# Elementary Affine Transformations

***Any affine transformation is equivalent to a combination of four elementary affine transformations***

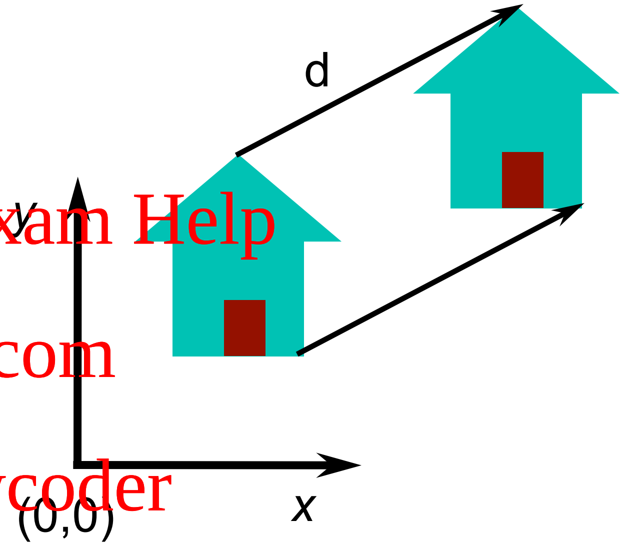


- Translation
- Scaling
- Rotation
- Shear

# Translation

$$Q = P + \mathbf{d}, \mathbf{d} = (T_x \; T_y)^T$$






$$Q_x = P_x + T_x$$

$$Q_y = P_y + T_y$$

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

# Scaling around the origin

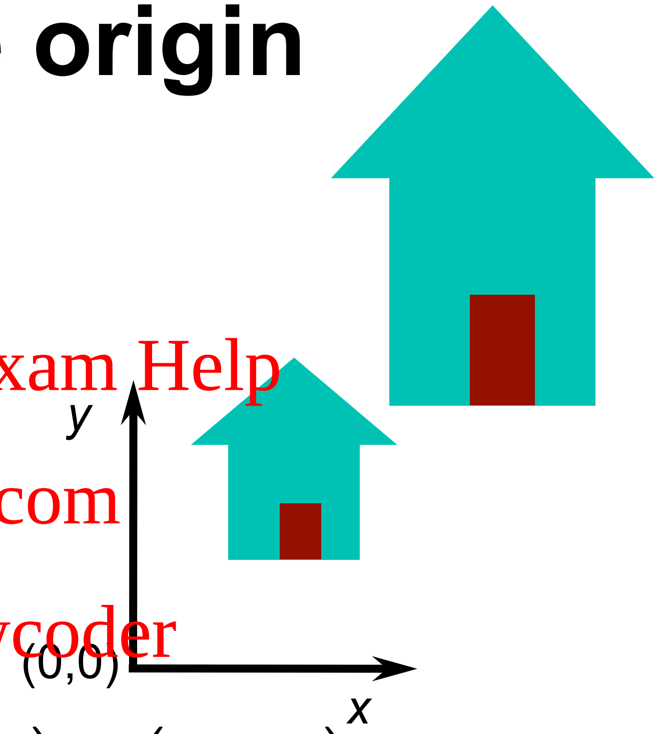


$$Q_x = s_x P_x$$



$$Q_y = s_y P_y$$

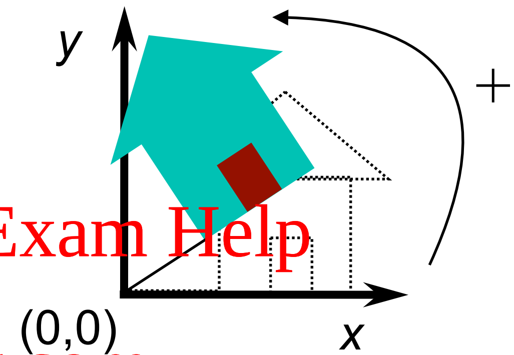





$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

$$Uniform : s_x = s_y$$

# Rotation around the origin






$$Q_x = cos\theta P_x - sin\theta P_y$$

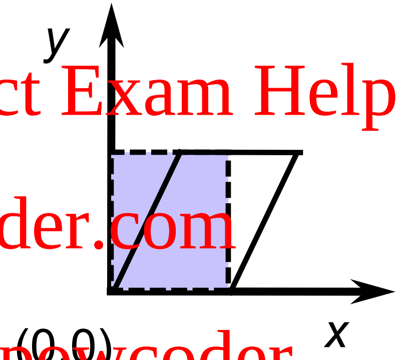
$$Q_y = sin\theta P_x + cos\theta P_y$$

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

# Shear around the origin

## *In the x-direction*

$$Q_x = P_x + aP_y$$
$$Q_y = P_y$$






$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

# Zero point

All elementary affine transformations have one or more 'zero' or 'identity' points **except translation**



