Recall

Summary of Lecture 19

How to prove a problem Z is NP-complete

1. Z is in NP
2. X $\leq_P$ Z, for some known NP-complete problem X. Use a ***many-one*** reduction.

Next: more NP-completeness proofs

Why use ~~a~~ many-one reductions?

- a many-one reduction is a special case of Turing reduction, so it is a stronger result to prove that there is a many-one reduction

- it gives more structure and will make your NP-completeness proofs easier to find and to prove correct

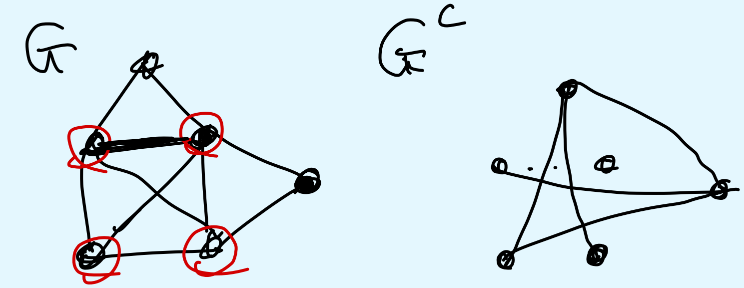
- convention

Is there always a many-one reduction to prove that a problem is NP-complete? i.e., if X, Z are in NP and $X \leq_P Z$ with a Turing reduction, then is there a many-one reduction $X \leq_P Z$?

This is an open question, but it holds in every known case.

Clique.
Input: Graph $G = (V,E)$, number k.
Question: Does G have a clique of size $\geq k$?

Recall: a clique is a set of vertices, every two joined by an edge.



Observe: $C \subseteq V$ is a clique in G iff
C is an independent set in G^c

Recall: G^c, the complement of G, has vertices V, edge (u,v) iff $(u,v) \notin E(G)$

Theorem. Clique is NP-complete.
Proof.

1. Clique is in NP.

certificate: the vertices, C of the clique

verification: check $\geq k$ vertices, every pair joined by edge.
This verifies iff C is a clique $\geq k$. Poly. time to verify.

2. [a known NP-complete problem] $\leq_P$ Clique

2. Independent Set $\leq_P$ Clique

Assume we have a polynomial time algorithm for Clique.
Make a polynomial time algorithm for Independent Set — use a many-one reduction.

Input for Independent Set: Graph $G = (V,E)$, number k.
Output: Does G have an independent set of size k?

- construct a graph G' and choose a number k' such that

  G has an independent set of size $\geq k$ iff G' has a clique of size $\geq k'$

- run the Clique algorithm on G', k'
- return its answer

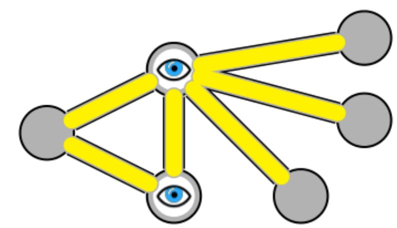
Construction: let $G' = G^c$ and $k' = k$

Runtime: clearly poly. time

Correctness: G has an ind. set of size $\geq k$ iff G^c has a clique of size $\geq k$.

Vertex Cover.
Input: Graph $G = (V,E)$, number k.
Question: Does G have a vertex cover of size $\leq k$?

A *vertex cover* is a set $S \subseteq V$ such that every edge $(u,v) \in E$ has u or v (or both) in S.



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W https://en.wikipedia.org/wiki/Vertex_cover

Observe: $S \subseteq V$ is a vertex cover in G iff
$V - S$ is an independent set in G

Theorem. Vertex Cover is NP-complete.

Proof.

1. Vertex Cover is in NP.

Exercise.

2. Ind. Set $\leq_p$ Vertex Cover

2. Independent Set $\leq_P$ Vertex Cover

Assume we have a polynomial time algorithm for Vertex Cover. Make a polynomial time algorithm for Independent Set — use a many-one reduction.

