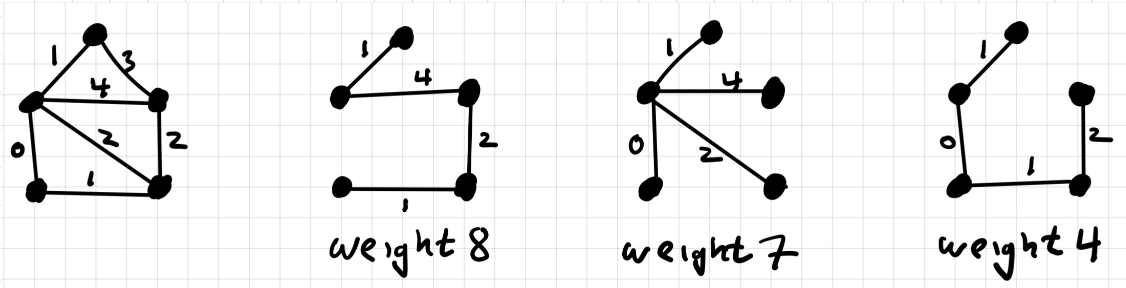
Minimum Spanning Tree

Problem: Given a connected graph $G = (V, E)$ with weights $w : E \to \mathbb{R}$ on the edges, find an edge subset of size $n - 1$ that connects all the vertices and has minimum weight.

E.g.



Recall: Any connected graph on n vertices and $n - 1$ edges is a tree.

The edge subset is called a minimum spanning tree..

Greedy algorithms will find minimum spanning trees.

In fact, there are several possible correct greedy approaches, with different implementation challenges. E.g.

- add cheapest edge first, never build a cycle
  Kruskal's algorithm
- grow connected graph from one vertex
  Prim's algorithm
- throw away expensive edges, never disconnect

Kruskal's Algorithm

Order edges by weight $e_1 \cdots e_m$

$$w(e_i) \le w(e_{i+1})$$

$T := \emptyset$

for i **from** 1 **to** m **do**

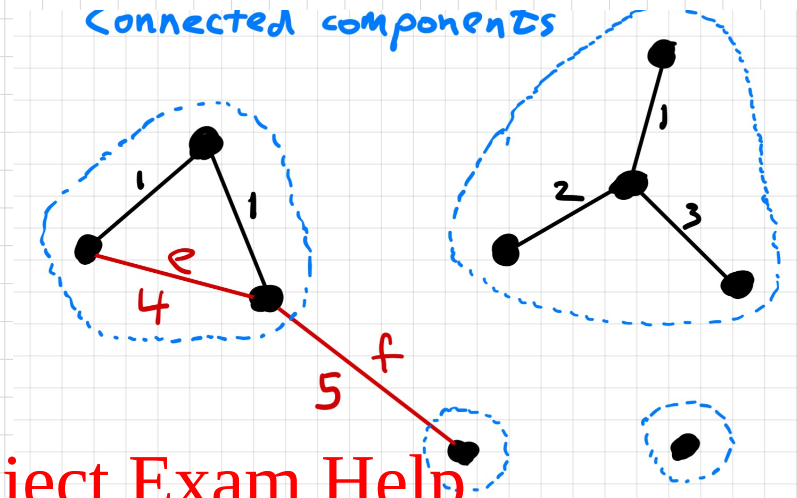
  if e_i does not make a cycle with T **then**

