Analysis of Algorithms

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CSCI 570

Lecture 5

University of Southern California

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Greedy Algorithms

Reading: chapter 4

Homework - 2

We do not have enough time to cover the master theorem this week. Therefore, you do not need to solve the last homework problem #5, it will be a part of the next Hw-3 assignment.

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The Minimum Spanning Tree

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Find a spanning tree of the minimum total weight.

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MST is fundamental problem with diverse applications.

Kruskal's Algorithm

algorithm builds a tree one EDGE at a time.

- Sort all edges by their weights.
- Loop:
  - Choose the minimum weight edge and join correspondent vertices (subject to cycles).
  - Go to the next edge.
  - Continue to grow the forest until all vertices are connected.

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Union-Find
O(1)
A.C.

Sorting edges – O(E log E)

Cycle detection – O(V) for each edge

binary heap

Total: O(V*E + E*log E)

Prim's Algorithm

algorithm builds a tree one VERTEX at a time.

- Start with an arbitrary vertex as a sub-tree C.
- Expand C by adding a vertex having the minimum weight edge of the graph having exactly one end point in C.
- Update distances from C to adjacent vertices.
- Continue to grow the tree until C gets all vertices.

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Fib. heap

deleteMin – $O(V)$, for each vertex

update(decreaseKey) – $O(\log V)$, for each edge

O(1) A.C.

Total: $O(V \log V + E \log V)$

Discussion Problem

1. Assume that an unsorted array is used instead of a binary heap. What would the running time of the Prim algorithm?

array : $O(V^2 + E) = O(V^2)$

heap : $O(V \cdot \log V + E \cdot \log V)$

binary

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$E = O(V^2)$

2. Assume that we need to find an MST in a dense graph using Prim's algorithm. Which implementation (heap or array) shall we use?

array: $O(V^2)$

heap: $O(V^2 \log V)$

MST: Proof of the correctness

A cut of a graph is a partition of its vertices into two disjoint sets (blue and green vertices below.)

(a, c)

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A crossing edge is an edge that connects a vertex in one set with a vertex in the other.

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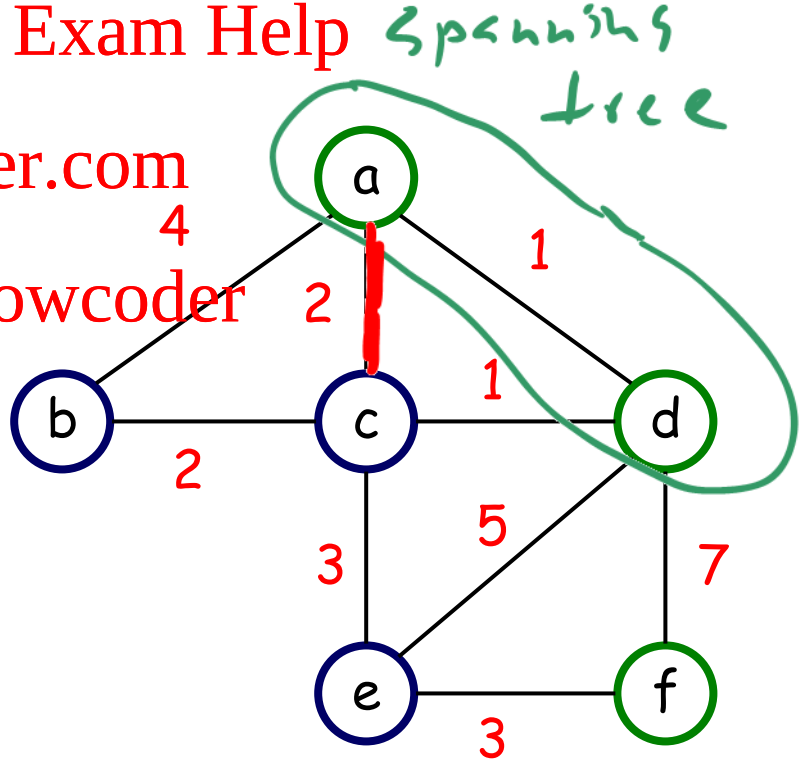
The smallest crossing edge must be in the MST.



MST: Proof of the correctness

Lemma: Given any cut in a weighted graph , the crossing edge of minimum weight is in the MST of the graph.

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By contradiction.

$e \notin MST$, $f \in MST$

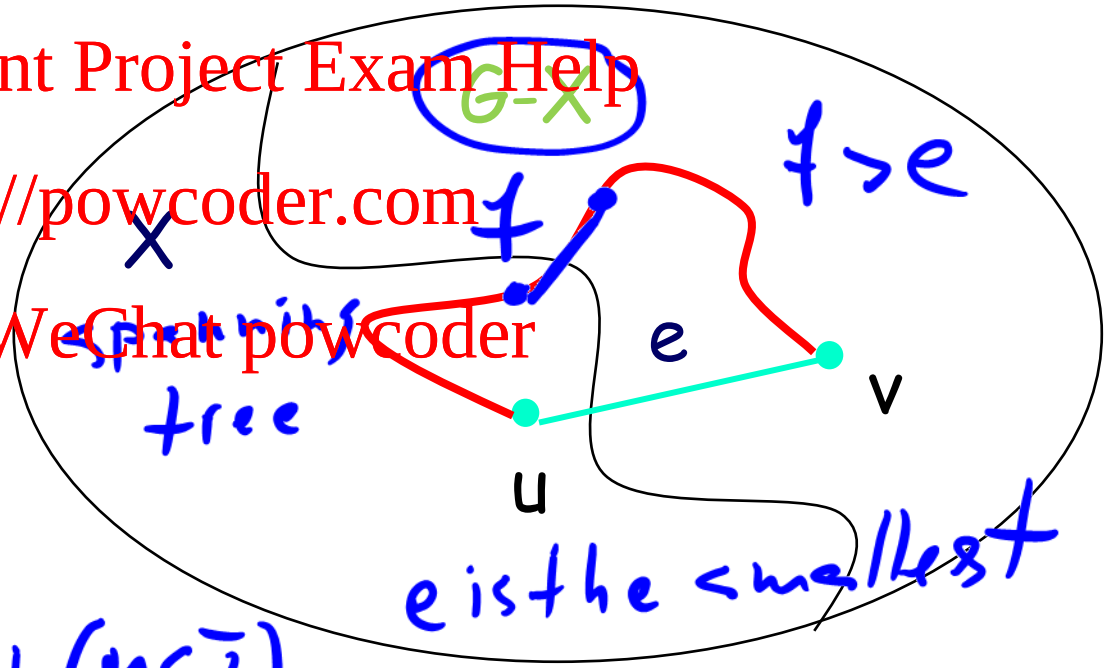
Add e to MST

MST will get a cycle

$MST_1 = MST - f + e$

$cost(MST_1) < cost(MST)$

contradiction



Review Questions

find an example

(T/F) The first edge added by Kruskal's algorithm can be the last edge added by Prim's algorithm.

find a counter example

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(T/F) Suppose we have a graph where each edge weight value appears at most twice. Then, there are at most two minimum spanning trees in this graph.

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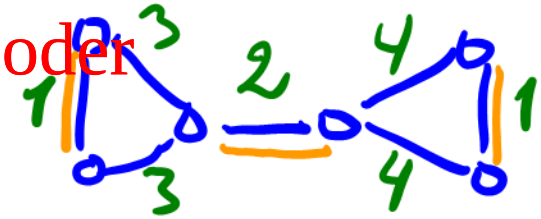
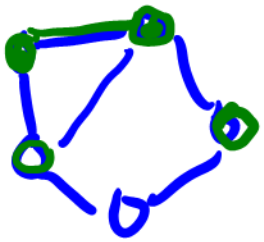


V -> ∞

E = V + 1

(T/F) If a connected undirected graph $G = (V, E)$ has $V + 100$ edges, we can find the minimum spanning tree of G in $O(V)$ runtime.

V = 5
E = 6



Prim's $O((E+V)\log V)$ ← heap size = 2

$|\log V| < \text{const}$

size of the heap

A*

The Shortest Path Problem

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Edsger Dijkstra
(1930-2002)

SSSP

The Shortest Path Problem

Given a positively weighted graph G with a source vertex s, find the shortest path from s to all other vertices in the graph.