Input for Independent Set: Graph $G = (V,E)$, number k.
Output: Does G have an independent set of size k?

- construct a graph G' and choose a number k' such that

(*) G has an independent set of size $\geq k$ iff G' has a vertex cover of size $\leq k'$

- run the Vertex Cover algorithm on G', k'
- return its answer

Construction: $G' = G \quad k' = n - k$

Runtime: poly. time

Correctness: Prove (*)

$\Rightarrow$ G has ind. set I of size $\geq k$. Then $V \setminus I$ is a vertex cover $|V \setminus I| \leq n-k$

$\Leftarrow$ G has a vertex cover S, $|S| \leq n-k$. Then $V \setminus S$ is an ind. set of size $\geq k$.

Road map of NP-completeness

✓ $\le_P$ Clique

✓ ✓

$\le_P$ Ind. Set $\le_P$ Vertex Cover $\le_P$ Set Cover

Circuit SAT $\le_P$ 3-SAT $\le_P$ Ham.cycle $\le_P$ TSP

$\le_P$ Subset Sum

later

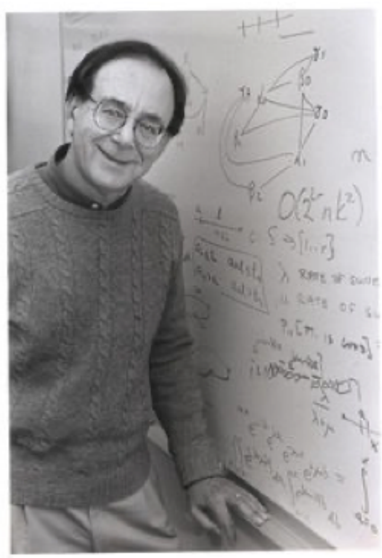
History of NP-completeness

Proof that 3-SAT is NP-complete due to Stephen Cook, U. Toronto, 1971, and independently to Leonid Levin

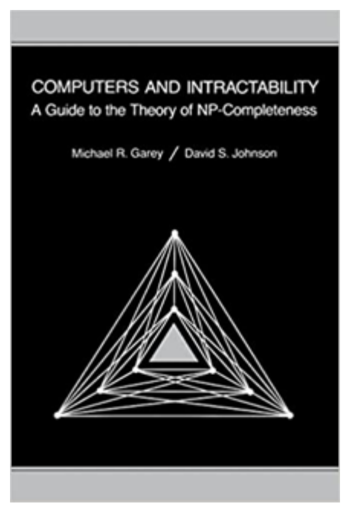
Stephen Cook, 1968

W https://en.wikipedia.org/wiki/Stephen_Cook

The other "first" NP-completeness proofs we cover are due to Richard Karp, UC Berkeley.



Richard Karp



Garey and Johnson's book, the "bible" of NP-complete problems, 1979.

https://ocul-wtl.primo.exlibrisgroup.com/permalink/01OCUL_WTL/156lh75/cdi_crossref_primary_10_2307_2273574

Summary of Lecture 20, Part 1

Clique and Vertex Cover are NP-complete

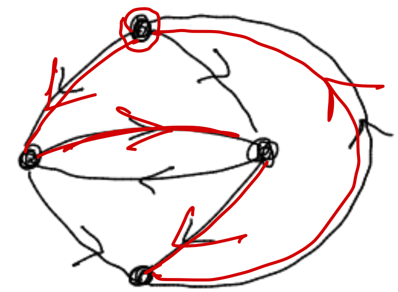
What you should know from Lecture 20, Part 1:

- how to prove a problem is NP-complete using a polynomial time many-one reduction

Next:

$$\begin{tikzcd}
 & & \text{Ind. Set} \arrow[r, phantom, "\le_P"] & \text{Vertex Cover} \arrow[r, phantom, "\le_P"] & \text{Set Cover} \\
\text{Circuit SAT} \arrow[r, phantom, "\le_P"] & \text{3-SAT} \arrow[ur, phantom, "\le_P"] \arrow[r, phantom, "\le_P"] \arrow[dr, phantom, "\le_P"] & \text{Ham.cycle} \arrow[r, phantom, "\le_P"] & \text{TSP} & \\
 & & \text{Subset Sum} & &
\end{tikzcd}$$

Directed Hamiltonian cycle.
Input: Directed graph G = (V,E).
Question: Does G have a directed Hamiltonian cycle?



directed Ham. cycle

Theorem. Directed Hamiltonian cycle is NP-complete.
Proof.

