COMP9334
Capacity Planning for Computer Systems and Networks

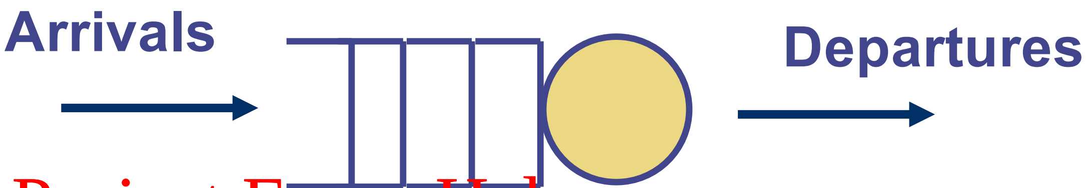
Week 4A: Non-markovian queueing models. Processor sharing.

Week 3A: Queues with Poisson arrivals (1)

- Single-server M/M/1

Exponential inter-arrivals (λ)

Exponential service time (μ)

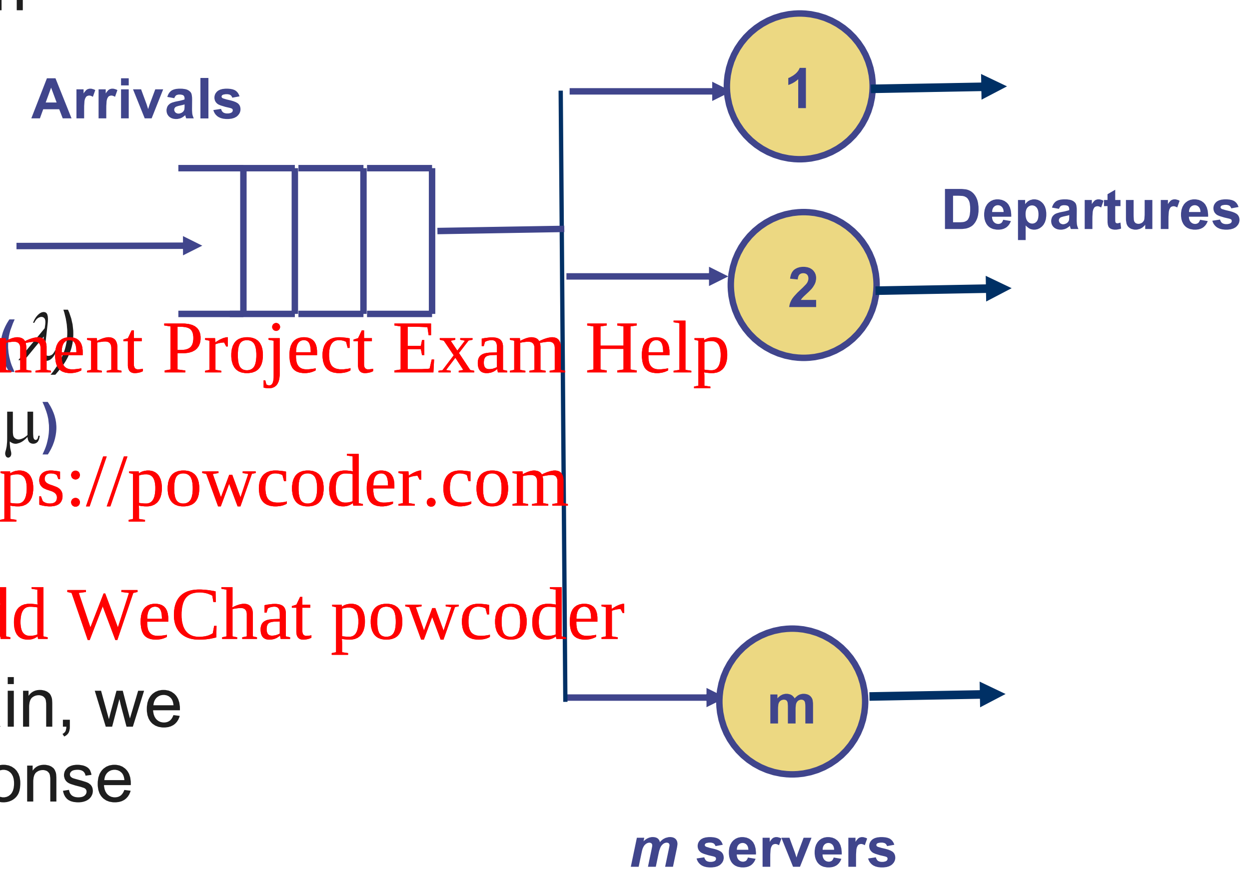


- By using a Markov chain, we can show that the mean response time is:

$$= \frac{1}{\mu - \lambda}$$

Week 3A: Queues with Poisson arrivals (2)