$$F(P) = P$$



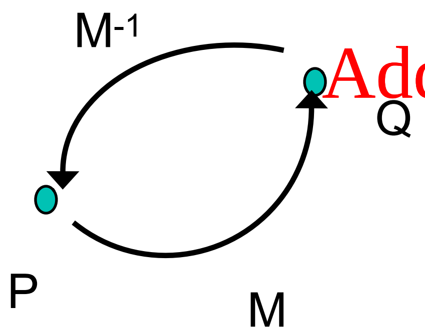
What are these points for Shear(), Scale() and Rotate() ?

# Inverse of a Transformation

## *Cramer's rule or we can be smarter*

- Inverse transformation: $Q = MP$, $P = M^{-1}Q$





# Inverse of Translation



$$Q = T(\mathbf{d})\mathbf{P} \rightarrow \mathbf{P} = \mathbf{T}(-\mathbf{d})\mathbf{Q}$$





$$\begin{pmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -T_x \\ 0 & 1 & -T_y \\ 0 & 0 & 1 \end{pmatrix}$$

# Inverse of Scaling



$$Q = S(\mathbf{s})P \rightarrow P = S(1/s_x, 1/s_y)Q$$





$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Inverse of Rotation

$$Q = R(\theta)P \rightarrow P = R(-\theta)Q$$



$$\begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Inverse of a Shear in x

$$Q = Sh_x(a)P \rightarrow P = Sh_x(-a)Q$$



$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Composing 2D Affine Transformations

***Composing two affine transformations produces an affine transformation***



$$Q = T_2(T_1(P))$$



In matrix form:



$$Q = M_2(M_1 P) = (M_2 M_1)P = MP$$

Which transformation happens first?