1. Directed Hamiltonian cycle is in NP. exercise

2. 3-SAT $\leq_P$ Directed Hamiltonian cycle

Assume we have a polynomial time algorithm for Directed Ham. cycle. Make a polynomial time algorithm for 3-SAT — use a many-one reduction.

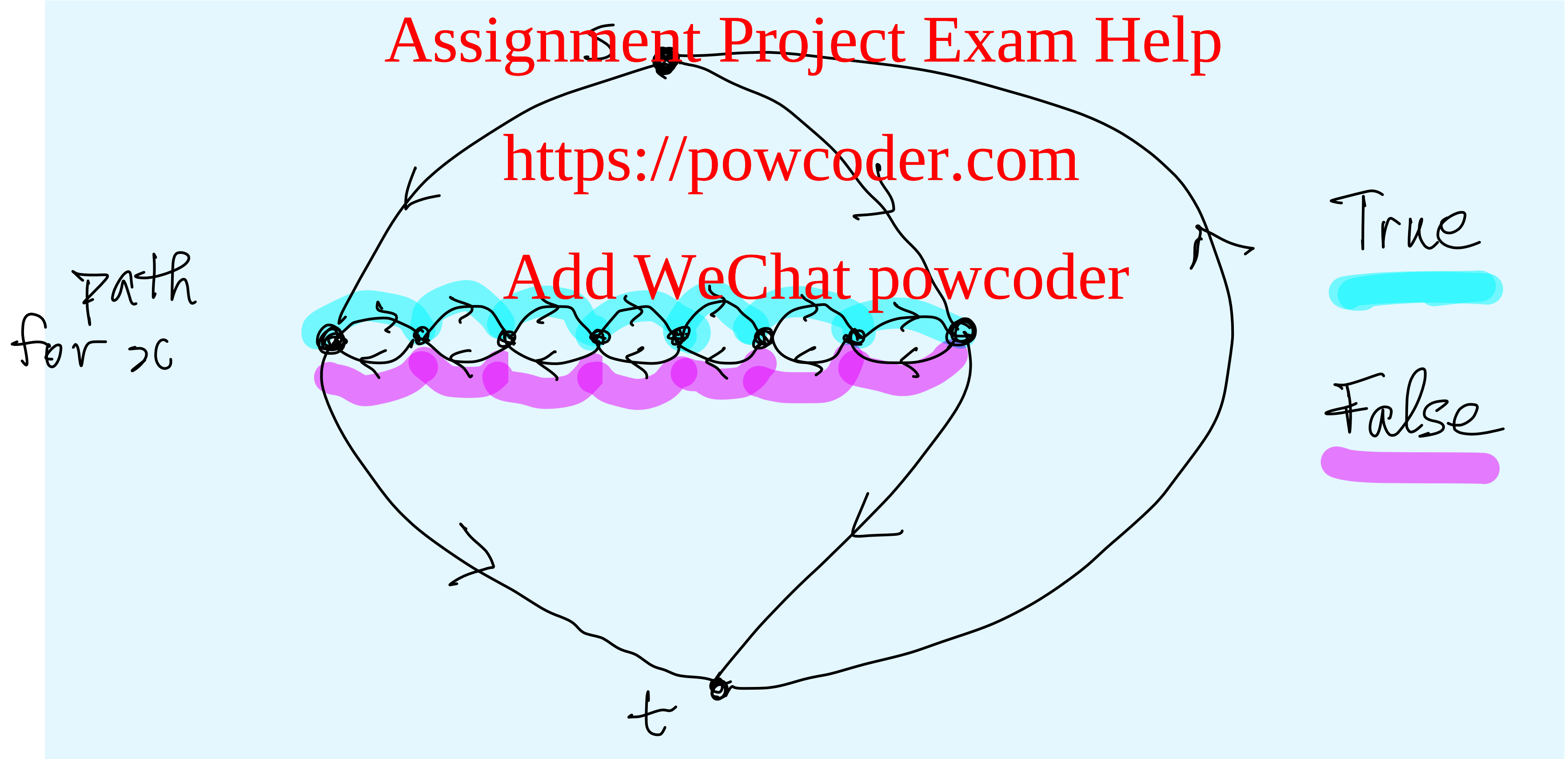
Input: A 3-SAT formula F with clauses $C_1 \ldots C_m$ on variables $x_1 \ldots x_n$
Output: Is F satisfiable?
- construct a directed graph G such that

