 Prove that the average value of

(a) n^2 as n varies over $nat+1$ according to probability 2^{-n} is 6 .

§ Lemma 0:

$\Sigma n: nat+1\cdot\ 2^{-n}$
$=\ (\Sigma n: nat\cdot\ 2^{-n}) - 2^{-0}$
$=\ (\Sigma n: nat+1\cdot\ 2^{-(n-1)}) - 1$
$=\ (\Sigma n: nat+1\cdot\ 2^{-n}\times 2) - 1$
$=\ 2\times(\Sigma n: nat+1\cdot\ 2^{-n}) - 1$

Lemma 1:

$\top$ use Lemma 0
$=\ (\Sigma n: nat+1\cdot\ 2^{-n})\ =\ 2\times(\Sigma n: nat+1\cdot\ 2^{-n}) - 1$
$=\ (\Sigma n: nat+1\cdot\ 2^{-n})\ =\ 1$

and therefore, as n varies over $nat+1$, 2^{-n} is a distribution.

Lemma 2:

$\Sigma n: nat+1\cdot\ n\times 2^{-n}$
$=\ \Sigma n: nat\cdot\ n\times 2^{-n}$
$=\ \Sigma n: nat+1\cdot\ (n-1)\times 2^{-(n-1)}$
$=\ 2\times(\Sigma n: nat+1\cdot\ n\times 2^{-n}) - 2\times(\Sigma n: nat+1\cdot\ 2^{-n})$ use Lemma 1
$=\ 2\times(\Sigma n: nat+1\cdot\ n\times 2^{-n}) - 2$

Lemma 3:

$\top$ use Lemma 2

$=\ (\Sigma n: nat+1\cdot\ n\times 2^{-n})\ =\ 2\times(\Sigma n: nat+1\cdot\ n\times 2^{-n}) - 2$
$=\ (\Sigma n: nat+1\cdot\ n\times 2^{-n})\ =\ 2$

Lemma 4:

$\Sigma n: nat+1\cdot\ n^2\times 2^{-n}$
$=\ \Sigma n: nat\cdot\ n^2\times 2^{-n}$
$=\ \Sigma n: nat+1\cdot\ (n-1)^2\times 2^{-(n-1)}$
$=\ 2\times(\Sigma n: nat+1\cdot\ n^2\times 2^{-n}) - 4\times(\Sigma n: nat+1\cdot\ n\times 2^{-n}) + 2\times(\Sigma n: nat+1\cdot\ 2^{-n})$ Lemmas 1 and 3

$=\ 2\times(\Sigma n: nat+1\cdot\ n^2\times 2^{-n}) - 4\times 2 + 2$
$=\ 2\times(\Sigma n: nat+1\cdot\ n^2\times 2^{-n}) - 6$

Lemma 5:

$\top$ use Lemma 4
$=\ (\Sigma n: nat+1\cdot\ n^2\times 2^{-n})\ =\ 2\times(\Sigma n: nat+1\cdot\ n^2\times 2^{-n}) - 6$
$=\ (\Sigma n: nat+1\cdot\ n^2\times 2^{-n})\ =\ 6$

The average value of n^2 as n varies over $nat+1$ according to distribution 2^{-n} is

$2^{-n'}.\ n^2$
$=\ \Sigma n'': nat+1\cdot\ 2^{-n''}\times n''^2$ use Lemma 5
$=\ 6$

(b) n as it varies over nat according to probability $(5/6)^n \times 1/6$ is 5 .

§ Lemma 6:

$\Sigma n: nat\cdot\ (5/6)^n$
$=\ 1 + \Sigma n: nat+1\cdot\ (5/6)^n$
$=\ 1 + \Sigma n: nat\cdot\ (5/6)^{n+1}$
$=\ 1\ +\ 5/6 \times \Sigma n: nat\cdot\ (5/6)^n$

Lemma 7:

$\top$ use Lemma 6
$=\ (\Sigma n: nat\cdot\ (5/6)^n)\ =\ 1\ +\ 5/6 \times (\Sigma n: nat\cdot\ (5/6)^n)$
$=\ (\Sigma n: nat\cdot\ (5/6)^n)\ =\ 1 / (1 - 5/6)$
$=\ (\Sigma n: nat\cdot\ (5/6)^n)\ =\ 6$

and therefore, as n varies over nat , $(5/6)^n \times 1/6$ is a distribution.

Lemma 8:

$\quad \Sigma n\!: nat\cdot\ (5/6)^n \times n$

$= \quad 0 + \Sigma n\!: nat{+}1\cdot\ (5/6)^n \times n$

$= \quad \Sigma n\!: nat\cdot\ (5/6)^{n+1} \times (n+1)$

$= \quad 5/6 \times (\Sigma n\!: nat\cdot\ (5/6)^n \times n) \ + \ 5/6 \times (\Sigma n\!: nat\cdot\ (5/6)^n)$ use Lemma 7

$= \quad 5/6 \times (\Sigma n\!: nat\cdot\ (5/6)^n \times n) \ + \ 5/6 \times 6$

$= \quad 5/6 \times (\Sigma n\!: nat\cdot\ (5/6)^n \times n) \ + \ 5$

Lemma 9:

$\quad \top$ use Lemma 8

$= \quad (\Sigma n\!: nat\cdot\ (5/6)^n \times n) \ = \ 5/6 \times (\Sigma n\!: nat\cdot\ (5/6)^n \times n) + 5$

$= \quad (\Sigma n\!: nat\cdot\ (5/6)^n \times n) \ = \ 5 / (1 - 5/6)$

$= \quad (\Sigma n\!: nat\cdot\ (5/6)^n \times n) \ = \ 30$

The average value of n as it varies over nat according to distribution $(5/6)^n \times 1/6$ is

$\quad (5/6)^{n'} \times 1/6.\ n$

$= \quad \Sigma n''\!: nat\cdot\ (5/6)^{n''} \times 1/6 \times n''$

$= \quad 1/6 \times \Sigma n''\!: nat\cdot\ (5/6)^{n''} \times n''$ use Lemma 9

$= \quad 1/6 \times 30$

$= \quad 5$