Perceptron and Online Perceptron

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Margins

Let S be a collection of labeled examples from $\mathbb{R}^d \times \{-1, +1\}$. We say S is *linearly separable* if there exists $w \in \mathbb{R}^d$ such that

$$\min_{(x,y) \in S} y \langle w, x \rangle > 0,$$

and we call w a *linear separator* for S.

The *(minimum) margin* of a linear separator w for S is the minimum distance from x to the hyperplane orthogonal to w, among all $(x, y) \in S$. Note that this notion of margin is invariant to positive scaling of w. If we rescale w so that

$$\min_{(x,y) \in S} y \langle w, x \rangle = 1,$$

then this minimum distance is $1/\|w\|_2$. Therefore, the linear separator with the largest minimum margin is described by the following mathematical optimization problem:

$$\begin{aligned} \min_{w \in \mathbb{R}^d} \quad & \frac{1}{2} \|w\|_2^2 \\ \text{s.t.} \quad & y \langle w, x \rangle \geq 1, \quad (x, y) \in S. \end{aligned}$$

Perceptron algorithm

The Perceptron algorithm is given as follows. The input to the algorithm is a collection S of labeled examples from $\mathbb{R}^d \times \{-1, +1\}$.

- Begin with $\hat{w}_1 := 0 \in \mathbb{R}^d$.
- For $t = 1, 2, \ldots$:
  - If there is a labeled example in S (call it (x_t, y_t)) such that $y_t \langle \hat{w}_t, x_t \rangle \leq 0$, then set $\hat{w}_{t+1} := \hat{w}_t + y_t x_t$.
  - Else, return $\hat{w}_t$.

Theorem. Let S be a collection of labeled examples from $\mathbb{R}^d \times \{-1, +1\}$. Suppose there exists a vector $w_\star \in \mathbb{R}^d$ such that

$$\min_{(x,y) \in S} y \langle w_\star, x \rangle = 1.$$

Then Perceptron on input S halts after at most $\|w_\star\|_2^2 L^2$ loop iterations, where $L := \max_{(x,y) \in S} \|x\|_2$.

Proof. Suppose Perceptron does not exit the loop in the t-th iteration. Then there is a labeled example $(x_t, y_t) \in S$ such that

$$\begin{aligned} y_t \langle w_\star, x_t \rangle &\geq 1, \\ y_t \langle \hat{w}_t, x_t \rangle &\leq 0. \end{aligned}$$

We bound $\langle w_\star, w_{t+1} \rangle$ from above and below to deduce a bound on the number of loop iterations. First, we bound $\langle w_\star, \hat{w}_t \rangle$ from below:

$$\langle w_\star, \hat{w}_{t+1} \rangle = \langle w_\star, \hat{w}_t \rangle + y_t \langle w_\star, x_t \rangle \geq \langle w_\star, \hat{w}_t \rangle + 1.$$

Since $\hat{w}_1 = 0$, we have

$$\langle w_\star, \hat{w}_{t+1} \rangle \geq t.$$

We now bound $\langle w_\star, \hat{w}_{t+1} \rangle$ from above. By Cauchy-Schwarz,

$$\langle w_\star, \hat{w}_{t+1} \rangle \leq \|w_\star\|_2 \|\hat{w}_{t+1}\|_2.$$

Also,

$$\|\hat{w}_{t+1}\|_2^2 = \|\hat{w}_t\|_2^2 + 2y_t \langle \hat{w}_t, x_t \rangle + y_t^2 \|x_t\|_2^2 \leq \|\hat{w}_t\|_2^2 + L^2.$$

Since $\|\hat{w}_1\|_2 = 0$, we have

$$\|\hat{w}_{t+1}\|_2^2 \leq L^2 t,$$

so

$$\langle w_\star, \hat{w}_{t+1} \rangle \leq \|w_\star\|_2 L \sqrt{t}.$$

Combining the upper and lower bounds on $\langle w_\star, \hat{w}_{t+1} \rangle$ shows that

$$t \leq \langle w_\star, \hat{w}_{t+1} \rangle \leq \|w_\star\|_2 L \sqrt{t},$$

which in turn implies the inequality $t \leq \|w_\star\|_2^2 L^2$. ∎

Online Perceptron algorithm

The Online Perceptron algorithm is given as follows. The input to the algorithm is a sequence $(x_1, y_1), (x_2, y_2), \ldots$ of labeled examples from $\mathbb{R}^d \times \{-1, +1\}$.

- Begin with $\hat{w}_1 := 0 \in \mathbb{R}^d$.
- For $t = 1, 2, \ldots$:
  - If $y_t \langle \hat{w}_t, x_t \rangle \leq 0$, then set $\hat{w}_{t+1} := \hat{w}_t + y_t x_t$.
  - Else, $\hat{w}_{t+1} := \hat{w}_t$.

We say that Online Perceptron makes a *mistake* in round t if $y_t \langle \hat{w}_t, x_t \rangle \leq 0$.

Theorem. Let $(x_1, y_1), (x_2, y_2), \ldots$ be a sequence of labeled examples from $\mathbb{R}^d \times \{-1, +1\}$ such that there exists a vector $w_\star \in \mathbb{R}^d$ satisfying

$$\min_{t=1,2,\ldots} y_t \langle w_\star, x_t \rangle = 1.$$

Then Online Perceptron on input $(x_1, y_1), (x_2, y_2), \ldots$ makes at most $\|w_\star\|_2^2 L^2$ mistakes, where $L := \max_{t=1,2,\ldots} \|x_t\|_2$.

