP \subseteq \text{PSPACE} = \text{NPSPACE} \subseteq \text{EXPTIME}$

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We don't know which of these inclusions are strict.

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But the Time Hierarchy Theorem says that  $P \subsetneq \text{EXPTIME}$ , so at least one of the inclusions must be strict.

# PSPACE-Completeness

Language  $B$  is PSPACE-hard if  $A \leq_P B$  for all  $A \in \text{PSPACE}$ .

This is a polynomial-time reduction; to be meaningful, we want the reduction to perform a fairly small amount of computation.

$B$  is PSPACE-complete if <https://powcoder.com>

- ①  $B \in \text{PSPACE}$ , and [Add WeChat powcoder](#)
- ②  $B$  is PSPACE-hard.

A fundamental PSPACE-complete problem is concerned with the truth of Quantified Boolean Formulas (QBFs).

# Boolean Quantification

exists

We can add existential ( $\exists$ , “there exists”) and universal ( $\forall$ , “for all”) quantification to a propositional formula. For example,

$\forall x \exists y [(x \vee y) \wedge (\neg x \vee \neg y)]$

If each variable is bound by a quantifier, the formula is **closed**, or **fully quantified**.

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# Quantifier evaluation

We can expand these quantifiers thus:

$$\exists x [E] = \overbrace{E_{\text{false}} \vee E_{\text{true}}}^{\exists x [E]} \quad \forall x [E] = \overbrace{E_{\text{false}} \wedge E_{\text{true}}}^{\forall x [E]}$$

where  $E_0$  is  $E$  with  $x$  replaced by *false* and  $E_1$  is  $E$  with  $x$  replaced by *true*.

A fully quantified formula is (therefore) either *false* or *true*.

$$TQBF = \{ \langle \varphi \rangle \mid \varphi \text{ is a true closed formula} \}$$

Order matters:

- $\forall x \exists y [(x \vee y) \wedge (\neg x \vee \neg y)] = \text{true}$
- $\exists y \forall x [(x \vee y) \wedge (\neg x \vee \neg y)] = \text{false}$

# TQBF is PSPACE-Complete

Here is a linear-space decision procedure for  $TQBF$ :

- ① If  $\varphi$  has no quantifiers, evaluate it, and accept iff  $\varphi$  is true.
- ② If  $\varphi = \exists x[E]$ , evaluate  $E_0$  recursively, then  $E_1$  recursively. Reject iff both lead to rejection (false).
- ③ If  $\varphi = \forall x[E]$ , evaluate  $E_0$  recursively, then  $E_1$  recursively. Accept iff both lead to acceptance (true).

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If there are  $n$  variables, the depth of recursion is  $n$ , and each activation record only needs to hold a single variable's value.

Hence  $TQBF \in \text{PSPACE}$ .

# TQBF is PSPACE-Complete

We need to show that  $TQBF$  is PSPACE-hard.

Let  $A$  be a language decided in space  $n^k$  for some  $k$ , by TM  $M$ . We describe a polynomial-time reduction to  $TQBF$ .

We want to produce a formula  $\phi$  which is true iff  $M$  accepts  $w$ .

Importantly, the formula, as well as the time to produce it, must be polynomial in the input size.

The ideas from the proof that  $SAT$  is  $NP$ -complete help us capture invariants about tape and so on, but we cannot afford to use a family of variables per **time-step**, as that may be an exponential number.

# TQBF is PSPACE-Complete

Instead, reachability of configurations guides us, similarly to the proof of Savitch's Theorem.

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Configuration represented by a sequence of variables  $y_{jA}$ ,  $0 \leq j \leq n^k$ , that assert that position  $j$  holds symbol  $A$  (a tape symbol or a state, a clever overloading of notation). Each configuration is polynomial in size.

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Notation convenience: If  $I = \{x_1, \dots, x_m\}$  then  $\exists I$  stands for  $\exists x_1 \dots \exists x_m$ .

Similarly for  $\forall I$ .

