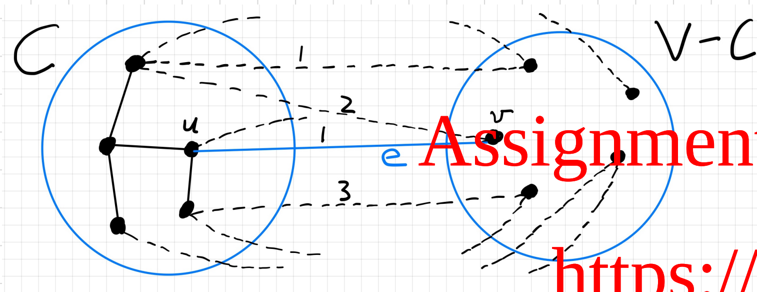
Recall: Minimum Spanning Tree (MST) Problem

last day: Kruskal's algorithm

today: a different greedy algorithm

<u>Prim's Algorithm</u>

Grow one connected component in a greedy fashion (i.e., by adding a vertex $v \in V - C$ that is one end of a minimum weight edge leaving C).



Choose vertex $v \in V - C$ connected to a minimum weight edge $e = (u, v)$ between C and $V - C$.

C = set of vertices reached by T so far

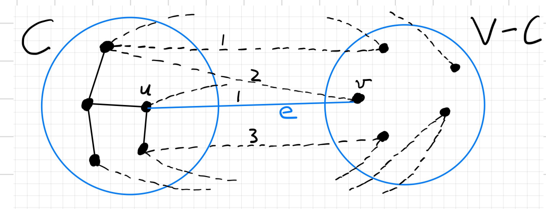
$C := \{s\}$
$T := \emptyset$
while $C \neq V$ **do**
  find vertex $v \in V - C$ such that there exists a $u \in C$
    with $e = (u, v)$ a minimum weight edge leaving C.
  $C := C \cup \{v\}$
  $T := T \cup \{e\}$
od

Correctness: The exact same exchange argument works.

And in fact, we could prove one lemma that gives correctness of both algorithms (see text).

Prim's Algorithm: Implementation

We need to find a vertex in $V - C$ connected to a minimum weight edge leaving C, the connected component of T.



For $v \in V - C$, define

$$\text{weight}(v) = \begin{cases} \infty & \text{if no edge } (u,v) \text{ with } u \in C \\ \min\{w(e) \mid e = (u,v) \in E \text{ and } u \in C\} & \text{otherwise} \end{cases}$$

Priority Queue using the heap data structure

Maintain set $V - C$ as an array in heap order, according to weight as defined above.

- ExtractMin( ) : remove and return vertex with minimal weight
- Insert$(v, \text{weight}(v))$: insert vertex v with weight(v)
- Delete(v) : delete vertex v

Can be implemented at $O(\log k)$ time per operations, $k = |V - C|$.

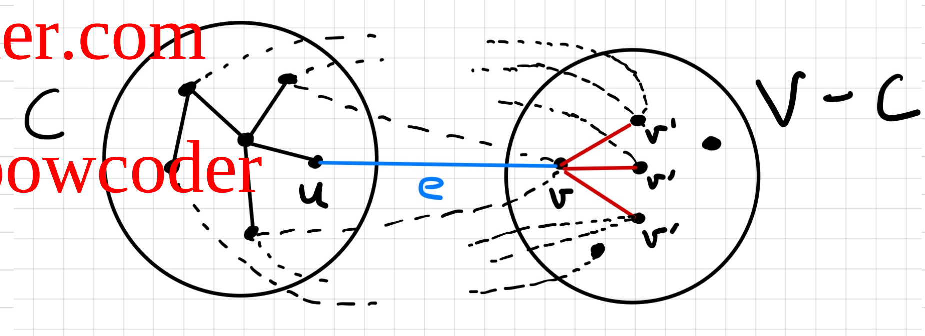
Note: For Delete, need to keep track of array of vertices $\bar{C}[1 \ldots n]$ with

$$\bar{C}[v] = \begin{cases} -1 & \text{if } v \notin V - C \\ \text{location of vertex } v \text{ in heap} & \text{otherwise} \end{cases}$$

Prim's Algorithm: Analysis

$C := \{s\}$
$T := \emptyset$
while $C \neq V$ **do**
  find vertex $v \in V - C$ such that there exists a $u \in C$
    with $e = (u, v)$ a minimum weight edge leaving C.
  $C := C \cup \{v\}$
  $T := T \cup \{e\}$
od

- One ExtractMin to find v.



- Scan through v's adjacency list to find
  - $e = (u, v)$ with $w(e) = \text{weight}(v)$.
  - all edges $e' = (v, v')$ with $v' \in V - C$
    reduce weight of v' in heap if necessary.

Size of heap is $O(n)$:

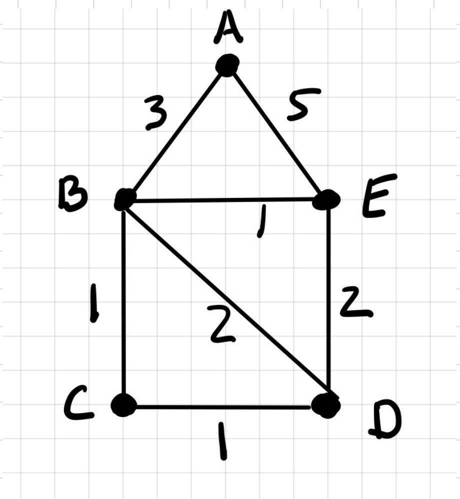
- $n - 1$ ExtractMin operations
- $O(m)$ "reduce weight" operations
  can be implemented using Delete + Insert

Total cost: $O(m \log n)$

Shortest Paths in Edge Weighted Graphs

Recall that BFS from v finds shortest paths from v in unweighted undirected graphs.

