# Variable Declaration

**var** $x$: $T\cdot P$ declare local state variable $x$ with type $T$ and scope $P$

$= \quad \exists x, x' : T \cdot P$

**var** $x$: $int\cdot$ $x := 2.\ y := x+z$



$= \quad \exists x, x' : int \cdot\ x'=2 \ \wedge\ y' = 2+z \ \wedge\ z'=z$ one-point for $x'$ and idempotent for $x$



$= \quad y' = 2+z \ \wedge\ z'=z$



**var** $x$: $int\cdot$ $y := x$

$= \quad \exists x, x' : int \cdot\ x'=x \wedge y'=x \wedge z'=z$ one-point for $x$ and $x'$

$= \quad z'=z$

**var** $x$: $int\cdot$ $y := x-x$

$= \quad y'=0 \wedge z'=z$

# Variable Declaration

$$\textbf{var}\ x\colon T \cdot P$$

$$= \quad \exists x\colon \mathit{undefined} \cdot \exists x'\colon T, \mathit{undefined} \cdot P$$

$$\textbf{var}\ x\colon T := e \cdot P$$

$$= \quad \exists x\colon e \cdot \exists x'\colon T \cdot P$$

# Variable Suspension

Suppose the state consists of variables $w$ , $x$ , $y$ , and $z$ .

**frame** $w, x\cdot P$ within $P$ , $y$ and $z$ are constants (no $y'$ and $z'$ )

$=$ $P \wedge y'=y \wedge z'=z$



$x := e$ $=$ **frame** $x\cdot\ x' = e$



$ok$ $=$ **frame**$\cdot\ \top$



$s := \Sigma L$ $\Leftarrow$ **frame** $s\cdot$ **var** $n$: $nat\cdot$ $s' = \Sigma L$

$s' = \Sigma L$ $\Leftarrow$

# Array

$A\ i := e \quad = \quad A'i{=}e \wedge (\forall j \cdot j{\neq}i \Rightarrow A'j{=}Aj) \wedge x'{=}x \wedge y'{=}y \wedge \ldots$

| | | |
|---|---|---|
| | $A2{:=}\ 3.\ \ i{:=}\ 2.\ \ Ai{:=}\ 4.\ \ Ai{=}A2$ | ✗ Substitution Law |
| = | $A2{:=}\ 3.\ \ i{:=}\ 2.\ \ 4{=}A2$ | ✓ Substitution Law |
| = | $A2{:=}\ 3.\ \ 4{=}A2$ | ✗ Substitution Law |
| = | $4{=}3$ | |
| = | $\bot$ ✗ | |



| | | |
|---|---|---|
| | $A2{:=}\ 2.\ \ A(A2){:=}\ 3.\ \ A2{=}2$ | ✗ Substitution Law |
| = | $A2{:=}\ 2.\ \ A2{=}2$ | ✗ Substitution Law |
| = | $2{=}2$ | |
| = | $\top$ ✗ | |

# Array

$$
\begin{aligned}
A\,i := e \quad &= \quad A'i{=}e \wedge (\forall j{\cdot}\ j{\neq}i \Rightarrow A'j{=}Aj) \wedge x'{=}x \wedge y'{=}y \wedge \ldots \\
&= \quad A' = i{\rightarrow}e \mid A \ \wedge\ x'{=}x \ \wedge\ y'{=}y \ \wedge \ldots \\
&= \quad A{:=}\ i{\rightarrow}e \mid A
\end{aligned}
$$



$A2{:=}\ 3.\ \ i{:=}\ 2.\ \ Ai{:=}\ 4.\ \ Ai{=}A2$



$=$ $\quad A{:=}\ 2{\rightarrow}3 \mid A.\ \ i{:=}\ 2.\ \ A{:=}\ i{\rightarrow}4 \mid A.\ \ Ai = A2$ Substitution Law



$=$ $\quad A{:=}\ 2{\rightarrow}3 \mid A.\ \ i{:=}\ 2.\ \ (i{\rightarrow}4 \mid A)i = (i{\rightarrow}4 \mid A)2$ Substitution Law

$=$ $\quad A{:=}\ 2{\rightarrow}3 \mid A.\ \ (2{\rightarrow}4 \mid A)2 = (2{\rightarrow}4 \mid A)2$ $=$ is reflexive

$=$ $\quad A{:=}\ 2{\rightarrow}3 \mid A.\ \ \top$ Substitution Law

$=$ $\quad \top$

# Array

$$
\begin{aligned}
A\ i := e \quad &= \quad A'i{=}e \wedge (\forall j\cdot\ j{\neq}i \Rightarrow A'j{=}Aj) \wedge x'{=}x \wedge y'{=}y \wedge \ldots \\
&= \quad A' = i{\rightarrow}e \mid A \ \wedge\ x'{=}x \ \wedge\ y'{=}y \ \wedge \ldots \\
&= \quad A{:=}\ i{\rightarrow}e \mid A
\end{aligned}
$$



$A2{:=}\ 2.\ \ A(A2){:=}\ 3.\ \ A2{=}2$



$=\quad A{:=}\ 2{\rightarrow}2 \mid A.\ \ A{:=}\ A2{\rightarrow}3 \mid A.\ \ A2{=}2$ Substitution Law



$=\quad A{:=}\ 2{\rightarrow}2 \mid A.\ \ (A2{\rightarrow}3 \mid A)2 = 2$ Substitution Law

$=\quad ((2{\rightarrow}2 \mid A)2{\rightarrow}3 \mid 2{\rightarrow}2 \mid A)\ 2 = 2$

$=\quad (2{\rightarrow}3 \mid 2{\rightarrow}2 \mid A)\ 2 = 2$

$=\quad 3 = 2$

$=\quad \bot$

# Array

remember

$Ai := e$ becomes $A := i \rightarrow e \mid A$



$Aij := e$ becomes $A := (i;j) \rightarrow e \mid A$

# Record

*person* = “name” → *text*

| “age” → *nat*

**var** *p*: *person*



*p*:= “name” → “Josh” | “age” → 17

*p* “age”:= 18

*p*:= “age” → 18 | *p*

# While Loop

$$W \Leftarrow \textbf{while } b \textbf{ do } P \textbf{ od}$$

means

$$W \Leftarrow \textbf{if } b \textbf{ then } P.\ W \textbf{ else } ok \textbf{ fi}$$



to prove

$$s' = s + \Sigma\, L\,[n;..\#L] \ \wedge\ t' = t + \#L - n \Leftarrow$$



$$\textbf{while } n{\neq}\#L \textbf{ do } s{:=}\, s + Ln.\ n{:=}\, n{+}1.\ t{:=}\, t{+}1 \textbf{ od}$$



prove instead

$$s' = s + \Sigma\, L\,[n;..\#L] \ \wedge\ t' = t + \#L - n \Leftarrow$$

$$\textbf{if } n{\neq}\#L \textbf{ then } \quad s{:=}\, s + Ln.\ n{:=}\, n{+}1.\ t{:=}\, t{+}1.$$

$$s' = s + \Sigma\, L\,[n;..\#L] \ \wedge\ t' = t + \#L - n$$

$$\textbf{else } ok \textbf{ fi}$$

# Exit Loop

$L \Leftarrow$ **do**

$A$.

**exit when** $b$.

$C$

**od**



means

$L \Leftarrow A.$ **if** $b$ **then** $ok$ **else** $C.$ $L$ **fi**

# Deep Exit

$P \Leftarrow$ **do**

$A$.

**do**

$B$.

**exit** 2 **when** $c$.

$D$

**od**.

$E$

**od**



means

$P \Leftarrow A.\ Q$

$Q \Leftarrow B.$ **if** $c$ **then** $ok$ **else** $D.\ Q$ **fi**

# Deep Exit

$P \Leftarrow$ **do**

$A$.

**exit** 1 **when** $b$.

$C$.

**do**

$D$.

**exit** 2 **when** $e$.

$F$.

