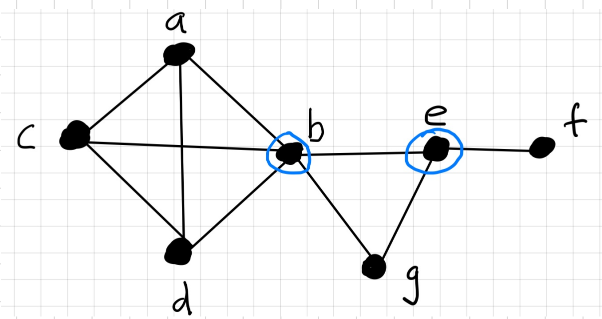
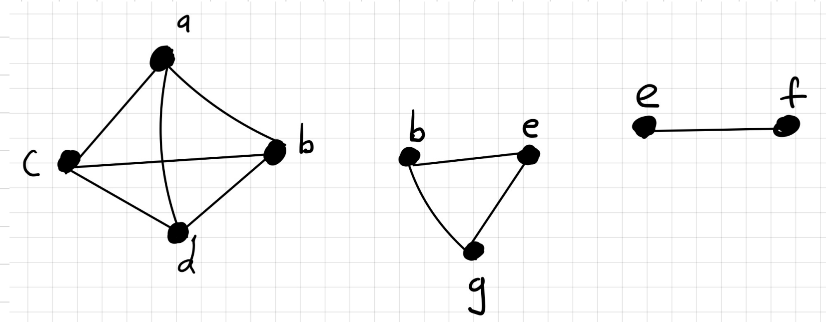
Recall: DFS to find 2-connected components



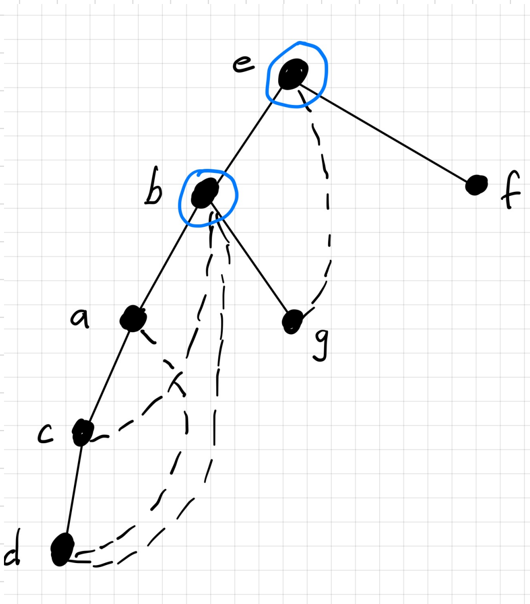
This graph is connected but removing one vertex b or e disconnects it.

Biconnected components



v is a cut vertex if removing v makes G disconnected. Cut vertices are bad in networks.

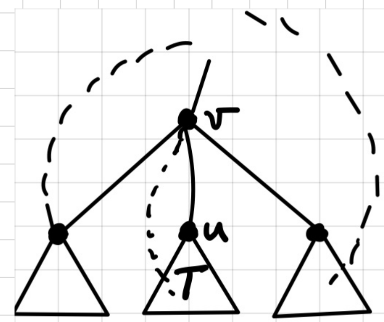
DFS from e



Characterizing cut vertices:

Claim The root is a cut vertex iff it has > 1 child.

Lemma A non-root v is a cut vertex iff v has a subtree T with no non-tree edge going to a proper ancestor of v.



Proof $\Leftarrow$ removing v separates T from rest of graph.

$\Rightarrow$ since removing v disconnects G, some subtree must get disconnected

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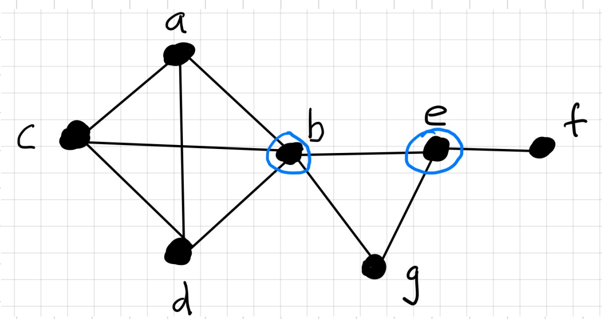

