6 Prove each of the following laws of Binary Theory using the proof format given in Subsection 1.0.1, and any laws listed in Section 11.4. Do not use the Completion Rule.

(a) $a \wedge b \Rightarrow a \vee b$

§	$a \wedge b$	specialization
$\Rightarrow$	a	generalization
$\Rightarrow$	$a \vee b$	

(b) $(a \wedge b) \vee (b \wedge c) \vee (a \wedge c) \;=\; (a \vee b) \wedge (b \vee c) \wedge (a \vee c)$

§	$(a \wedge b) \vee (b \wedge c) \vee (a \wedge c)$	distribute
$=$	$(a \vee b \vee a) \wedge (a \vee b \vee c) \wedge (a \vee c \vee a) \wedge (a \vee c \vee c) \wedge (b \vee b \vee a) \wedge (b \vee b \vee c) \wedge (b \vee c \vee a) \wedge (b \vee c \vee c)$	symmetry and idempotence
$=$	$\underline{(a \vee b) \wedge (a \vee b \vee c)} \wedge (a \vee c) \wedge (b \vee c)$	absorption
$=$	$(a \vee b) \wedge (b \vee c) \wedge (a \vee c)$	

(c) $\neg a \Rightarrow (a \Rightarrow b)$

§	$\neg a \Rightarrow (a \Rightarrow b)$	portation
$=$	$\neg a \wedge a \Rightarrow b$	noncontradiction
$=$	$\bot \Rightarrow b$	base
$=$	$\top$	

(d) $a = (b \Rightarrow a) \;=\; a \vee b$

§	$(\underline{a = (b \Rightarrow a)} \;=\; a \vee b)$	symmetry of $=$
$=$	$((b \Rightarrow a) = a \;=\; a \vee b)$	associativity of $=$
$=$	$((b \Rightarrow a) \;=\; \underline{a = (a \vee b)})$	symmetry of $=$ and $\vee$
$=$	$((b \Rightarrow a) \;=\; (b \vee a) = a)$	inclusion
$=$	$\top$	

(e) $a = (a \Rightarrow b) \;=\; a \wedge b$

§	$(\underline{a = (a \Rightarrow b)} \;=\; a \wedge b)$	symmetry of $=$
$=$	$((a \Rightarrow b) = a \;=\; a \wedge b)$	associativity of $=$
$=$	$((a \Rightarrow b) \;=\; \underline{a = (a \wedge b)})$	symmetry of $=$
$=$	$((a \Rightarrow b) \;=\; (a \wedge b) = a)$	inclusion
$=$	$\top$	

(f) $(a \Rightarrow c) \wedge (b \Rightarrow \neg c) \Rightarrow \neg(a \wedge b)$

§	$(a \Rightarrow c) \wedge (b \Rightarrow \neg c)$	law of conflation
$\Rightarrow$	$a \wedge b \;\Rightarrow\; c \wedge \neg c$	contrapositive law
$=$	$\neg(c \wedge \neg c) \;\Rightarrow\; \neg(a \wedge b)$	antecedent is law of noncontradiction
$=$	$\top \;\Rightarrow\; \neg(a \wedge b)$	identity for $\Rightarrow$
$=$	$\neg(a \wedge b)$	

(g) $a \wedge \neg b \Rightarrow a \vee b$

§	$a \wedge \neg b$	specialization
$\Rightarrow$	a	generalization
$\Rightarrow$	$a \vee b$	

(h) $(a \Rightarrow b) \wedge (c \Rightarrow d) \wedge (a \vee c) \Rightarrow (b \vee d)$

§	$(a \Rightarrow b) \wedge (c \Rightarrow d) \wedge (a \vee c) \Rightarrow (b \vee d)$	portation
$=$	$(a \Rightarrow b) \wedge (c \Rightarrow d) \Rightarrow (a \vee c \Rightarrow b \vee d)$	conflation
$=$	$\top$	