**exit** 1 **when** $g$.

$H$

**od**.

$I$

**od**

means

$P \Leftarrow A$. **if** $b$ **then** $ok$ **else** $C$. $Q$ **fi**

$Q \Leftarrow D$. **if** $e$ **then** $ok$ **else** $F$. **if** $g$ **then** $I$. $P$ **else** $H$. $Q$ **fi fi**

# Two-Dimensional Search

$P$ = **if** $x$: $A\ (0,..n)\ (0,..m)$ **then** $x = A\ i'\ j'$ **else** $i'=n \wedge j'=m$ **fi**

$Q$ = **if** $x$: $A\ (i,..n)\ (0,..m)$ **then** $x = A\ i'\ j'$ **else** $i'=n \wedge j'=m$ **fi**

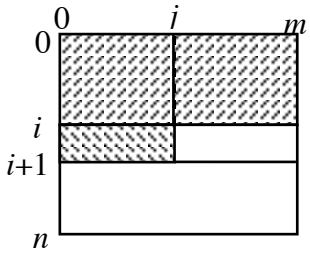
$R$ = **if** $x$: $A\ i\ (j,..m)$, $A\ (i+1,..n)\ (0,..m)$ **then** $x = A\ i'\ j'$ **else** $i'=n \wedge j'=m$ **fi**



$P \Leftarrow i:= 0.\ i \le n \Rightarrow Q$

$i \le n \Rightarrow Q \Leftarrow$ **if** $i=n$ **then** $j:= m$ **else** $i<n \Rightarrow Q$ **fi**

$i<n \Rightarrow Q \Leftarrow j:= 0.\ i<n \wedge j \le m \Rightarrow R$




$i<n \wedge j \le m \Rightarrow R \Leftarrow$ **if** $j=m$ **then** $i:= i+1.\ i \le n \Rightarrow Q$ **else** $i<n \wedge j<m \Rightarrow R$ **fi**

$i<n \wedge j<m \Rightarrow R \Leftarrow$ **if** $A\ i\ j = x$ **then** $ok$ **else** $j:= j+1.\ i<n \wedge j \le m \Rightarrow R$ **fi**

$t' \le t + n{\times}m \;\Leftarrow\; i{:=}\,0.\; i{\le}n \;\Rightarrow\; t' \le t + (n{-}i){\times}m$

$i{\le}n \;\Rightarrow\; t' \le t + (n{-}i){\times}m \;\Leftarrow\;$ **if** $i{=}n$ **then** $j{:=}\,m$ **else** $i{<}n \;\Rightarrow\; t' \le t + (n{-}i){\times}m$ **fi**

$i{<}n \;\Rightarrow\; t' \le t + (n{-}i){\times}m \;\Leftarrow\; j{:=}\,0.\; i{<}n \wedge j{\le}m \;\Rightarrow\; t' \le t + (n{-}i){\times}m - j$



$i{<}n \wedge j{\le}m \;\Rightarrow\; t' \le t + (n{-}i){\times}m - j \;\Leftarrow$



$t{:=}\,t{+}1.$

**if** $j{=}m$ **then** $i{:=}\,i{+}1.\; i{\le}n \;\Rightarrow\; t' \le t + (n{-}i){\times}m$



**else** $i{<}n \wedge j{<}m \;\Rightarrow\; t' \le t + (n{-}i){\times}m - j$ **fi**

$i{<}n \wedge j{<}m \;\Rightarrow\; t' \le t + (n{-}i){\times}m - j \;\Leftarrow$

**if** $A\; i\; j = x$ **then** $ok$ **else** $j{:=}\,j{+}1.\; i{<}n \wedge j{\le}m \;\Rightarrow\; t' \le t + (n{-}i){\times}m - j$ **fi**

$P \Leftarrow i := 0.\ L0$

$L0 \Leftarrow$ **if** $i=n$ **then** $j := m$

**else** $j := 0.\ L1$ **fi**

$L1 \Leftarrow$ **if** $j=m$ **then** $i := i+1.\ L0$

**else if** $A\ i\ j = x$ **then** $ok$

**else** $j := j+1.\ L1$ **fi fi**



in C:

```
P:   i = 0;
L0:  if (i==n) j = m;
     else {      j = 0;
        L1: if (j==m) {i = i+1;  goto L0;}
            else if (A[i][j]==x);
            else {j = j+1;  goto L1;}
     }
```

# For Loop

**for** $i$:= $m$;..$n$ **do** $P$ **od**

$i$ is a fresh name (a local constant)

$m$ and $n$ are integer expressions such that $m \leq n$

the number of iterations is $n-m$

$P$ is a specification

# For Loop

$Fmn \Leftarrow$ **for** $i := m;..n$ **do** $P$ **od**

means

$Fii \Leftarrow m \leq i \leq n \wedge ok$

$Fi(i+1) \Leftarrow m \leq i < n \wedge P$

$Fik \Leftarrow m \leq i < j < k \leq n \wedge (Fij.\ Fjk)$

# For Loop

example: $x'=2^n$

$$F \;=\; \langle i, j: nat \rightarrow x' = x \times 2^{j-i} \rangle$$

$$x'=2^n \;\Leftarrow\; x:= 1.\ F0n$$



proof

| | | |
|---|---|---|
| | $x:= 1.\ F0n$ | expand $F0n$ |
| = | $x:= 1.\ x' = x \times 2^{n-0}$ | simplify and Substitution Law |
| = | $x'=2^n$ | |

# For Loop

example: $x'=2^n$

$$F \;=\; \langle i, j: nat \rightarrow x' = x \times 2^{j-i} \rangle$$

$$x'=2^n \;\Leftarrow\; x:= 1.\ F0n$$

$$F0n \;\Leftarrow\; \textbf{for}\ i:= 0;..n\ \textbf{do}\ x:= 2\times x\ \textbf{od}$$



proof

| | | |
|---|---|---|
| | $F\ i\ i$ | |
| = | $x' = x \times 2^{i-i}$ | law of exponents |
| = | $x' = x \times 1$ | simplify |
| = | $x' = x$ | |
| $\Leftarrow$ | $ok$ | |

# For Loop

example: $x'=2^n$

$$F \;=\; \langle i, j: nat \rightarrow x' = x \times 2^{j-i} \rangle$$

$$x'=2^n \;\Leftarrow\; x:= 1.\ F0n$$

$$F0n \;\Leftarrow\; \textbf{for}\ i:= 0;..n\ \textbf{do}\ x:= 2\times x\ \textbf{od}$$



proof

$Fi(i+1)$ expand

$=$ $x' = x \times 2^{i+1-i}$

$\Leftarrow$ $x:= 2\times x$

# For Loop

example: $x'=2^n$

$$F \;=\; \langle i, j: nat \rightarrow x' = x \times 2^{j-i} \rangle$$

$$x'=2^n \;\Leftarrow\; x:= 1.\ F0n$$

$$F0n \;\Leftarrow\; \textbf{for}\ i:= 0;..n\ \textbf{do}\ x:= 2\times x\ \textbf{od}$$



proof

$$F\ i\ j.\ F\ j\ k$$

$$= \quad x' = x \times 2^{j-i}.\ x' = x \times 2^{k-j} \qquad \text{dependent composition}$$

$$= \quad x' = x \times 2^{j-i} \times 2^{k-j} \qquad \text{law of exponents}$$

$$= \quad x' = x \times 2^{(j-i)+(k-j)} \qquad \text{simplify}$$

$$= \quad x' = x \times 2^{k-i}$$

# For Loop

example:

$$t' = t + \Sigma i{:}\ m,..n{\cdot}\ fi \quad \Leftarrow \quad \textbf{for}\ i := m;..n\ \textbf{do}\ t' = t+fi\ \textbf{od}$$

If $fi = c$ (a constant) then

$$t' = t + (n-m)\times c \quad \Leftarrow \quad \textbf{for}\ i := m;..n\ \textbf{do}\ t' = t+c\ \textbf{od}$$