- Multi-server M/M/m



Exponential inter-arrivals (λ)

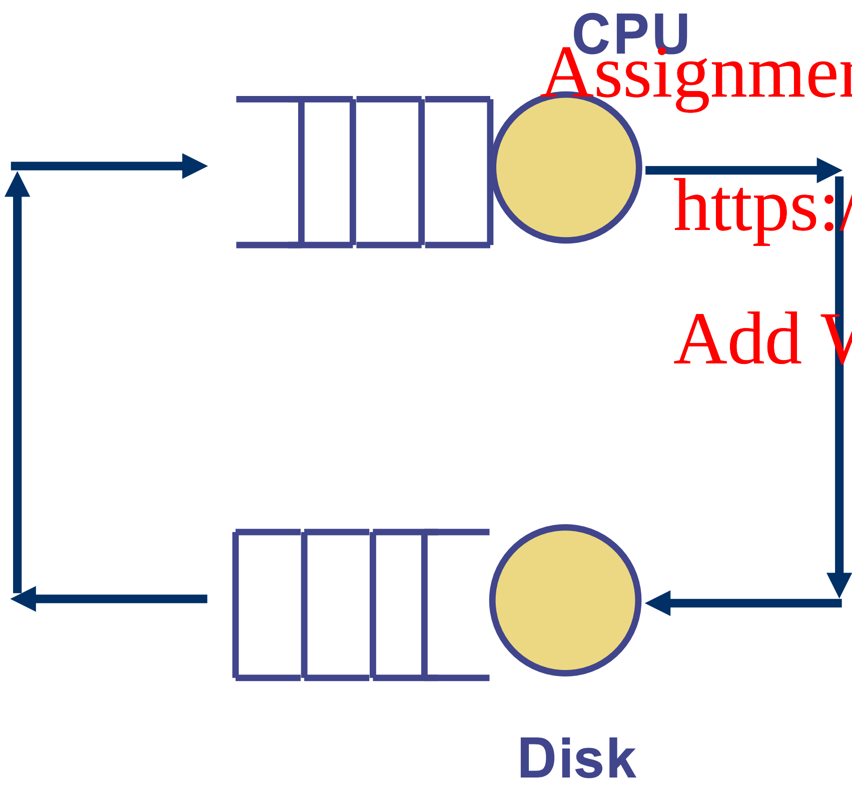
Exponential service time (μ)

- By using Markov chain, we know the mean response time is

$$T = \frac{C(\rho, m)}{m\mu(1-\rho)} + \frac{1}{\mu} \qquad \rho = \frac{\lambda}{m\mu} \qquad C(\rho, m) = \frac{\frac{(m\rho)^m}{m!}}{(1-\rho)\sum_{k=0}^{m-1}\frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!}}$$

Week 3B: Closed-queueing networks

- Analyse closed-queueing network with Markov chain
  - The transition between states is caused by an arrival or a departure according to exponential distribution

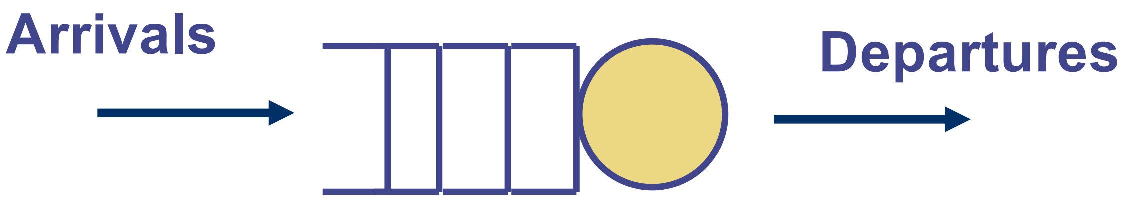


- General procedure
  - Identify the states
  - Find the state transition rates
  - Set up the balance equations
  - Solve for the steady state probabilities
  - Find the response time etc.

This lecture: Road Map

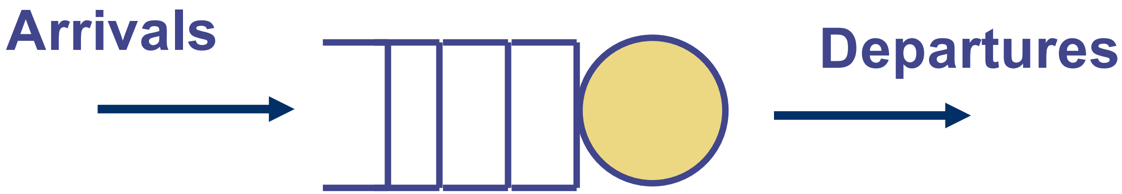
- Single-server queues
  - What if the arrival rate and/or the service rate is not exponentially distributed
- Multi-server queues
  - What if the arrival rate and/or the service rate is not exponentially distributed
- Processor sharing

General single-server queues



- Need to specify the
  - Inter-arrival time probability distribution
  - Service time probability distribution
- Independence assumptions:
  - All inter-arrival times are independent
  - All service times are independent
    - The amount of service of customer A needs is independent of the amount of time customer B needs
  - The inter-arrival time and service time are independent of each other
- Under the independence assumption, we can analyse a number of types of single server queues
  - Without the independence assumption, queueing problems are very difficult to solve!