General input: directed or undirected graph with weights on edges



- Shortest path A to D is ABD, weight 5.
- Shortest path A to E is ABE, weight 4.

Note: Does a MST always contain the shortest paths?
No, e.g. above: shortest path E to D is edge (E, D), weight 2.

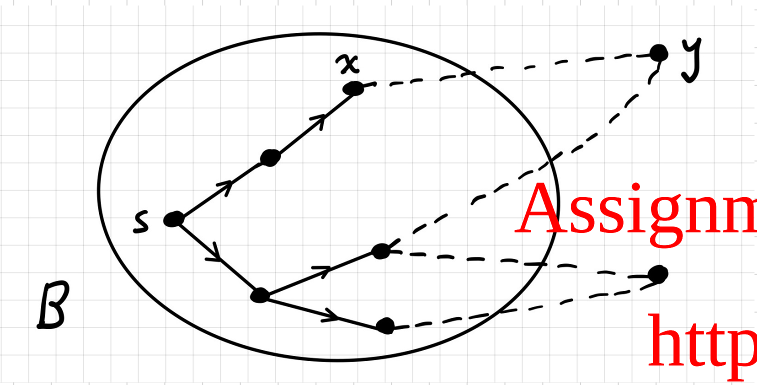
We will study several shortest path algorithms.
Today: Dijkstra's algorithm.

Dijkstra's Algorithm 1959

Input: graph or digraph $G = (V, E)$, $w : E \to \mathbb{R}^{\geq 0}$, $s \in V$

Output: shortest path from s to every other vertex v.

Idea: Grow tree of shortest paths starting from s.



General step: We have tree of shortest paths to all vertices in set B. Initially $B = \{s\}$.

Choose edge (x, y), $x \in B$, $y \notin B$ to minimize $\underline{d(s, x) + w(x, y)}$, where $d(s, x)$ is the (known) minimum distance from s to x.

Note similarity and differences to Prim's algorithm.

Call this minimum d.

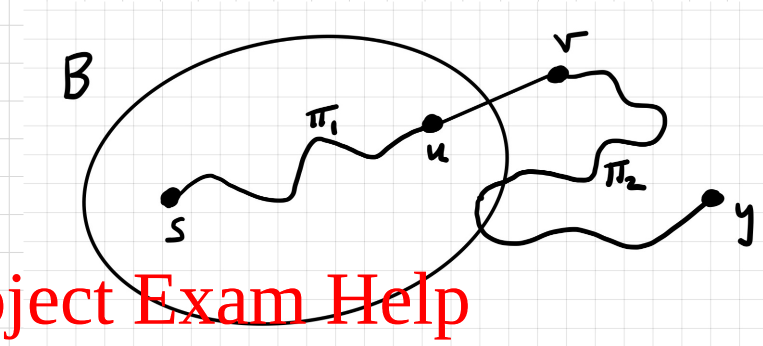
$d(s, y) := d$

add (x, y) to tree (Parent$(y) := x$)

This approach is greedy in the sense that we always add the vertex with the next minimum distance from s.

Claim d is the minimum distance from s to y.
(this justifies the output of the problem being a tree)

Proof Any path π from s to y consists of



- π_1 — initial part of path in B
- $e = (u, v)$ — first edge leaving B
- π_2 — rest of path

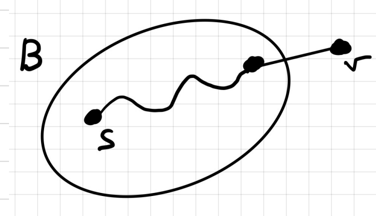
$$w(\pi) \geq w(\pi_1) + w(u, v) \geq d(s, u) + w(u, v) \geq d$$

using that $w(\pi_2) \geq 0$

Note: the proof breaks down for negative weight cycles

Therefore, by induction on $|B|$, the algorithm correctly finds $d(s, v)$ for all v.

Dijkstra's Algorithm: Implementation



Keep "tentative distance" $d(v)\ \forall\ v \notin B$.
$d(v)$ = minimum weight path from s to v with all but last edge in B

$d(v) := \infty\ \forall\ v \neq s$
$d(s) := 0$
$B := \emptyset$
while $|B| < n$ **do**
    $y :=$ vertex of $V - B$ with minimum d value — from heap
    for each edge (y, z) **do**
        if $d(y) + w(y, z) < d(z)$ **then**
            $d(z) := d(y) + w(y, z)$ — and update heap
            $\text{Parent}(z) := y$
        fi
    od
od

Store the d values in a heap of size $\leq n$.
Modifying a d value takes $O(\log n)$ to adjust heap.
Total time, assuming G connected:

$$O(\underbrace{n \log n}_{\text{find min}} + \underbrace{m \log n}_{\text{adjust heap}}) = O(m \log n)$$

Actually, there is a fancier "Fibonacci heap" that gives $O(n \log n + m)$ (see CLRS)

Dijkstra was known for many contributions to computer science, e.g., structured programming, concurrent programming. He designed the above algorithm to demonstrate the capabilities of a new computer (to find railway journeys in the Netherlands). At that time (the 50's) the result was not considered important. He wrote:

> At the time, algorithms were hardly considered a scientific topic. I wouldn't have known where to publish it... The mathematical culture of the day was very much identified with the continuum and infinity. Could a finite discrete problem be of any interest? The number of paths from here to there on a finite graph is finite; each path is a finite length; you must search for the minimum of a finite set. Any finite set has a minimum — next problem, please. It was not considered mathematically respectable.