NUMERICAL OPTIMISATION
ASSIGNMENT 1: SOLUTION

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EXERCISE 1.

(a) Function and its contours.

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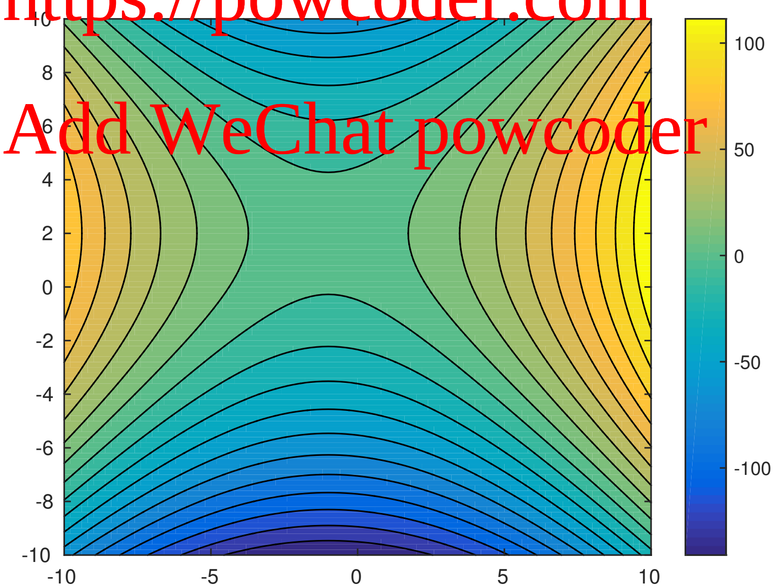
n = 100;
x = linspace(-1,1,n+1); y = x;
[X,Y] = meshgrid(x,y);

% Plot the function of two variables
alpha = 10;
figure; contourf(alpha*X, alpha*Y, f(alpha*X,alpha*Y), 20)
colorbar();
saveas(gcf, 'contours', 'epsc');
```

(b)

$$f(x,y) = 2x + 4y + x^2 - 2y^2 = (x+1)^2 - 2(y-1)^2 + 1$$

For $c \in \mathbb{R}$ we have the contour sets

$$S_c = \{(x,y) : (x+1)^2 - 2(y-1)^2 + 1 = c\}.$$

For $c = 1$ we have the point

$$S_1 = \{(-1,1)\}.$$

Finally, for $c \neq 1$ we have that the contour set is a hyperbola that can be represented implicitly by

$$S_c = \{(x, y) : \frac{(x+1)^2}{c-1} - \frac{(y-1)^2}{\frac{1}{2}(c-1)} = 1\}.$$

(c) Gradient.

$$\frac{\partial f}{\partial x} = 2x + 2, \qquad \frac{\partial f}{\partial y} = -4y + 4$$

The only stationary point is $x_0 = (-1, 1)$. To classify it we consider the Hessian at this point.

$$\nabla^2 f(x_0) = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$$

Since the eigenvalues of the Hessian are $\{2, -4\}$, we have a saddle point.

EXERCISE 2.

(a) We need to show that the eigenvalues of A are non-negative. Consider an eigenvalue λ. The eigenvalue problem has the form

$$Ax = \lambda x, \quad x \neq 0,$$

with λ the eigenvalue and $x \neq 0$ the corresponding eigenvector. We left-multiply each side of this equation by x^T, to obtain

$$x^T Ax = \lambda x^T x.$$

Dividing by $x^T x$ we obtain the Rayleigh quotient expression for the eigenvalues

$$\lambda = \frac{x^T Ax}{x^T x}.$$

Plugging in $A = B^T B$ we obtain

$$\lambda = \frac{x^T B^T Bx}{x^T x} = \frac{||Bx||^2}{||x||^2} \geq 0$$

which implies that all eigenvalues are non-negative and hence A is positive semidefinite.

(b) We would like to show that

$$f(y + \alpha(x - y)) - \alpha f(x) - (1 - \alpha) f(y) \leq 0$$

We have $f(x) = x^T Qx$ for Q symmetric positive semidefinite:

$$\begin{aligned}
&(y + \alpha(x-y))^T Q(y + \alpha(x-y)) - \alpha x^T Qx - (1-\alpha) y^T Qy = \\
&= y^T Qy - (1-\alpha) y^T Qy - \alpha x^T Qx + 2\alpha x^T Qy - 2\alpha y^T Qy - \alpha^2 x^T Qx - \alpha^2 x^T Qy + \alpha^2 y^T Qy = \\
&= y^T Qy \alpha(\alpha - 1) + x^T Qx \alpha(\alpha - 1) + 2\alpha x^T Qy (1 - \alpha) = \\
&= \alpha(\alpha-1)[y^T Qy + x^T Qx - 2x^T Qy] = \\
&= \alpha(\alpha-1)(x-y)^T Q(x-y).
\end{aligned}$$

Now, since $\alpha \in [0, 1]$ and Q is positive semidefinite, we have

$$\underbrace{\alpha(\alpha-1)}_{\leq 0} \underbrace{(x-y)^T Q(x-y)}_{\geq 0} \leq 0.$$