  G has a directed Ham. cycle iff F is satisfiable
- run the Directed Ham. cycle algorithm on G
- return its answer

This seems tricky! The problems seem so different!

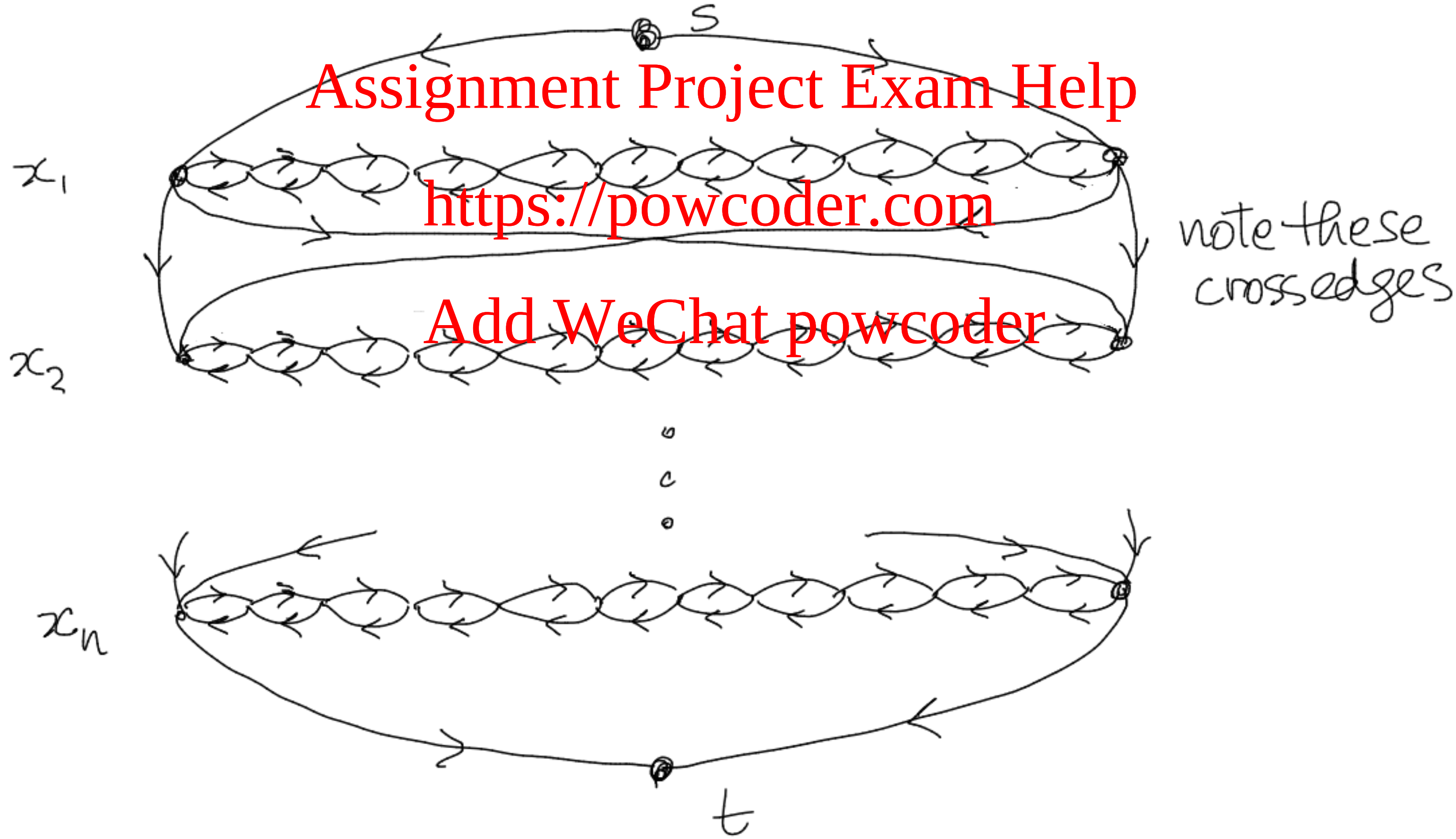
Input: A 3-SAT formula F with clauses $C_1 \ldots C_m$ on variables $x_1 \ldots x_n$
Construct a directed graph G such that
G has a directed Ham. cycle iff F is satisfiable