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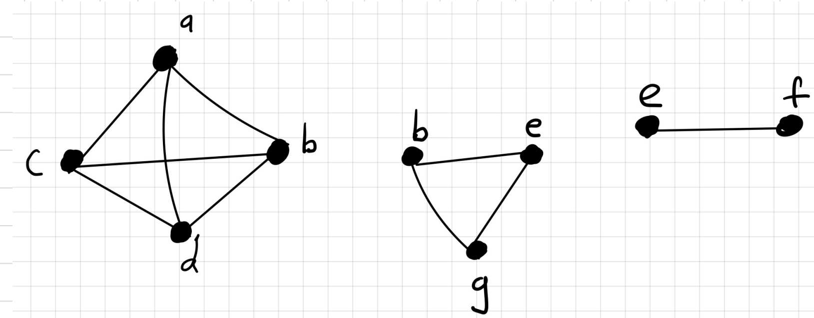

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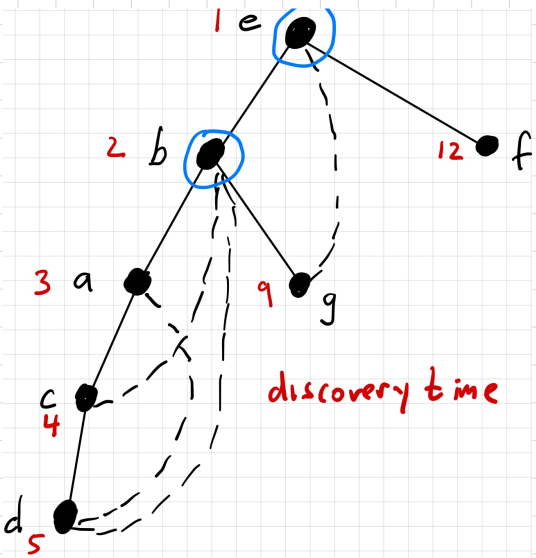

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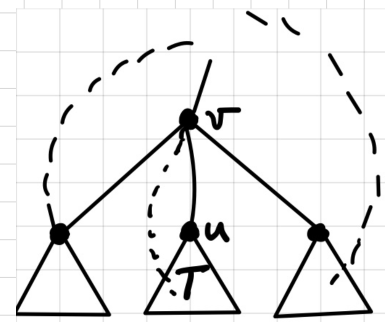

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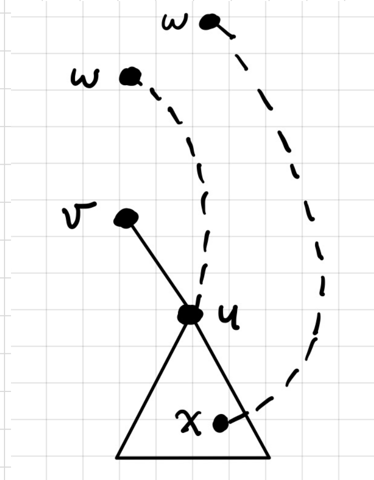
Making the lemma into an algorithm

Define: $\mathrm{low}(u) = \min\{d(w) \ : \ x \text{ a descendant of } u \text{ and } (x, w) \text{ an edge}\}$

Convention: u is a descendant of u



$\mathrm{low}(u)$ = how high in tree we can get to from u by going down (0 or more) and then up 1 edge

Note: it does not hurt to look at all edges, not just non-tree edges

Fact: non-root v is a cut vertex iff v has a child u with $\mathrm{low}(u) \geq d(v)$

We can compute low recursively:

$$\mathrm{low}(u) = \min \begin{cases} \min\{d(w) \ : \ (u, w) \in E\} \\ \min\{\mathrm{low}(x) \ : \ x \text{ a child of } u\} \end{cases} \tag{1}$$