Proof. The proof of this theorem is essentially the same as the proof of the iteration bound for Perceptron. ∎

Online Perceptron may be applied to a collection of labeled examples S by considering the labeled examples in S in any (e.g., random) order. If S is linearly separable, then the number of mistakes made by Online Perceptron can be bounded using the theorem.

However, Online Perceptron is also useful when S is not linearly separable. This is especially notable in comparison to Perceptron, which never terminates if S is not linearly separable.

Theorem. Let $(x_1, y_1), (x_2, y_2), \ldots$ be a sequence of labeled examples from $\mathbb{R}^d \times \{-1, +1\}$. Online Perceptron on input $(x_1, y_1), (x_2, y_2), \ldots$ makes at most

$$\min_{w_\star \in \mathbb{R}^d} \left[\|w_\star\|_2^2 L^2 + \|w_\star\|_2 L \sqrt{\sum_{t \in \mathcal{M}} \ell(\langle w_\star, x_t \rangle, y_t)} + \sum_{t \in \mathcal{M}} \ell(\langle w_\star, x_t \rangle, y_t) \right]$$

mistakes, where $L := \max_{t=1,2,\ldots} \|x_t\|_2$, $\mathcal{M}$ is the set of rounds on which Online Perceptron makes a mistake, and $\ell(\hat{y}, y) := [1 - \hat{y}y]_+ = \max\{0, 1 - \hat{y}y\}$ is the *hinge loss* of $\hat{y}$ when y is the correct label.

Proof. Fix any $w_\star \in \mathbb{R}^d$. Consider any round t in which Online Perceptron makes a mistake. Let $\mathcal{M}_t := \{1, \ldots, t\} \cap \mathcal{M}$ and $M_t := |\mathcal{M}_t|$. We will bound $\langle w_\star, \hat{w}_{t+1} \rangle$ from above and below to deduce a bound on M_t, the number of mistakes made by Online Perceptron through the first t rounds. First we bound $\langle w_\star, \hat{w}_{t+1} \rangle$ from above. By Cauchy-Schwarz,

$$\langle w_\star, \hat{w}_{t+1} \rangle \le \|w_\star\|_2 \|\hat{w}_{t+1}\|_2.$$

Moreover,

$$\|\hat{w}_{t+1}\|_2^2 = \|\hat{w}_t\|_2^2 + 2y_t \langle \hat{w}_t, x_t \rangle + y_t^2 \|x_t\|_2^2 \le \|\hat{w}_t\|_2^2 + L^2.$$

Since $\hat{w}_1 = 0$, we have

$$\|\hat{w}_{t+1}\|_2^2 \le L^2 M_t,$$

and thus

$$\langle w_\star, \hat{w}_{t+1} \rangle \le \|w_\star\|_2 L \sqrt{M_t}.$$

We now bound $\langle w_\star, w_{t+1} \rangle$ from below:

$$\begin{aligned}
\langle w_\star, \hat{w}_{t+1} \rangle &= \langle w_\star, \hat{w}_t \rangle + 1 - [1 - y_t \langle w_\star, x_t \rangle] \\
&\ge \langle w_\star, \hat{w}_t \rangle + 1 - [1 - y_t \langle w_\star, x_t \rangle]_+ \\
&= \langle w_\star, \hat{w}_t \rangle + 1 - \ell(\langle w_\star, x_t \rangle, y_t),
\end{aligned}$$

Since $\hat{w}_1 = 0$,

$$\langle w_\star, \hat{w}_{t+1} \rangle \ge M_t - H_t,$$

where

$$H_t := \sum_{t \in \mathcal{M}_t} \ell(\langle w_\star, x_t \rangle, y_t).$$

Combining the upper and lower bounds on $\langle w_\star, \hat{w}_{t+1} \rangle$ shows that

$$M_t - H_t \le \langle w_\star, \hat{w}_{t+1} \rangle \le \|w_\star\|_2 L \sqrt{M_t},$$

i.e.,

$$M_t - \|w_\star\|_2 L \sqrt{M_t} - H_t \le 0.$$

This inequality is quadratic in $\sqrt{M_t}$. By solving it[1], we deduce the bound

$$M_t \le \frac{1}{2} \|w_\star\|_2^2 L^2 + \frac{1}{2} \|w_\star\|_2 L \sqrt{\|w_\star\|_2^2 L^2 + 4H_t} + H_t,$$

which can be further loosened to the following (slightly more interpretable) bound:

$$M_t \le \|w_\star\|_2^2 L^2 + \|w_\star\|_2 L \sqrt{H_t} + H_t.$$

The claim follows. ∎

[1] The inequality is of the form $x^2 - bx - c \le 0$ for some non-negative b and c. This implies that $x \le (b + \sqrt{b^2 + 4c})/2$, and hence $x^2 \le (b^2 + 2b\sqrt{b^2 + 4c} + b^2 + 4c)/4$. We can then use the fact that $\sqrt{A + B} \le \sqrt{A} + \sqrt{B}$ for any non-negative A and B to deduce $x^2 \le b^2 + b\sqrt{c} + c$.