**COMS 4236: Introduction to Computational Complexity, Spring 2018**

**Problem Set 3, due Thursday March 8, 11:59pm on Courseworks**

**Please follow the homework submission guidelines posted on Courseworks.**

**Problem 1.** [15 points]
a. Consider the following problem, called CIRCUIT-SAT-20.
Input: A Boolean circuit C with 20 input variables.
Question: Is C satisfiable, i.e. does there exist an assignment to the input variables of C for which the value of C is 1?

Show that CIRCUIT-SAT-20 is in P.

b. Suppose that *A, B, C* are three languages such that (1) $A \leq_p B$, (2) $B \leq_p C$, (3) *A* is NP-complete, and (4) *C* is NP-complete. Show that *B* is also NP-complete.



**Problem 2**. [20 points]
a. Show that if a language *L* is complete for a class C under log-space or polynomial time reductions then its complement $\bar{L}$ is complete for the class coC under the same type of reductions. (Recall that coC = { $\bar{L}$ | $L \in$ C }.)

b. We say that a class C is *closed* under a type of reductions (eg. log-space or polynomial time reductions) if, whenever L reduces to L' and L' is in C, then also L is in C.
Suppose that the class C is closed under a type of reductions and the language *M* is complete for C. Show that *M* is in coC if and only if C = coC.

**Problem 3**. [20 points]
a. Show that NP, coNP, and EXP=$\bigcup_{c>0} TIME(2^{n^c})$ are closed under polynomial time reductions.

b. Show that the class $E = \bigcup_{c>0} TIME(2^{cn})$ is not closed under polynomial time reductions.
That is, there are languages L, L' such that $L \leq_p L'$ and $L' \in E$, but $L \notin E$.

(*Hint*: Use a padding function and the time hierarchy theorem.)

**Problem 4**. [25 points]

a. Consider the following transformation that maps a directed graph G=(V,E) with m nodes to another directed graph G'=(V',E') with $m^2$ nodes. The set of nodes of G' is V'={ [v,i] | $v \in V$, $1 \le i \le m$ }, and the set of edges is E'= { ([u,i],[v,i+1]) | $(u,v) \in E$, $1 \le i \le m-1$ } $\cup$ { ([v,i],[v,i+1]) | $v \in V$, $1 \le i \le m-1$ }.

Use this transformation to show that the Graph Reachability problem is NL-complete (under log-space reductions) even when the input is restricted to acyclic graphs and the nodes are ordered topologically.

Specifically, show that the following *DAG Reachability* problem is NL-complete.

Input: A directed acyclic graph H=(N,A) with set of nodes N={1,…,n} in topological order, i.e. all edges $(i,j) \in A$ satisfy $i < j$; Two nodes s, t.

Question: Is there a path in H from s to t?

b. The *Graph Cyclicity* problem is as follows.

Input: A directed graph G.

Question: Does the graph contain a cycle?

Show that the Graph Cyclicity problem is NL-complete.

c. Is the *Graph Acyclicity* problem (Given a directed graph, is it acyclic?) NL-complete? Justify your answer.



**Problem 5.** [20 points]

a. Consider the following set of inequalities: $a \le b$, $a \le c$, $b+c \le a+1$, $0 \le a$.

Show that for every assignment of values 0 or 1 to $b$ and $c$, there is a unique value of $a$ over the real numbers that satisfies the inequalities, namely the value $a = b \wedge c$, i.e. $a = 1$ if both $b$, $c$ are 1, and $a = 0$ otherwise.

b. Give a set of inequalities in variables $a$, $b$, $c$ that has the analogous property for $a = b \vee c$, i.e., for every assignment of values 0 or 1 to $b$ and $c$, there is a unique real value of $a$ that satisfies the inequalities, namely the value $a = b \vee c$.

c. Show that the following "*Linear Inequalities*" problem is P-hard under log-space reductions. (The problem is also in P but you do not have to show this.)

*Linear Inequalities:*

Input*:* A system of linear inequalities in a set of variables.

Question*:* Does the system have a solution over the reals, i.e. is there an assignment of real values to the variables that satisfies all the inequalities?

(*Hint:* Reduce from the Circuit Value problem with fan-in 2. Introduce variables for the gates and the inputs of the circuit and include appropriate inequalities for the gates and the given input assignment to the circuit.)