# For Loop

example: add 1 to each item in a list

$$\#L'=\#L \;\wedge\; \forall i{:}\; 0,..\#L\cdot\; L'i=Li+1$$

*Fik* describes an arbitrary segment of iterations:

$$\begin{aligned} Fik \;=\; & \#L'=\#L \\ & \wedge\; (\forall j{:}\; i,..k\cdot\; L'j=Lj+1) \\ & \wedge\; (\forall j{:}\; (0,..i),(k,..\#L)\cdot\; L'j=Lj) \end{aligned}$$



$$F\ 0\ (\#L) \;\Leftarrow\; \textbf{for}\ i{:=}\ 0;..\#L\ \textbf{do}\ L{:=}\ i\rightarrow Li+1 \mid L\ \textbf{od}$$

prove

$$Fii \;\Leftarrow\; 0\le i\le \#L \wedge ok$$

$$Fi(i+1) \;\Leftarrow\; 0\le i<\#L \wedge (L{:=}\ i\rightarrow Li+1 \mid L)$$

$$Fik \;\Leftarrow\; 0\le i<j<k\le \#L \wedge (Fij.\ Fjk)$$

# For Loop

special case: invariant

$$Im \Rightarrow I'n \quad \Leftarrow \quad \textbf{for } i := m;..n \textbf{ do } m \le i < n \wedge Ii \Rightarrow I'(i+1) \textbf{ od}$$

means

$$Ii \Rightarrow I'i \quad \Leftarrow \quad m \le i \le n \wedge ok$$

$$Ii \Rightarrow I'(i+1) \quad \Leftarrow \quad m \le i < n \wedge (m \le i < n \wedge Ii \Rightarrow I'(i+1))$$

$$Ii \Rightarrow I'k \quad \Leftarrow \quad m \le i < j < k \le n \wedge (Ii \Rightarrow I'j \,.\, Ij \Rightarrow I'k)$$

# For Loop

special case: invariant

$$Im \Rightarrow I'n \quad \Leftarrow \quad \textbf{for } i := m;..n \textbf{ do } m \le i < n \wedge Ii \Rightarrow I'(i+1) \textbf{ od}$$

example: $x' = 2^n$

$$Ii \quad = \quad x = 2^i$$



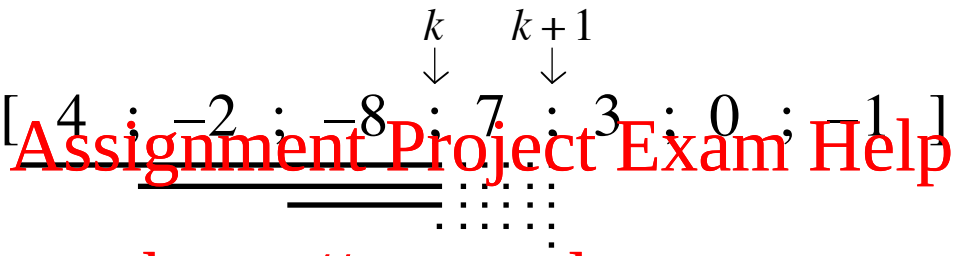
$$x' = 2^n \quad \Leftarrow \quad x := 1 \,.\; I0 \Rightarrow I'n$$

$$I0 \Rightarrow I'n \quad \Leftarrow \quad \textbf{for } i := 0;..n \textbf{ do } Ii \Rightarrow I'(i+1) \textbf{ od}$$

$$Ii \Rightarrow I'(i+1) \quad \Leftarrow \quad x := 2 \times x$$

# Minimum Sum Segment

Given a list $L$ of integers, possibly including negatives, write a program to find the minimum sum of any segment.






$$s' = MIN\ i, j \cdot \Sigma\ L\ [i;..j] \Leftarrow s := 0.\ c := 0.\ I\ 0 \Rightarrow I'(\#L)$$

$$I0 \Rightarrow I'(\#L) \Leftarrow \textbf{for}\ k := 0;..\#L\ \textbf{do}\ Ik \Rightarrow I'(k+1)\ \textbf{od}$$

$$Ik \Rightarrow I'(k+1) \Leftarrow c := min\ (c + L\ k)\ 0.\ s := min\ c\ s$$

$$I\ k = s = (MIN\ i: 0,..k+1 \cdot MIN\ j: i,..k+1 \cdot \Sigma\ L\ [i;..j])$$
$$\wedge\ c = (MIN\ i: 0,..k+1 \cdot \Sigma\ L\ [i;..k])$$

# Review

| | | | |
|---|---|---|---|
| Binary Theory | laws | proof | |
| Number Theory | Character Theory | | |
| Bunches | Sets | Strings | Lists |
| Functions | Quantifiers | | |
| Specification | Refinement | exact precondition | exact postcondition |
| Program Development | Time Calculation | real-time | recursive time |
| Space Calculation | maximum space | average space | |
| Scope | variable declaration | frame | |
| Data Structures | array element assignment | | |
| Control Structures | **while**-loop | loop with **exit** | **for**-loop |

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

constant $n$: *nat* variables $x, y$: *nat*



$x'=n^3 \Leftarrow x:= n.\ x'=x\times n.\ x'=x\times n$



$x'=x\times n \Leftarrow y:= x.\ x:=0.\ x' = x + y\times n$

$x' = x + y\times n \Leftarrow$ **if** $y=0$ **then** *ok* **else** $x:= x+n.\ y:= y-1.\ x' = x + y\times n$ **fi**



In C

```
void P (void) { x = n;  Q( );  Q( );}

void Q (void) { y = x;  x = 0;  R( );}

void R (void) { if (y==0); else {x += n;  y--;  R( );}}
```

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

constant $n$: *nat* variables $x, y$: *nat*



$x'=n^3 \Leftarrow x:= n.\ x'=x\times n.\ x'=x\times n$



$x'=x\times n \Leftarrow y:= x.\ x:=0.\ x' = x + y\times n$

$x' = x + y\times n \Leftarrow$ **if** $y=0$ **then** *ok* **else** $x:= x+n.\ y:= y-1.\ x' = x + y\times n$ **fi**



In C

```
void P (void) { x = n;  Q( );  Q( );}

void Q (void) { y = x;  x = 0;  R: if (y==0); else {x += n;  y--;  goto R;}}
```

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

constant $n$: *nat* variables $x, y$: *nat*



$x'=n^3 \Leftarrow x:= n.\ x'=x\times n.\ x'=x\times n$



$x'=x\times n \Leftarrow y:= x.\ x:=0.\ x'=x+y\times n$

$x' = x + y\times n \Leftarrow$ **if** $y=0$ **then** *ok* **else** $x:= x+n.\ y:= y-1.\ x' = x + y\times n$ **fi**



In C

```
void P (void) { x = n;  Q( );  Q( );}

void Q (void) { y = x;  x = 0;  while (y!=0) {x += n;  y--;}}
```

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

constant $n$: *nat* variables $x, y$: *nat*



$x'=n^3 \Leftarrow x:= n.\ x'=x\times n.\ x'=x\times n$



$x'=x\times n \Leftarrow y:= x.\ x:=0.\ x'=x + y\times n$

$x' = x + y\times n \Leftarrow$ **if** $y=0$ **then** *ok* **else** $x:= x+n.\ y:= y-1.\ x' = x + y\times n$ **fi**



In C

```
x = n;

y = x;  x = 0;  while (y!=0) {x += n;  y--;}

y = x;  x = 0;  while (y!=0) {x += n;  y--;}
```

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

constant $n$: *nat* variables $x, y$: *nat* time $t$: *xnat*



$x'=n^3$ $\Leftarrow$ $x:= n.\ x'=x\times n$ . $x'=x\times n$



$x'=x\times n$ $\Leftarrow$ $y:= x.\ x:= 0.\ x' = x + y\times n \wedge t'=t+y$



$x' = x + y\times n \wedge t'=t+y$ $\Leftarrow$

**if** $y=0$ **then** *ok* **else** $x:= x+n.\ y:= y-1.\ t:= t+1.\quad x' = x + y\times n \wedge t'=t+y$ **fi**

## proof

| | | |
|---|---|---|
| | $y=0 \wedge ok$ | expand *ok* |
| = | $y=0 \wedge x'=x \wedge y'=y \wedge t'=t$ | context |
| = | $y=0 \wedge x' = x + y\times n \wedge y'=y \wedge t'=t+y$ | specialize |
| $\Rightarrow$ | $x' = x + y\times n \wedge t'=t+y$ | |