Classification of single-server queues



- Recall Kendall's notation: "M/M/1" means
  - "M" in the 1st place means inter-arrival time is exponentially distributed
  - "M" in the 2nd place means service time probability is exponentially distributed
  - "1" in 3rd position means 1 server
- We use a "G" to denote a general probability distribution
  - Meaning any probability distribution
- Classification of single-server queues:

		Service time Distribution:	
		Exponential	General
Inter-arrival time distribution:	Exponential	M/M/1	M/G/1
	General	G/M/1	G/G/1

Example M/G/1 queue problem

- Consider an e-mailer server
- E-mails arrive at the mail server with a Poisson distribution with mean arrival rate of 1.2 messages/s
- The service time distribution of the emails are:
  - 30% of messages processed in 0.1 s, 50% in 0.3 s, 20% in 2 s
- What is
  - Average waiting time for a message?
  - Average response time for a message?
  - Average number of messages in the mail system?
- This is an M/G/1 queue problem
  - Arrival is Poisson
  - Service time is not exponential
- In order to solve an M/G/1 queue, we need to understand what the moment of a probability distribution is.

Revision: moment of a probability distribution (1)

- Consider a discrete probability distribution
  - There are *n* possible outcomes: $x_1, x_2, \ldots, x_n$
  - The probability that x_i occurs is p_i
- Example: For a fair die
  - The possible outcomes are 1,2,…, 6
  - The probability that each outcome occurs is 1/6

- The first moment (also known as the mean or expected value) is

$$E[X] = \sum_{i=1}^{n} x_i p_i$$

- For a fair die, the first moment is

= 1 * 1/6 + 2 * 1/6 + … + 6 * 1/6 = 3.5

Revision: moment of a probability distribution (2)

- The second moment of a discrete probability distribution is

$$E[X^2] = \sum_{i=1}^{n} x_i^2 p_i$$

- For a fair die, the second moment is

= 1^2 * 1/6 + 2^2 * 1/6 + … + 6^2 * 1/6

- You can prove that

  - Second moment of X = $(E[X])^2$ + Variance of X

- Note: The above definitions are for discrete probability distribution. We will look at continuous probability distribution a moment later

Solution to M/G/1 queue

- M/G/1 analysis is still tractable
- M/G/1 is no longer a Markov chain
- For a M/G/1 queue with the characteristics
  - Arrival is Poisson with rate λ
  - Service time S has
    - Mean = 1/ μ = E[S] = First moment
    - Second moment = E[S^2]
- The mean waiting time W of a M/G/1 queue is given by the Pollaczek-Khinchin (P-K) formula:

$$W = \frac{\lambda E[S^2]}{2(1-\rho)} \quad \text{where} \quad \rho = \frac{\lambda}{\mu}$$

Back to our example queueing problem (1)

- Consider an e-mailer server
- E-mails arrive at the mail server with a Poisson distribution with mean arrival rate of 1.2 messages/s
- The service time distribution of the emails are:
  - 30% of messages processed in 0.1 s, 50% in 0.3 s, 20% in 2 s

- *Exercise*: In order to find the mean waiting time using the P-K formula, we need to know
  - Mean arrival rate,
  - Mean service time, and,
  - Second moment of service time.
- Can you find them?

Back to our example queueing problem (2)

- Consider an e-mailer server
- E-mails arrive at the mail server with a Poisson distribution with mean arrival rate of 1.2 messages/s
- The service time distribution of the emails are:
  - 30% of messages processed in 0.1 s, 50% in 0.3 s, 20% in 2 s

- Solution
  - Mean arrival rate =

  - Mean service time
    =
  - Second moment of the service time
    =
- You now have everything you need to compute the mean waiting time using the P-K formula

Back to our example queueing problem (3)

- Since
  - Mean arrival rate λ = 1.2 messages/s
  - Mean service time (E[S] or 1 / μ) = 0.58s
  - Second moment of mean service time $E[S^2]$ = 0.848 s^2
- Utilisation $\rho = \lambda / \mu = \lambda E[S] = 1.2 * 0.58 = 0.696$
- Substituting these values in the P-K formula

$$W = \frac{\lambda E[S^2]}{2(1-\rho)}$$

W = 1.673s.

