423 A binary tree can be stored as a list of nodes in breadth order. Traditionally, the root is at index 1, the node at index n has its left child at index $2{\times}n$ and its right child at index $2{\times}n+1$. Suppose the user's variable is $x\colon X$, and the implementer's variables are $s\colon [*X]$ and $p\colon nat+1$, and the operations are

$$\begin{array}{lcl} goHome & = & p{:=}\ 1 \\ goLeft & = & p{:=}\ 2{\times}p \\ goRight & = & p{:=}\ 2{\times}p+1 \\ goUp & = & p{:=}\ div\ p\ 2 \\ put & = & s{:=}\ p{\to}x \mid s \\ get & = & x{:=}\ s\ p \end{array}$$

Now suppose we decide to move the entire list down one index so that we do not waste index 0. The root is at index 0, its children are at indexes 1 and 2, and so on. Develop the necessary data transform, and use it to transform the operations.

§ The new implementer's variables are $r\colon [*X]$ and $q\colon nat$. The transform is

$$r = s[1;..\#s] \ \wedge\ p = q+1$$

For each of the transformations, it will be easy enough to eliminate the three variables p, s', and p' by one-point. The trick to eliminate s is explained after the transformations.

Transform $goHome$:

$\forall s, p\cdot \quad r = s[1;..\#s] \wedge p = q+1$
$\qquad \Rightarrow\ \exists s', p'\cdot\ r' = s'[1;..\#s'] \wedge p' = q'+1 \wedge x'=x \wedge s'=s \wedge p'=1$
$=\quad x'=x \wedge r'=r \ \wedge\ 1 = q'+1$
$=\quad q{:=}\ 0$

Transform $goLeft$:

$\forall s, p\cdot \quad r = s[1;..\#s] \wedge p = q+1$
$\qquad \Rightarrow\ \exists s', p'\cdot\ r' = s'[1;..\#s'] \wedge p' = q'+1 \wedge x'=x \wedge s'=s \wedge p'=2{\times}p$
$=\quad x'=x \wedge r'=r \ \wedge\ 2{\times}(q+1) = q'+1$
$=\quad q{:=}\ 2{\times}q + 1$

Transform $goRight$:

$\forall s, p\cdot \quad r = s[1;..\#s] \wedge p = q+1$
$\qquad \Rightarrow\ \exists s', p'\cdot\ r' = s'[1;..\#s'] \wedge p' = q'+1 \wedge x'=x \wedge s'=s \wedge p'=2{\times}p+1$
$=\quad x'=x \wedge r'=r \ \wedge\ 2{\times}(q+1) + 1 = q'+1$
$=\quad q{:=}\ 2{\times}q + 2$

Transform $goUp$:

$\forall s, p\cdot \quad r = s[1;..\#s] \wedge p = q+1$
$\qquad \Rightarrow\ \exists s', p'\cdot\ r' = s'[1;..\#s'] \wedge p' = q'+1 \wedge x'=x \wedge s'=s \wedge p' = div\ p\ 2$
$=\quad x'=x \wedge r'=r \ \wedge\ div\ (q+1)\ 2 = q'+1$
$=\quad q{:=}\ div\ (q+1)\ 2 - 1$
$=\quad q{:=}\ div\ (q-1)\ 2$

Transform put :

$\forall s, p\cdot \quad r = s[1;..\#s] \wedge p = q+1$
$\qquad \Rightarrow\ \exists s', p'\cdot\ r' = s'[1;..\#s'] \wedge p' = q'+1 \wedge x'=x \wedge s'=p{\to}x \mid s \wedge p'=p$
$=\quad x'=x \wedge r' = q{\to}x \mid r \ \wedge\ q'+1=q+1$
$=\quad r{:=}\ q{\to}x \mid r$

Transform *get* :

$$\forall s, p\cdot \quad r = s[1;..\#s] \land p = q{+}1$$
$$\Rightarrow \quad \exists s', p'\cdot\ r' = s'[1;..\#s'] \land p' = q'{+}1 \land x'{=}s\ p \land s'{=}s \land p'{=}p$$
$$= \quad x'{=}r\ q \land r'{=}r \ \land\ q'{+}1{=}q{+}1$$
$$= \quad x{:=}\ r\ q$$

To transform *put* we start with

$$\forall s, p\cdot \quad r = s[1;..\#s] \land p = q{+}1$$
$$\Rightarrow \quad \exists s', p'\cdot\ r' = s'[1;..\#s'] \land p' = q'{+}1 \land x'{=}x \land s'{=}p{\to}x \mid s \land p'{=}p$$

The three variables p , s' , and p' are easy to eliminate by one-point. We get

$$= \quad \forall s\cdot\ \ r = s[1;..\#s] \ \Rightarrow\ r' = (q{+}1{\to}x \mid s)[1;..\#(q{+}1{\to}x \mid s)] \land q{+}1 = q'{+}1 \land x'{=}x$$

The problem is to get rid of s because we don't have s=something . We have

$$r = s[1;..\#s]$$

From this we see that $\#r = \#s{-}1$ and $s = [i];;r$ for some unknown item i . I'll use that to eliminate s .

$$= \quad r' = (q{+}1{\to}x \mid [i];;r)[1;..\#(q{+}1{\to}x \mid [i];;r)] \land q{+}1 = q'{+}1 \land x'{=}x$$

We can simplify $\#(q{+}1{\to}x \mid [i];;r)$ ro $\#r{+}1$ and simplify $q{+}1 = q'{+}1$ to $q'{=}q$.

$$= \quad r' = (q{+}1{\to}x \mid [i];;r)[1;..\#r{+}1] \land q'{=}q \land x'{=}x$$

Now we need to simplify $(q{+}1{\to}x \mid [i];;r)[1;..\ \#r{+}1]$. We have a list $(q{+}1{\to}x \mid [i];;r)$ of length $\#r{+}1$, and in this list at index $q{+}1$ the item is x . Now we index with the list $[1;..\ \#r{+}1]$, which shifts all the indexes down 1 . So now at index q the item is x .

$$= \quad x'{=}x \land r' = q{\to}x \mid r \ \land\ q'{=}q$$
$$= \quad r{:=}\ q{\to}x \mid r$$

I wish I could see a nice series of formal steps.