Function Theory

$\langle v: D \to b \rangle$ "map v in D to b"

body (may use v)

domain, type (a bunch)

variable, parameter (a fresh name)

$v: D$ is a local axiom within b

$\langle n: nat \to n+1 \rangle$

$$\begin{array}{ccc} 0 & \searrow & 0 \\ 1 & \searrow & 1 \\ 2 & \searrow & 2 \\ 3 & \searrow & 3 \\ \vdots & & 4 \\ & & \vdots \end{array}$$

Renaming

$\langle n\text{: } nat \rightarrow n{+}1\rangle \;=\; \langle m\text{: } nat \rightarrow m{+}1\rangle$

Domain

□f “domain of f”

□$\langle n\text{: } nat \rightarrow n{+}1\rangle \;=\; nat$

Application

$f\,x$ “f applied to x” or “f of x”

$f\,(x)$ $(f)x$ $f\,(x{+}1)$ $-x$ $\neg x$

$\langle n\text{: } nat \rightarrow n{+}1\rangle\, 3 \;=\; 3{+}1 \;=\; 4$

two variables

$$max \;=\; \langle x: xrat \rightarrow \langle y: xrat \rightarrow \textbf{if } x{\ge}y \textbf{ then } x \textbf{ else } y \textbf{ fi}\rangle\rangle$$

$$max\ 3 \;=\; \langle y: xrat \rightarrow \textbf{if } 3{\ge}y \textbf{ then } 3 \textbf{ else } y \textbf{ fi}\rangle$$

$$max\ 3\ 5 \;=\; \textbf{if } 3{\ge}5 \textbf{ then } 3 \textbf{ else } 5 \textbf{ fi} \;=\; 5$$

$$max(3, 5) \;=\; max\ 3, max\ 5$$

predicate

function with binary result

$$even \;=\; \langle i: int \rightarrow i/2: int\rangle$$

relation

function with predicate result

$$divides \;=\; \langle n: nat{+}1 \rightarrow \langle i: int \rightarrow i/n: int\rangle\rangle$$

$$even \;=\; divides\ 2$$

selective union

$f \mid g$ "f otherwise g"

$\square(f \mid g) \;=\; \square f, \square g$

$(f \mid g)\; x \;=\;$ **if** x: $\square f$ **then** $f\,x$ **else** $g\,x$ **fi**

abbreviated function notations

$\langle x\text{: } xrat \rightarrow \langle y\text{: } xrat \rightarrow$ **if** $x \ge y$ **then** x **else** y **fi**$\rangle\rangle \;=\; \langle x, y\text{: } xrat \rightarrow$ **if** $x \ge y$ **then** x **else** y **fi**$\rangle$

$\langle n\text{: } nat \rightarrow n+1\rangle \;=\; \langle n \rightarrow n+1\rangle$

$\langle n\text{: } 2 \rightarrow 3\rangle \;=\; 2 \rightarrow 3$ scope brackets go with variable

$\langle x\text{: } int \rightarrow \langle y\text{: } int \rightarrow x+3\rangle\rangle \;=\; x+3$? but we can't apply it

Scope and Substitution

local bound, hidden, private

introduction is inside the expression (formal)

nonlocal global, free, visible, public

introduction is outside the expression (formal or informal)

$\langle x \rightarrow \quad x \quad y \rangle (\quad x \quad y \quad)$

Scope and Substitution

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$$\langle x \rightarrow \quad x \quad \langle x \rightarrow \quad x \quad \rangle \quad x \quad \rangle \; 3$$

$$= \quad (\quad 3 \quad \langle x \rightarrow \quad x \quad \rangle \quad 3 \quad)$$

Scope and Substitution

local bound, hidden, private

introduction is inside the expression (formal)

nonlocal global, free, visible, public

introduction is outside the expression (formal or informal)

$$\langle y \rightarrow \; x \;\; y \;\; \langle x \rightarrow \; x \;\; y \; \rangle \;\; x \;\; y \; \rangle \, x$$ rename inner x to z

$$= \langle y \rightarrow \; x \;\; y \;\; \langle z \rightarrow \; z \;\; y \; \rangle \;\; x \;\; y \; \rangle \, x$$ now apply

$$= (\;\; x \;\; x \;\; \langle z \rightarrow \; z \;\; x \; \rangle \;\; x \;\; x \;)$$

Quantifiers

A quantifier is an operator that applies to a function.

It is defined from a two-operand symmetric associative operator.