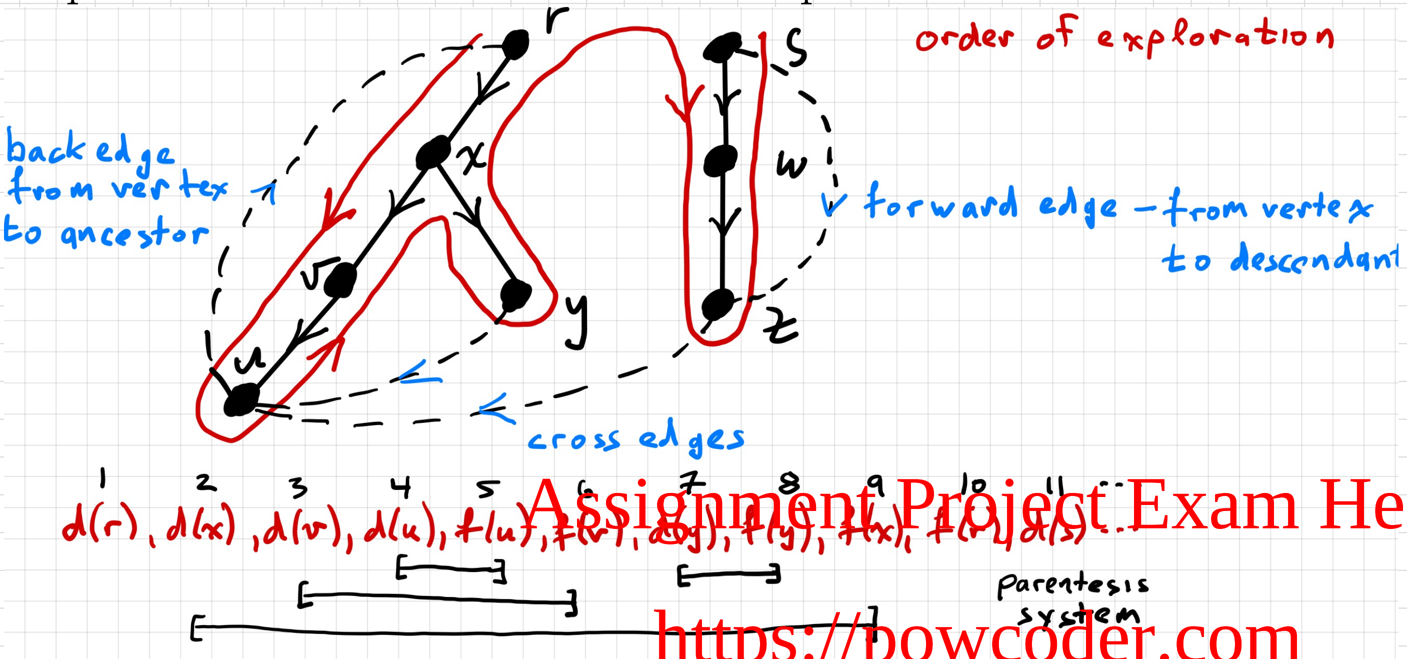
Algorithm to compute all cut vertices

- Enhance DFS code to compute low, OR
- Run DFS to compute discover times $d(\cdot)$.
  Then, for every vertex u in finish time order use (1) to compute $\mathrm{low}(u)$.

For every non-root v: if v has a child u with $\mathrm{low}(u) \geq d(v)$ then v is a cut vertex.

Also handle the root.

Depth First Search on Directed Graphs



DFS(v)
 mark(v) := discovered
 $d(v)$:= time; time := time + 1
 for $u \in$ AdjacencyList(v) **do**
  if u is undiscovered **then**
   DFS(u); (v, u) is a tree edge
  else
   ⊛
  fi
 od
 mark(v) := finished
 $f(v)$:= time; time := time + 1

⊛ # label back, forward, cross edges
if u not finished **then**
 (v, u) is a back edge
elif $d(u) > d(v)$ **then**
 (v, u) is a forward edge
else # $d(u) < d(v)$
 (v, u) is a cross edge
fi

DFS takes $O(n + m)$

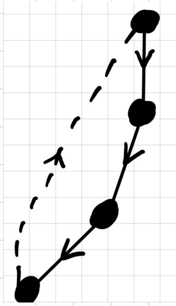
Note: result depends on vertex ordering.