$e \in \mathbb{R}^+$

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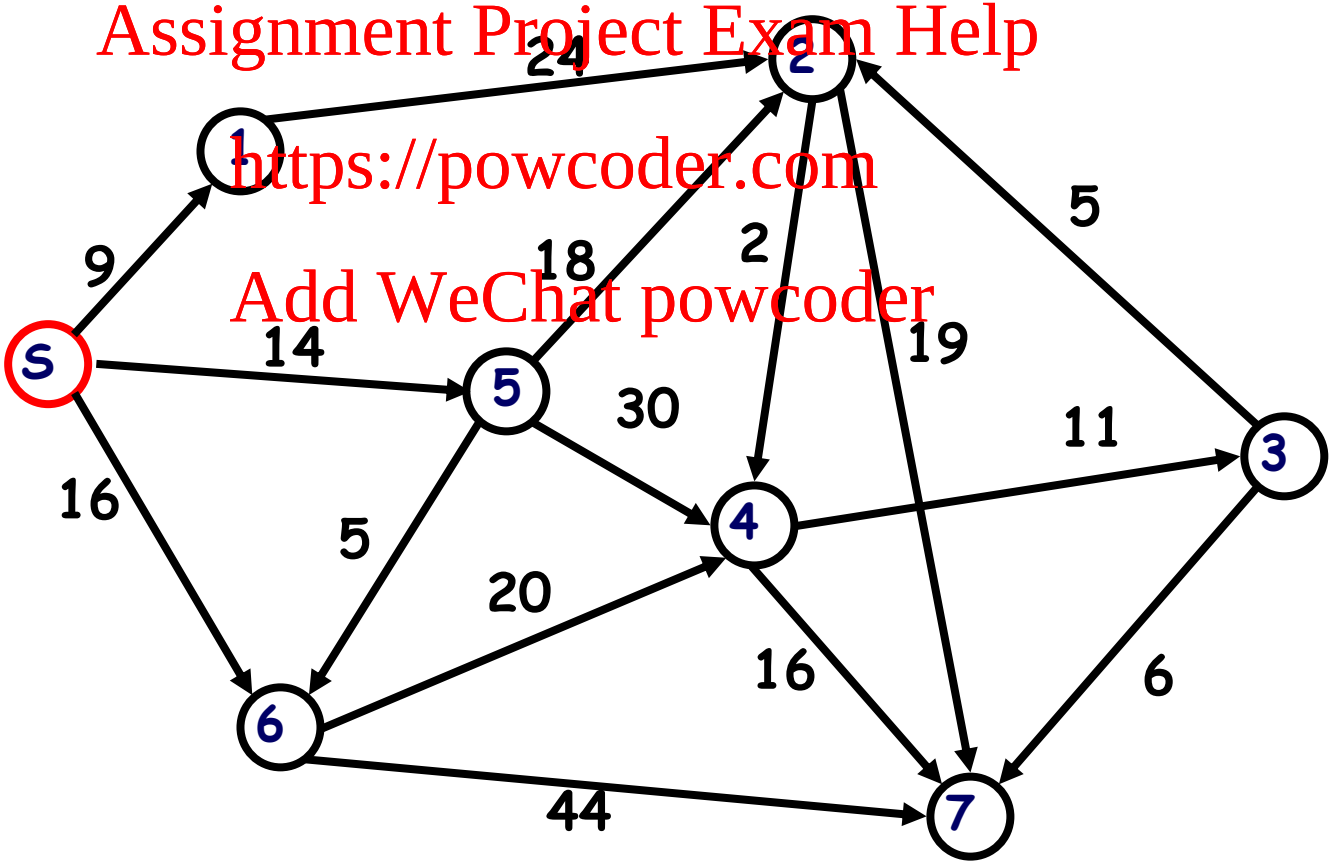
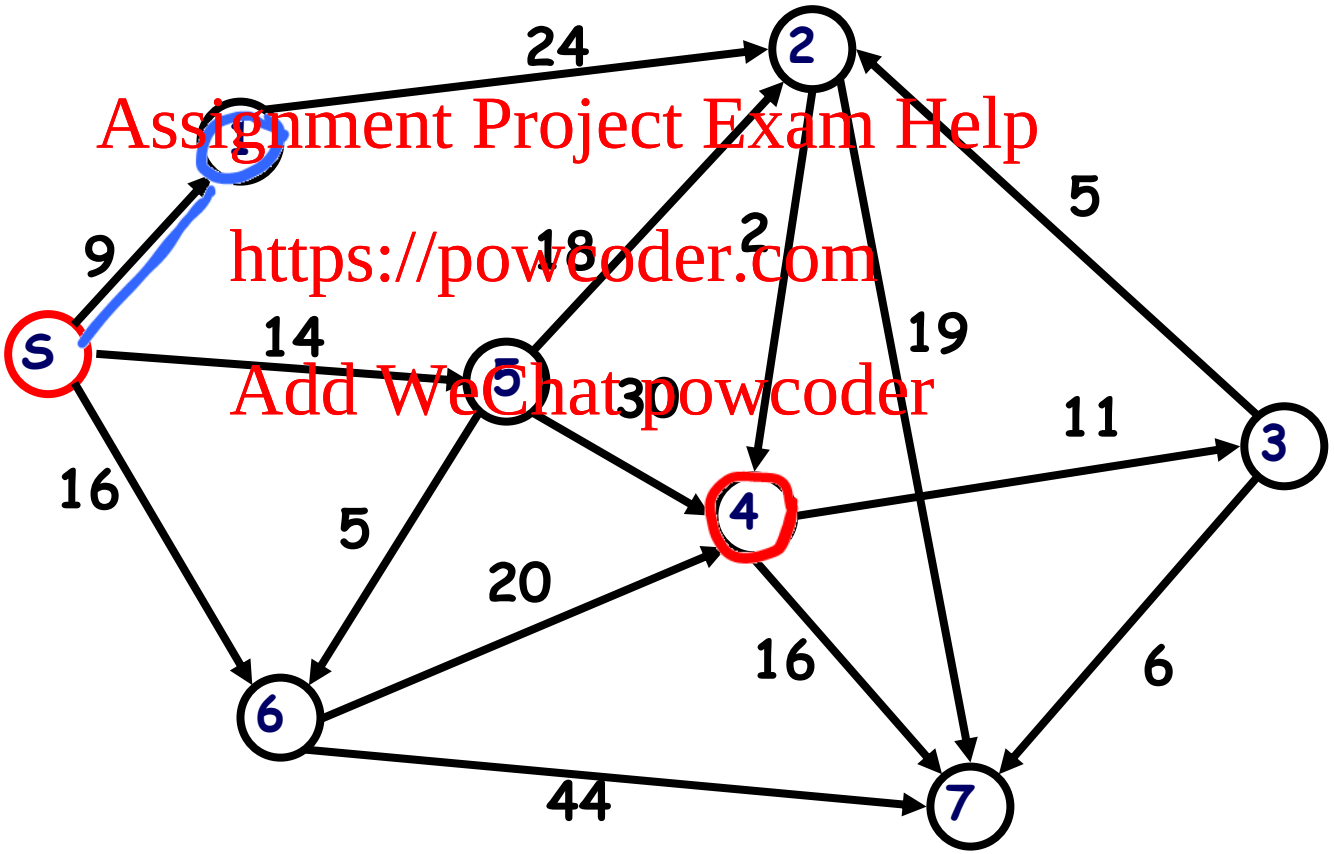


The Shortest Path Problem

What is the shortest distance from s to 4?



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Greedy Approach

Prim's

When algorithm proceeds all vertices are divided into two groups

- vertices whose shortest path from the source is known
- vertices whose shortest path from the source is NOT discovered yet

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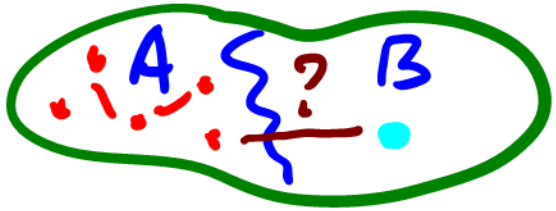
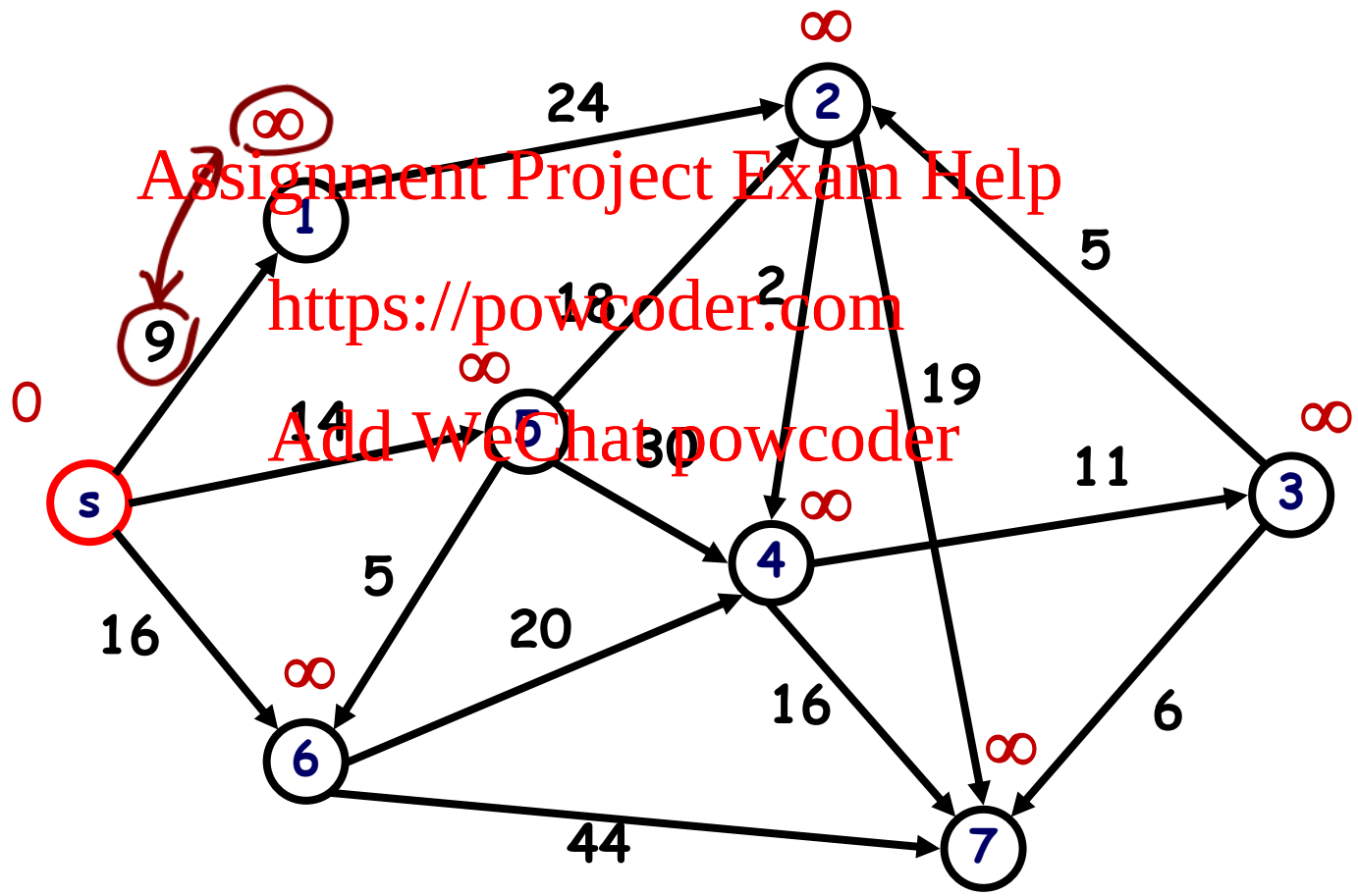
Move vertices one at a time from the undiscovered set of vertices to the known set of the shortest distances, based on the shortest distance from the source.