# TQBF is PSPACE-Complete

The formula  $\varphi$  will have the form

$$\exists I_s \exists I_f [S \wedge N \wedge F]$$

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where

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- $I_s$  represents the initial (start) configuration.
- $I_f$  represents accepting (final) configuration.
- $S$  expresses that the simulation starts correctly.
- $N$  expresses that  $M$  can take  $I_s$  to  $I_f$  in less than  $i$  steps.
- $F$  expresses that the simulation finishes correctly, that is,  $I_f$  is accepting.

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# TQBF is PSPACE-Complete

$S$  is a conjunction ( $\wedge$ ) of literals in  $I_s$ .

$S$  has  $y_{jA}$  if the  $j$ th position of the initial configuration (on input  $w$ ) is  $A$ .

It has  $\neg y_{jA}$  otherwise.

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Since the machine runs in polynomial space, this is a polynomial-time reduction.

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$F$  can be written as a disjunction of the variables  $y_{jA}$ , in  $I_f$ , for which  $A$  represents not a symbol, but an accepting state.

# TQBF is PSPACE-Complete

$N$  is constructed recursively.

This recursion doubles the number of “steps” considered, each time only adding only  $O(n^k)$  symbols to the formula per level of recursion.

Pairs of configurations  $I, J$ : if  $I$  has variables  $y_{jA}$  and  $J$  has variables  $z_{jA}$ , then we let  $I = J$  stand for

$$\text{Add WeChat powcoder} \wedge \bigwedge_{j,A} (y_{jA} \leftrightarrow z_{jA})$$

We will construct formulas  $N_i(I, J)$ , for  $i = 1, 2, 4, 8, \dots$  to express that  $J$  can be reached from  $I$  in  $i$  or fewer steps (of machine  $M$ ).

# TQBF is PSPACE-Complete

$N_1(I, J)$  asserts that either  $I = J$ , or  $M$  can take  $I$  to  $J$  in one step.

This can be expressed by a polynomial-sized formula, generated (automatically) from  $M$ 's transition function  $\delta$ . For example, if

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we add the conjunct <https://powcoder.com>

$$\bigwedge_{i=0}^{n^k-1} ((y_{iq} \wedge y_{(i+1)u}) \Rightarrow (z_{iv} \wedge z_{(i+1)p}))$$

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That is, for all tape prefixes  $w$  and suffixes  $w'$ , configuration  $w q u w'$  goes to  $w v p w'$ .

(But the formula contains only a constant number of literals per position-transition combination.)

# TQBF is PSPACE-Complete

How to construct  $N_{2i}$  from  $N_i$ ? The obvious formula is

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$$N_{2i}(I, J) = \exists K [N_i(I, K) \wedge N_i(K, J)].$$

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Unfortunately, this would double the size of the formula each time we halve  $i$ . Since  $M$  might run for an exponential number of time steps, this can result in an exponential size formula, which is infeasible.

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# TQBF is PSPACE-Complete

The trick is to apply quantifiers cleverly:

This is one of the most subtle slides in this subject!

$$N_{2i}(I, J) = \exists K \forall I' \forall J' \left[ \left[ \begin{array}{c} (I' = I \wedge J' = J) \\ \wedge \\ (\neg(I' = K \wedge J' = J)) \end{array} \right] \vee N_i(I', J') \right]$$

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Recalling that  $\neg a \vee b$  is equivalent to  $a \Rightarrow b$ , the core of the quantified formula says

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- If  $(I', J') = (I, K)$  then  $N_i(I', J')$ , and
- If  $(I', J') = (K, J)$  then  $N_i(I', J')$ .

Writing  $N_{2i}$  takes only the time to write  $N_i$ , plus  $O(n^k)$  additional work.

# TQBF is PSPACE-Complete

Max number of time steps is  $2^{dn^k}$  where  $d$  is some constant. Writing  $N_{2i}$  takes the time to write  $M_i$ , plus  $O(n^k)$  additional work, so the time taken to write  $N_{2^{dn^k}}(I_s, I_f)$  is  $\log 2^{dn^k} O(n^k) = O(n^{2k})$ .

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All in all, the formula  $\exists I_s \exists I_f [S \wedge N \wedge F]$  can be constructed in polynomial time.

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The formula is true iff  $M$  accepts  $w$ .

