A Karush-Kuhn-Tucker Example

It's only for very simple problems that we can use the Karush-Kuhn-Tucker conditions to solve a nonlinear programming problem. Consider the following problem:

$$\begin{array}{ll} \text{maximize} & f(x,y) = xy \\ \text{subject to} & x + y^2 \leq 2 \\ & x, y \geq 0 \end{array}$$

Note that the feasible region is bounded, so a global maximum must exist: a continuous function on a closed and bounded set has a maximum there.

We write the constraints as $g_1(x,y) = x + y^2 \leq 2$, $g_2(x,y) = -x \leq 0$, $g_3(x,y) = -y \leq 0$. Thus the KKT conditions can be written as

$$\begin{aligned} y - \lambda_1 + \lambda_2 &= 0 \\ x - 2y\lambda_1 + \lambda_3 &= 0 \\ \lambda_1(2 - x - y^2) &= 0 \\ \lambda_2 x &= 0 \\ \lambda_3 y &= 0 \\ x + y^2 &\leq 2 \\ x, y, \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

In each of the "complementary slackness" equations $\lambda_i(b_i - g_i(x_1, \ldots, x_n)) = 0$, at least one of the two factors must be 0. With n such conditions, there would potentially be 2^n possible cases to consider. However, with some thought we might be able to reduce that considerably.

Case 1: Suppose $\lambda_1 = 0$. Then the first KKT condition says $y + \lambda_2 = 0$ and the second says $x + \lambda_3 = 0$. Since each term is nonnegative, the only way that can happen is if $x = y = \lambda_2 = \lambda_3 = 0$. Indeed, the KKT conditions are satisfied when $x = y = \lambda_1 = \lambda_2 = \lambda_3 = 0$ (although clearly this is not a local maximum since $f(0,0) = 0$ while $f(x,y) > 0$ at points in the interior of the feasible region).

Case 2: Suppose $x + y^2 = 2$. Now at least one of $x = 2 - y^2$ and y must be positive.

Case 2a: Suppose $x > 0$. Then $\lambda_2 = 0$. The first KKT condition says $\lambda_1 = y$. The second KKT condition then says $x - 2y\lambda_1 + \lambda_3 = 2 - 3y^2 + \lambda_3 = 0$, so $3y^2 = 2 + \lambda_3 > 0$, and $\lambda_3 = 0$. Thus $y = \sqrt{2/3}$, and $x = 2 - 2/3 = 4/3$. Again all the KKT conditions are satisfied.

Case 2b: Suppose $x = 0$, i.e. $y = \sqrt{2}$. Since $y > 0$ we have $\lambda_3 = 0$. From the second KKT condition we must have $\lambda_1 = 0$. But that takes us back to Case 1.

We conclude there are only two candidates for a local max: $(0,0)$ and $(4/3, \sqrt{2/3})$. The global maximum is at $(4/3, \sqrt{2/3})$.