F = A + B



solution tree = s
heap = {1, 2, 3, 4, 5, 6, 7}



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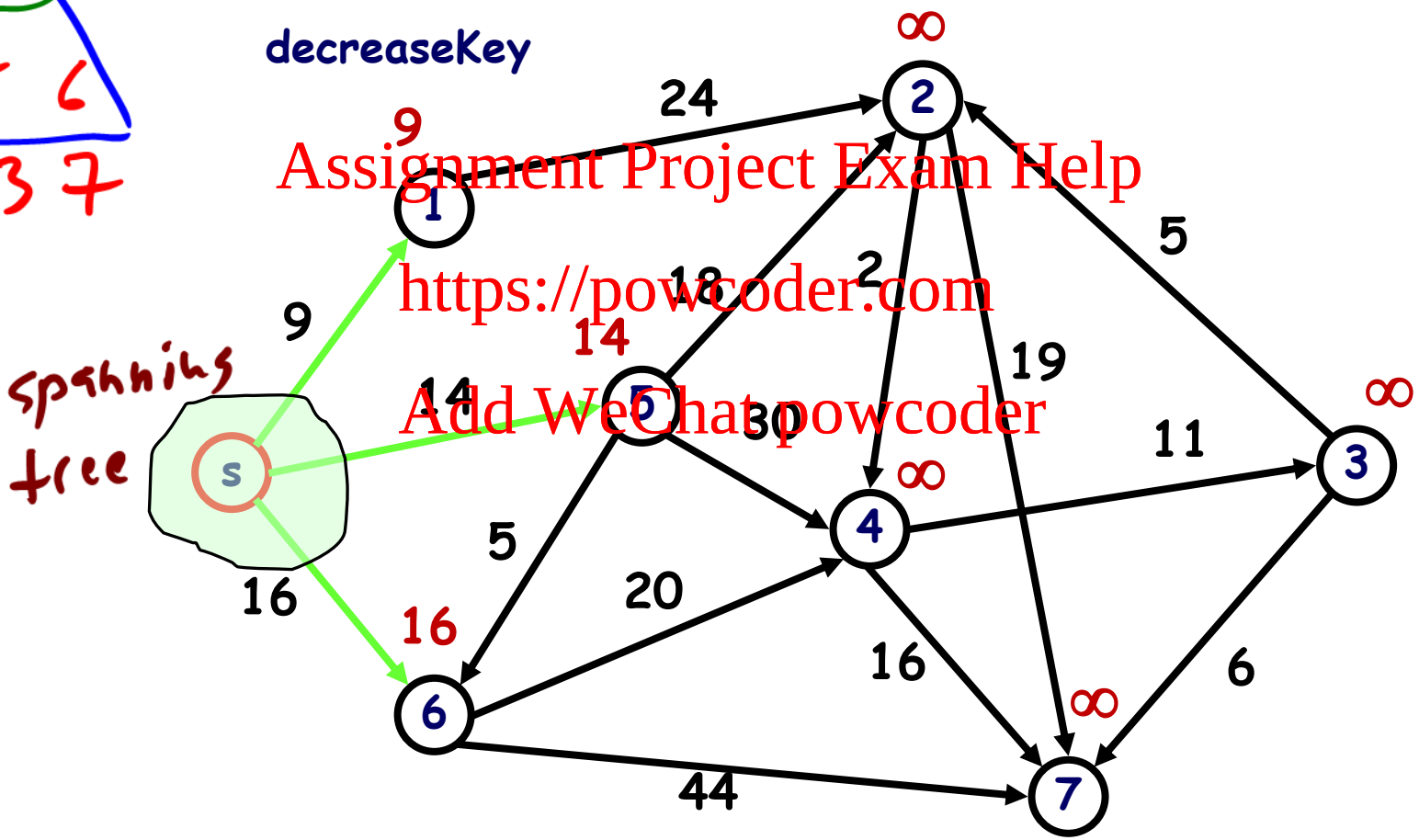
solution tree = { s }
heap = {1, 2, 3, 4, 5, 6, 7}



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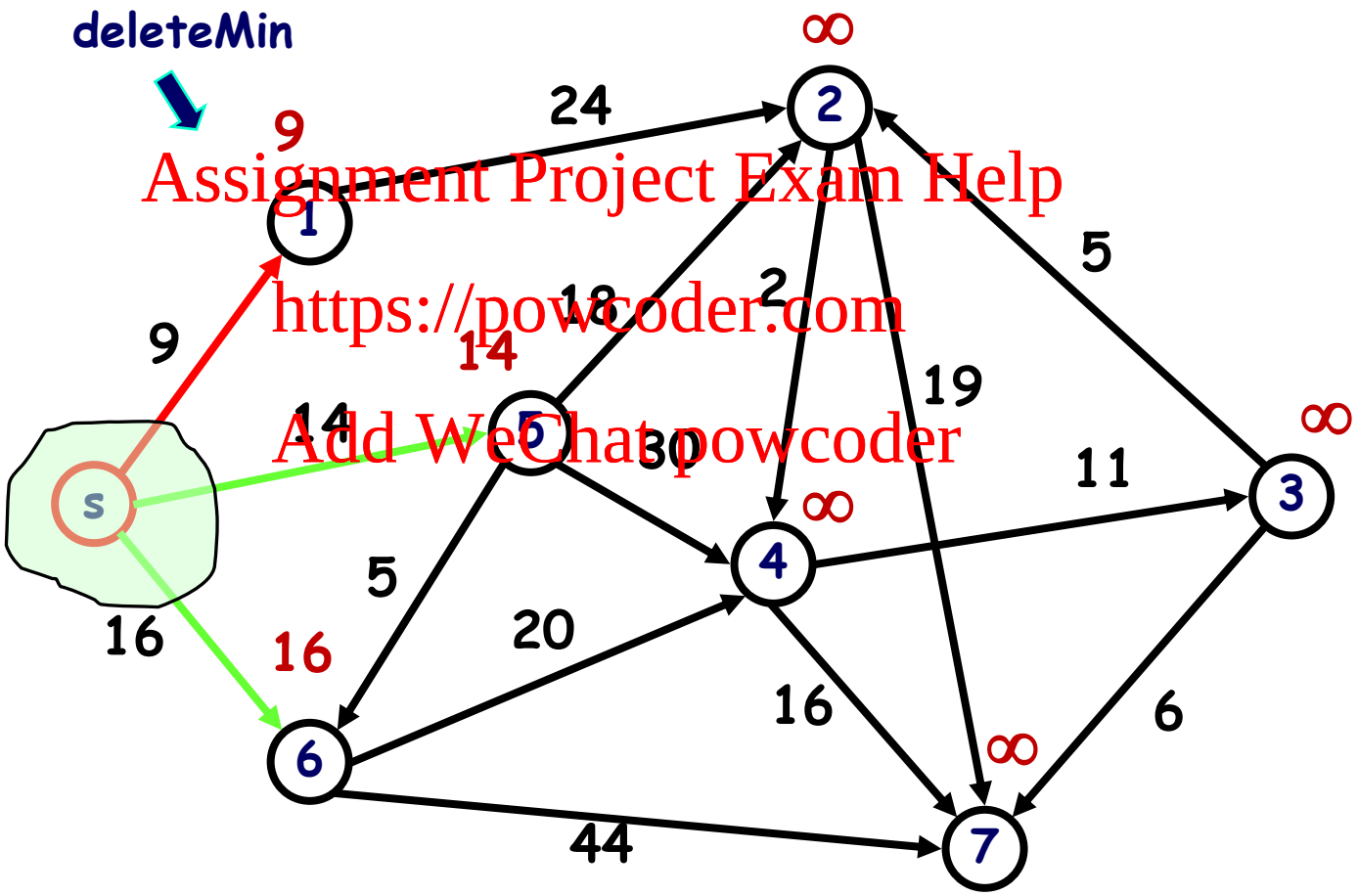
solution tree = { s }
heap = {1, 2, 3, 4, 5, 6, 7}



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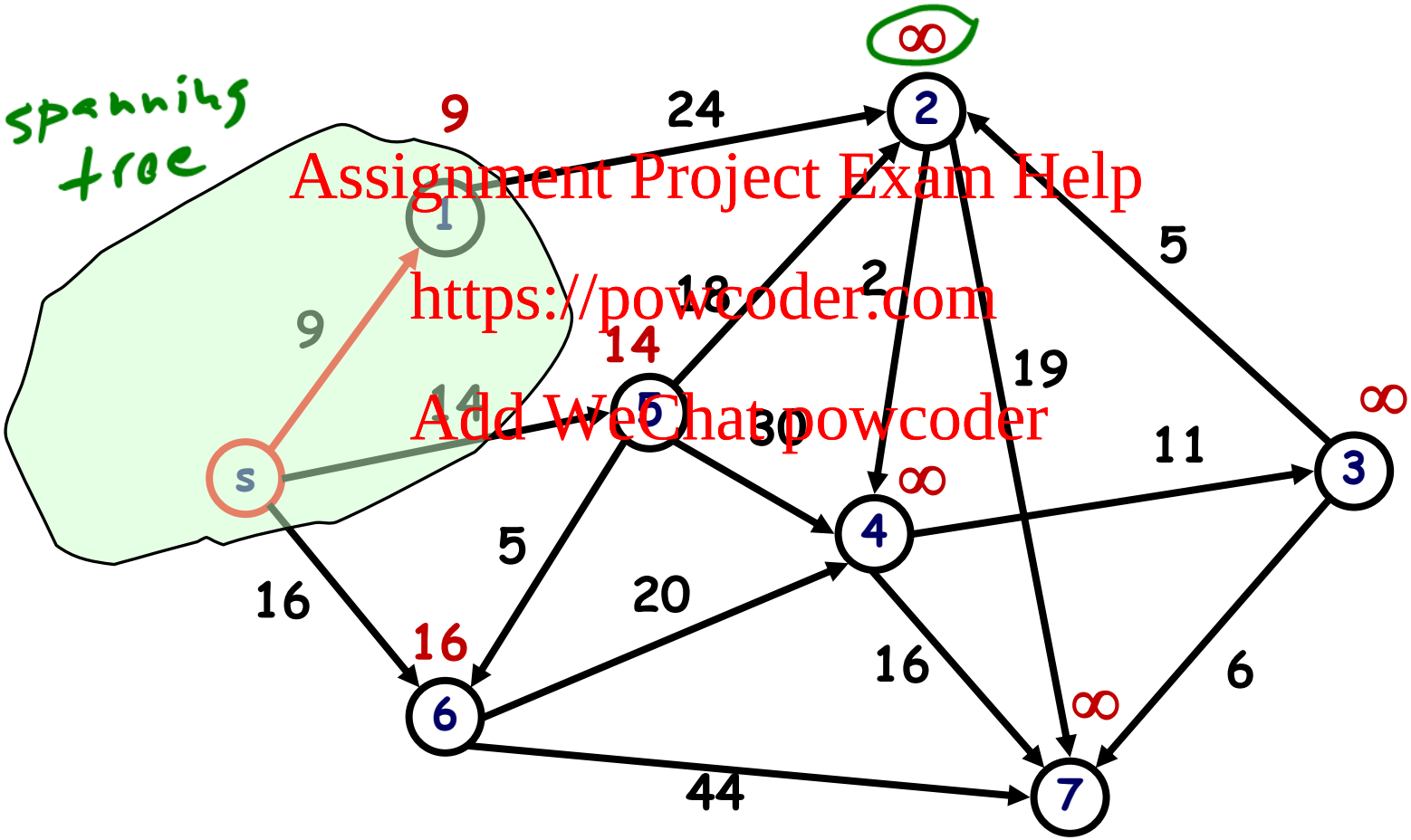
solution tree = { s, 1 }
heap = {2, 3, 4, 5, 6, 7}