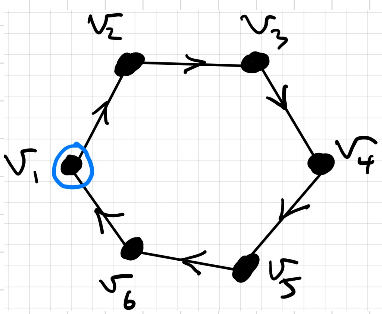
Applications of DFS

(1) Detecting cycles in directed graphs.

Lemma A directed graph has a (directed) cycle iff DFS has a back edge.

Proof

$\Longleftarrow$  back edge gives directed cycle

$\Longrightarrow$ 

Suppose there is a directed cycle. Let v_1 be first vertex discovered in DFS. Number vertices of cycle $v_1 \cdots v_k$.

Claim (v_k, v_1) is a back edge.

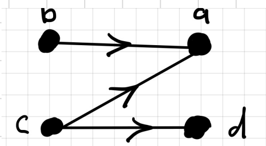
Proof Because we must discover & explore all v_i before we finish v_1, when we test edge (v_k, v_1) we label it a back edge.

Applications of DFS

(2) Topological sort of directed acyclic graph (acyclic $\equiv$ no directed cycle)

Edge (a, b) means a must come before b (e.g., job scheduling).

Find a linear order of vertices satisfying all edges (possible iff no directed cycle).

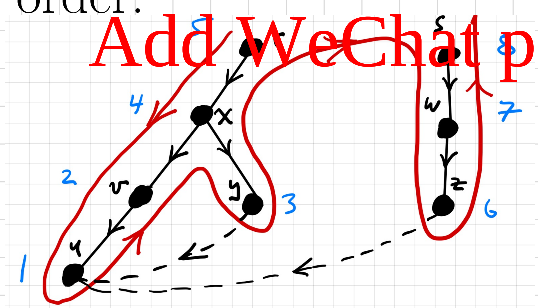
Example: 

topological sort: $b\,c\,a\,d$ or $c\,d\,b\,a$ or $\ldots$

<u>One solution:</u> Find vertex v with no in-edge. Remove v and repeat.

Solution using DFS: $O(n + m)$
use reverse of finish order.

Example
(first example
without back edges)



finish order
reverse finish order s, w, z, r, x, y, v, u

This is a topological order.

Proof that this works.

<u>Claim</u> For every directed edge (u, v), finish$(u) >$ finish(v)

case 1 u discovered before v. Then because of edge (u, v), v is discovered and finished before u is finished.

case 2 v discovered before u. Because G has no directed cycle, we can't reach u in DFS(v). So v finished before u is discovered and finished.

Applications of DFS

(3) Finding strongly connected components in a directed graph.

strongly connected $\equiv$ for all vertices u, v there is a path $u \to v$

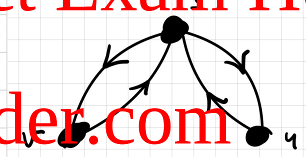
Easy to test if G is strongly connected because we don't need to test all pairs u, v.

Here's how: Let s be a vertex

<u>Claim</u> G is strongly connected iff for all vertices v, there is a path $s \to v$ and a path $v \to s$.

<u>Proof</u> $\Rightarrow$ clear

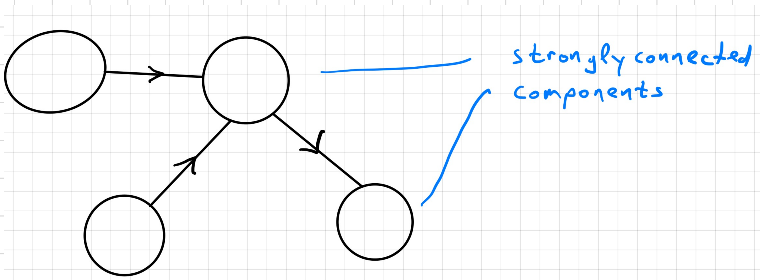
$\Leftarrow$ to get from $u \to v$: $u \to s \to v$



To test if there's a path $s \to v$ $\forall v$ — do DFS(s).

How can we test if there's a path $v \to s$ $\forall v$? Reverse edge directions and do DFS(s). Neat!

More generally, the structure of a digraph is



Contracting strongly connected components gives an <u>acyclic</u> graph (think about why).