(i)		$a \wedge \neg a \Rightarrow b$	
§		$a \wedge \neg a \Rightarrow b$	noncontradiction
	$=$	$\bot \Rightarrow b$	base
	$=$	$\top$	
(j)		$(a \Rightarrow b) \vee (b \Rightarrow a)$	
§		$(\underline{a \Rightarrow b}) \vee (\underline{b \Rightarrow a})$	inclusion, twice
	$=$	$\neg a \vee b \vee \neg b \vee a$	symmetry of $\vee$
	$=$	$\underline{a \vee \neg a} \vee \underline{b \vee \neg b}$	excluded middle, twice
	$=$	$\top \vee \top$	idempotence of $\vee$, or base law
	$=$	$\top$	
(k)√		$\neg(a \wedge \neg(a \vee b))$	
(l)		$(\neg a \Rightarrow \neg b) \wedge (a \ne b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	
§		$(\neg a \Rightarrow \neg b) \wedge (a \ne b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	law of exclusion
	$=$	$(\neg a \Rightarrow \neg b) \wedge (a = \neg b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	use $a = \neg b$ to replace $\neg b$ with a
	$=$	$(\neg a \Rightarrow a) \wedge (a = \neg b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	indirect proof
	$=$	$a \wedge (a = \neg b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	context to replace second a by $\top$, and identity
	$=$	$a \wedge \neg b \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	duality and double negation
	$=$	$\neg(\neg a \vee b) \vee (a \wedge c \Rightarrow b \wedge c)$	inclusion
	$=$	$\neg(a \Rightarrow b) \vee (a \wedge c \Rightarrow b \wedge c)$	
			assume $a \Rightarrow b$ to simplify the right disjunct; strengthen b to a
	$\Leftarrow$	$\neg(a \Rightarrow b) \vee (a \wedge c \Rightarrow a \wedge c)$	reflexivity and base
	$=$	$\top$	
	Alternative proof		
		$(\neg a \Rightarrow \neg b) \wedge (a \ne b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	contrapositive, monotonicity
	$\Leftarrow$	$(b \Rightarrow a) \wedge (a \ne b) \;\vee\; (a \Rightarrow b)$	distributivity
	$=$	$((b \Rightarrow a) \vee (a \Rightarrow b)) \wedge ((a \ne b) \vee (a \Rightarrow b))$	material implication twice
	$=$	$(\neg b \vee a \vee \neg a \vee b) \wedge ((a \ne b) \vee (a \Rightarrow b))$	symmetry and associativity
	$=$	$(a \vee \neg a \vee b \vee \neg b) \wedge ((a \ne b) \vee (a \Rightarrow b))$	excluded middle twice, base or idempotent
	$=$	$(a \ne b) \vee (a \Rightarrow b)$	generic unequality
	$=$	$\neg(a = b) \vee (a \Rightarrow b)$	material implication
	$=$	$(a = b) \Rightarrow (a \Rightarrow b)$	antisymmetry (double implication)
	$=$	$(a \Rightarrow b) \wedge (b \Rightarrow a) \Rightarrow (a \Rightarrow b)$	specialization
	$=$	$\top$	
(m)		$(a \Rightarrow \neg a) \Rightarrow \neg a$	
§		$(\underline{a \Rightarrow \neg a}) \Rightarrow \neg a$	material implication
	$=$	$(\underline{\neg a \vee \neg a}) \Rightarrow \neg a$	idempotent
	$=$	$\neg a \Rightarrow \neg a$	reflexive
	$=$	$\top$	
(n)		$(a \Rightarrow b) \wedge (\neg a \Rightarrow b) = b$	
§		$(a \Rightarrow b) \wedge (\neg a \Rightarrow b)$	antidistributive law
	$=$	$a \vee \neg a \Rightarrow b$	antecedent is law of excluded middle
	$=$	$\top \Rightarrow b$	identity for $\Rightarrow$
	$=$	b	
	OR	$(a \Rightarrow b) \wedge (\neg a \Rightarrow b)$	case analysis
	$=$	**if** a **then** b **else** b **fi**	generic case idempotent
	$=$	b	

