Homework 8

Due date: 3/10/2021 right before class

Problem 1

The problem of finding a Clique of size k in a graph $G = (V, E)$ requires finding a set of vertices of size k such that every pair of vertices in the set is adjacent. The problem of finding an Independent-Set of size k in a graph $G = (V, E)$ requires finding a set of vertices of size k such that no pair of vertices in the set is adjacent.

Given that the Clique problem is NP-Complete, Show that the Independent-Set problem is NP-Complete as well.

Solution 1

In the discussion slides.

Problem 2

A Hamiltonian Cycle in an undirected graph G is a simple cycle that goes through all nodes. A Hamiltonian Path in G is a simple Path that goes through all nodes.

Given that the Hamiltonian path problem is NP-Complete, show that the Hamiltonian cycle problem is NP-Complete as well.

Solution 2

a. HC is in NP. Given a certificate (a proposed solution) we can iterate the graph in the order it states in polynomial time to verify.

b. Given a graph $G = (V, E)$ we reduce the HP problem to the HC problem by adding a special vertex v^* and connect all vertices in the original graph to it. Now if the original graph had a HP, in the new graph we will be able to find a HC. The reduction is of course polynomial since we can create a new vertex and connect all vertices to it in linear time.

Problem 3. Page 425 Q19

You've periodically helped the medical consulting firm Doctors Without Weekends on various hospital scheduling issues, and they've just come to you with a new problem. For each of the next n days, the hospital has determined the number of doctors they want on hand; thus, on day i, they have a requirement that exactly p_i doctors be present. There are k doctors, and each is asked to provide a list of days on which he or she is willing to work. Thus doctor j provides a set L_j of days on which he or she is willing to work. The system produced by the consulting firm should take these lists and try to return to each doctor j a list L'_j with the following properties.
(A) L'_j is a subset of L_j, so that doctor j only works on days he or she finds acceptable.
(B) If we consider the whole set of lists $L'_1, ..., L'_k$, it causes exactly p_i doctors to be present on day i, for $i = 1, 2, ..., n$.

a. Describe a polynomial-time algorithm that implements this system. Specifically, give a polynomial-time algorithm that takes the numbers $p_1, p_2, ..., p_n$, and the lists $L_1, ..., L_k$, and does one of the following two things.

– Return lists $L'_1, L'_2, ..., L'_k$ satisfying properties (A) and (B); or
– Report (correctly) that there is no set of lists $L'_1, L'_2, ..., L'_k$ that satisfies both properties (A) and (B).

b. The hospital finds that the doctors tend to submit lists that are much too restrictive, and so it often happens that the system reports (correctly, but unfortunately) that no acceptable set of lists $L'_1, L'_2, ..., L'_k$ exists. Thus the hospital relaxes the requirements as follows. They add a new parameter $c > 0$, and the system now should try to return to each doctor j a list L'_j with the following properties.

(A*) L'_j contains at most c days that do not appear on the list L_j.
(B) (Same as before) If we consider the whole set of lists $L'_1, ..., L'_k$, it causes exactly p_i doctors to be present on day i, for $i = 1, 2, ..., n$. Describe a polynomial-time algorithm that implements this revised system. It should take the numbers $p_1, p_2, ..., p_n$, the lists $L_1, ..., L_k$, and the parameter $c > 0$, and do one of the following two things.
- Return lists $L'_1, L'_2, ..., L'_k$ satisfying properties (A*) and (B); or
- Report (correctly) that there is no set of lists $L'_1, L'_2, ..., L'_k$ that satisfies both properties (A*) and (B).

Solution 3

In the end.

Problem 4. Page 435 Q33

Let $G = (V, E)$ be a directed graph, and suppose that for each node v, the number of edges into v is equal to the number of edges out of v. That is, for all v,

$|(u, v) : (u, v) \in E| = |(v, w) : (v, w) \in E|.$

Let x, y be two nodes of G, and suppose that there exist k mutually edge-disjoint paths from x to y. Under these conditions, does it follow that there exist k mutually edge-disjoint paths from y to x? Give a proof or a counterexample with explanation.

Solution 4

In the end.