# Textbook Questions

Questions you can try at this point:

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Exercise 8.6

Problem 8.8

Problem 8.9

Problem 8.11

Problem 8.12

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# The Complexity of Games

Interestingly, **TQBF** gives us a handle on the complexity of various games (or their winning strategies) such as (generalised) Go and many noughts-and-crosses type games.

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Such games often have a polynomial-time reduction from **TQBF**.

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# The Formula Game

In fact, we can associate a simple game with QBFs:

$$\exists x \forall y \exists z [(x \vee y) \wedge (y \vee z) \wedge (\neg y \vee \neg z)]$$

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Players  $E$  and  $A$  take turns to select values for variables.

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$E$  chooses values for the existentially quantified variables and wins if the formula (without quantifiers) evaluates to *true*.

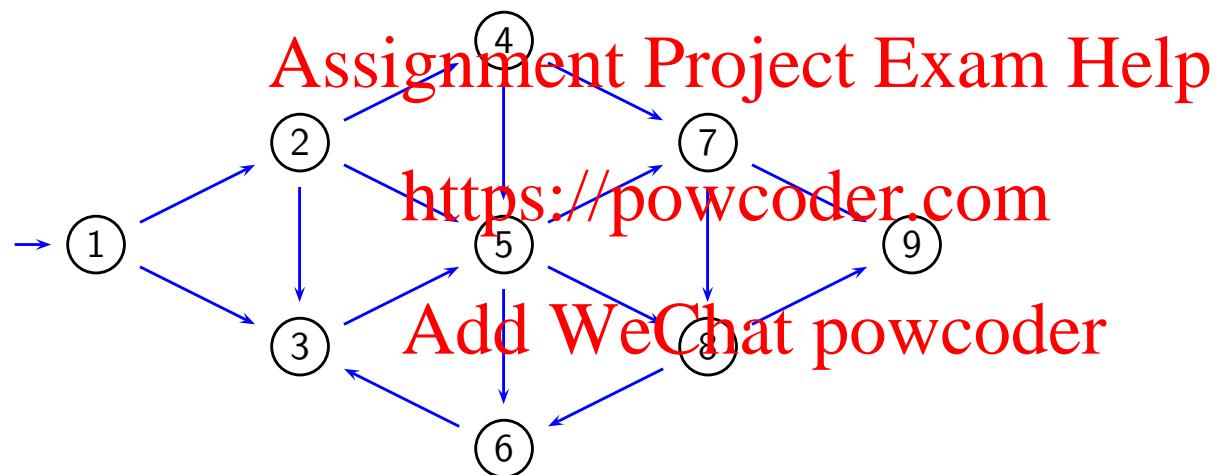
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$A$  chooses values for the universally quantified variables and wins if the formula (without quantifiers) evaluates to *false*.

This game is just *TQBF* in disguise.

# Other Games

The formula game can be reduced to a number of more realistic two-player games in PSPACE, such as Generalised Geography:



showing that deciding winning strategies for such games is PSPACE-complete.

# Textbook Questions

Questions you can try at this point:

Problem 8.10

Problem 8.14

Problem 8.15

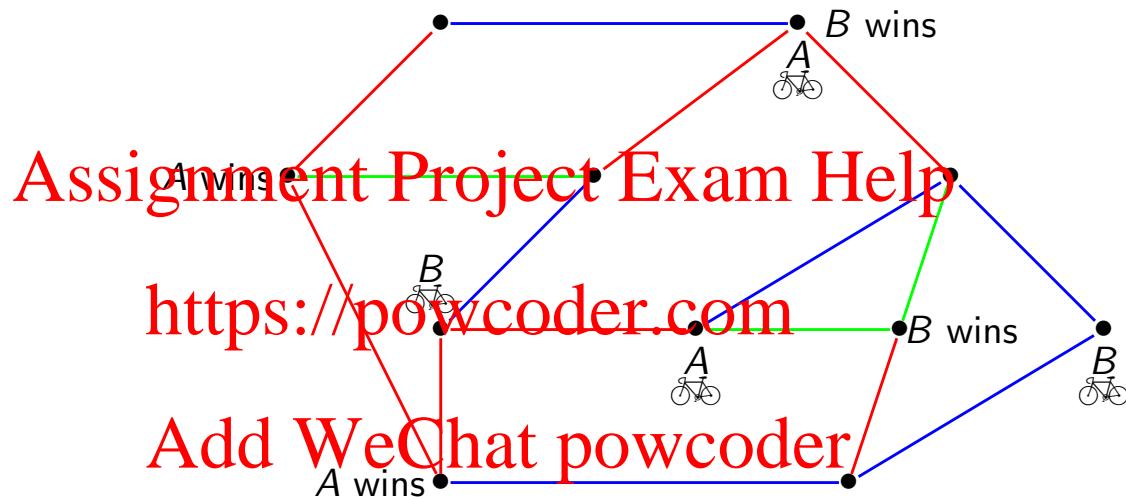
Problem 8.16

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# Provable Intractability: “Roadblock”

Is there a winning strategy for A? Not even in EXPTIME!



The rules: There is a board with towns and roads connecting towns. Each player has a fleet of bicycles, initially placed in towns, and aims to get one bicycle to some “home” town for the player. Roads have colours, and a bicycle can, in one move, travel some number of segments of the same colour, unless blocked by some other bicycle.

# Even Greater Intractability

The class 2-EXPTIME is the class of problems solvable in  $2^{2^{p(n)}}$  time, deterministically, for some polynomial  $p$ .

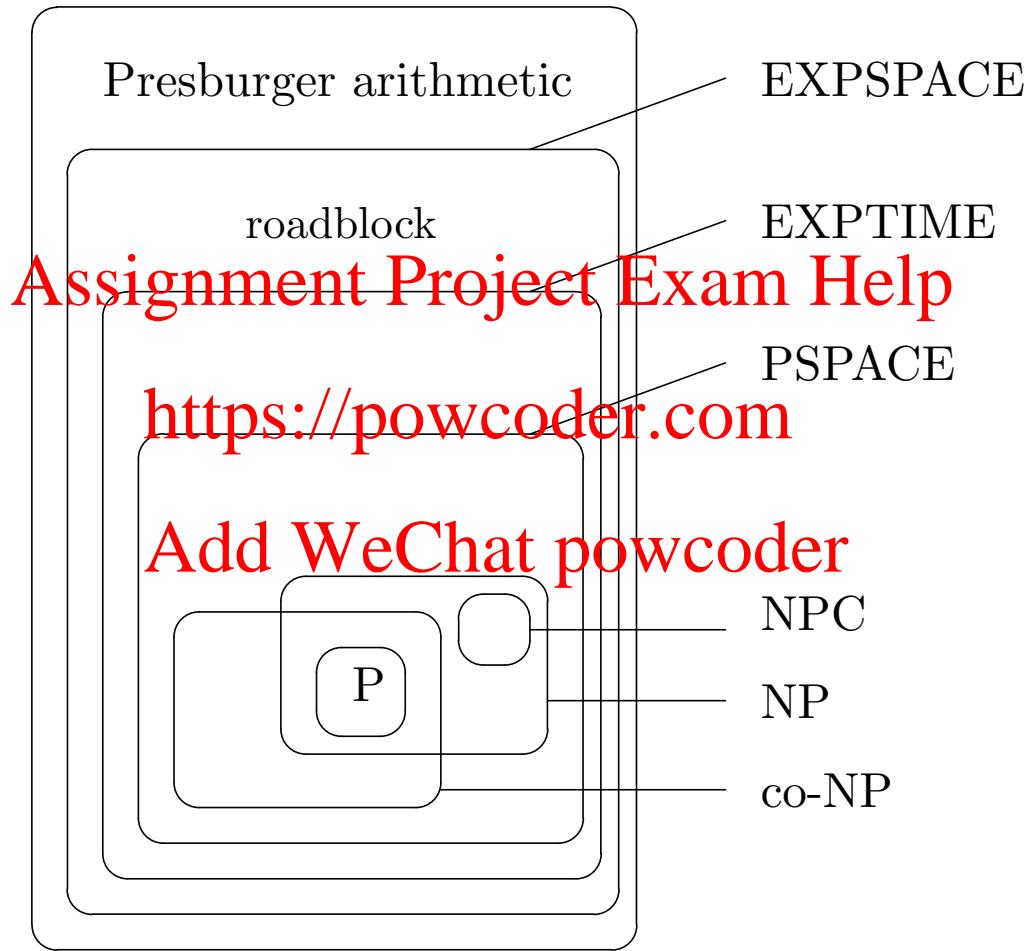
This generalises to k-EXPTIME.