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

constant $n$: *nat* variables $x, y$: *nat* time $t$: *xnat*



$x'=n^3$ $\Leftarrow$ $x:= n.\ x'=x\times n$ . $x'=x\times n$



$x'=x\times n$ $\Leftarrow$ $y:= x.\ x:= 0.\ x' = x + y\times n \wedge t'=t+y$



$x' = x + y\times n \wedge t'=t+y$ $\Leftarrow$

**if** $y=0$ **then** *ok* **else** $x:= x+n.\ y:= y-1.\ t:= t+1.\ x' = x + y\times n \wedge t'=t+y$ **fi**

## proof

| | | |
|---|---|---|
| | $y\neq 0 \ \wedge\ (x:= x+n.\ y:= y-1.\ t:= t+1.\ x' = x + y\times n \wedge t'=t+y)$ | substitution law |
| $=$ | $y\neq 0 \ \wedge\ x' = x + n + (y-1)\times n \ \wedge\ t'=t+1+y-1$ | simplify and specialize |
| $\Rightarrow$ | $x' = x + y\times n \wedge t'=t+y$ | |

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

constant $n$: *nat* variables $x, y$: *nat* time $t$: *xnat*



$x'=n^3 \wedge t'=t+n^2+n \Leftarrow x:= n.\ x'=x\times n \wedge t'=t+x\ .\ x'=x\times n \wedge t'=t+x$



$x'=x\times n \wedge t'=t+x \Leftarrow y:= x.\ x:= 0.\ x' = x + y\times n \wedge t'=t+y$

$x' = x + y\times n \wedge t'=t+y \Leftarrow$



**if** $y=0$ **then** *ok* **else** $x:= x+n.\ y:= y-1.\ t:= t+1.\quad x' = x + y\times n \wedge t'=t+y$ **fi**

## proof

$x:= n.\ x'=x\times n \wedge t'=t+x.\ x'=x\times n \wedge t'=t+x$ substitution law

$=$ $x'=n^2 \wedge t'=t+n.\ x'=x\times n \wedge t'=t+x$ dependent composition

$=$ $\exists x'', y'', t''\cdot x''=n^2 \wedge t''=t+n \wedge x'=x''\times n \wedge t'=t''+x''$ 1-pt for $x''$, $t''$, idempotent for $y''$

$=$ $x'=n^3 \wedge t'=t+n^2+n$

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

$n^3 = (n-1)^3 + 3\times n^2 - 3\times n + 1$

$n^2 = (n-1)^2 + 2\times n - 1$



variables $x, y, n$: *nat*



$x'=n^3 \Leftarrow x'=n^3 \wedge y'=n^2$



$x'=n^3 \wedge y'=n^2 \Leftarrow$

**if** $n=0$ **then** $x:= 0.\ y:= 0$

**else** $n:= n-1.\ x'=n^3 \wedge y'=n^2.$

$y:= y + n + n - 1.$

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

$n^3 = (n-1)^3 + 3\times n^2 - 3\times n + 1$

$n^2 = (n-1)^2 + 2\times n - 1$



variables $x, y, n$: *nat*



$x'=n^3 \Leftarrow x'=n^3 \wedge y'=n^2 \wedge n'=n$



$x'=n^3 \wedge y'=n^2 \wedge n'=n \Leftarrow$

**if** $n=0$ **then** $x:= 0.\ y:= 0$

**else** $n:= n-1.\ x'=n^3 \wedge y'=n^2 \wedge n'=n.$

$y:= y + n + n - 1.$

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

$n^3 = (n-1)^3 + 3\times n^2 - 3\times n + 1$

$n^2 = (n-1)^2 + 2\times n - 1$



variables $x, y, n$: *nat*



$x'=n^3 \wedge t'=t+n \Leftarrow x'=n^3 \wedge y'=n^2 \wedge n'=n \wedge t'=t+n$



$x'=n^3 \wedge y'=n^2 \wedge n'=n \wedge t'=t+n \Leftarrow$

**if** $n=0$ **then** $x:= 0.\ y:= 0$

**else** $n:= n-1.\ t:= t+1.\ x'=n^3 \wedge y'=n^2 \wedge n'=n \wedge t'=t+n.\ n:= n+1.$

$y:= y + n + n - 1.\ x:= x + y + y + y - n - n - n + 1$ **fi**

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

$x'=n^3 \Leftarrow x:=0.\ I0 \Rightarrow I'n$

$I0 \Rightarrow I'n \Leftarrow$ **for** $k:=0;..n$ **do** $Ik \Rightarrow I'(k+1)$ **od**

$Ik \Rightarrow I'(k+1) \Leftarrow x:=x+?$



$Ik = x=k^3$

$Ik \Rightarrow I'(k+1)$

$= x=k^3 \Rightarrow x'=(k+1)^3$

$= x=k^3 \Rightarrow x' = k^3 + 3\times k^2 + 3\times k + 1$

$\Leftarrow x:= x + 3\times k^2 + 3\times k + 1$

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

$x'=n^3 \Leftarrow x:=0.\ y:=1.\ I0 \Rightarrow I'n$

$I0 \Rightarrow I'n \Leftarrow$ **for** $k:=0;..n$ **do** $Ik \Rightarrow I'(k+1)$ **od**

$Ik \Rightarrow I'(k+1) \Leftarrow x:=x+y$



$Ik = x=k^3 \wedge y = 3\times k^2 + 3\times k + 1$

$Ik \Rightarrow I'(k+1)$

$= x=k^3 \wedge y = 3\times k^2 + 3\times k + 1 \Rightarrow x'=(k+1)^3 \wedge y' = 3\times(k+1)^2 + 3\times(k+1) + 1$

$= x=k^3 \wedge y = 3\times k^2 + 3\times k + 1 \Rightarrow x' = x+y \wedge y' = 3\times k^2 + 9\times k + 7$

$= x=k^3 \wedge y = 3\times k^2 + 3\times k + 1 \Rightarrow x' = x+y \wedge y' = y + 6\times k + 6$

$\Leftarrow x:=x+y.\ y:=y+k+k+k+k+k+k+6$

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

$$x'=n^3 \;\Leftarrow\; x:=0.\ y:=1.\ z:=6.\ I0 \Rightarrow I'n$$

$$I0 \Rightarrow I'n \;\Leftarrow\; \textbf{for}\ k:=0;..n\ \textbf{do}\ Ik \Rightarrow I'(k+1)\ \textbf{od}$$

$$Ik \Rightarrow I'(k+1) \;\Leftarrow\; x:=x+y.\ y:=y+z$$



$$Ik \;=\; x=k^3 \ \wedge\ y=3\times k^2+3\times k+1 \ \wedge\ z=6\times k+6$$



$Ik \Rightarrow I'(k+1)$

$=$ $\quad x=k^3 \ \wedge\ y=3\times k^2+3\times k+1 \ \wedge\ z=6\times k+6$

$\Rightarrow$ $\quad x'=(k+1)^3 \ \wedge\ y'=3\times(k+1)^2+3\times(k+1)+1 \ \wedge\ z'=6\times(k+1)+6$

$\Leftarrow$ $\quad x'=x+y \ \wedge\ y'=y+z \ \wedge\ z'=z+6$

$=$ $\quad x:=x+y.\ y:=y+z.\ z:=z+6$

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

$x'=n^3 \Leftarrow x:=0.\ y:=1.\ z:=6.\ I0 \Rightarrow I'n$

$I0 \Rightarrow I'n \Leftarrow \textbf{for}\ k:=0;..n\ \textbf{do}\ Ik \Rightarrow I'(k+1)\ \textbf{od}$



$Ik \Rightarrow I'(k+1) \Leftarrow x:=x+y.\ y:=y+z.\ z:=z+6$



$Ik = x=k^3 \wedge y=3\times k^2+3\times k+1 \wedge z=6\times k+6$



$x:=0.\ y:=1.\ z:=6.\ \textbf{for}\ k:=0;..n\ \textbf{do}\ x:=x+y.\ y:=y+z.\ z:=z+6\ \textbf{od}$

# Cube

Write a program that cubes using only addition, subtraction, and test for zero.