# Main Points

***Any affine transformation can be performed as series of elementary transformations.***



***Affine transformations are the main modeling tool in graphics.***



***Make sure you understand the order.***

# Examples

*Reflection*

*Rotation about an arbitrary pivot point*

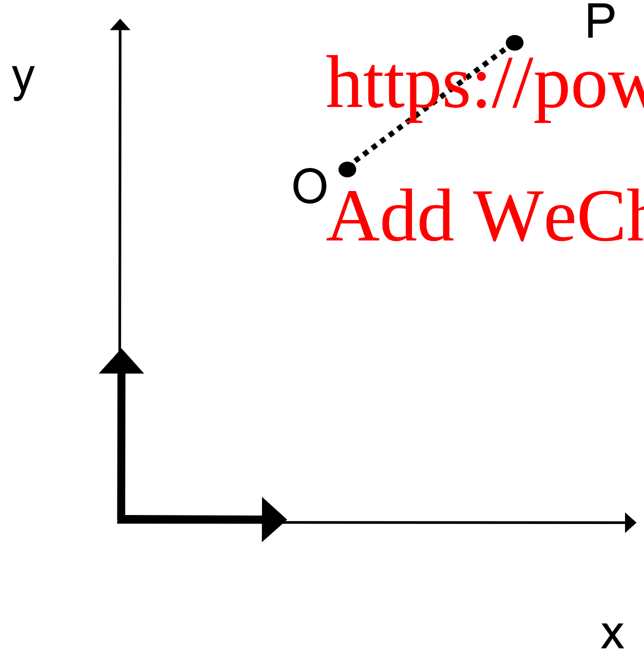
*Scaling around an arbitrary point*

*Reflection about a tilted line*

# Example of 2D transformation

***Rotate around an arbitrary point O by -90 degrees:***

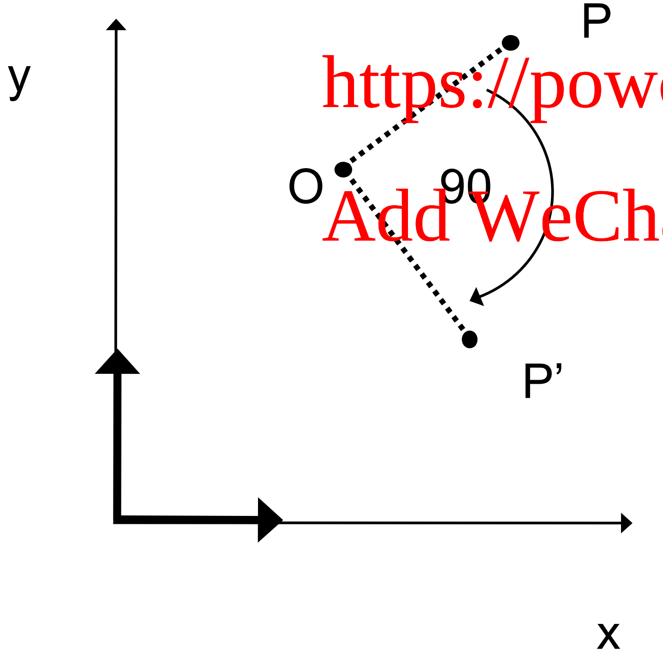




# Rotate around an arbitrary point

***Rotate around an arbitrary point O by -90 degrees:***

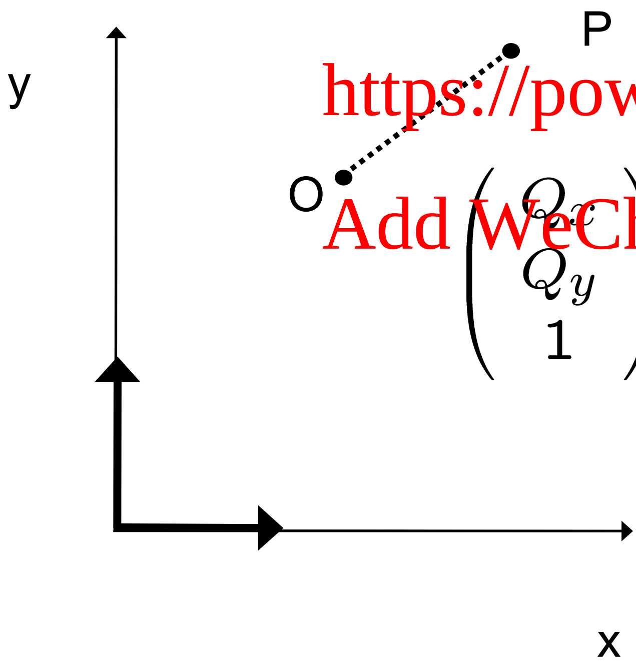