- How about:
  - Average response time for a message
  - Average number of messages in the mail system

Back to our example queueing problem (4)

Since the mean waiting time W = 1.673s.

The mean response time T is

T =

Average # messages in the system

Exercise: Can you use mean waiting time and Little's Law to determine the mean number of messages in the queue?

Understanding the P-K formula

- Since the Second moment of S = $E[S]^2$ + Variance of S
- We can write the P-K formula as
  - Meaning waiting time =

$$W = \frac{\lambda(E[S]^2 + \sigma_S^2)}{2(1-\rho)}$$

- Smaller variance in service time ➔ smaller waiting time
- M/D/1 is a special case of M/G/1
  - "D" stands for deterministic: Constant service time E[S] and Variance of S = 0
  - For the same value of ρ and E[S], deterministic has the smallest mean response time

Moments for continuous probability density

- Exponential function is a continuous probability density
- If a random variable X has continuous probability density function f(x), then its
  - first moment (= mean, expected value) E[X] and
  - second moment E[X^2]

  are given by

$$E[X] = \int x f(x) dx$$

$$E[X^2] = \int x^2 f(x) dx$$

- If the service time S is exponential with rate μ, then
  - E[S] = 1/ μ
  - E[S^2] = 2 / μ^2

M/M/1 as a special case of M/G/1

- Let us apply the result of the M/G/1 queue to exponential service time
  - Let us put $E[S] = 1/\mu$ and $E[S^2] = 2/\mu^2$ in the P-K formula:

$$W = \frac{\lambda E[S^2]}{2(1-\rho)}$$

- We get

$$W = \frac{\rho}{\mu(1-\rho)}$$

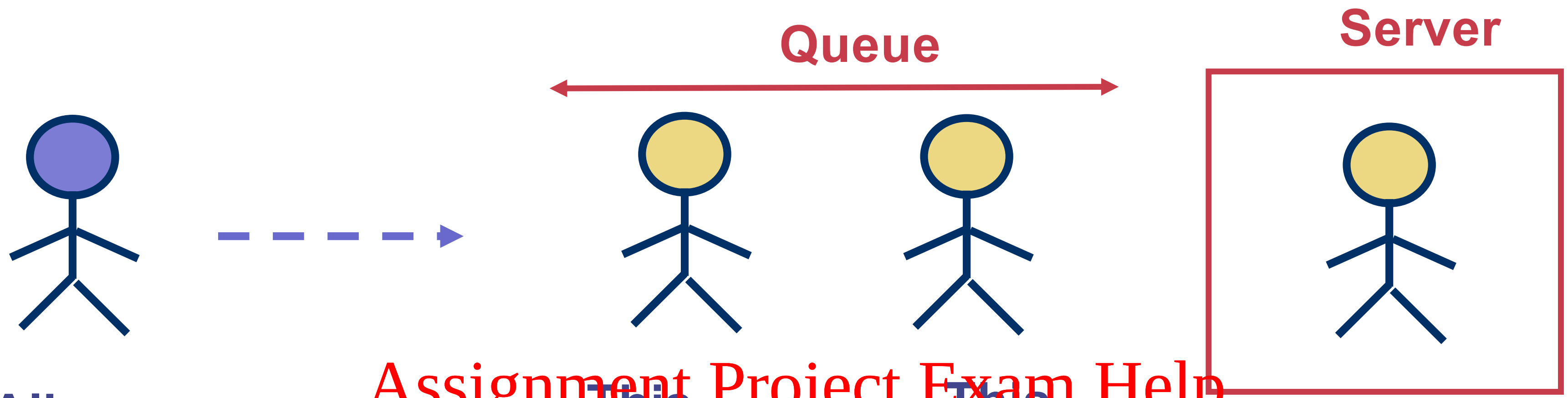
- Which is the same as the M/M/1 queue waiting time formula that we derive in Week 3A

Remark on M/G/1

$$W = \frac{\lambda E[S^2]}{2(1-\rho)}$$

- $\rho \to 1$, $W \to \infty$

Deriving the P-K formula (1)



Alice: An arriving customer

This customer needs 5 min

This customer needs 6 min

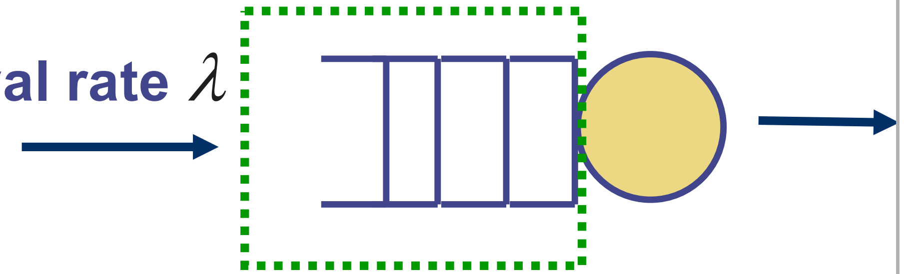
This customer still needs 3 min when Alice joins the queue

Residual Service Time

- How long does Alice (the arriving customer) need to wait before she gets served?