$x'=n^3 \wedge t'=t+n \quad \Leftarrow \quad x:=0.\ \ y:=1.\ \ z:=6.\ \ Q \wedge t'=t+n$

$Q \wedge t'=t+n \quad \Leftarrow$

**if** $n=0$ **then** $ok$ **else** $x:=x+y.\ \ y:=y+z.\ \ z:=z+6.\ \ n:=n-1.\ \ t:=t+1.\ \ Q \wedge t'=t+n$ **fi**



$Q \quad = \quad \forall k: nat \cdot\ x=k^3 \wedge y = 3{\times}k^2 + 3{\times}k + 1 \wedge z = 6{\times}k + 6 \quad \Rightarrow \quad x' = (k+n)^3$

$Q \quad = \quad y = 3{\times}x^{2/3} + 3{\times}x^{1/3} + 1 \ \wedge \ z = 6{\times}x^{1/3} + 6 \quad \Rightarrow \quad x' = (x^{1/3}+n)^3$

x = 0;  y = 1;  z = 6;

Q:  if (n!=0) {x+=y;  y+=z.  z+=6;  n--; goto Q;}

# Time Dependence

| | | |
|---|---|---|
| *deadline*:= *t* + 5 | | no problem |
| **if** *t* < *deadline* **then** ... **else** ... **fi** | | no problem |
| *t*:= 5 | | problem: unimplementable |
| **wait until** *w* | = | *t*:= *max t w* busy-wait loop |
| **wait until** *w* | ⇐ | **if** *t*≥*w* **then** *ok* **else** *t*:= *t*+1. **wait until** *w* **fi** |



**proof**

| | |
|---|---|
| | *t*≥*w* ∧ *ok* |
| = | *t*≥*w* ∧ (*t*:= *t*) |
| = | *t*≥*w* ∧ (*t*:= *max t w*) |
| ⇒ | **wait until** *w* |

# Time Dependence

| | | |
|---|---|---|
| *deadline*:= $t + 5$ | | no problem |
| **if** $t <$ *deadline* **then** ... **else** ... **fi** | | no problem |
| $t$:= 5 | | problem: unimplementable |
| **wait until** $w$ | = $t$:= *max* $t$ $w$ | busy-wait loop |
| **wait until** $w$ | ⇐ **if** $t \geq w$ **then** *ok* **else** $t$:= $t$+1. **wait until** $w$ **fi** | |



**proof**

| | |
|---|---|
| | $t<w \wedge (t:= t+1.$ **wait until** $w)$ |
| = | $t<w \wedge (t:= t+1.\ t:= max\ t\ w)$ |
| = | $t+1 \leq w \wedge (t:= max\ (t+1)\ w)$ |
| = | $t<w \wedge (t:= w)$ |
| = | $t<w \wedge (t:= max\ t\ w)$ |
| ⇒ | **wait until** $w$ |

# Space Dependence

| | |
|---|---|
| **if** $s < 1000000$ **then** ... **else** ... **fi** | no problem |
| $s := 5$ | problem |

assignments to $s$ must account for space

| | |
|---|---|
| real space | implementation dependent |

# Assertions

**assert** $b$

= "I believe $b$ is true"

= **precondition** $b$

= **postcondition** $b$

= **invariant** $b$

= **if** $b$ **then** *ok* **else** *print* "error: ..." ; **wait until** $\infty$ **fi**

redundant, adds robustness, costs execution time



**ensure** $b$

= "make $b$ be true without doing anything"

= **if** $b$ **then** *ok* **else** $b' \wedge ok$ **fi**

= $b' \wedge ok$

unimplementable by itself, but may be used in some contexts

**nondeterministic choice** (a programming notation):

$P$ **or** $Q$ = $P \vee Q$

$x{:=}\ 0$ **or** $x{:=}\ 1$. **ensure** $x{=}1$

= $x'{=}0 \wedge y'{=}y \ \vee \ x'{=}1 \wedge y'{=}y.\ x'{=}1 \wedge x'{=}x \wedge y'{=}y$

= $\exists x'', y''\cdot\ (x''{=}0 \wedge y''{=}y \ \vee \ x''{=}1 \wedge y''{=}y) \wedge x'{=}1 \wedge x'{=}x'' \wedge y'{=}y''$

= $(x'{=}0 \wedge y'{=}y \ \vee \ x'{=}1 \wedge y'{=}y) \wedge x'{=}1$

= $x'{=}1 \wedge y'{=}y$

= $x{:=}\ 1$



implementation: **backtracking**

**natural square root** Given natural $n$ find natural $s$ satisfying $s^2 \le n < (s{+}1)^2$

$s{:=}\ 0,..n{+}1$. **ensure** $s^2 \le n < (s{+}1)^2$

# Result Expression

*P* **result** *e*  execute *P* then evaluate *e* but no state change

**var** *term*, *sum*: *rat* := 1·

**for** *i*:= 1;..15 **do** *term*:= *term*/*i*. *sum*:= *sum*+*term* **od**

**result** *sum*



**axiom** *P* (*P* **result** *e*)=*e* but don't double-prime (*P* **result** *e*)
and don't substitute in (*P* **result** *e*)

| | | |
|---|---|---|
| | ⊤ | **result** axiom |
| = | *x*:= *x*+1. (*x*:= *x*+1 **result** *x*)=*x* | Substitution Law but ... |
| = | (*x*:= *x*+1 **result** *x*) = *x*+1 | |

# Result Expression

$P$ **result** $e$ execute $P$ then evaluate $e$ but no state change

**var** *term*, *sum*: *rat* := 1·

**for** $i$:= 1;..15 **do** *term*:= *term*/$i$. *sum*:= *sum*+*term* **od**

**result** *sum*



**axiom** $P$ (*P* **result** $e$)= $e$ but don't double-prime ($P$ **result** $e$)

don't substitute in ($P$ **result** $e$)

$y$:= ($x$:= $x$+1 **result** $x$) assignment

= $y'$ = ($x$:= $x$+1 **result** $x$) ∧ $x'$=$x$ previous calculation

= $y'$ = $x$+1 ∧ $x'$=$x$ assignment

= $y$:= $x$+1

# Result Expression

*P* **result** *e* execute *P* then evaluate *e* but no state change

## implementation



Replace each nonlocal variable within *P* and *e* that is assigned within *P*



by a fresh local variable initialized to the value of the nonlocal variable.

Then execute *P* and evaluate *e*.



but some language implementations don't introduce local variables

so expression evaluation can cause state change

# Side Effects

$x = x$ ?  not if there are side-effects !

$x + x = 2 \times x$ ?  not if there are side-effects !

for reasoning



$x := (P \textbf{ result } e)$ becomes $P.\ x := e$



$x := (P \textbf{ result } e) + y$ becomes $(\textbf{var } z := y \cdot\ P.\ x := e+z)$



Don't neglect the time for expression evaluation.

# Function

```
int bexp (int n)
{  int r = 1;
   int i;
   for (i=0; i<n; i++) r = r*2;
   return r; }
```

*bexp* = ⟨ *n*: *int* → **var** *r*: *int* := 1· **for** *i*:= 0;..*n* **do** *r*:= *r*×2 **od**. **assert** *r*: *int* **result** *r* ⟩



C function = assertion about the result
+ name
+ parameters
+ scope control
+ result expression

# Procedure

procedure = name of procedure

\+ parameters

\+ scope control



$P = \langle x: int \rightarrow a' < x < b' \rangle$



$P\ 3 = a' < 3 < b'$ 

$P\ (a+1) = a' < a+1 < b'$

$a' < x < b' \Leftarrow a := x-1.\ b := x+1$

$\langle p: D \rightarrow B \rangle\ a = (\textbf{var}\ p: D := a \cdot\ B)$ if $B$ doesn't use $p'$ or $p:=$

# Procedure

**reference parameter** **var** parameter

$\langle \textbf{var}\ x{:}\ int \rightarrow a := 3.\ b := 4.\ x := 5 \rangle\ a$ $\langle \textbf{var}\ x{:}\ int \rightarrow x := 5.\ b := 4.\ a := 3 \rangle\ a$