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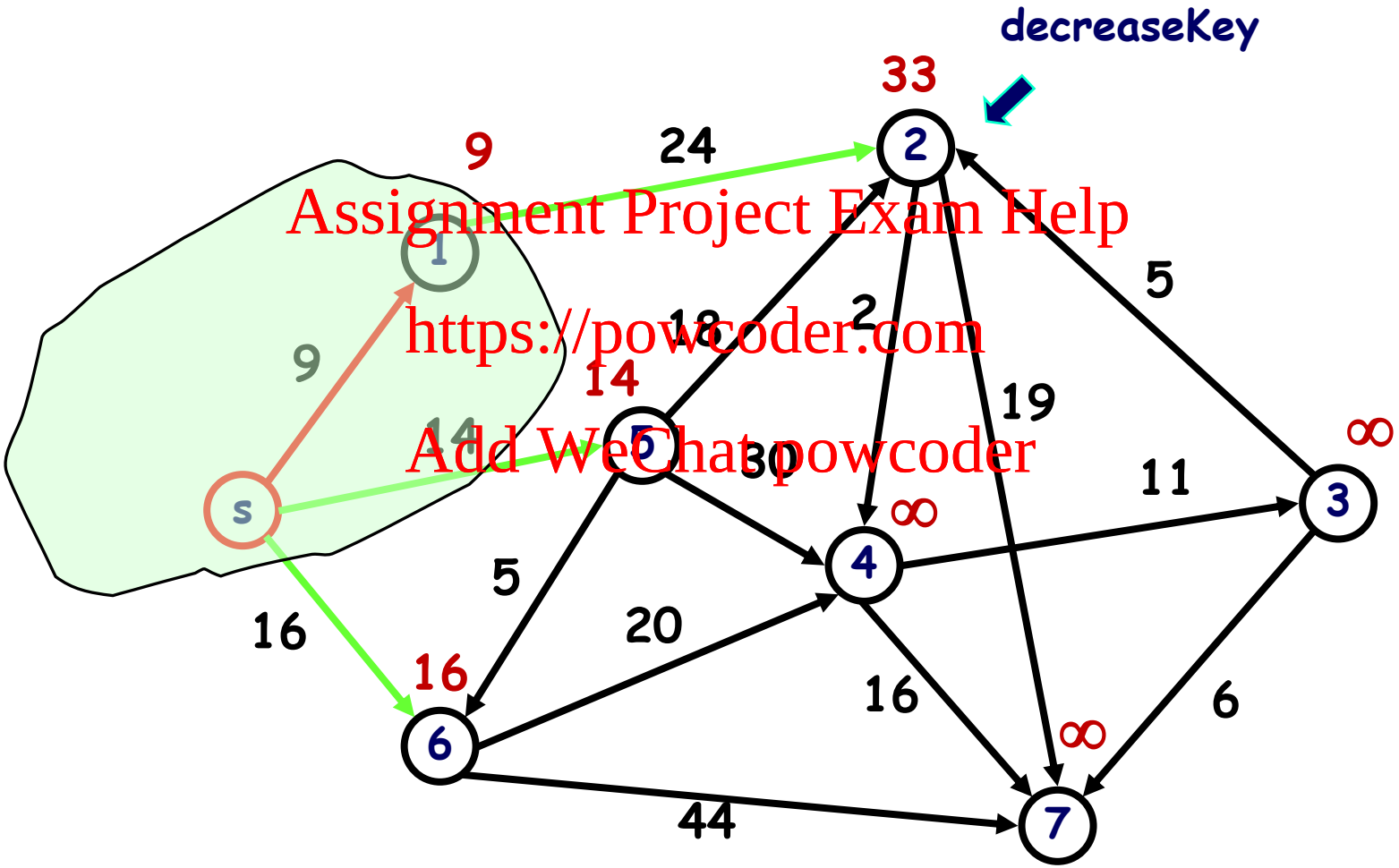
solution tree = { s, 1 }
heap = {2, 3, 4, 5, 6, 7}



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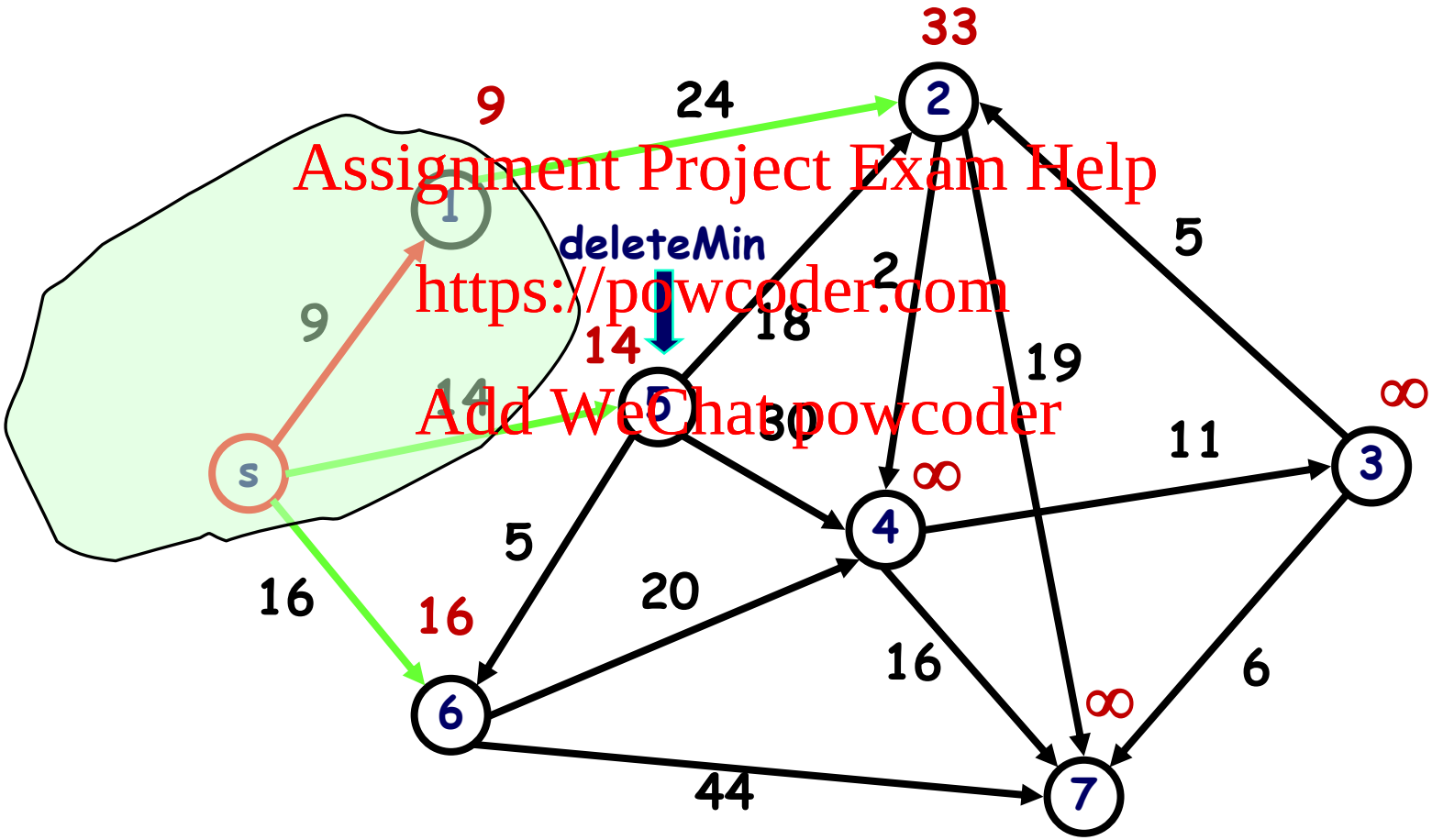
solution tree = { s, 1 }
heap = {2, 3, 4, 5, 6, 7}