Deriving the P-K formula (2)

- Let
  - W = Mean waiting time
  - N = Mean number of customers in the queue
  - 1/ μ = Mean service time
  - R = Mean residual service time
- We can prove that
  - W = N * (1/ μ) + R



- Applying Little's Law to the queue
  - N = λ W

Substitution

$$W = \lambda \times W \times \frac{1}{\mu} + R \Rightarrow W = \frac{R}{1-\rho}$$

where $\rho = \frac{\lambda}{\mu}$

Deriving P-K formula (3)

- We have just showed that the mean waiting time in a M/G/1 queue is

$$W = \frac{R}{1-\rho}$$

- The P-K formula says

$$W = \frac{\lambda E[S^2]}{2(1-\rho)}$$

- We can prove the P-K formula if we can show that the mean residual time R is

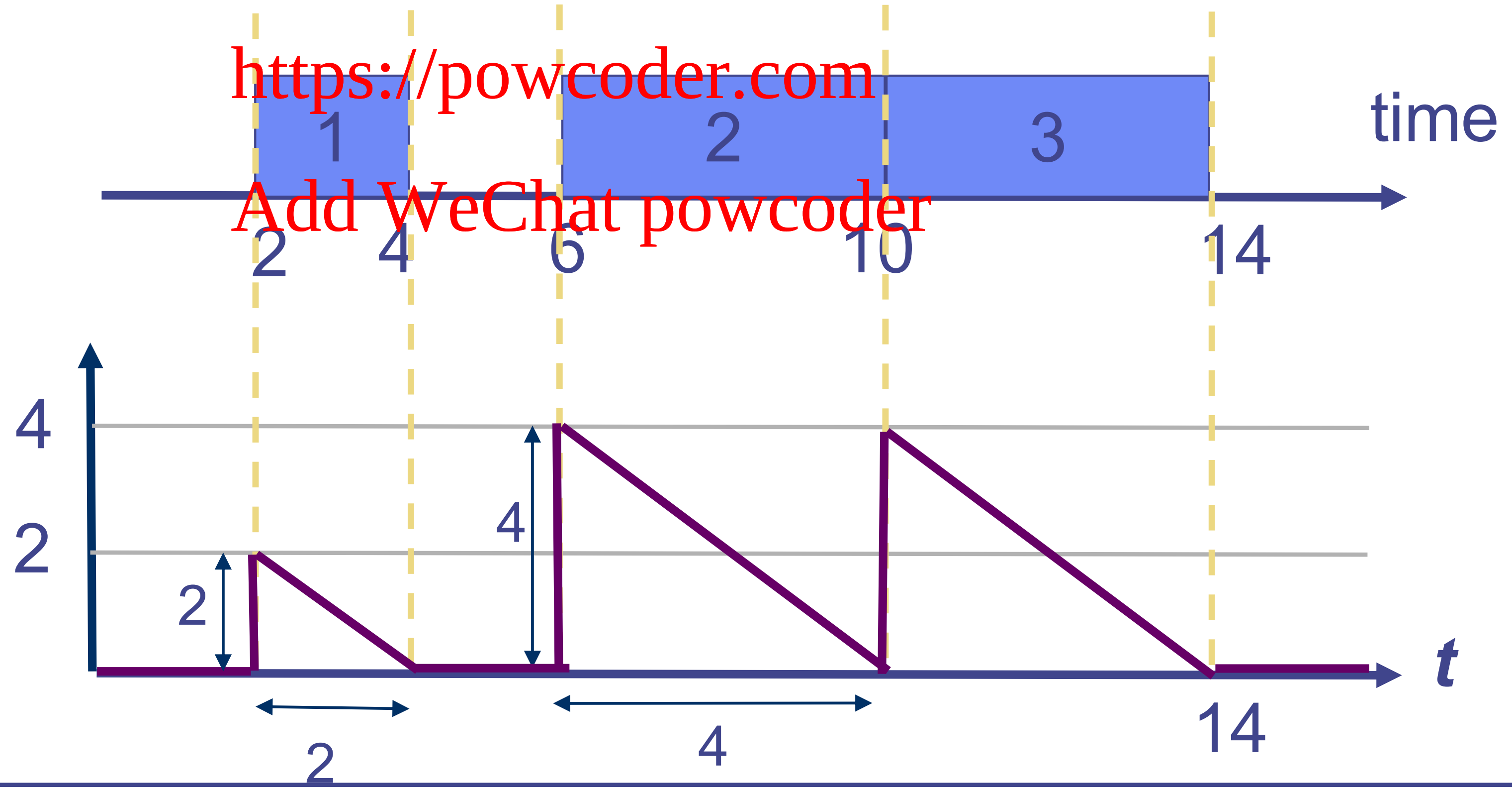
$$R = \frac{1}{2}\lambda E[S^2]$$

How residual service time changes over time?