Idea: for each variable x_i, there is a part of G (a "variable gadget") that chooses whether x_i is True or False



Input: A 3-SAT formula F with clauses $C_1 \ldots C_m$ on variables $x_1 \ldots x_n$
Construct a directed graph G such that
G has a directed Ham. cycle iff F is satisfiable

All the variable gadgets together:

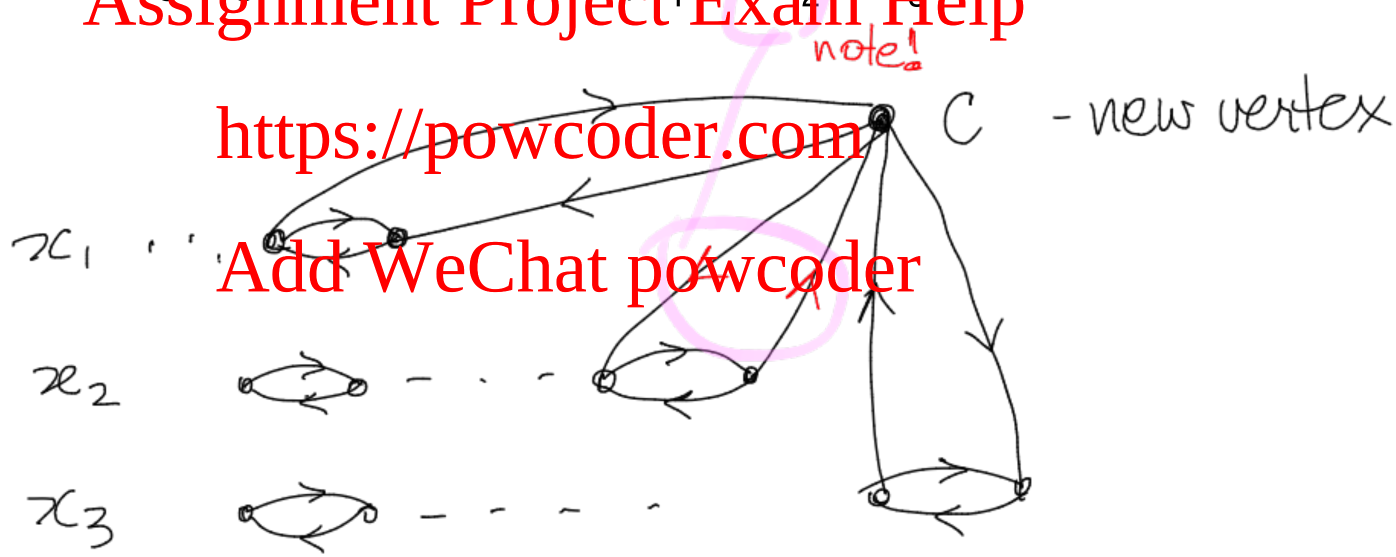


Input: A 3-SAT formula F with clauses $C_1 \ldots C_m$ on variables $x_1 \ldots x_n$
Construct a directed graph G such that

G has a directed Ham. cycle iff F is satisfiable

For each clause C_j we must make a "clause gadget" such that the cycle can go through the clause gadget iff one of the literals is True.

