COMS 4236: Introduction to Computational Complexity, Spring 2018

Problem Set 2, due Thursday February 22, 11:59pm on Courseworks

Please follow the homework submission guidelines posted on Courseworks.

Problem 1. [20 points]

a. Let $f(n) \geq n$ be any nondecreasing function. Show that NTIME($f(n)$) is closed under union, intersection and concatenation. That is, if L_1, L_2 are two languages in NTIME($f(n)$) then $L_1 \cup L_2, L_1 \cap L_2$ and $L_1 \cdot L_2$ are also in NTIME($f(n)$). Recall that the concatenation of two languages L_1, L_2 is $L_1 \cdot L_2 = \{xy \mid x \in L_1, y \in L_2\}$, i.e. the set of strings that can be written as the concatenation of a string in L_1 and a string in L_2.

b. The *Kleene star* of a language L is defined to be the set of all strings that can be written as the concatenation of any number (0, 1, or more) strings of L, i.e. $L^* = \{x_1 \cdots x_k \mid k \geq 0 \text{ and } x_1 \in L, \cdots, x_k \in L\}$. For example, if L={a,ab,ba}, then ε (the empty string), aba and ababa are in L^* but abbba is not.

Show that NTIME(n) is closed under Kleene star, i.e. if $L \in$NTIME(n) then also $L^* \in$ NTIME(n).

Problem 2. [15 points] Use the theorems that we learned on the relations between complexity classes to prove the following.

a. TIME(2^n) $\subsetneq$ TIME(2^{2n})

b. NTIME(n^2) $\subsetneq$ SPACE(n^3)

(*Note*: $\subsetneq$ means that the left hand side is *contained* but *is not equal* to the right hand side. Make sure to show both facts.)

Problem 3. [15 points]

a. An undirected graph G=(N,E) is called *3-colorable* if its set N of nodes can be partitioned into 3 disjoint subsets N_1, N_2, N_3 (i.e. $N_1 \cup N_2 \cup N_3$ = N) such that every edge connects nodes in different subsets.

Show that the language 3-COLORABLE= { G | G is a 3-colorable graph } is in NP.

You can assume that a graph G is encoded as a string that gives its number n of nodes (in binary), followed by a list of its edges, i.e. a sequence of pairs (u,v) of nodes, where each node is a number (in binary) in $\{1,\ldots,n\}$.

b. An undirected graph G=(N,E) is called *bipartite* (or *2-colorable*) if its set N of nodes can be partitioned into 2 disjoint subsets N_1, N_2 (i.e. $N_1 \cup N_2$ = N) such that every edge

connects nodes in different subsets. An equivalent characterization is given by the following property: A graph is bipartite if and only if it does not contain a cycle with an odd number of nodes. (You do not have to show this fact).
Show that the language BIPARTITE = { G | G is a bipartite graph } is in NL.

Problem 4. [15 points] Do Problem 7.4.4, parts (a,b,f) in the book. We reproduce the problem below for convenience.

7.4.4 Problem: Let C be a *class of functions* from nonnegative integers to nonnegative integers. We say that C is closed under *left polynomial composition* if $f(n) \in C$ implies $p(f(n)) = \mathcal{O}(g(n))$ for some $g(n) \in C$, for all polynomials $p(n)$. We say that C is closed under *right polynomial composition* if $f(n) \in C$ implies $f(p(n)) = \mathcal{O}(g(n))$ for some $g(n) \in C$, for all polynomials $p(n)$.

Intuitively, the first closure property implies that the corresponding complexity class is "computational model-independent," that is, it is robust under reasonable changes in the underlying model of computation (from RAM's to Turing machines, to multistring Turing machines, etc.) while closure under right polynomial composition suggests closure under *reductions* (see the next chapter).

Which of the following classes of functions are closed under left polynomial composition, and which under right polynomial composition?
(a) $\{n^k : k > 0\}$.
(b) $\{k \cdot n : k > 0\}$.

(f) $\{\log n\}$.

Problem 5. [20 points] Let A be an alphabet, # a symbol not in A, and N the set of natural numbers. A *padding function* is a function *pad*: $A^* \times N \to (A \cup \{\#\})^*$ defined as $pad(w,l) = w\#^j$, where j=max(0, l-|w|); that is, if the length of a string w is less than l then we pad it with enough #'s so that the resulting string has length l. For any language $L \subseteq A^*$ and function $f : N \to N$ define the language $pad(L,f(m)) = \{\, pad(w,f(|w|)) \mid w \in L \,\}$.

a. Prove that a language L is in TIME(n^2) if and only if $pad(L, m^2)$ is in TIME(n).

b. Argue similarly that L is in NTIME(n^2) if and only if $pad(L, m^2)$ is in NTIME(n).

c. Prove that if TIME(n) = NTIME(n) then TIME(n^2) = NTIME(n^2).

(*Note:* This is an instance of a general translation property: Equality of complexity classes translates upwards to higher resource bounds. You do not have to prove this.)

Problem 6. [15 points] Show that P ≠ SPACE(n).
(*Hint*: Assume P= SPACE(n). Use the space hierarchy theorem and a padding function to get a contradiction.)
Note: We do not know how these two classes, P and SPACE(n), compare, i.e. if either one contains the other (we believe neither does), but we know they are not equal.