

Knapsack Problem

You're going on a 5 day canoeing trip to Algonquin Park.

You want to pack your knapsack to maximize value and minimize weight.

Given n items, item i has weight w_i and value v_i . Weight limit of knapsack is W . Put items in knapsack, sum of weight $\leq W$, maximize sum of values.

[Notation: find $S \subseteq \{1, \dots, n\}$, $\sum\{w_i \mid i \in S\} \leq W$ and maximize $\sum\{v_i \mid i \in S\}$.]

Two versions of the problem:

- 0-1 knapsack. Items are indivisible (tent, flashlight)
- fractional knapsack. Can use fractions of items (oatmeal, cheese)

We'll see a dynamic programming algorithm for 0-1 knapsack, but (in some sense) the algorithm is not efficient and the problem is hard.

Today: a greedy algorithm for the fractional knapsack.

Example:

i	v_i	w_i	v_i/w_i
1	12	4	3
2	7	3	$2\frac{1}{3}$
3	6	3	2

$W = 6$

Note: it makes sense to order items by value per weight.

For the 0-1 case, greedy gives item 1, value 12 (nothing else fits) but taking items 2 and 3 gives value 13.

For fractional case, greedy takes item 1, leaving weight of 2 free, so take $\frac{2}{3}$ of item 2. Value: $12 + \frac{2}{3} \cdot 7$.

Greedy Algorithm

x_i - weight of item i that we take

$\text{free}W := W$

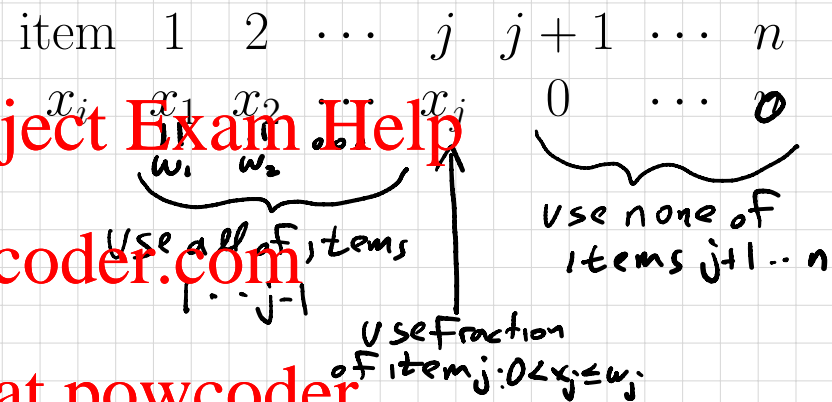
for $i = 1$ **to** n **do**

$x_i := \min\{w_i, \text{free}W\}$

$\text{free}W := \text{free}W - x_i$

od

Note that the solution will look like:



Final weight: $\sum x_i = W$ (if $\sum w_i \geq W$)

Final value: $\sum \left(\frac{v_i}{w_i} \right) x_i$

Running time $O(n \log n)$ to sort by v_i/w_i .

Claim: The greedy algorithm gives the optimal solution to the fractional knapsack problem.

Proof: greedy solution $x_1 \ x_2 \ \cdots \ x_{k-1} \ x_k \ \cdots \ x_\ell \ \cdots \ x_n$
 optimal solution $y_1 \ y_2 \ \cdots \ y_{k-1} \ y_k \ \cdots \ y_\ell \ \cdots \ y_n$

Suppose y is an optimal solutions that matches x on maximum # of indices, say M . Assume, to arrive at a contradiction, that greedy solution is not optimal: $M < n$. We will show that there exists an optimal solution that matches x on at least $M + 1$ indices.

Let k be the first index where $x_k \neq y_k$.

Then $x_k > y_k$ since greedy maximizes x_k .

Since $\sum y = \sum x = W$, there is a later index $\ell > k$ with $y_\ell > x_\ell$.

Exchange weight Δ of item ℓ for equal weight of item k in optimal solution.

$$y'_k \leftarrow y_k + \Delta$$

$$y'_\ell \leftarrow y_\ell - \Delta$$

Choose $\Delta \leftarrow \min\{y_\ell - x_\ell, x_k - y_k\}$. Then $x_k = y'_k$ or $x_\ell = y'_\ell$ and $\Delta > 0$.

change in value

$$\Delta \left(\frac{v_k}{w_k} \right) - \Delta \left(\frac{v_\ell}{w_\ell} \right) = \Delta \left(\frac{v_k}{w_k} - \frac{v_\ell}{w_\ell} \right)$$

This is non-negative because $\frac{v_k}{w_k} \geq \frac{v_\ell}{w_\ell}$ (we sorted this way)

But y was an optimal solution, so this can't be better.

Therefore it's a new optimal solution that matches x on (at least) one more index (k or ℓ).