Clause gadget for clause $C = (x_1 \vee \neg x_2 \vee x_3)$



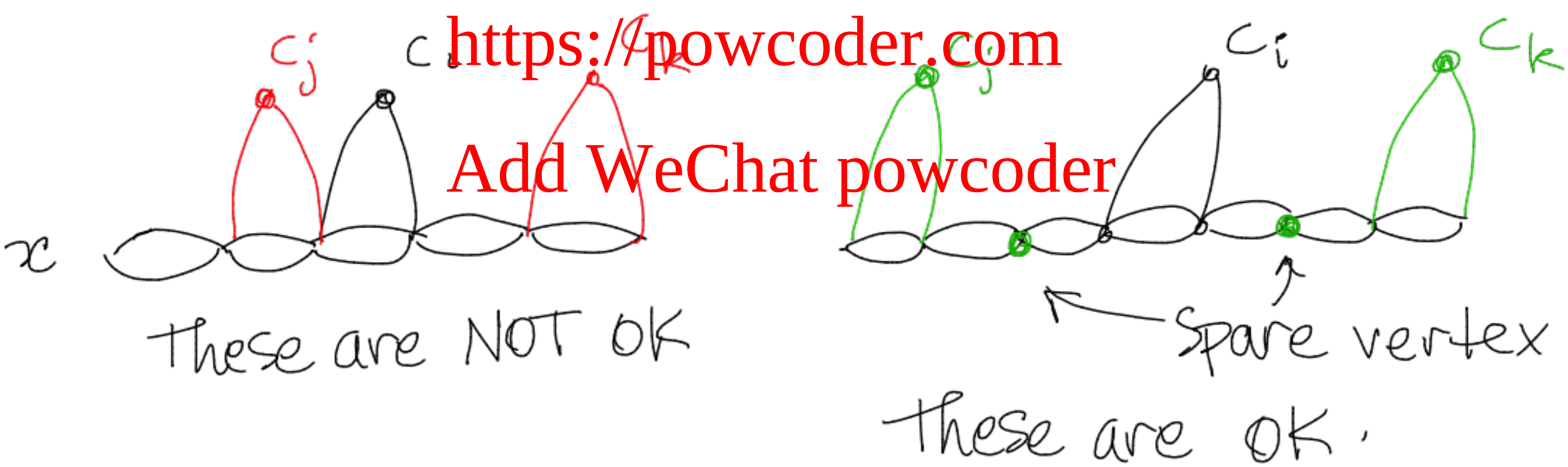
Idea: visit vertex C by detouring off the x_1 True path OR the x_2 False path OR the x_3 True path

Input: A 3-SAT formula F with clauses $C_1 \ldots C_m$ on variables $x_1 \ldots x_n$
Construct a directed graph G such that

G has a directed Ham. cycle iff F is satisfiable

For each clause C_j we must make a "clause gadget" such that the cycle can go through the clause gadget iff one of the literals is True.

Note: make sure to leave a spare vertex between two clause detours



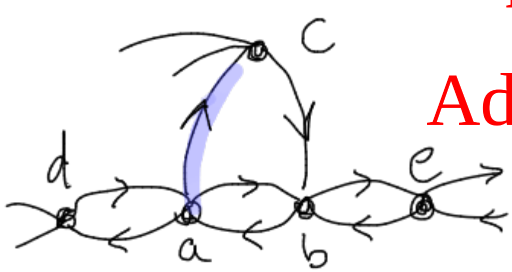
Claim. *G* has a directed Ham. cycle iff *F* is satisfiable

Proof.