$= \quad a := 3.\ b := 4.\ a := 5$ $= \quad a := 5.\ b := 4.\ a := 3$

$= \quad a' = 5 \wedge b' = 4$ $= \quad a' = 3 \wedge b' = 4$

$\langle \textbf{var}\ x{:}\ int \rightarrow a' = 3 \wedge b' = 4 \wedge x' = 5 \rangle\ a' = 3$



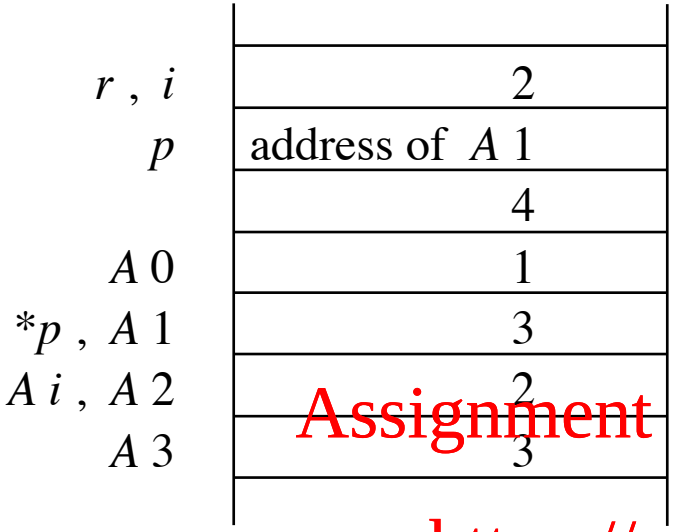
**warning** Use only for programs, not for arbitrary specifications.

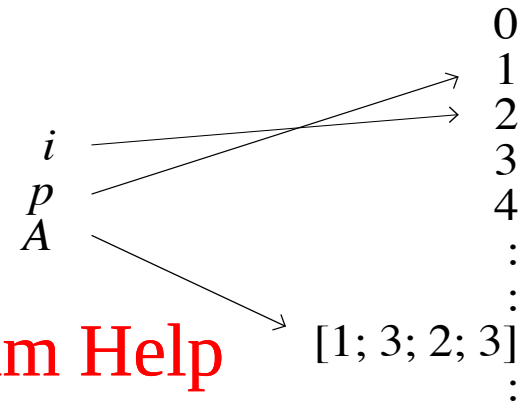
Do not manipulate the procedure body.

Substitute arguments for parameters before any other manipulations.

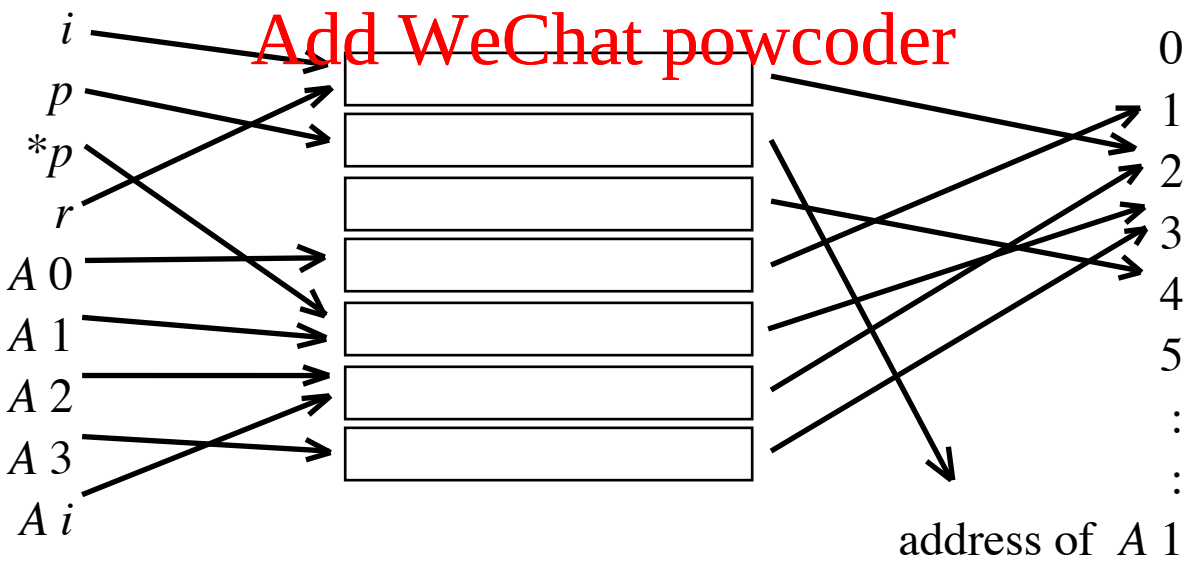
Apply programming theory separately for each call.

# Alias





# Probabilistic Programming

**probability** real number between 0 and 1

$prob = \S r\colon real \cdot 0 \le r \le 1$ $\qquad \top = 1$ $\qquad \bot = 0$

$\neg x = 1 - x$ $\qquad x \wedge y = x \times y$ $\qquad x \vee y = x - x \times y + y$



**distribution** value is a probability, sum is 1

says the probability of each state





$2^{-n}$ is a distribution of $n\colon nat+1$ because $(\forall n\colon nat+1 \cdot 2^{-n}\colon prob) \wedge (\Sigma n\colon nat+1 \cdot 2^{-n}) = 1$

$2^{-n}$ says $n=3$ with probability 1/8

$2^{-n-m}$ is a distribution of $n, m\colon nat+1$ because

$(\forall n, m\colon nat+1 \cdot 2^{-n-m}\colon prob) \wedge (\Sigma n, m\colon nat+1 \cdot 2^{-n-m}) = 1$

$2^{-n-m}$ says $n=3 \wedge m=1$ with probability 1/16

# Probabilistic Programming

$n' = n+1$ says: if $n=5$ then $n'=6$ with probability 1

and $n'=7$ with probability 0

$(\forall n, n' : nat \cdot n' = n+1 : prob) \ \wedge \ (\Sigma n, n' \cdot \ n' = n+1) = \infty$

so $n' = n+1$ is not a distribution of $n$ and $n'$



$(\forall n' : nat \cdot n' = n+1 : prob) \ \wedge \ (\Sigma n' \cdot \ n' = n+1) = 1$



so (for any value of $n$) $n' = n+1$ is a one-point distribution of $n'$

Any implementable deterministic specification is a one-point distribution of the final state.

# Probabilistic Programming

$ok \quad = \quad (x'=x) \times \quad (y'=y) \times \quad \ldots$

$x := e \quad = \quad (x'=e) \times \quad (y'=y) \times \quad \ldots$



**if** $b$ **then** $P$ **else** $Q$ **fi** $\quad = \quad b \times P \; + \; (1-b) \times Q$





$P \,.\, Q \quad = \quad \Sigma\, x'', y'', \ldots \cdot$ (for $x', y', \ldots$ substitute $x'', y'', \ldots$ in $P$ )

$\times$ (for $x, y, \ldots$ substitute $x'', y'', \ldots$ in $Q$ )

## example

**if** 1/3 **then** $x$:= 0 **else** $x$:= 1 **fi**

= $1/3 \times (x'=0) + (1 - 1/3) \times (x'=1)$

evaluate using 0 for $x'$

= $1/3 \times (0=0) + (1 - 1/3) \times (0=1)$

= $1/3 \times 1 + 2/3 \times 0$

= $1/3$



evaluate using 1 for $x'$

= $1/3 \times (1=0) + (1 - 1/3) \times (1=1)$



= $1/3 \times 0 + 2/3 \times 1$

= $2/3$

evaluate using 2 for $x'$

= $1/3 \times (2=0) + (1 - 1/3) \times (2=1)$

= $1/3 \times 0 + 2/3 \times 0$

= $0$

**example** in one integer variable $x$

**if** 1/3 **then** $x$:= 0 **else** $x$:= 1 **fi**.

**if** $x$=0 **then if** 1/2 **then** $x$:= $x$+2 **else** $x$:= $x$+3 **fi**

**else if** 1/4 **then** $x$:= $x$+4 **else** $x$:= $x$+5 **fi fi**



$$= \quad \Sigma x'' \cdot \quad ((x''=0)/3 + (x''=1)\times 2/3)$$

$$\times \quad ( \quad (x''=0) \times ((x'=x''+2)/2 + (x'=x''+3)/2)$$

$$+ \quad (x''\neq 0) \times ((x'=x''+4)/4 + (x'=x''+5)\times 3/4))$$





$$= \quad (x'=2)/6 + (x'=3)/6 + (x'=5)/6 + (x'=6)/2$$

# Average

after $P$ , average value of $e$ is $P.e$

as $n$ varies over $nat{+}1$ according to distribution $2^{-n}$ the average value of $n^2$ is

$2^{-n'}.\ n^2$

$= \quad \Sigma n'' : nat{+}1 \cdot\ 2^{-n''} \times n''^2$

$= \quad 6$

# Average

after $P$ , average value of $e$ is $P.e$

**if** 1/3 **then** $x$:= 0 **else** $x$:= 1 **fi**.

**if** $x$=0 **then if** 1/2 **then** $x$:= $x$+2 **else** $x$:= $x$+3 **fi**
**else if** 1/4 **then** $x$:= $x$+4 **else** $x$:= $x$+5 **fi fi**.

