172 (combinations) Write a program to find the number of ways to partition $a+b$ things into a things in the left part and b things in the right part. Include recursive time.

§ The number of ways to partition $a+b$ things into a things and b things is $(a+b)! / (a!\times b!)$ where $!$ is the factorial function. First without time.

$x:= (a+b)! / (a!\times b!) \;\Leftarrow$
  if $a=0$ **then** $x:= 1$
  else $a:= a-1.\; x:= (a+b)! / (a!\times b!).\; a:= a+1.\; x:= x\times(a+b)/a$ **fi**

The assignment $x:= (a+b)! / (a!\, b!)$ means $x' = (a+b)! / (a!\, b!)) \land a'=a \land b'=b$. On the right side it is a recursive call. Stating it as an assignment makes the proof easy: just use the substitution law and simplify. The proof is by cases. First case:

$a=0 \land (x:= 1) \;\Rightarrow\; (x:= (a+b)! / (a!\times b!))$ — definition of assignment
$=\; a=0 \land x'=1 \land a'=a \land b'=b \;\Rightarrow\; x' = (a+b)! / (a!\, b!)) \land a'=a \land b'=b$ — use $0!=1$
$=\; \top$

Second case, starting with the right side:

$a\ne0 \land (a:= a-1.\; x:= (a+b)! / (a!\times b!).\; a:= a+1.\; x:= x\times(a+b)/a)$ — assignment
$=\; a\ne0 \land (a:= a-1.\; x:= (a+b)! / (a!\times b!).\; a:= a+1.\; x'=x\times(a+b)/a \land a'=a \land b'=b)$ — substitution law 3 times
$=\; a\ne0 \land x' = (a-1+b)!/((a-1)!\times b!)\times(a+b)/a \land a'=a \land b'=b$ — simplify
$=\; a\ne0 \land x' = (a+b)! / (a!\times b!) \land a'=a \land b'=b$ — specialization
$\Rightarrow\; x:= (a+b)! / (a!\times b!)$

Now the time.

$t' = t+a \;\Leftarrow$ **if** $a=0$ **then** $x:= 1$
  else $a:= a-1.\; t:= t+1.\; t' = t+a.\; a:= a+1.\; x:= x\times(a+b)/a$ **fi**

Proof by cases. First case:

$a=0 \land (x:= 1) \Rightarrow t' = t+a$ — definition of assignment
$=\; a=0 \land x'=1 \land a'=a \land b'=b \land t'=t \;\Rightarrow\; t' = t+a$
$=\; \top$

Second case, starting with the right side:

$a\ne0 \land (a:= a-1.\; t:= t+1.\; t' = t+a.\; a:= a+1.\; x:= x\times(a+b)/a)$ — assignment
$=\; a\ne0 \land (a:= a-1.\; t:= t+1.\; t' = t+a.\; a:= a+1.\; x'=x\times(a+b)/a \land a'=a \land b'=b \land t'=t)$ — substitution law 3 times
$=\; a\ne0 \land (t' = t+a.\; x'=x\times(a+1+b)/(a+1) \land a'=a+1 \land b'=b \land t'=t)$ — dependent comp
$=\; a\ne0 \land (\exists x'', a'', b'', t''\cdot\; t''=t+a \land x'=x''\times(a''+1+b'')/(a''+1) \land a'=a''+1 \land b'=b'' \land t'=t'')$ — one point 4 times
$=\; a\ne0 \land t' = t+a$ — specialization
$\Rightarrow\; t' = t+a$

When refining $x:= (a+b)! / (a!\times b!)$, there was no time variable. Adding the time variable, we cannot write this as an assignment, because that would mean $t'=t$. We can put the result and the timing together as

$x' = (a+b)! / (a!\, b!)) \land a'=a \land b'=b \land t'=t+a$

or as

$x:= (a+b)! / (a!\times b!).\; t:= t+a$

Here is a solution that is symmetric in a and b .

$x:= (a+b)! / (a!\times b!) \;\Leftarrow$
  if $a=0 \lor b=0$ **then** $x:= 1$
  else $a:= a-1.\; b:= b-1.\; x:= (a+b)! / (a!\times b!).$
    $a:= a+1.\; b:= b+1.\; x:= x/a/b\times(a+b-1)\times(a+b)$ **fi**

And its execution time is smaller: $min\ a\ b$.

Here is a solution with the same execution time and its recursion does not require a stack.

$$x' = (a+b)!/(a!\times b!) \land t' = t + \mathit{min}\ a\ b \quad \Leftarrow$$
$$x:= 1.\ \ x' = x \times (a+b)!/(a!\times b!) \land t' = t + \mathit{min}\ a\ b$$

$$x' = x \times (a+b)!/(a!\times b!) \land t' = t + \mathit{min}\ a\ b \quad \Leftarrow$$
$$\textbf{if}\ a{=}0 \lor b{=}0\ \textbf{then}\ \mathit{ok}$$
$$\textbf{else}\ x:= x/a/b\times(a+b-1)\times(a+b).\ \ a:= a-1.\ \ b:= b-1.\ \ t:= t+1.$$
$$x' = x \times (a+b)!/(a!\times b!) \land t' = t + \mathit{min}\ a\ b\ \textbf{fi}$$

Now, here is a **for**-loop solution. Define

$$Ik \;=\; x = (a+k)!/(a!\times k!)$$

Then

$$x' = (a+b)!/(a!\times b!) \quad \Leftarrow \quad x:= 1.\ \ I0 \Rightarrow I'b$$
$$I0 \Rightarrow I'b \quad \Leftarrow \quad \textbf{for}\ k:= 0;..b\ \textbf{do}\ Ik \Rightarrow I'(k+1)\ \textbf{od}$$
$$Ik \Rightarrow I'(k+1) \quad \Leftarrow \quad x:= x\times(a+k+1)/(k+1)$$

with timing $t' = t+b$.

Finally, here are two functional solutions. Define

$$f \;=\; \langle a, b\colon \mathit{nat} \rightarrow (a+b)!/(a!\times b!)\rangle$$

Then

$$f\ a\ b \;=\; \textbf{if}\ a{=}0\ \textbf{then}\ 1\ \textbf{else}\ f\ (a-1)\ b \times (a+b)\ /\ a\ \textbf{fi}$$

with execution time a . For execution time $\mathit{min}\ a\ b$

$$f\ a\ b \;=\; \textbf{if}\ a{=}0 \lor b{=}0\ \textbf{then}\ 1\ \textbf{else}\ f\ (a-1)\ (b-1) \times (a+b-1) \times (a+b)\ /\ a\ /\ b\ \textbf{fi}$$