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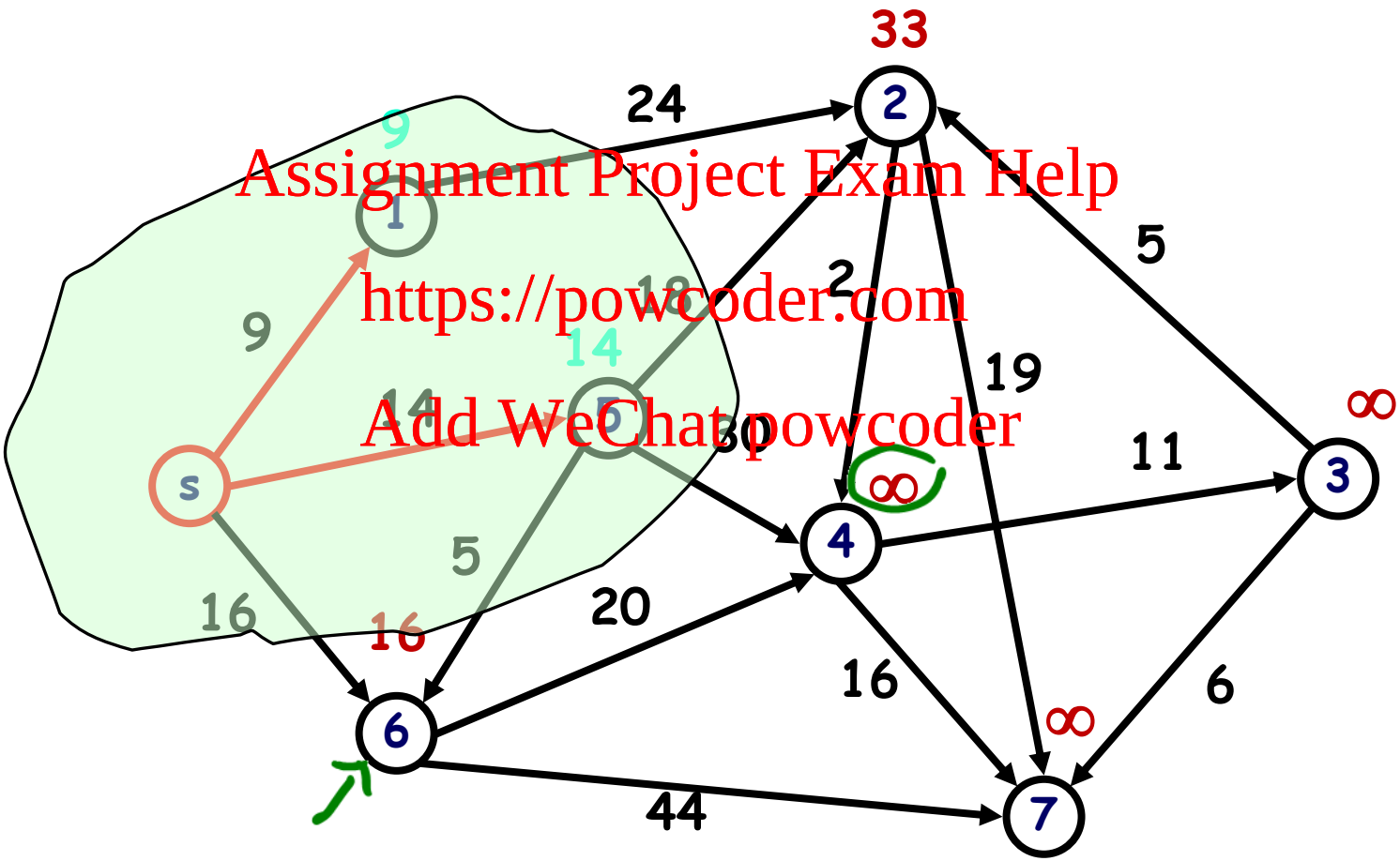
solution tree = { s, 1, 5 }
heap = {2, 3, 4, 6, 7}



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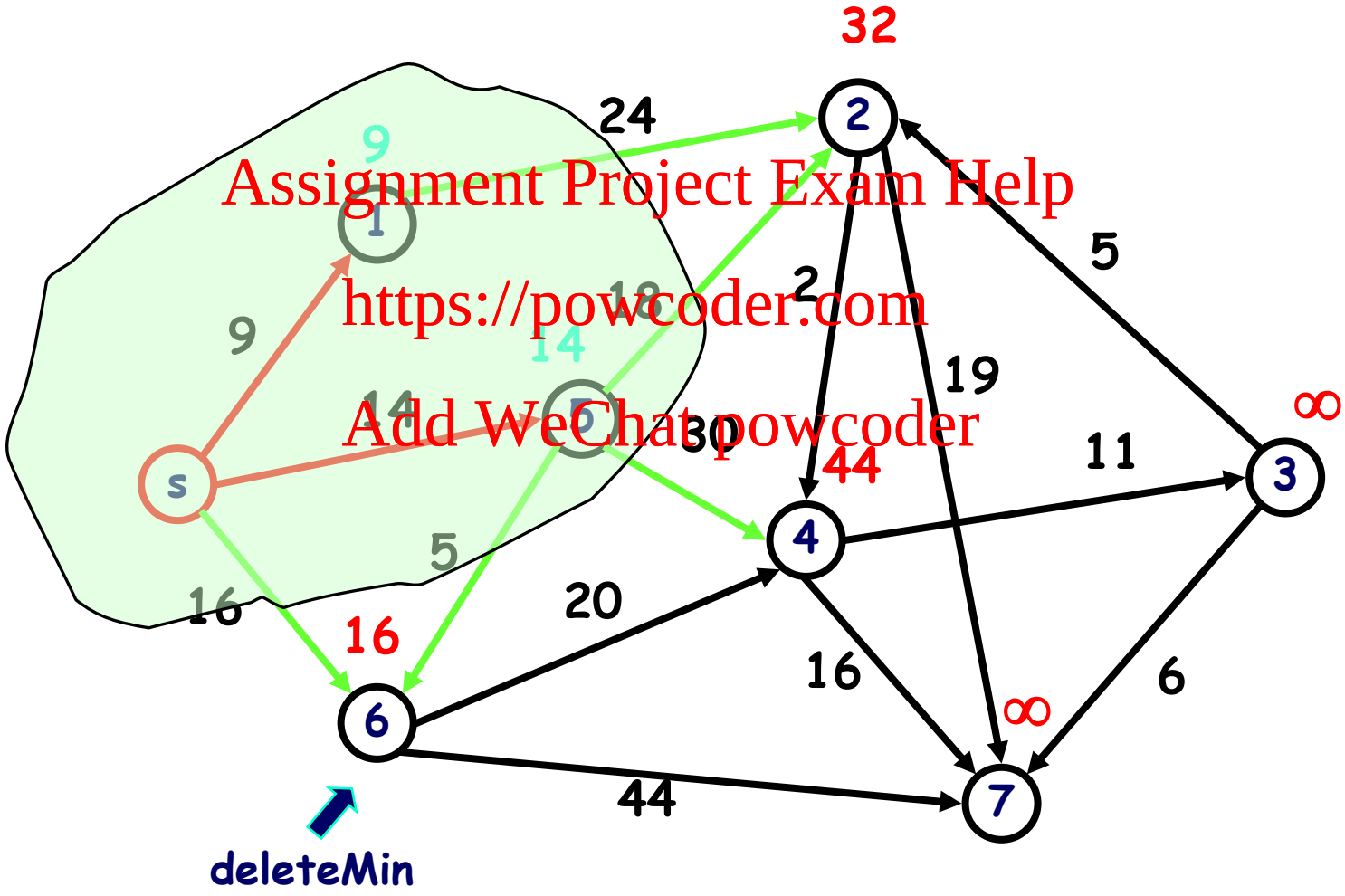
solution tree = { s, 1, 5 }
heap = {2, 3, 4, 6, 7}



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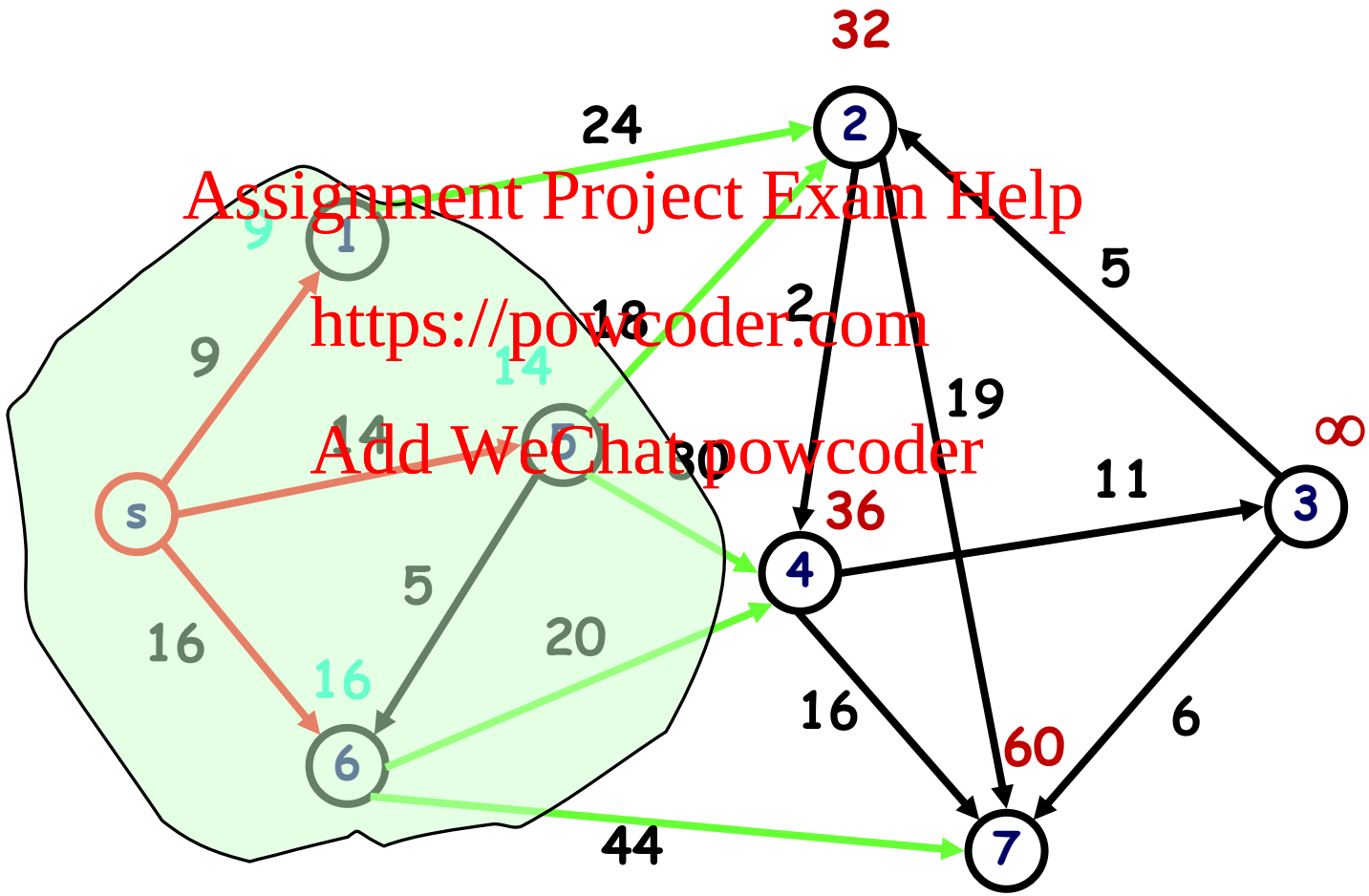
solution tree = { s, 1, 5, 6 }
heap = {2, 3, 4, 7}