The theory of natural numbers with addition and inequality (Presburger arithmetic) is decidable, but lies outside 2-EXPTIME.

Somewhat easier is the problem of real numbers with addition, which is still NEXPTIME-hard.

(Contrary to what we were told in school, integers are more difficult than reals!)

# The Bigger Picture



## Recap

Reachability of configurations helps us prove that there is only a quadratic space blowup in simulating nondeterminism, and that TQBF is PSPACE-complete.

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## Preparation

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Read through the proof that GENERALIZED GEOGRAPHY is PSPACE-complete. Add WeChat powcoder

# Sub-Linear Space

At first glance, it makes no sense to talk about a Turing machine using space that is less than linear in its input. How do we even store the input!

However, we can refine the Turing machine model by considering that the TM has an input tape that is **separate** from its working tape.

The input tape is read-only (like a CD-ROM). Does anyone know what that is any more?

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If the Turing machine also has a separate write-only output tape, we call it a **transducer**.

This separation allows us to consider space complexity in terms of **working space** only.

# The Classes L and NL

Reading “decidable in logarithmic space” to mean “in logarithmic **working** space”, we can define

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$$\begin{array}{c} L \\ \subseteq \\ \text{SPACE}(\log n) \\ \hline \text{NL} \\ \subseteq \\ \text{NSPACE}(\log n) \end{array}$$

The focus on logarithmic behaviour is again justified by robustness results:  
Logarithmic behaviour is preserved amongst under the standard variant forms of the Turing machine.

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# NL Example: PATH

We previously looked at the language

$$PATH = \left\{ \langle G, s, t \rangle \mid \begin{array}{l} G \text{ is a directed graph} \\ \text{with a path from } s \text{ to } t \end{array} \right\}$$

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and noticed that  $PATH$  is in  $P$ .

A nondeterministic TM can decide  $PATH$  in logarithmic space.

(Think of it as “guessing” a path from  $s$  to  $t$  as it goes, only storing the “current” node and “next” nodes on its working tape. It can stop after  $n$  guesses, where  $n$  is the number of nodes in  $G$ .)

So  $PATH$  is in  $NL$ , but we don’t know whether it is in  $L$ .

In fact,  $UPATH$  is in  $L$ : For this result, Reingold won best paper in STOC 2005.

# Textbook Questions

Questions you can try at this point:

Exercise 8.1

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Exercise 8.5

Exercise 8.7

Problems 8.17 – 8.25

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# Log-Space Reduction

In analogy with NP-completeness, we can try to isolate the core of “most difficult” problems in  $\mathcal{N}$ .

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We need a concept of reducibility, but polynomial-time reductions will not be very helpful.

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Suppose we reduce  $A$  to  $B$  and we have a log-space decision procedure for  $B$ . Unless the reduction itself is log-space, that does not give us a log-space decision procedure for  $A$ .

# Log-Space Reduction

$A$  is **log-space (mapping) reducible** to  $B$ ,  $A \leq_L B$ , iff there is some log-space computable function  $f: \Sigma^* \rightarrow \Sigma^*$  such that for all  $w$ ,

$w \in A$  iff  $f(w) \in B$ .

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The function  $f$  should be realised by a transducer (that writes to a write-only tape) and we say that  $f$  is a **log-space reduction** of  $A$  to  $B$ .

**Theorem:** If  $A \leq_L B$  and  $B \in L$  then  $A \in L$ .

# Log-Space Reduction

Many of the polynomial-time reductions in previous examples are in fact log-space reductions, or it is possible to find log-space replacements for them.

