Tutorial 6

PS 6-6: Gaussian Classifier w/ shared covariance

$p(y=j)=\pi_j \qquad y\in\{1,\dots,C\}$

$$p(x|y=j) = \mathcal{N}(x|\underbrace{\mu_j}_{\text{different mean for each class}}, \underbrace{\Sigma}_{\text{shared covariance among classes}})$$

a) $$g(x) = \operatorname*{argmax}_j \underbrace{\log p(x|y=j)+\log p(y=j)}_{g_j(x)\ \leftarrow\ \text{discriminant function}}$$

$$g_j(x) = \log p(x|y=j)+\log p(y=j)$$

$$= \underbrace{-\frac{1}{2}\|x-\mu_j\|^2_\Sigma}_{=} \underbrace{-\frac{1}{2}\log|\Sigma|}_{\times} \underbrace{-\frac{D}{2}\log 2\pi}_{\times} + \log\pi_j$$

$$= -\frac{1}{2}(x-\mu_j)^T\Sigma^{-1}(x-\mu_j)+\log\pi_j$$

$$= \underbrace{-\frac{1}{2}x^T\Sigma^{-1}x}_{\times} -\frac{1}{2}(-2)\mu_j^T\Sigma^{-1}x - \frac{1}{2}\mu_j^T\Sigma^{-1}\mu_j+\log\pi_j$$

$$= \underbrace{\mu_j^T\Sigma^{-1}}_{w_j^T}\underbrace{x}\ \underbrace{-\frac{1}{2}\mu_j^T\Sigma^{-1}\mu_j+\log\pi_j}_{b_j}$$

$$\boxed{g_j(x) = w_j^T x + b \quad \text{(linear function in } x) \quad \text{where } w_j = \Sigma^{-1}\mu_j,\ \ b_j = -\frac{1}{2}\mu_j^T\Sigma^{-1}\mu_j + \log\pi_j}$$

b) Decision boundary between class i & j is:

$$g_j(x) = g_i(x)$$

$$w_j^Tx+b_j = \underbrace{w_i^Tx+b_i}$$ ↓ subtract RHS

$$w_j^Tx+b_j-w_i^Tx-b_i=0$$

$$\underbrace{(w_j-w_i)^T}_{w}x + \underbrace{b_j-b_i}_{b}=0$$

$$\Rightarrow w = w_j-w_i = \Sigma^{-1}\mu_j - \Sigma^{-1}\mu_i \boxed{=\Sigma^{-1}(\mu_j-\mu_i)}$$

$$\Rightarrow b = b_j-b_i = \underbrace{-\frac{1}{2}\mu_j^T\Sigma^{-1}\mu_j}+\log\pi_j - \left(\underbrace{-\frac{1}{2}\mu_i^T\Sigma^{-1}\mu_i}+\log\pi_i\right)$$

$$= -\frac{1}{2}\left(\underbrace{\mu_j^T\Sigma^{-1}\mu_j - \mu_i^T\Sigma^{-1}\mu_i}\right)+\log\frac{\pi_j}{\pi_i}$$

$$x^2-y^2=(x+y)(x-y)$$

$$a^Ta-b^Tb=(a+b)^T(a-b)$$

$$=a^Ta+\underbrace{b^Ta-a^Tb}_{\times}-b^Tb$$

$$\boxed{b = -\frac{1}{2}(\mu_j+\mu_i)^T\Sigma^{-1}(\mu_j-\mu_i)+\log\frac{\pi_j}{\pi_i}}$$

Decision boundary: $\boxed{w^Tx+b=0}$

Note: $\underbrace{-w^T}_{\hat{w}}x\ \underbrace{-b}_{\hat{b}}=0$

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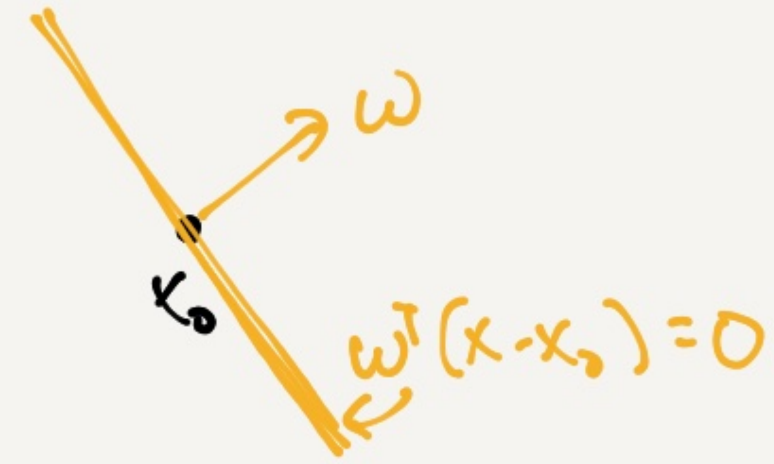
c) write in form

$$w^T(x - x_0) = 0 \quad \Rightarrow$$



$$\Rightarrow \quad w^T x \underbrace{- w^T x_0}_{+b} = 0$$

$$w = \Sigma_i^{-1}(\mu_j - \mu_i)$$

$$-w^T x_0 = b = -\frac{1}{2}(\mu_j + \mu_i)^T \Sigma_i^{-1}(\mu_j - \mu_i) + \log\frac{\pi_j}{\pi_i}$$