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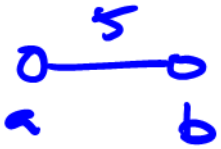
solution tree = { s, 1, 5, 6, 2, 4, 3, 7 }

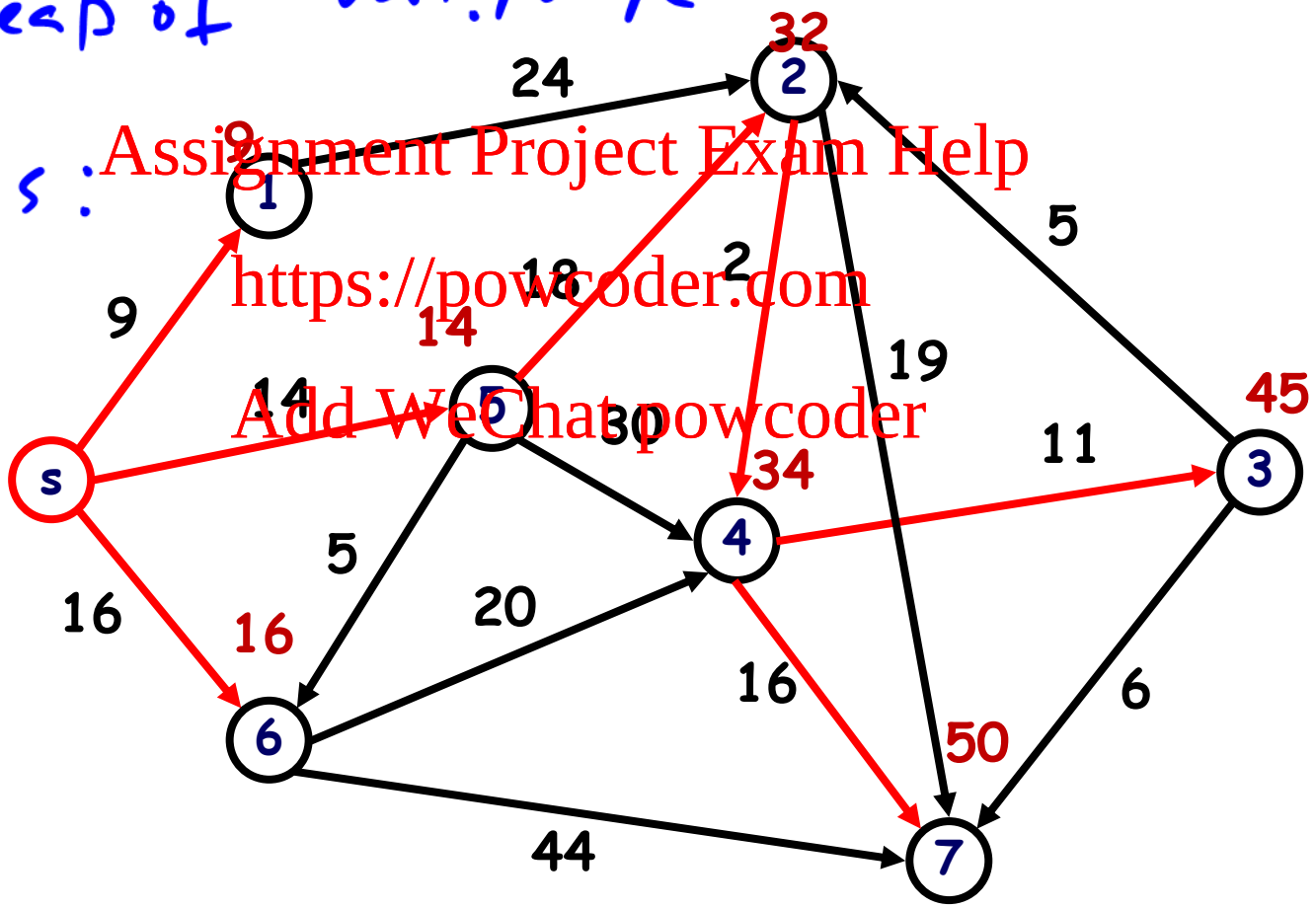
heap = {}



spanning tree ≠ MST

Prim's : heap of vert./edge

Dijkstra's : heap of vert./dist.



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Complexity

Let D(v) denote a length from the source s to vertex v.
We store distances D(v) in a binary heap.

INIT: $D(s)=0$, $D(v)=\infty$ for $\forall v \neq s$

LOOP: deleteMin, $O(\log V)$ for each vertex

update distances in the heap, $O(\log V)$ for each edge

for all u in adj(v)

$\min(D(u), D(v) + \text{weight}(u,v))$

Runtime: $O(V \log V + E \log V)$, same as Prim's

Fibonacci: $O(V \log V + E \cdot 1)$ amortized

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Discussion Problem

Assume that an unsorted array is used instead of a binary heap. What would the running time of the Dijkstra algorithm?

$O(V \cdot V + E \cdot 1)$

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dense graph, $E = O(V^2)$

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$O(V^2)$

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Fibonacci dense graph $O(V \log V + E)$, use a heap

Proof of Correctness

Lemma. For each node $u \in S$ (solution tree), $d(u)$ is the shortest s-u path.

Induction on $|S|$

Base case: $|S| = 1$, $d(s) = 0$

IH: $|S| = k$ vertices

IS: prove it for $(k+1)$ vertices

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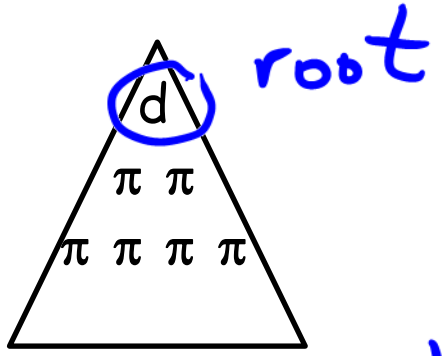
$\pi(x) = \min\left(d(v),\ d(u) + w(u,x)\right)$



$\pi(v) \geq d(v)$

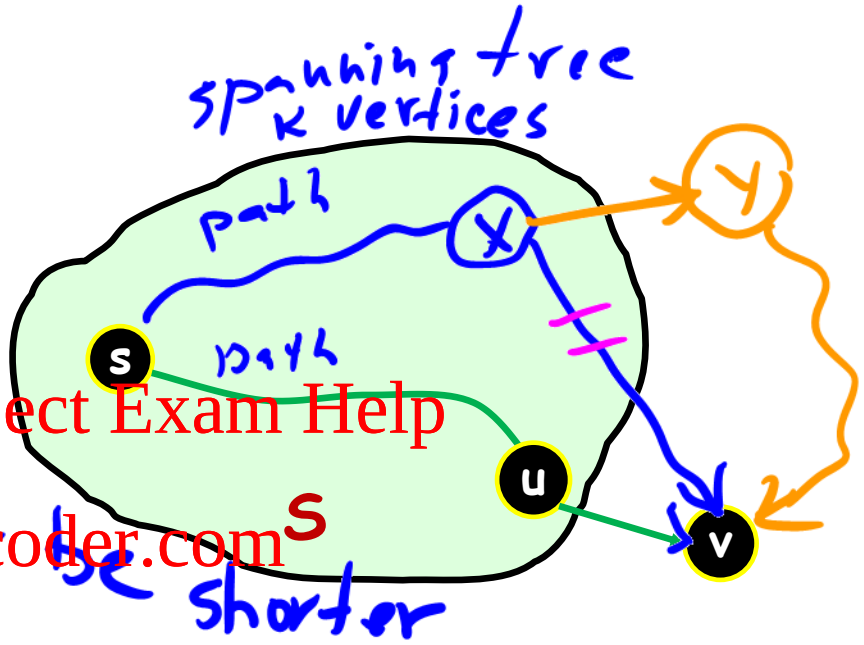
weights $\in \mathbb{R}^+$

Proof of Correctness

$d(u)$ is the shortest

$\pi(v) = d(u) + \text{edge}(u,v)$

Assume $\pi(v) \neq d(v)$



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Case 1.
distance through x cannot be shorter

Case 2. path $s - x - y - v$
is it the shortest?
our ALG would choose edge (x, y)
but (u,v);

Discussion Problem 1

You are given a graph representing the several career paths available in industry. Each node represents a position and there is an edge from node v to node u if and only if v is a pre-requisite for u. Top positions are the ones which are not pre-requisites for any positions. Start positions are the ones which have no pre-requisites. The cost of an edge (v,u) is the effort required to go from one position v to position u. Salma wants to start a career and achieve a top position with minimum effort. Using the given graph can you provide an algorithm with the same run time complexity as Dijkstra's algorithm?