$\forall p$ is defined from $\wedge$	$\forall\langle r\text{: } rat \rightarrow r<0 \vee r=0 \vee r>0\rangle$
$\exists p$ is defined from $\vee$	$\exists\langle n\text{: } nat \rightarrow n=0\rangle$
Σf is defined from $+$	$\Sigma\langle n\text{: } nat+1 \rightarrow 1/2^n\rangle$
Πf is defined from $\times$	$\Pi\langle n\text{: } nat+1 \rightarrow (4\times n^2)/(4\times n^2-1)\rangle$

abbreviations

$\forall r\text{: } rat\cdot\ r<0 \vee r=0 \vee r>0$

$\Sigma n\text{: } nat+1\cdot\ 1/2^n$

$\forall x, y\text{: } rat\cdot\ x = y+1 \ \Rightarrow\ x > y \quad = \quad \forall x\text{: } rat\cdot\ \forall y\text{: } rat\cdot\ x = y+1 \ \Rightarrow\ x > y$

$\Sigma n, m\text{: } 0,..10\cdot\ n\times m \quad = \quad \Sigma n\text{: } 0,..10\cdot\ \Sigma m\text{: } 0,..10\cdot\ n\times m$

$\forall v\text{: } null\cdot\ b \ = \ \top$

$\forall v\text{: } x\cdot\ b \ = \ \langle v\text{: } x \rightarrow b\rangle\ x$

$\forall v\text{: } A,B\cdot\ b \ = \ (\forall v\text{: } A\cdot\ b) \wedge (\forall v\text{: } B\cdot\ b)$

$\exists v\text{: } null\cdot\ b \ = \ \bot$

$\exists v\text{: } x\cdot\ b \ = \ \langle v\text{: } x \rightarrow b\rangle\ x$

$\exists v\text{: } A,B\cdot\ b \ = \ (\exists v\text{: } A\cdot\ b) \vee (\exists v\text{: } B\cdot\ b)$

$\Sigma v\text{: } null\cdot\ n \ = \ 0$

$\Sigma v\text{: } x\cdot\ n \ = \ \langle v\text{: } x \rightarrow n\rangle\ x$

$(\Sigma v\text{: } A,B\cdot\ n) + (\Sigma v\text{: } A`B\cdot\ n) \ = \ (\Sigma v\text{: } A\cdot\ n) + (\Sigma v\text{: } B\cdot\ n)$

$\Pi v\text{: } null\cdot\ n \ = \ 1$

$\Pi v\text{: } x\cdot\ n \ = \ \langle v\text{: } x \rightarrow n\rangle\ x$

$(\Pi v\text{: } A,B\cdot\ n) \times (\Pi v\text{: } A`B\cdot\ n) \ = \ (\Pi v\text{: } A\cdot\ n) \times (\Pi v\text{: } B\cdot\ n)$

build your own

$$MAX\ \ x\text{: } rat\cdot\ 4{\times}x - x^2\ \ =\ \ 4$$

$$MAX\ v\text{: } null\cdot\ n\ \ =\ \ -\infty$$

$$MAX\ v\text{: } x\cdot\ n\ \ =\ \ \langle v\text{: } x \to n\rangle\ x$$

$$MAX\ v\text{: } A{,}B\cdot\ n\ \ =\ \ max\ (MAX\ v\text{: } A\cdot\ n)\ (MAX\ v\text{: } B\cdot\ n)$$

$$x \wedge \top\ \ =\ \ x$$

$$x \vee \bot\ \ =\ \ x$$

$$x + 0\ \ =\ \ x$$

$$x \times 1\ \ =\ \ x$$

$$max\ x\ (-\infty)\ \ =\ \ x$$

Solution Quantifier

$\S p$ is the (bunch of) solutions of predicate p

$\S v\colon null\cdot\ b \ = \ null$

$\S v\colon x\cdot\ b \ = \ \textbf{if}\ \langle v\colon x \to b\rangle\ x\ \textbf{then}\ x\ \textbf{else}\ null\ \textbf{fi}$

$\S v\colon A,B\cdot\ b \ = \ (\S v\colon A\cdot\ b),\ (\S v\colon B\cdot\ b)$

$\{\ \S i\colon int\cdot\ i^2{=}4\ \} \quad = \ \{-2, 2\ \}$

$\{\ \S n\colon nat\cdot\ n{<}3\ \} \quad = \ \{0,..3\ \}$

An expression talks about its nonlocal variables.

$$\exists n: nat \cdot x = 2 \times n$$

says

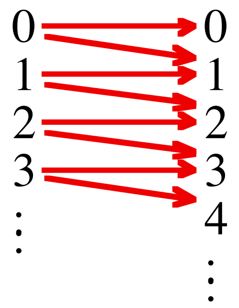
" x is an even natural"

Function Fine Points

partial:	sometimes no result
total:	always at least one result
deterministic:	always at most one result
nondeterministic:	sometimes more than one result

$\langle n: nat \cdot n, n{+}1 \rangle$ total and nondeterministic

$\langle n: nat \cdot n, n{+}1 \rangle\ 3 \;=\; 3, 4$



distribution

$$(f, g)\, x \;=\; fx, gx$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n\colon nat \to n{+}n\rangle$$

$$double\ 2 \;=\; 4$$

$$double\ (2, 3) \;=\; double\ 2, double\ 3 \;=\; 4, 6$$

$$double\ (2, 3) \;\neq\; (2, 3) + (2, 3) \;=\; 4, 5, 6$$

$$tiny \;=\; \langle S\colon \text{⁄}nat \to \$S < 3\rangle$$

$$tiny\ \{null\} \;=\; \top$$

$$tiny\ \{0, 1, 2, 3\} \;=\; \bot$$

$$tiny\ null \;=\; null$$

function inclusion

$$f: g \;\; = \;\; \square g: \square f \;\wedge\; \forall x: \square g \cdot fx: gx$$

$$A \to B \;\; = \;\; \langle a: A \to B \rangle$$

$A \to B$ is a function whose domain is A and whose result is B .

$$f: A \to B \;\; = \;\; A: \square f \;\wedge\; \forall a: A \cdot fa: B$$

$A \to B$ is all those functions whose domain is at least A and whose result is at most B .