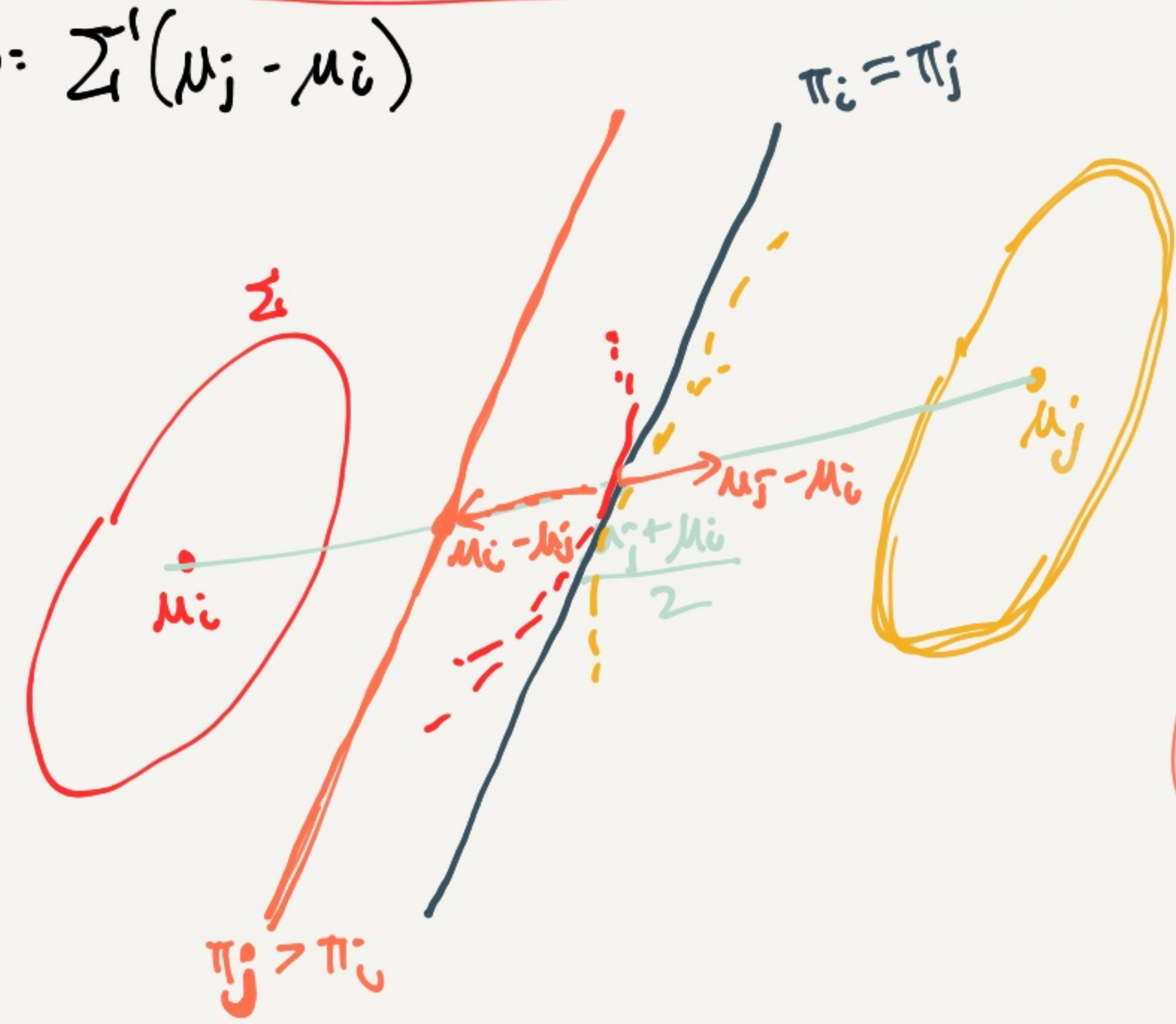
$$\underbrace{-(\mu_j - \mu_i)^T \Sigma_i^{-1}}_{w^T} x_0 = -\frac{1}{2}\underbrace{(\mu_j - \mu_i)^T \Sigma_i^{-1}}_{w^T}(\mu_j + \mu_i) + \log\frac{\pi_j}{\pi_i}$$

$$1 = \frac{\boxed{(\mu_j - \mu_i)^T \Sigma_i^{-1}}(\mu_j - \mu_i)}{(\mu_j - \mu_i)^T \Sigma_i^{-1}(\mu_j - \mu_i)}$$

$$= -w^T\left(\frac{\mu_j + \mu_i}{2} - \frac{(\mu_j - \mu_i)}{\|\mu_j - \mu_i\|^2_{\Sigma_i}}\log\frac{\pi_j}{\pi_i}\right)$$

$$\Rightarrow \quad \boxed{x_0 = \frac{\mu_j + \mu_i}{2} - \frac{(\mu_j - \mu_i)}{\|\mu_j - \mu_i\|^2_{\Sigma_i}}\log\left(\frac{\pi_j}{\pi_i}\right)}$$

$$w = \Sigma_i^{-1}(\mu_j - \mu_i)$$



if $\pi_j > \pi_i$

$\log\frac{\pi_j}{\pi_i} > 0$