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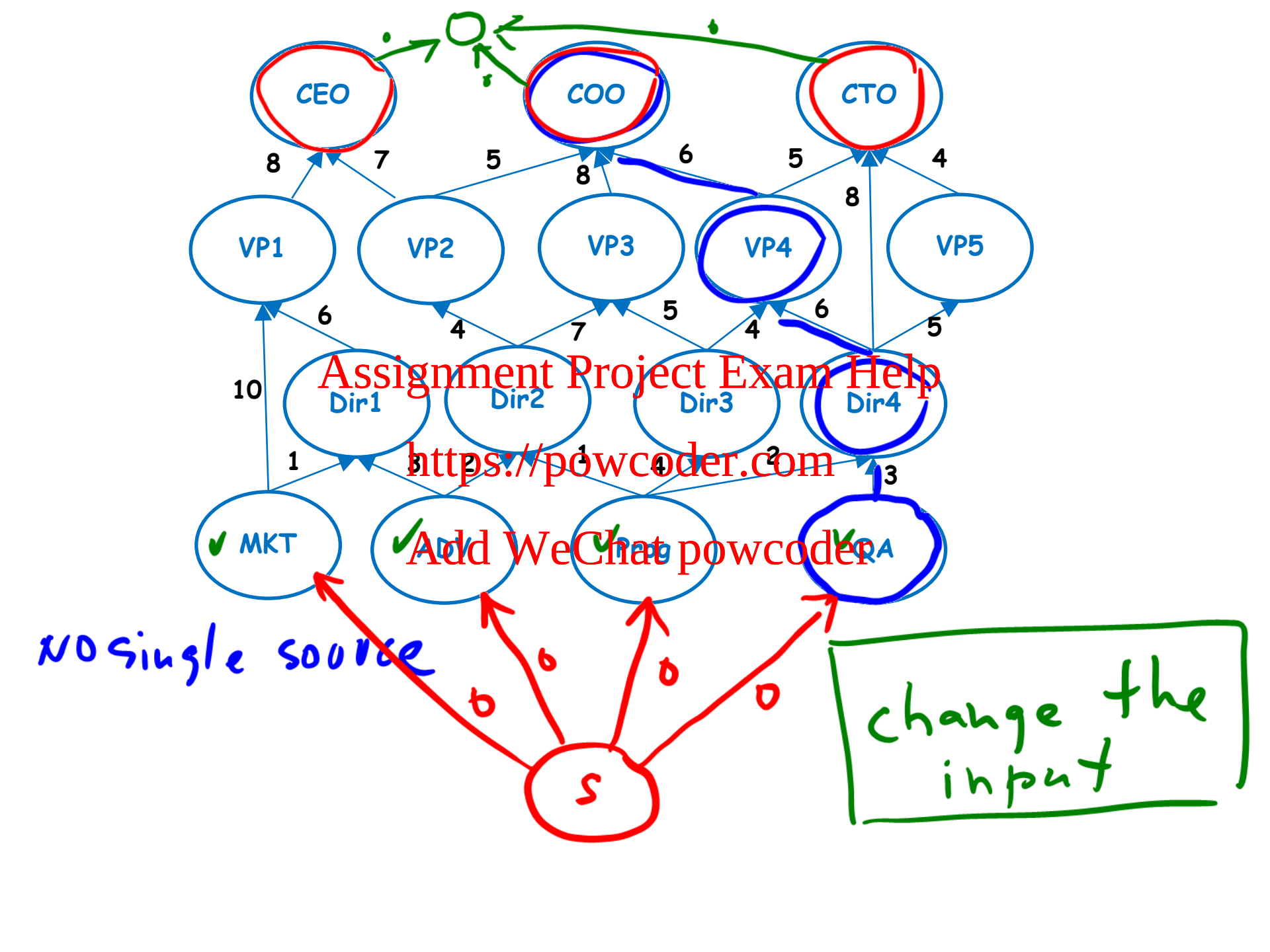
SSSP



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Discussion Problem 2

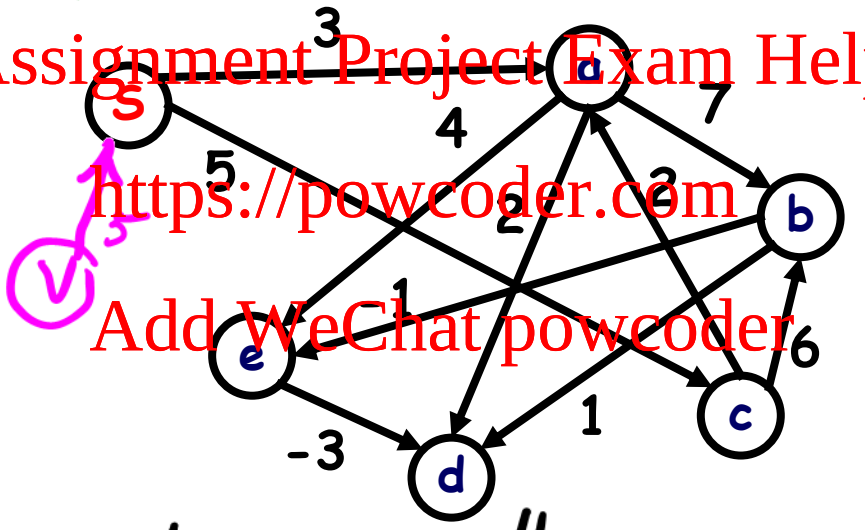
No Dijkstra

Design a linear time algorithm to find shortest distances in a DAG.

a) traversal, BFS

b) topological sort

How topological sort help as to solve the problem?



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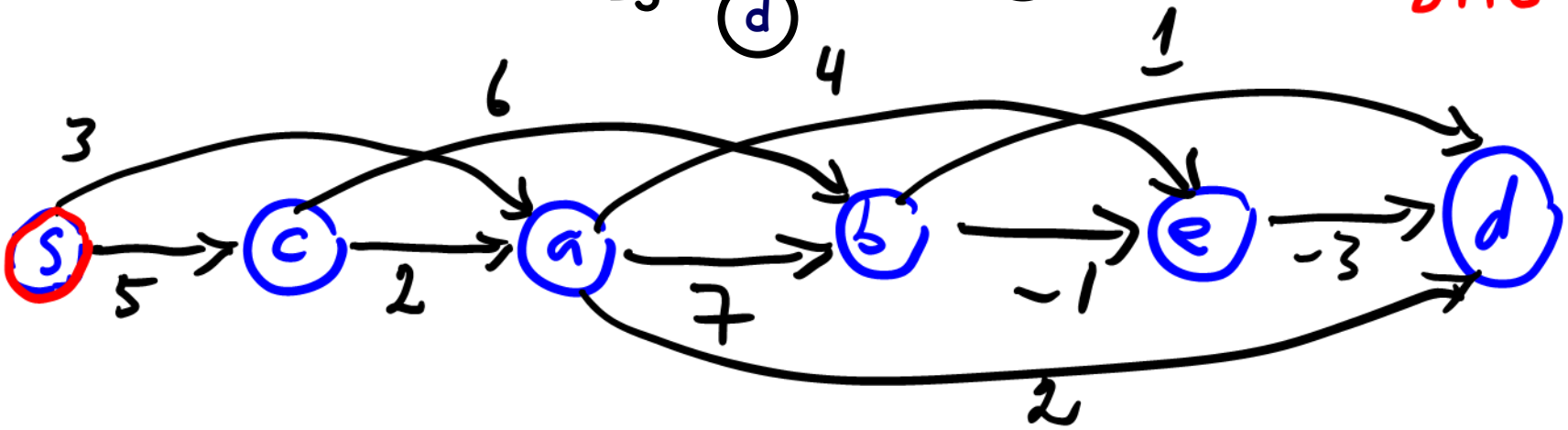
DAG



s	a	b	c	d	e	
0	3	-	5	-	-	s
	3	5+6	5	-	-	s, c
	3	10	5	5	7	s, c, a
	3	10	5	5	7	s, c, a, b
0	3	10	5	5	7	s, c, a, b, e

Runtime
O(E)

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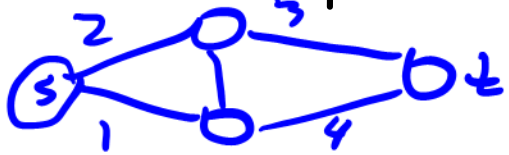
Review Questions

(T/F) If all edges in a connected undirected graph have distinct positive weights, the shortest path between any two vertices is unique. example



(T/F) Suppose we have calculated the shortest paths from a source to all other vertices. If we modify the original graph, G, such that weights of all edges are increased by 2 then the shortest path tree of G is also the shortest path tree of the modified graph.

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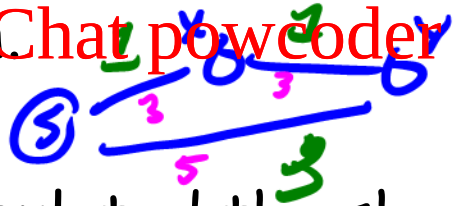
+2

s-x-y , s-y



(T/F) Suppose we have calculated the shortest paths from a source to all other vertices. If we modify the original graph G such that weights of all edges are doubled, then the shortest path tree of G is also the shortest path tree of the modified graph.

$x+y+z < a+b$

$2(x+y+z) < 2(a+b)$

Discussion Problem 3

Why Dijkstra's greedy algorithm does not work on graphs with negative weights?

Run Dijkstra

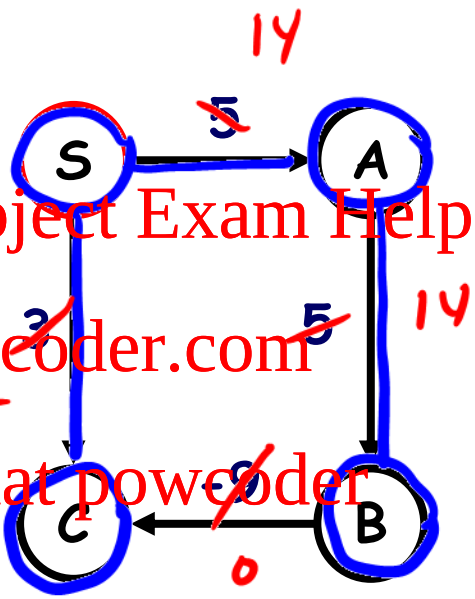
$d(S)=0$, $d(C)=3$

$d(A)=5$, $d(B)=10$

STOP

OPT: $d(C)=1$

S-A-B-C



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Run Dijkstra

$d(C)=12$

How to fix Dijkstra?

a) Re-weight

add 9

it does not work!

b) DP

Discussion Problem 4

In this problem you are to find the most efficient way to deliver power to a network of n cities. It costs p_i to open up a power plant at city i. It costs $c_{ij} \geq 0$ to put a cable between cities i and j. A city is said to have power if either it has a power plant, or it is connected by a series of cables to some other city with a power plant. Devise an efficient algorithm for finding the *minimum cost* to power all the cities.