$x$

$= (x'=2)/6 + (x'=3)/6 + (x'=5)/6 + (x'=6)/2 \times x$

$= \Sigma x'' \cdot ((x''=2)/6 + (x''=3)/6 + (x''=5)/6 + (x''=6)/2) \times x''$

$= 1/6 \times 2 + 1/6 \times 3 + 1/6 \times 5 + 1/2 \times 6$

$= 4 + 2/3$

# Average

after $P$ , average value of $e$ is $P.e$

after $P$ , probability that $b$ is true is $P.b$

Probability is just the average value of a binary expression.



**if** 1/3 **then** $x$:= 0 **else** $x$:= 1 **fi**.

**if** $x$=0 **then if** 1/2 **then** $x$:= $x$+2 **else** $x$:= $x$+3 **fi**

**else if** 1/4 **then** $x$:= $x$+4 **else** $x$:= $x$+5 **fi fi**.

$x>3$

$=$ $(x'=2)/6 + (x'=3)/6 + (x'=5)/6 + (x'=6)/2.\ \ x>3$

$=$ $\Sigma x''\cdot\ ((x''=2)/6 + (x''=3)/6 + (x''=5)/6 + (x''=6)/2)\ \times\ (x''>3)$

$=$ $1/6 \times (2>3)\ +\ 1/6 \times (3>3)\ +\ 1/6 \times (5>3)\ +\ 1/2 \times (6>3)$

$=$ $2/3$

# Random Number Generator

*rand n* has value $r$ with probability $(r: 0,..n) / n$

$x=x$ therefore *rand n* = *rand n* ?

$x+x = 2\times x$ therefore *rand n* + *rand n* = $2\times$ *rand n* ?



Replace *rand n* with $r$: *int* with distribution $(r: 0,..n) / n$

Replace *rand n* with $r: 0,..n$ with distribution $1/n$

| | | |
|---|---|---|
| | $x:=$ *rand* 2. $x:= x +$ *rand* 3 | replace one *rand* with $r$ and one with $s$ |
| = | $\Sigma r: 0,..2\cdot \Sigma s: 0,..3\cdot (x:= r)/2.\ (x:= x + s)/3$ | Substitution Law |
| = | $\Sigma r: 0,..2\cdot \Sigma s: 0,..3\cdot (x' = r+s) / 6$ | sum |
| = | $((x' = 0+0) + (x' = 0+1) + (x' = 0+2) + (x' = 1+0) + (x' = 1+1) + (x' = 1+2)) / 6$ | |
| = | $(x'=0) / 6 + (x'=1) / 3 + (x'=2) / 3 + (x'=3) / 6$ | |

# Random Number Generator

*rand n* has value $r$ with probability $(r: 0,..n) / n$

$x{=}x$ therefore *rand n* = *rand n* ?

$x{+}x = 2{\times}x$ therefore *rand n* + *rand n* = 2 × *rand n* ?



Replace *rand n* with $r$: *int* with distribution $(r: 0,..n) / n$

Replace *rand n* with $r: 0,..n$ with distribution $1/n$

| | | |
|---|---|---|
| | $x{:=}$ *rand* 2. $x{:=}\ x + $ *rand* 3 | replace *rand* |
| = | $(x': 0,..2)/2.\ (x': x{+}(0,..3))/3$ | dependent composition |
| = | $\Sigma x''\cdot (x'': 0,..2)/2 \times (x': x''{+}(0,..3))/3$ | sum |
| = | $1/2 \times (x': 0,..3)/3\ +\ 1/2 \times (x': 1,..4)/3$ | |
| = | $(x'{=}0) / 6\ +\ (x'{=}1) / 3 + (x'{=}2) / 3\ +\ (x'{=}3) / 6$ | |

# Blackjack

You are dealt a card from a deck; its value is in the range 1 to 13 inclusive. You may stop with just one card, or have a second card if you want. Your object is to get a total as near as possible to 14 , but not over 14 . Your strategy is to take a second card if the first is under 7 .



$x := (rand\ 13) + 1.$ **if** $x<7$ **then** $x := x + (rand\ 13) + 1$ **else** *ok* **fi** replace *rand* and *ok*

$= (x' : (0,..13)+1)/13.$ **if** $x<7$ **then** $(x' : x+(0,..13)+1)/13$ **else** $x'=x$ **fi** replace . and **if**

$= \Sigma x'' \cdot (x'' : 1,..14)/13 \times ((x''<7)\times(x' : x''+1,..x''+14)/13 + (x''\geq 7)\times(x'=x''))$

by several omitted steps

$= ((2\leq x'<7)\times(x'-1) + (7\leq x'<14)\times 19 + (14\leq x'<20)\times(20-x')) / 169$

Player $x$ plays "under $n$" and player $y$ plays "under $n+1$"

$c := (rand\ 13) + 1.\ \ d := (rand\ 13) + 1.$

**if** $c < n$ **then** $x := c+d$ **else** $x := c$ **fi**. **if** $c < n+1$ **then** $y := c+d$ **else** $y := c$ **fi**.

$y < x \leq 14 \vee x \leq 14 < y$

replace *rand*

$=$ $(c' : (0,..13)+1 \ \wedge\ d' : (0,..13)+1 \ \wedge\ x' = x \ \wedge\ y' = y) / 13 / 13.$

**if** $c < n$ **then** $x := c+d$ **else** $x := c$ **fi**. **if** $c < n+1$ **then** $y := c+d$ **else** $y := c$ **fi**.

$y < x \leq 14 \vee x \leq 14 < y$

4 omitted steps

$=$ $(n-1) / 169$

probability that $x$ wins is $(n-1) / 169$

probability that $y$ wins is $(14-n) / 169$

probability of a tie is 12/13

"under 8" beats both "under 7" and "under 9"

# Dice

If you repeatedly throw a pair of six-sided dice until they are equal, how long does it take?

$$R \Leftarrow u := (rand\ 6) + 1.\ v := (rand\ 6) + 1.\ \textbf{if}\ u{=}v\ \textbf{then}\ ok\ \textbf{else}\ t := t{+}1.\ R\ \textbf{fi}$$

$u := (rand\ 6) + 1.\ v := (rand\ 6) + 1.$ replace *rand*

**if** $u{=}v$ **then** $t'{=}t$ **else** $t := t{+}1.\ (t' \geq t) \times (5/6)^{t'-t} \times 1/6$ **fi** Substitution Law

$=$ $(u' : 1,..7\ \wedge\ v'{=}v\ \wedge\ t'{=}t)/6.\ (u'{=}u\ \wedge\ v' : 1,..7\ \wedge\ t'{=}t)/6.$ replace first .

**if** $u{=}v$ **then** $t'{=}t$ **else** $(t' \geq t{+}1) \times (5/6)^{t'-t-1} / 6$ **fi**

$=$ $(u', v' : 1,..7\ \wedge\ t'{=}t)/36.$ replace .