$A \to B$ is antimonotonic in A and monotonic in B .

	$suc: nat \to nat$	function inclusion
$=$	$nat: \square suc \;\wedge\; \forall n: nat \cdot suc\ n: nat$	definition of suc
$=$	$nat: nat \;\wedge\; \forall n: nat \cdot n+1: nat$	reflexivity and definition of nat
$=$	$\top$	

function inclusion

$f \colon g \;\; = \;\; \square g \colon \square f \;\wedge\; \forall x \colon \square g \cdot f x \colon g x$

$suc \colon nat \rightarrow nat$

$even \colon int \rightarrow bin$

$max \colon xrat \rightarrow xrat \rightarrow xrat$

$A \colon B \;\wedge\; f \colon B \rightarrow C \;\wedge\; C \colon D \;\;\Rightarrow\;\; f \colon A \rightarrow D$

$(0,..10) \colon nat \;\wedge\; suc \colon nat \rightarrow nat \;\wedge\; nat \colon int \;\;\Rightarrow\;\; suc \colon (0,..10) \rightarrow int$

$\langle f \colon (0,..10) \rightarrow int \cdot \forall n \colon 0,..10 \cdot even\ (f\ n) \rangle\ suc$

$= \quad \forall n \colon 0,..10 \cdot even\ (suc\ n)$

$= \quad \bot$

function equality

$$f = g \quad = \quad \square f = \square g \;\land\; \forall x\colon\ \square f\cdot\ fx = gx$$

function composition

If ¬ *f*: □*g* then

□(*g f*) = §*x*: □*f*· *fx*: □*g*

(*g f*) *x* = *g* (*f x*)

□(*even suc*)

= §*x*: □*suc*· *suc x*: □*even*

= §*x*: *nat*· *x*+1: *int*

= *nat*

(*even suc*) 3 = *even* (*suc* 3) = *even* 4 = ⊤

(–*suc*) 3 = –(*suc* 3) = –4

(¬*even*) 3 = ¬(*even* 3) = ¬⊥ = ⊤

function composition

Suppose x, y: *int*

f, g: $int \to int$

h: $int \to int \to int$

Then

$$
\begin{array}{ll}
 & h\ f\ x\ g\ y \\
= & (((h\ f)\ x)\ g)\ y \\
= & ((h\ (f\ x))\ g)\ y \\
= & (h\ (f\ x))\ (g\ y) \\
= & h\ (f\ x)\ (g\ y)
\end{array}
$$

list as function

If m: 0,..#L then

$$\langle n\colon 0,..\#L \to L n\rangle\ m \;=\; L\ m$$

function ≈ list

application ≈ indexing

function composition ≈ list composition

$-\,[3;\,5;\,2] \;=\; [-3;\,-5;\,-2]$

$suc\ [3;\,5;\,2] \;=\; [4;\,6;\,3]$

$1\to 21\ |\ [10;\,11;\,12] \;=\; [10;\,21;\,12]$

$\Sigma L \;=\; \Sigma n\colon 0,..\#L\cdot L n$

limit

f: $nat \to rat$

$f0; f1; f2; \ldots$ is a sequence of rationals

$(MAX\ m \cdot\ MIN\ n \cdot\ f(m+n)) \ \le\ (LIM\ f) \ \le\ (MIN\ m \cdot\ MAX\ n \cdot\ f(m+n))$

$LIM\ n \cdot\ 1/(n+1) \quad = \ 0$

$-1 \le\ (LIM\ n \cdot\ (-1)^n\ ) \ \le\ 1$

$(MIN\ f) \le (LIM\ f) \le (MAX\ f)$

$xreal \quad = \quad LIM\ (nat \to rat)$

limit

$p: nat \to bin$

$p0; p1; p2; \ldots$ is a sequence of binary values

$\exists m \cdot \forall n \cdot p(m+n) \quad \Rightarrow \quad LIM\ p \quad \Rightarrow \quad \forall m \cdot \exists n \cdot p(m+n)$

$\exists m \cdot \forall i \cdot i \ge m \Rightarrow pi \quad \Rightarrow \quad LIM\ p$

$\exists m \cdot \forall i \cdot i \ge m \Rightarrow \neg pi \quad \Rightarrow \quad \neg\ LIM\ p$

$LIM\ n \cdot 1/(n+1) = 0 \quad = \quad \bot$