(o)	$(a\Rightarrow b)\Rightarrow a \;=\; a$	
§	$(a\Rightarrow b)\Rightarrow a$	use main consequent to simplify antecedent
$=$	$(\underline{\bot\Rightarrow b})\Rightarrow a$	base
$=$	$\top \Rightarrow a$	identity
$=$	a	
or	$(\underline{a\Rightarrow b})\Rightarrow a$	inclusion
$=$	$(\neg a \lor b) \Rightarrow a$	inclusion
$=$	$\underline{\neg(\neg a \lor b)} \lor a$	duality
$=$	$(\underline{\neg\neg a} \land \neg b) \;\lor\; a$	double negation
$=$	$(a \land \neg b) \;\lor\; a$	symmetry of $\lor$
$=$	$a \;\lor\; (a \land \neg b)$	absorption
$=$	a	
(p)	$a{=}b \lor a{=}c \lor b{=}c$	
§	$a{=}b \lor a{=}c \lor b{=}\underline{c}$	identity and reflexive laws for $=$
$=$	$a{=}b \lor a{=}c \lor \underline{b{=}((a{=}a){=}c)}$	symmetry and associative laws for $=$
$=$	$(a{=}b \lor a{=}c) \lor (a{=}b){=}(a{=}c)$	main $\lor$ distributes over $=$
$=$	$(a{=}b \lor a{=}c \lor a{=}b) = (a{=}b \lor a{=}c \lor a{=}c)$	symmetry, associativity, and idempotence of $\lor$ twice
$=$	$(a{=}b \lor a{=}c) = (a{=}b \lor a{=}c)$	$=$ is reflexive
$=$	$\top$	
Here's another solution.		
	$a{=}b \lor a{=}c \lor b{=}c$	double negation
$=$	$\neg\neg(a{=}b) \lor a{=}c \lor b{=}c$	material implication
$=$	$\neg(a{=}b) \Rightarrow a{=}c \lor b{=}c$	unequality
$=$	$a{\ne}b \Rightarrow a{=}c \lor b{=}c$	exclusion
$=$	$a{=}\neg b \Rightarrow a{=}c \lor b{=}c$	context: assume antecedent
$=$	$a{=}\neg b \Rightarrow \neg b{=}c \lor b{=}c$	exclusion
$=$	$a{=}\neg b \Rightarrow b{\ne}c \lor b{=}c$	unequality
$=$	$a{=}\neg b \Rightarrow \neg(b{=}c) \lor b{=}c$	symmetry, excluded middle
$=$	$a{=}\neg b \Rightarrow \top$	base
$=$	$\top$	
(q)	$a\land b \lor a\land\neg b \;=\; a$	
§	$a\land b \lor a\land\neg b \;=\; a$	factor
$=$	$a\land(b \lor \neg b) \;=\; a$	excluded middle
$=$	$a\land\top \;=\; a$	symmetry
$=$	$\top\land a \;=\; a$	identity
$=$	$\top$	
(r)	$a\Rightarrow(b\Rightarrow a)$	
§	$a\Rightarrow(b\Rightarrow a)$	portation
$=$	$a\land b \Rightarrow a$	specialization
$=$	$\top$	
(s)	$a \Rightarrow a \land b \;=\; a \Rightarrow b \;=\; a \lor b \Rightarrow b$	
§	$(a \Rightarrow a \land b \;=\; a \Rightarrow b \;=\; a \lor b \Rightarrow b)$	distribute $\Rightarrow$ over $\land$ in first part; antidistribute $\Rightarrow$ over $\lor$ in last part
$=$	$((a \Rightarrow a) \land (a \Rightarrow b) \;=\; a \Rightarrow b \;=\; (a \Rightarrow b) \land (b \Rightarrow b))$	reflexivity of $\Rightarrow$ and identity of $\land$
$=$	$((a \Rightarrow b) \;=\; a \Rightarrow b \;=\; (a \Rightarrow b))$	reflexivity of $=$
$=$	$\top$	

(t) $(a \Rightarrow a \wedge b) \vee (b \Rightarrow a \wedge b)$

§	$(a \Rightarrow a \wedge b) \vee (b \Rightarrow a \wedge b)$	anti-distributive
$=$	$a \wedge b \Rightarrow a \wedge b$	reflexive
$=$	$\top$	

(u) $(a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p) \;=\; p=(x \vee \neg a)$

§	$p=(x \vee \neg a)$	case idempotent law
$=$	**if** a **then** $p=(x \vee \neg a)$ **else** $p=(x \vee \neg a)$ **fi**	context
$=$	**if** a **then** $p=(x \vee \neg \top)$ **else** $p=(x \vee \neg \bot)$ **fi**	
$=$	**if** a **then** $p=(x \vee \bot)$ **else** $p=(x \vee \top)$ **fi**	
$=$	**if** a **then** $p=x$ **else** p **fi**	case analysis
$=$	$(a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p)$	