    $T := T \cup \{e_i\}$

  fi

od

General situation



e makes a cycle with T iff e joins vertices in same connected component

e.g. edge e makes a cycle $\rightarrow$ throw it out

edge f does not $\rightarrow$ add f to T

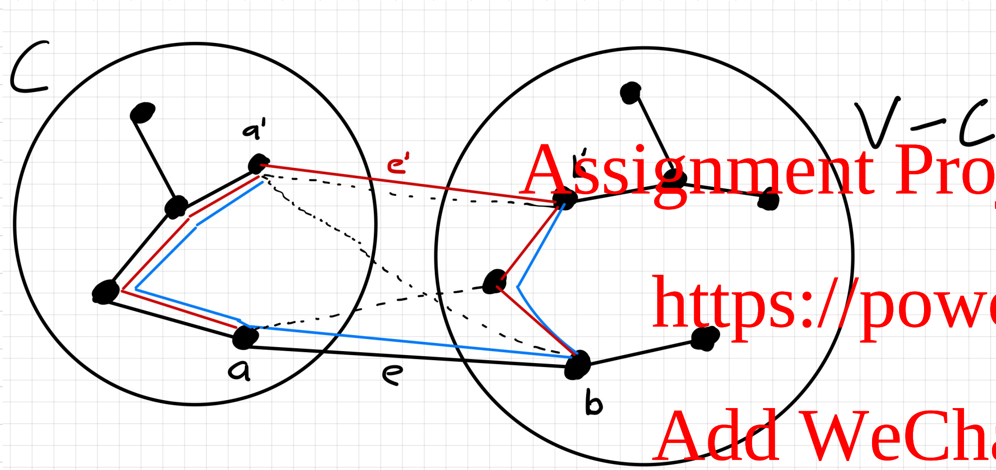
Correctness — an exchange proof. Let T have edges $t_1 \cdots t_{n-1}$.
Prove by induction on i that there is a MST matching T on the first i edges.

Base case: $i = 0$. (Trivially true.)

Assume by induction that there is a MST M matching T on the first $i-1$ edges.

alg.	T	t_1	$\cdots$	t_{i-1}	t_i	$\cdots$	t_{n-1}
		$\parallel$	$\cdots$	$\parallel$	$\nparallel$		
MST	M	m_1	$\cdots$	m_{i-1}	m_i	$\cdots$	m_{n-1}

Let $t_i = e = (a,b)$ and let C be the connected component of T containing a.



Note: When the algorithm considers t_i all edges of weight $< w(t_i)$ have been considered, and none of them go from C to $V-C$.

Look at red path in M from a to b. It must cross from C to $V-C$, say on edge e' .
Then $w(e) \leq w(e')$ by note above, so e' is later in ordering.
Exchange: Let $M' = (M - \{e'\}) \cup \{e\}$.

Claim M' is a MST. (Then we're done, since M' matches T on i edges.)

Proof 1. M' is a spanning tree because it connects all vertices (replace e' by blue path from a' to a in M, edge e, from b to b' in M) and has same number of edges.

2. $w(M') = w(M) - w(e') + w(e) \leq w(M)$ so M' is a minimum spanning tree.

[Use fact: Any connected graph on n vertices and $n-1$ edges is a tree.]

Implementing and analyzing Kruskal's Algorithm

Graph $G = (V, e)$ $\quad |V| = n \quad |E| = m$

$O(m \log m)$ to sort edges $= O(m \log n)$ because $m \leq n^2$ so $\log m$ is $O(\log n)$.

Then we need to maintain connected components as we add edges. Also test

- if (a, b) has a, b in same component (don't add edge), or
- if (a, b) have different components (do add edge).

Union-Find Problem

Maintain a collection of disjoint sets. Operations:

- Find(x) — which set contains element x?
- Union — unite two sets

In our case the elements are vertices and the sets are connected components of T, the tree so far. This Abstract Data Type has a very simple implementation that gives $O(m \log n)$ for Kruskal.

Aside: There is a fancier implementation — CS 466. Algorithm is pretty simple, analysis is hard and true run time involves the very slowly growing inverse Ackerman's function.

Simple implementation of Union-Find.

Keep array $S[1 \cdots n]$, $S[i]$ = components of element i
and keep linked list of elements in each set.

$$\text{e.g.} \quad \begin{array}{lcl} C_1 & : & 1, 3, 5, 6 \\ C_2 & : & 2, 4 \\ C_3 & : & 7 \end{array}$$

S	1	2	3	4	5	6	7
	1	2	1	2	1	1	3

Find is $O(1)$. Union — must join 2 linked lists $O(1)$ and must rename one of the two sets so $O(n)$ in worst case.

But renaming smaller set does better!
e.g., to unite C_1 and C_2 do $C_1 \leftarrow C_1 \cup C_2$ and must update $S(2) = 1$ and $S(4) = 1$

If an element's set number changes, then its set (more than) doubles.
This happens $\leq \log n$ times. Therefore total renaming work is $O(n \log n)$.

Total run time:

$$\underbrace{O(m \log n)}_{\text{sort}} + \underbrace{O(m)}_{\text{finds}} + \underbrace{O(n \log n)}_{\text{unions}}$$

so $O(m \log n)$ assuming G is connected so $m \geq n - 1$.