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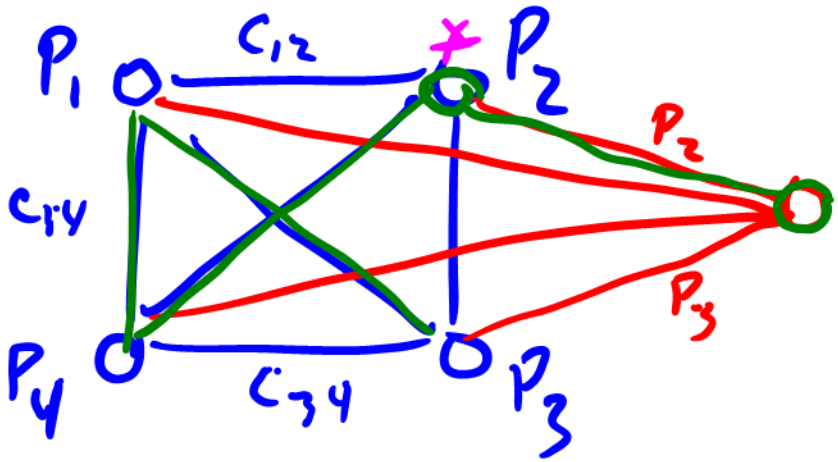
MST



Run MST algo.

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Why our solution will have at least one power plant?

The solution must have at least one red edge...

Two ways to solve (in general)

① change the input

② change the algorithm

Discussion Problem 5

Hardy decides to start running to work in San Francisco to get in shape. He prefers a route to work that goes first entirely uphill and then entirely downhill. To guide his run, he prints out a detailed map of the roads between home and work. Each road segment has a positive length, and each intersection has a distinct elevation. Assuming that every road segment is either fully uphill or fully downhill, give an efficient algorithm to find the shortest path that meets Hardy's specifications.

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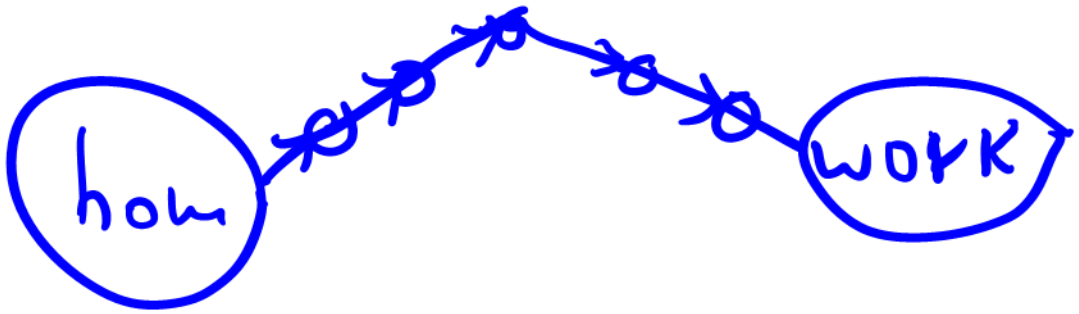
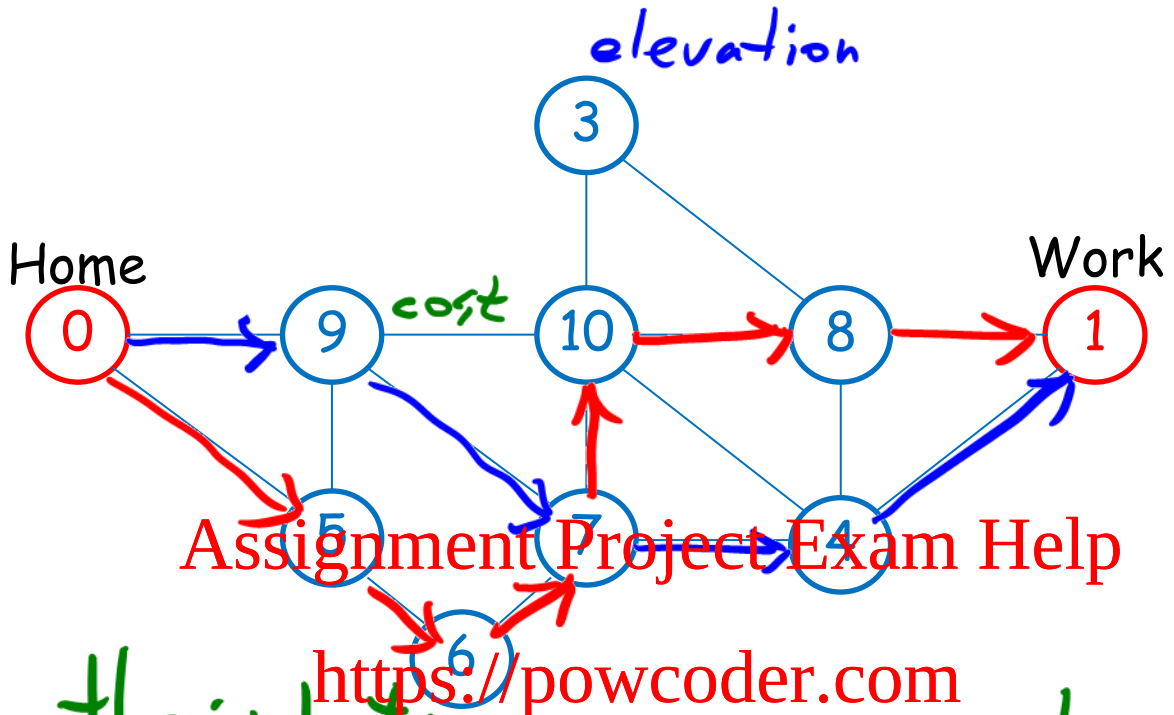
Dijkstra





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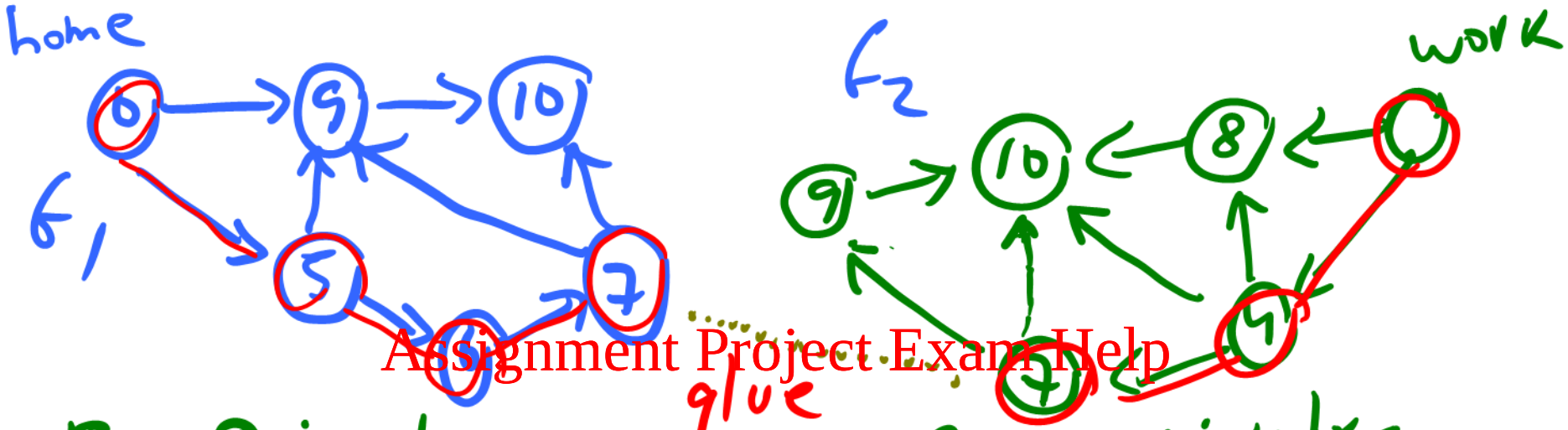
Change the input:

1. create uphill graph from home

2. create uphill graph from work

Uphill graph



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Run Dijkstra from home

glue two paths

Run Dijkstra from work

Complexity: 1. construction G_1, G_2
2. run Dijkstra (+ twice)
3. find common vertices

Discussion Problem 6

$V \to \infty$, $E \to \infty$

Given a graph G=(V,E) whose edge weights are integers in the range [0, W], where W is a relatively small integer number compare to the number of vertices, $W = O(1)$. We could run Dijkstra's algorithm to find the shortest distances from the start vertex to all other vertices. Design a new linear time algorithm that will outperform Dijkstra's algorithm.

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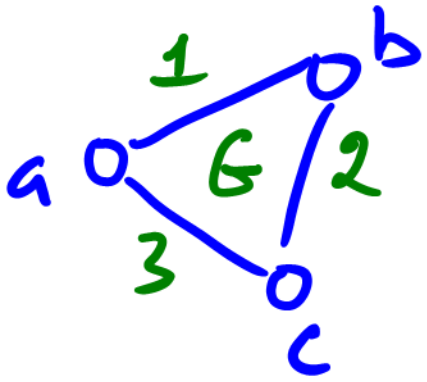
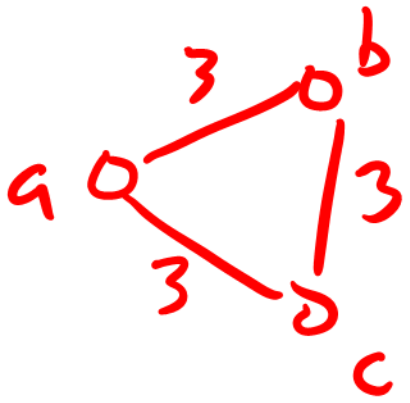
traversal
BFS

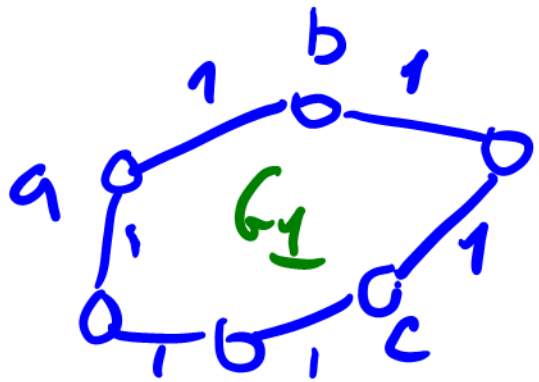
Change the input !!



W = 3, worst-case



Solution



Run BFS!

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Runtime, $G=(V,E)$ ← Input

$G_1=(V_1,E_1)$

$BFS(G_1)=O(V_1+E_1)$

DIY verify

$V_1 = E+W\cdot V$, $E_1 = W\cdot E$ verify!

$BFS(G_1)=O(W\cdot V+W\cdot E)=O(V+E)$ if $W=O(1)$