⇐ Suppose F is satisfiable. Traverse the variable paths in the True/False directions. For each clause *C*, at least one literal is True — take the detour from that path to vertex *C*. This gives a directed Ham. cycle.

⇒ Suppose G has a directed Hamiltonian path.

Claim. The only way to visit C is by detouring off a variable path.



d,e spare vertices

Suppose we use (a,C). Show must use (C,b). (e.g. can't have e,b,a,C then to different chain). Can't use (a,d) so must enter d from left. Must use (d,a). Can't use (b,a). Must use (b,e). Must use (C,b).

Thus the Hamiltonian cycle must traverse a True or False path for each variable, and must visit each clause vertex off such a path. So this corresponds to a satisfying truth-value assignment.

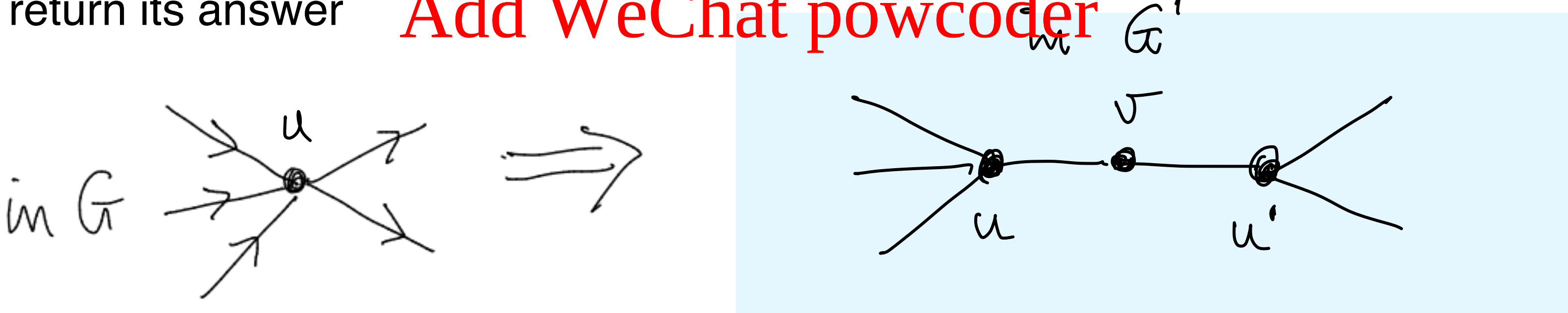
Claim. This construction takes polynomial time.

Theorem. [undirected] Hamiltonian cycle is NP-complete.
Proof.

1. Hamiltonian cycle is in NP.

2. Directed Hamiltonian cycle $\leq_P$ Hamiltonian cycle
   Assume we have a polynomial time algorithm for Ham. cycle. Make a polynomial time algorithm for Directed Ham. cycle — use a many-one reduction.

   Input: A directed graph G.
   Output: Does G have a directed Ham. cycle?
   - construct an undirected graph G' such that
     G has a directed Ham. cycle iff G' has a Ham. cycle (*)
   - run the Ham. cycle algorithm on G'
   - return its answer



check out correctness (*) and poly. time.

Ex. Show that it is wrong to omit v and just use edge (u, u')

Exercises.

Theorem. Travelling Salesman Problem (directed or undirected) is NP-complete.

Theorem. Hamiltonian **path** (directed or undirected) is NP-complete.

Summary of Lecture 20

NP-completeness of Independent Set, Vertex Cover, Hamiltonian cycle, TSP

What you should know from Lecture 20:

- how to prove a problem is NP-complete using a polynomial time many-one reduction

Next:

Ind. Set $\le_P$ Vertex Cover $\le_P$ Set Cover

$\le_P$

Circuit SAT $\le_P$ 3-SAT $\le_P$ Ham.cycle $\le_P$ TSP

$\le_P$ Subset Sum