**if** $u{=}v$ **then** $t'{=}t$ **else** $(t' \geq t{+}1) \times (5/6)^{t'-t-1} / 6$ **fi** replace **if**

$=$ $\Sigma u'', v'' : 1,..7 \cdot \Sigma t'' \cdot\ \ (t''{=}t)/36 \times (\ \ (u''{=}v'') \times (t'{=}t'')$

$+\ \ (u''{\neq}v'') \times (t' \geq t''{+}1) \times (5/6)^{t'-t''-1} / 6)$ sum

$=$ $(6 \times (t'{=}t)\ +\ 30 \times (t' \geq t{+}1) \times (5/6)^{t'-t-1} / 6) / 36$ combine

$=$ $(t' \geq t) \times (5/6)^{t'-t} \times 1/6$

# Dice

If you repeatedly throw a pair of six-sided dice until they are equal, how long does it take?

$$R \Leftarrow u := (rand\ 6) + 1.\ v := (rand\ 6) + 1.\ \textbf{if}\ u{=}v\ \textbf{then}\ ok\ \textbf{else}\ t := t{+}1.\ R\ \textbf{fi}$$

The average value of $t'$ is $(t' \geq t) \times (5/6)^{t'-t} \times 1/6.\ t = t{+}5$

# Functional Programming

– assignment

– dependent composition

\+ functions



specification = function from input to output



program = implemented specification



application ✔

composition ✔

selective union ✔

quantifiers ✗

program + inputs = function + arguments

**example** specification $\langle L\colon [{*}rat] \to \Sigma L \rangle$

$$\Sigma L \;=\; \langle n\colon 0,..\#L{+}1 \to \Sigma\, L\, [n;..\#L]\rangle\, 0$$

$$\langle n\colon 0,..\#L{+}1 \to \Sigma\, L\, [n;..\#L]\rangle$$

$$= \quad \langle n\colon 0,..\#L \;,\; \#L \to \Sigma\, L\, [n;..\#L]\rangle$$

$$= \quad \langle n\colon 0,..\#L \to \Sigma\, L\, [n;..\#L]\rangle \mid \langle n\colon \#L \to \Sigma\, L\, [n;..\#L]\rangle$$



$$\langle n\colon 0,..\#L \to \Sigma\, L\, [n;..\#L]\rangle$$



$$= \quad \langle n\colon 0,..\#L \to Ln + \Sigma\, L\, [n{+}1;..\#L]\rangle$$

$$\langle n\colon \#L \to \Sigma\, L\, [n;..\#L]\rangle \quad = \quad \langle n\colon \#L \to 0\rangle$$

$$\Sigma\, L\, [n{+}1;..\#L] \quad = \quad \langle n\colon 0,..\#L{+}1 \to \Sigma\, L\, [n;..\#L]\rangle\, (n{+}1)$$

**time** specification $\langle L\colon [*rat] \to \#L \rangle$

$$\#L \;=\; \langle n\colon 0,..\#L{+}1 \to \#L{-}n \rangle\, 0$$

$$\langle n\colon 0,..\#L{+}1 \to \#L{-}n \rangle$$

$$= \quad \langle n\colon 0,..\#L \;,\; \#L \to \#L{-}n \rangle$$

$$= \quad \langle n\colon 0,..\#L \to \#L{-}n \rangle \mid \langle n\colon \#L \to \#L{-}n \rangle$$



$$\langle n\colon 0,..\#L \to \#L{-}n \rangle$$

$$= \quad \langle n\colon 0,..\#L \to 1 + \#L{-}n{-}1 \rangle$$

$$\langle n\colon \#L \to \#L{-}n \rangle \;=\; \langle n\colon \#L \to 0 \rangle$$

$$\#L{-}n{-}1 \;=\; \langle n\colon 0,..\#L{+}1 \to \#L{-}n \rangle\, (n{+}1)$$

**time** specification $\langle L\colon [*rat] \to \#L \rangle$

$$\#L \;=\; \langle n\colon 0,..\#L{+}1 \to \#L{-}n \rangle\, 0$$

$$\langle n\colon 0,..\#L{+}1 \to \#L{-}n \rangle$$

$$= \quad \langle n\colon 0,..\#L \;,\; \#L \to \#L{-}n \rangle$$

$$= \quad \langle n\colon 0,..\#L \to \#L{-}n \rangle \mid \langle n\colon \#L \to \#L{-}n \rangle$$



$$\langle n\colon 0,..\#L \to \#L{-}n \rangle$$

$$= \quad \langle n\colon 0,..\#L \to \#L{-}n \rangle$$

$$\langle n\colon \#L \to \#L{-}n \rangle \;=\; \langle n\colon \#L \to 0 \rangle$$

$$\#L{-}n \;=\; 1 + \langle n\colon 0,..\#L{+}1 \to \#L{-}n \rangle\, (n{+}1)$$

# Function Refinement

| | |
|---|---|
| Specification $S$ is unsatisfiable for domain element $x$: | $\notin S\ x < 1$ |
| Specification $S$ is satisfiable for domain element $x$: | $\notin S\ x \geq 1$ |
| Specification $S$ is deterministic for domain element $x$: | $\notin S\ x \leq 1$ |
| Specification $S$ is nondeterministic for domain element $x$: | $\notin S\ x > 1$ |



| | |
|---|---|
| Specification $S$ is satisfiable for domain element $x$: | $\exists y \cdot y : S\ x$ |



| | |
|---|---|
| Specification $S$ is implementable: | $\forall x \cdot \exists y \cdot y : S\ x$ |
| | $\forall x \cdot S\ x \neq null$ |
| $S$ is refined by $P$ | $P : S$ |
| | $S :: P$ |

**example** search for an item in a list

$\langle L\colon [*int] \to \langle x\colon int \to \S n\colon 0,..\#L\cdot\ Ln = x\ \rangle\rangle$ unimplementable

$\langle L\colon [*int] \to \langle x\colon int \to$ **if** $x\colon L\ (0,..\#L)$ **then** $\S n\colon 0,..\#L\cdot\ Ln = x$ **else** $\#L,..\infty$ **fi**$\rangle\rangle$



**if** $x\colon L\ (0,..\#L)$ **then** $\S n\colon 0,..\#L\cdot\ Ln = x$ **else** $\#L,..\infty$ **fi** ::



$\langle i\colon nat \to$ **if** $x\colon L\ (i,..\#L)$ **then** $\S n\colon i,..\#L\cdot\ Ln = x$ **else** $\#L,..\infty$ **fi**$\rangle\ 0$



**if** $x\colon L\ (i,..\#L)$ **then** $\S n\colon i,..\#L\cdot\ Ln = x$ **else** $\#L,..\infty$ **fi** ::

**if** $i = \#L$ **then** $\#L$

**else if** $x = Li$ **then** $i$

**else** $\langle i\colon nat \to$ **if** $x\colon L\ (i,..\#L)$ **then** $\S n\colon i,..\#L\cdot\ Ln = x$ **else** $\#L,..\infty$ **fi**$\rangle\ (i{+}1)$ **fi**

## recursive timing $\langle L \to \langle x \to 0,..\#L+1\rangle\rangle$

$0,..\#L+1 :: \quad \langle i \to 0,..\#L-i+1\rangle\, 0$

$0,..\#L-i+1 ::$ **if** $i = \#L$ **then** $0$

**else if** $x = Li$ **then** $0$

**else** $1 + \langle i \to 0,..\#L-i+1\rangle\,(i+1)$ **fi fi**



$1 + \langle i \to 0,..\#L-i+1\rangle\,(i+1)$

$= \quad 1 + (0,..\#L-(i+1)+1)$

$= \quad 1 + (0,..\#L-i)$

$= \quad 1,..\#L-i+1$

$: \quad 0,..\#L-i+1$

# functional versus imperative

same programming steps, different notation

functional programming has Application Axiom

$$\langle v{:}\ D{\cdot}\ b\rangle\, x \quad = \quad (\text{for } v \text{ substitute } x \text{ in } b\ )$$



imperative programming has Substitution Law



$$x{:=}\ e\,.\,P \quad = \quad (\text{for } x \text{ substitute } e \text{ in } P\ )$$