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In complexity theory papers, you will often see that authors prefer to give (the stronger) log-space reductions if they can, even if they are trying to establish, say, NP-completeness.

# NL-Completeness

$B$  is **NL-hard** iff every  $A \in NL$  is reducible to  $B$  in logarithmic space.

$B$  is **NL-complete** iff

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- ①  $B \in NL$ , and
- ②  $B$  is NL-hard.

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**Theorem:** If  $B \in L$  is NL-complete then  $L = NL$ .  
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**Theorem:** If  $B$  is NL-complete and  $B \leq_L C$  for  $C \in NL$  then  $C$  is NL-complete.

# NL-Completeness

It can be shown that  $PATH$  is NL-complete.

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Given NTM  $M$  that decides  $A$  in  $O(\log n)$  space and input  $w$ , we can construct  $\langle G, s, t \rangle$  in  $\log$  working space, where  $G$  is a directed graph that contains a path from  $s$  to  $t$  iff  $M$  accepts  $w$ .

Nodes = configurations, edges = transitions. Each can be calculated in  $O(\log n)$  space.

# NL and other complexity classes

Since a log-space reducer is a polynomial-time reducer, every language in  $NL$  can be reduced, in polynomial time, to  $PATH$ , which is in  $P$ .

Hence we can extend our chain of inclusions to

$$L \subseteq NL \subseteq P \subseteq NP \subseteq \text{PSPACE} = \text{NPSPACE} \subseteq \text{EXPTIME}$$

It is known that  $NL \subsetneq \text{PSPACE}$ , so in the chain

$$NL \subseteq P \subseteq NP \subseteq \text{PSPACE}$$

at least one of the inclusions is proper.

# Regular Grammar Non-Emptiness Is $NL$ -Complete

The language

$$NE_{RG} = \left\{ \langle V, \Sigma, R, S \rangle \mid \begin{array}{l} (V, \Sigma, R, S) \text{ is a regular} \\ \text{grammar with } L(G) \neq \emptyset \end{array} \right\}$$

is in  $NL$ . Namely, for each rule  $A \rightarrow w B$  in  $R$  (here  $w \in \Sigma^*$ ), we can build a directed graph with an edge from node  $A$  to node  $B$ .

The language generated by the grammar is non-empty iff there is a path from  $S$  to some node  $C$ , where  $C \rightarrow w$  is a rule in  $R$  having only terminals (alphabet symbols) in its right-hand side.

This is essentially the *PATH* problem, which, as we have seen, can be solved in nondeterministic logarithmic space.

# Regular Grammar Non-Emptiness Is $NL$ -Complete

To show  $NL$ -hardness, we reduce  $PATH$  to  $NE_{RG}$ .

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Given a  $PATH$  instance  $\langle G, S, T \rangle$  we simply construct the regular grammar that has  $S$  as its start symbol, a rule  $A \rightarrow B$  for each edge  $(A, B) \in G$ , plus the rule  $T \rightarrow \epsilon$ .

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This grammar generates  $\{\epsilon\}$ , the empty string, if  $G$  has a path from  $S$  to  $T$ , and  $\emptyset$  (that is, no strings) if no such path exists.

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Questions you can try at this point:

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Problems 8.26 – 8.34

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# NL and coNL

It follows from Savitch's theorem that  $NPSPACE$  is closed under complement, as are all the “higher” nondeterministic space classes. But we do not know whether  $L = NL$ .

The following result, known as the Immerman-Szelepcsenyi Theorem, came as a surprise in 1988:

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$NL = coNL$

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(For those interested, Sipser ends his Chapter 8 with the proof.) This result won the authors the Gödel prize in 1995.

We can conclude that the **emptiness** problem for regular grammars also is  $NL$ -complete, as is the problem of whether, given a graph  $G$  and nodes  $s$  and  $t$ , **no** path exists in  $G$  from  $s$  to  $t$ .

## Recap

The intuition for space is different from time.

Yet we can instead of polytime reductions focus on log-space reductions.

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Again, we can model computation with computations and reachability, so

NL-completeness of PATH isn't such a surprise.  
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Further reading

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The fact that  $NL = co\text{-}NL$  is unexpected: read the proof in the textbook.