Job index	Arrival time	Processing time required
1	2	2
2	6	4
3	8	4

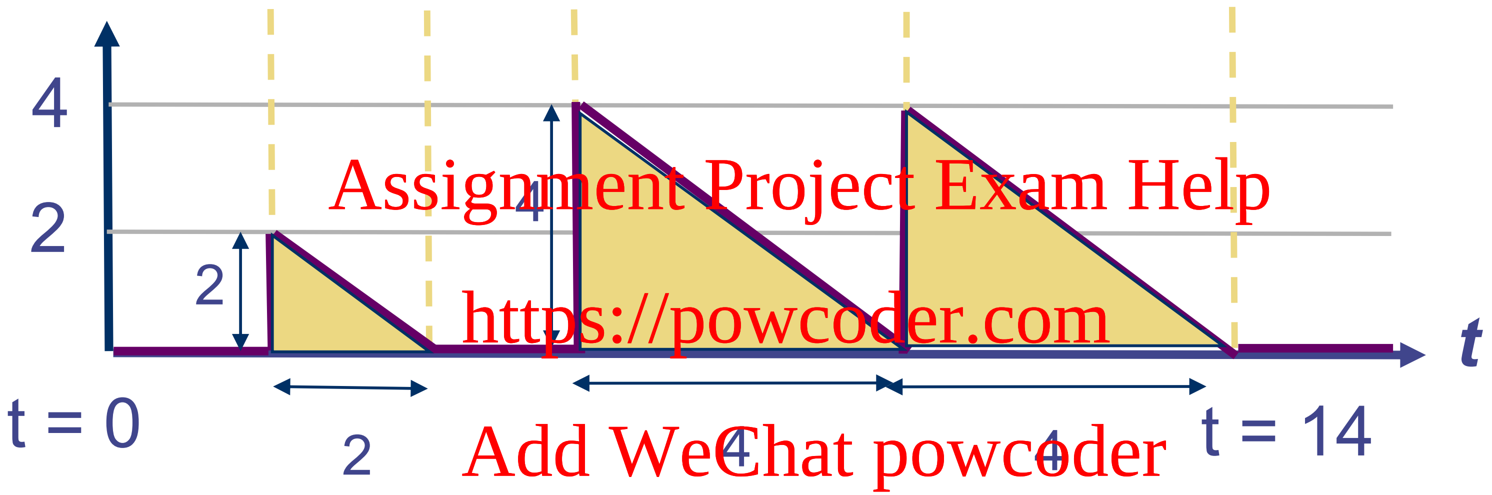
Time when each job is being served:

Residual service time seen by a customer arriving at time *t*



What is the mean residual time …

Residual service time seen by a customer arriving at time *t*



Mean residual time seen by an arriving customer over time [0,14]

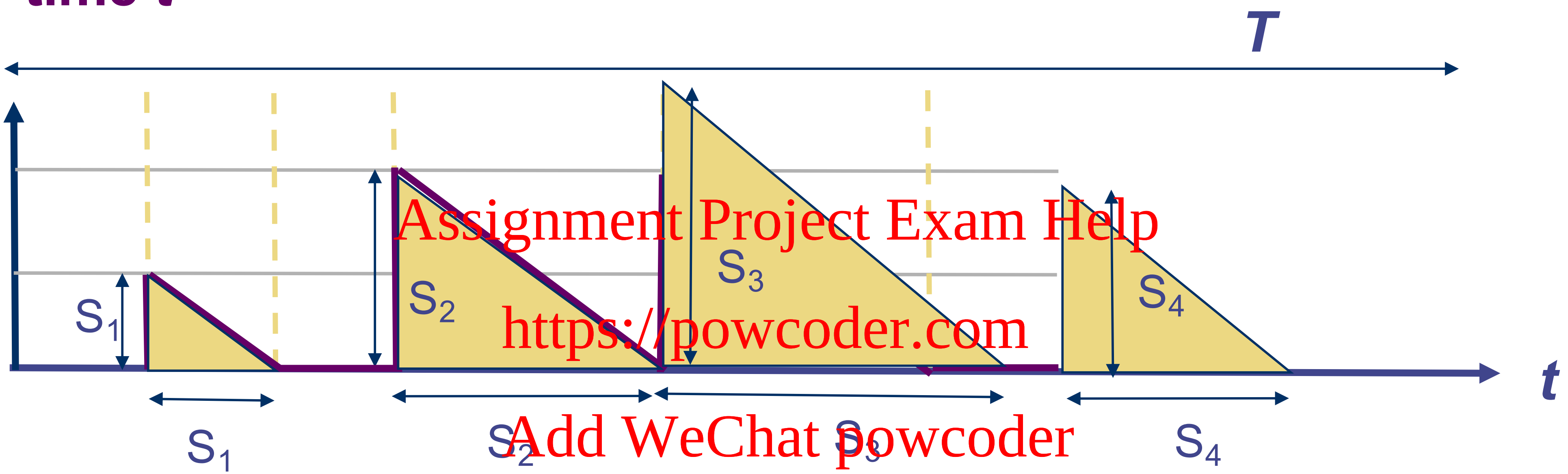
$$= \frac{\text{Area under the curve over } [0,14]}{14}$$

