356 Let $A \setminus B$ be the difference between bunch A and bunch B. The operator $\setminus$ has precedence level 4, and is defined by the axiom

$$x{:}\ A \setminus B \;=\; x{:}\ A \ \wedge\ \neg\, x{:}\ B$$

For each of the following fixed-point equations, what does recursive construction yield? Does it satisfy the fixed-point equation?

(a) $Q \;=\; nat \setminus (Q{+}3)$

§

$$Q_0 \;=\; null$$
$$Q_1 \;=\; nat \setminus (null{+}3) \;=\; nat \setminus null \;=\; nat$$
$$Q_2 \;=\; nat \setminus (nat{+}3) \;=\; 0, 1, 2$$
$$Q_3 \;=\; nat \setminus ((0, 1, 2){+}3) \;=\; nat \setminus (3, 4, 5) \;=\; 0, 1, 2, nat{+}6$$
$$Q_4 \;=\; nat \setminus ((0, 1, 2, nat{+}6){+}3) \;=\; nat \setminus (3, 4, 5, nat{+}9) \;=\; 0, 1, 2, 6, 7, 8$$
$$Q_5 \;=\; nat \setminus ((0, 1, 2, 6, 7, 8){+}3) \;=\; nat \setminus (3, 4, 5, 9, 10, 11)$$
$$\;=\; 0, 1, 2, 6, 7, 8, nat{+}12$$

Time for a guess. It looks like there are two patterns: the even index pattern and the odd index pattern. So I guess

$$Q_{2\times n} \;=\; 6\times(0,..n) + (0,..3)$$
$$Q_{2\times n+1} \;=\; 6\times(0,..n) + (0,..3),\ (6\times n,..\infty)$$

From the even case, I propose

$$Q_\infty \;=\; 6\times nat + (0,..3)$$

and now I have to check whether it satisfies the equation. Starting with the right side,

$nat \setminus (Q_\infty{+}3)$

$=\ nat \setminus (6\times nat + (0,..3) + 3)$

$=\ nat \setminus (6\times nat + (3,..6))$ — here is an informal expansion

$=\ nat \setminus ((0, 6, 12, 18, 24, ...) + (3,..6))$ — and an informal addition

$=\ nat \setminus (3, 4, 5, 9, 10, 11, 15, 16, 17, 21, 22, 23, 27, 28, 29, ...)$

$=\ 0, 1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, 24, 25, 26, ...$

$=\ (0, 6, 12, 18, 24, ...) + (0,..3)$

$=\ 6\times nat + (0,..3)$

$=\ Q_\infty$

So it does satisfy the equation. From the odd case, we can't make a proposal because we can't simplify $\infty,..\infty$.

(b) $D \;=\; 0, (D{+}1) \setminus (D{-}1)$

§

$$D_0 \;=\; null$$
$$D_1 \;=\; 0, (D_0{+}1) \setminus (D_0{-}1)$$
$$= 0, (null{+}1) \setminus (null{-}1)$$
$$= 0, null \setminus null$$
$$= 0$$
$$D_2 \;=\; 0, (D_1{+}1) \setminus (D_1{-}1)$$
$$=\; 0, (0{+}1) \setminus (0{-}1)$$
$$=\; 0, 1 \setminus {-}1$$
$$=\; 0, 1$$
$$D_3 \;=\; 0, (D_2{+}1) \setminus (D_2{-}1)$$
$$=\; 0, ((0, 1){+}1) \setminus ((0, 1){-}1)$$
$$=\; 0, (1, 2) \setminus ({-}1, 0)$$
$$=\; 0, 1, 2$$
$$D_4 \;=\; 0, (D_3{+}1) \setminus (D_3{-}1)$$
$$=\; 0, ((0, 1, 2){+}1) \setminus ((0, 1, 2){-}1)$$
$$=\; 0, (1, 2, 3) \setminus ({-}1, 0, 1)$$
$$=\; 0, 2, 3$$

$$
\begin{aligned}
D_5 &= 0, (D_4 + 1) \setminus (D_4 - 1)\\
&= 0, ((0, 2, 3)+1) \setminus ((0, 2, 3)-1)\\
&= 0, (1, 3, 4) \setminus (-1, 1, 2)\\
&= 0, 3, 4\\
D_6 &= 0, (D_5 + 1) \setminus (D_5 - 1)\\
&= 0, ((0, 3, 4)+1) \setminus ((0, 3, 4)-1)\\
&= 0, (1, 4, 5) \setminus (-1, 2, 3)\\
&= 0, 1, 4, 5\\
D_7 &= 0, (D_6 + 1) \setminus (D_6 - 1)\\
&= 0, ((0, 1, 4, 5)+1) \setminus ((0, 1, 4, 5)-1)\\
&= 0, (1, 2, 5, 6) \setminus (-1, 0, 3, 4)\\
&= 0, 1, 2, 5, 6\\
D_8 &= 0, (D_7 + 1) \setminus (D_7 - 1)\\
&= 0, ((0, 1, 2, 5, 6)+1) \setminus ((0, 1, 2, 5, 6)-1)\\
&= 0, (1, 2, 3, 6, 7) \setminus (-1, 0, 1, 4, 5)\\
&= 0, 2, 3, 6, 7
\end{aligned}
$$

It's still hard to see the patterns, so maybe we have to go a bit farther. Then we see

$$
\begin{aligned}
D_{4\times n+1} &= 0, 4\times(0,..n) + (3, 4)\\
D_{4\times n+2} &= 0, 1, 4\times(0,..n) + (4, 5)\\
D_{4\times n+3} &= 0, 1, 2, 4\times(0,..n) + (5, 6)\\
D_{4\times n+4} &= 0, 2, 3, 4\times(0,..n) + (6, 7)
\end{aligned}
$$

We have a choice of four possible answers for D_∞, but none of them satisfies the equation. Recursive construction fails.

(c) $E = nat \setminus (E+1)$

§

$$
\begin{aligned}
E_0 &= null\\
E_1 &= nat\\
E_2 &= 0\\
E_3 &= 0, nat+2\\
E_4 &= 0, 2\\
E_5 &= 0, 2, nat+4\\
E_{2\times n} &= 2\times(0,..n)\\
E_{2\times n+1} &= 2\times(0,..n), nat+2\times n
\end{aligned}
$$

From the even case, we propose

$E_\infty = 2\times nat$

which satisfies the equation. From the odd case, we propose

$E_\infty = 2\times nat, \infty$

which does not satisfy the equation.

(d) $F = 0, (nat \setminus F)+1$

§

$$
\begin{aligned}
F_0 &= null\\
F_1 &= nat\\
F_2 &= 0\\
F_3 &= 0, nat+2\\
F_4 &= 0, 2\\
F_5 &= 0, 2, nat+4\\
F_{2\times n} &= 2\times(0,..n)\\
F_{2\times n+1} &= 2\times(0,..n), nat+2\times n
\end{aligned}
$$

From the even case, we propose

$F_\infty = 2\times nat$

which satisfies the fixed-point equation. From the odd case, we propose

$F_{\infty} = 2 \times nat, \infty$

which does not satisfy the fixed-point equation.