NUMERICAL OPTIMISATION
ASSIGNMENT 8

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EXERCISE 1

Consider a problem to minimise the function

$$\min_x f(x) = \frac{1}{2}x^T G x + c^T x$$

subject to the constraint

$$Ax \le b,$$

where $G \in \mathbb{R}^{n\times n}$ symmetric positive semidefinite, $A \in \mathbb{R}^{m\times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m$.

(a) State the KKT conditions for this problem.
We can rewrite $Ax \le b$ as $Ax - b \le 0$. The Lagrangian for this problem is

$$\mathcal{L}(x,\lambda,\nu) = \frac{1}{2}x^T G x + c^T x + \lambda^T (Ax - b),$$

therefore, the KKT conditions can be written as

$$\begin{aligned} \nabla_x \mathcal{L}(x,\lambda,\nu) = Gx + c + A^T\lambda = 0, \\ Ax - b \le 0, \\ \lambda \ge 0, \\ \lambda_i \cdot [Ax - b]_i = 0 \quad i = 1\ldots m. \end{aligned}$$

(b) Rewrite the constraint using a vector of slack variables $y \in \mathbb{R}^m, y \ge 0$ and give the corresponding KKT conditions.
We set $Ax - b + y = 0$ and $y \ge 0$. The Lagrangian for this problem is

$$\mathcal{L}(x,\lambda,\nu) = \frac{1}{2}x^T G x + c^T x - \lambda^T y + \nu^T (Ax - b + y)$$

and the KKT conditions can be expressed as

$$\begin{aligned} \nabla_x \mathcal{L}(x,\lambda,\nu) = Gx + c + A^T\nu = 0, \\ \nabla_y \mathcal{L}(x,\lambda,\nu) = \lambda - \nu = 0 \Rightarrow \lambda = \nu, \\ Ax - b + y = 0, \\ y \ge 0, \\ \lambda \ge 0, \\ \lambda_i \cdot y_i = 0 \quad i = 1\ldots m. \end{aligned}$$

(c) Formulate the dual to the problems in (b) and discuss its properties.
Since the gradient of the Lagrangian for (b) is

$$\nabla_x \mathcal{L}(x,\lambda,\nu) = Gx + c + A^T\nu,$$

and $\lambda = \nu$ (according to the KKT conditions) we find the dual for problem (a) with Lagrangian

$$\nabla_x \mathcal{L}(x, \lambda, \nu) = Gx + c + A^T\lambda.$$

This function has a unique zero that corresponds to the minimum of the quadratic form

$$x^* = -G^{-1}(A^T\lambda + c).$$

We substitute it into the Lagrangian to obtain the dual problem:

$$\begin{aligned}
\mathcal{L}(x^*, \lambda, \nu) &= \frac{1}{2}[-G^{-1}(A^T\lambda + c)]^T G[-G^{-1}(A^T\lambda + c)] + c^T(-G^{-1}(A^T\lambda + c)) + \lambda^T[A(-G^{-1}(A^T\lambda + c)) - b] = \\
&= \frac{1}{2}(A^T\lambda + c)^T G^{-1}(A^T\lambda + c) - (A^T\lambda + c)^T G^{-1}(A^T\lambda + c) - \lambda^T b = \\
&= -\frac{1}{2}(A^T\lambda + c)^T G^{-1}(A^T\lambda + c) - \lambda^T b
\end{aligned}$$

Therefore the dual problem is

$$\max_{\lambda} -\frac{1}{2}(A^T\lambda + c)^T G^{-1}(A^T\lambda + c) - \lambda^T b$$
$$\text{subject to } \lambda \geq 0$$

The Lagrangian for (b) is the same as for (a).

EXERCISE 2

Solve the following constraint minimisation problem:

$$\min_{(x,y)} f(x, y) = (x - 2y)^2 + (x - 2)^2, \quad x - y = 4$$

(a) Formulate the KKT system.
This problem can be reformulated as

$$\min_{(x,y)} f(x, y) = \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 4$$
$$\text{subject to } \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 4 = 0$$

Therefore, the Lagrangian is

$$\mathcal{L}(x, \lambda, \nu) = \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 4 + \nu\left(\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 4\right),$$

and the KKT conditions for this problem are

$$\nabla_x \mathcal{L}(x, \lambda, \nu) = \begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \end{pmatrix} + \nu\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$$
$$\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 4 = 0$$

(b) Solve the KKT system with a method of your choice. Explain briefly your results We have a problem of the form

$$\min_x \frac{1}{2}x^T P x + q^T x + r$$

subject to $Ax = b$. Its unique solution satisfies

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

In our problem

$$P = \begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix}, \quad q = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad b = 4$$

Then we have

$$\begin{pmatrix} 4 & -4 & 1 \\ -4 & 8 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x^* \\ \nu^* \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix}.$$

Solving for x^*, ν^* we obtain

$$x^* = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \quad \nu^* = -12.$$