Service time!

$$= \frac{\frac{1}{2} \times 2^2 + \frac{1}{2} \times 4^2 + \frac{1}{2} \times 4^2}{14}$$

In general

Residual service time seen by a customer arriving at time *t*



Assuming M jobs are completed in time T

Mean residual time

$$= \frac{\sum_{i=1}^{M} \frac{1}{2} S_i^2}{T} = \frac{1}{2} \frac{\sum_{i=1}^{M} S_i^2}{M} \frac{M}{T} = \frac{1}{2} E[S^2] \lambda$$

The P-K formula

- Thus, the mean residual time R is

$$R = \frac{1}{2}\lambda E[S^2]$$

- By substituting this into $W = \frac{R}{1-\rho}$

- We get the P-K formula

- This derivation also shows that the waiting time is proportional to the residual service time
- The residual service time is proportional to the 2nd moment of service time

G/G/1 queue

- G/G/1 queue are harder to analyse
- Generally, we cannot find an explicit formula for the the waiting time or response time for a G/G/1 queue
- Results on G/G/1 queue include
  - Approximation results
  - Bounds on waiting time

Approximate G/G/1 waiting time

- There are many different methods to find the approximate waiting time for a G/G/1 queue
- Most of the approximation works well when the traffic is heavy, i.e. when the utilisation ρ is high
- Let
  - Mean arrival rate = λ
  - Variance of inter-arrival time = σ_a^2
  - Service time S has mean $1/\mu = E[S]$
  - Variance of service time = σ_s^2
- The approximate waiting time for a G/G/1 queue is

$$W \approx \frac{\lambda^2(\sigma_a^2 + \sigma_s^2)}{1 + \lambda^2 \sigma_s^2} \frac{\lambda(E[S]^2 + \sigma_s^2)}{2(1-\rho)} \quad \text{where} \quad \rho = \frac{\lambda}{\mu}$$

- Note: $\rho \to 1$, $W \to \infty$
- Large variance means large waiting time

Bounds for G/G/1 waiting time

- Let
  - Mean arrival rate = λ
  - Variance of inter-arrival time = σ_a^2
  - Service time S has mean $1/\mu$ = E[S]
  - Variance of service time = σ_s^2
- A bound for the waiting time for a G/G/1 queue is

$$W \leq \frac{\lambda(\sigma_a^2 + \sigma_s^2)}{2(1-\rho)}$$

- Note that the bound suggests that large variance means large waiting time

Approximation for G/G/m queue

- Only approximate waiting time available for G/G/m
- The waiting time is

$$W_{G/G/m} = W_{M/M/m} \frac{C_a^2 + C_s^2}{2}$$

where $W_{M/M/m} = \text{Waiting time of M/M/m queue}$

$C_a = \text{Coeff of variation of inter-arrival time}$

$C_b = \text{Coeff of variation of service time}$

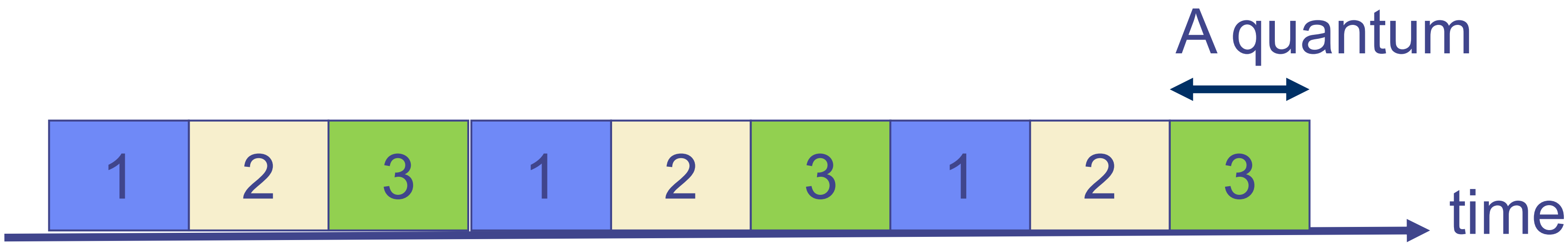
- Coefficient of variation of a random variable *X*

= Standard deviation of *X* / mean of *X*

Note: Variance in arrival or service time increases queueing

Processor sharing (PS)

- We have so far assumed that the processor performs work on a first-come-first-serve basis
- However, this is not how CPUs perform tasks
- Consider an example: a CPU has a job queue with three tasks called Tasks 1, 2 and 3 in it
  - CPU works on Task 1 for a certain amount of time (called a quantum) and then returns the task to the job queue if it is not yet finished
  - CPU works on Task 2 for a quantum and returns the task to the job queue if it is not yet finished
  - CPU works on Task 3 for a quantum and returns the task to the job queue if it is not yet finished



Modelling processor sharing

- We assume the context switching time is negligible
- We assume the quantum is small compared with the length of the task, we can think about continuous processing instead of discrete processing
- In a duration of time when there are n jobs in the job queue, each job receives 1/n of the service

PS: Example 1

- Example 1:
  - At time 0, there are 2 jobs in the job queue
  - Job 1 still needs 5 seconds of service
  - Job 2 still needs 3 seconds of service

  - Assuming no more jobs will arrive, determine the time at which the jobs will be completed

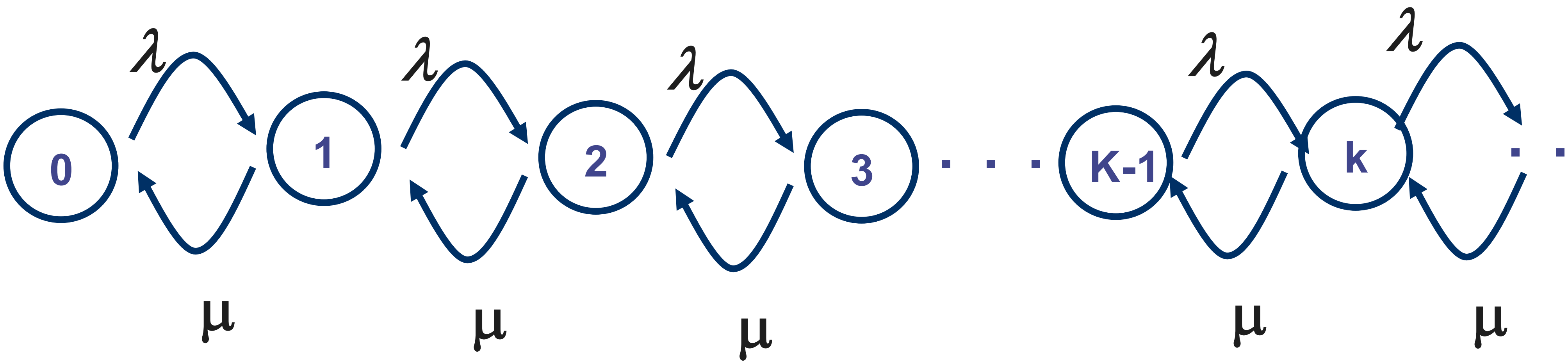
PS: Example 2

- Example 2:
  - At time 0, there are 2 jobs in the job queue
    - Job 1 still needs 5 seconds of service
    - Job 2 still needs 3 seconds of service
  - Job 3 arrives at time = 1 second and requires 4 seconds of service
  - Job 4 arrives at time = 2 second and requires 1 second of service
  - No more jobs will arrive after Job 4
  - Questions:
    - Without computing the finished times for Jobs 1 and 3, are you able to tell which of these two jobs will finish first?
    - Determine the time at which the jobs will be completed

M/M/1/PS queues

- Jobs arrive according to Poisson distribution
- Exponential service time
- One processor using processor sharing

- State n = there are n jobs in the job queue
- State diagram: same as M/M/1 queue and there is a reason for that



Summary

- We have studied a few types of non-Markovian queues
  - M/G/1, G/G/1, G/G/m
- Key method to derive the M/G/1 waiting time is via the *residual service time*
- Processor sharing (PS)

References

- Recommended reading
  - Bertsekas and Gallager, "Data Networks"
    - Section 3.5 for M/G/1 queue
    - The result on G/G/1 bound is taken from Section 3.5.4
  - Processing sharing
    - Harchol-Balter Section 22.2.2