# Rotate around an arbitrary point

***We know how to rotate around the origin***

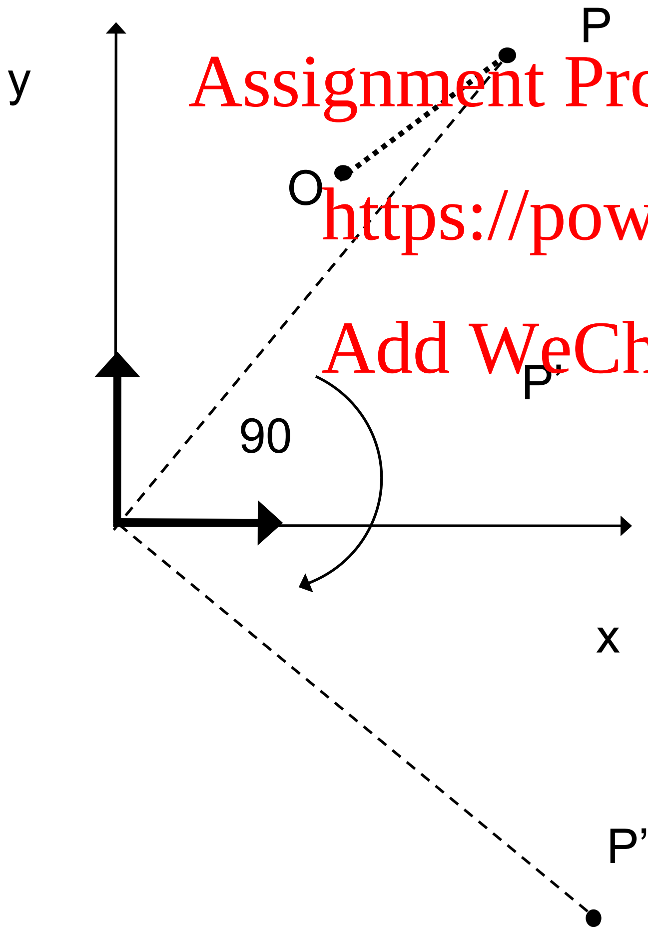





$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

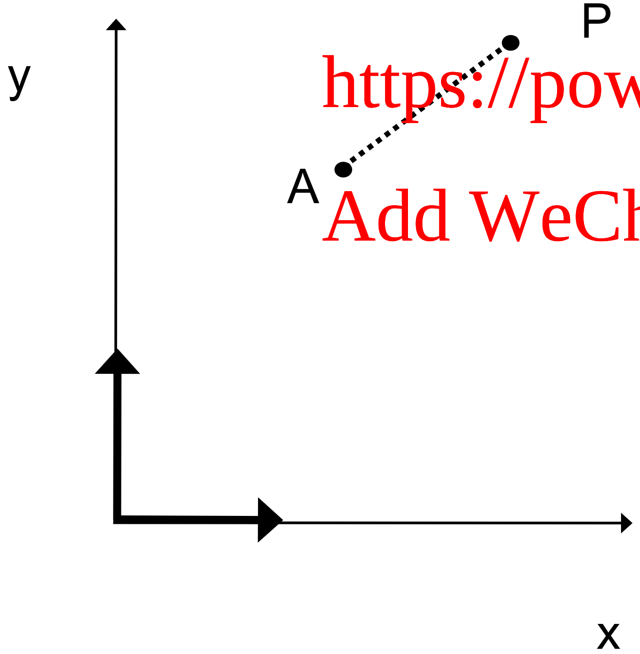
# Rotate around an arbitrary point

***…but that is not what we want to do!***



# So what do we do?

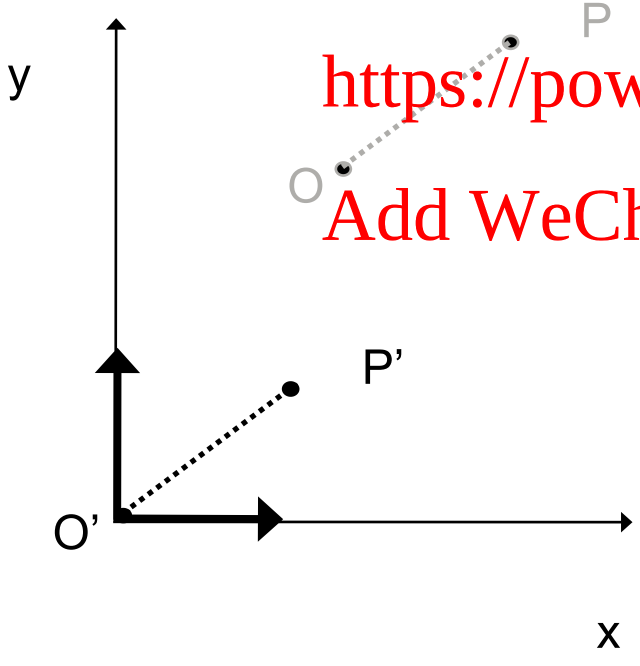




# Transform it to a known case

*Translate(-Ox,-Oy)*



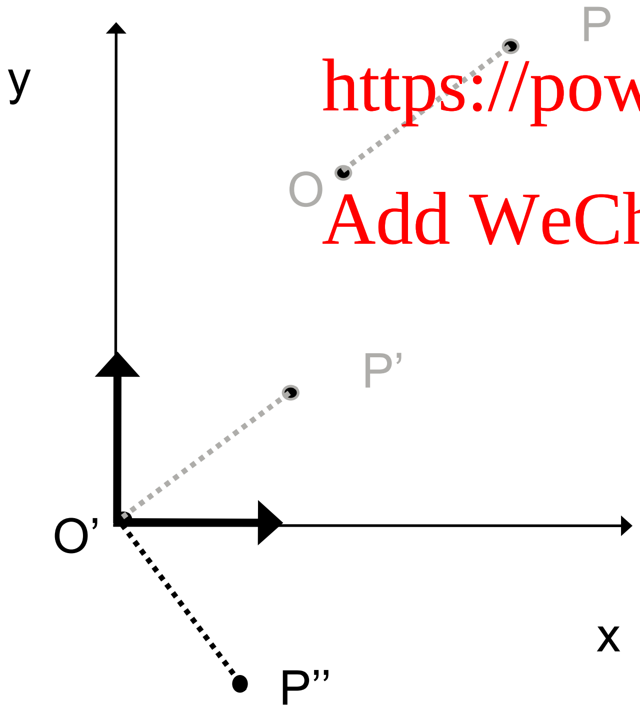


# Second step: Rotation

***Translate(-Ox,-Oy)***

***Rotate(-90)***





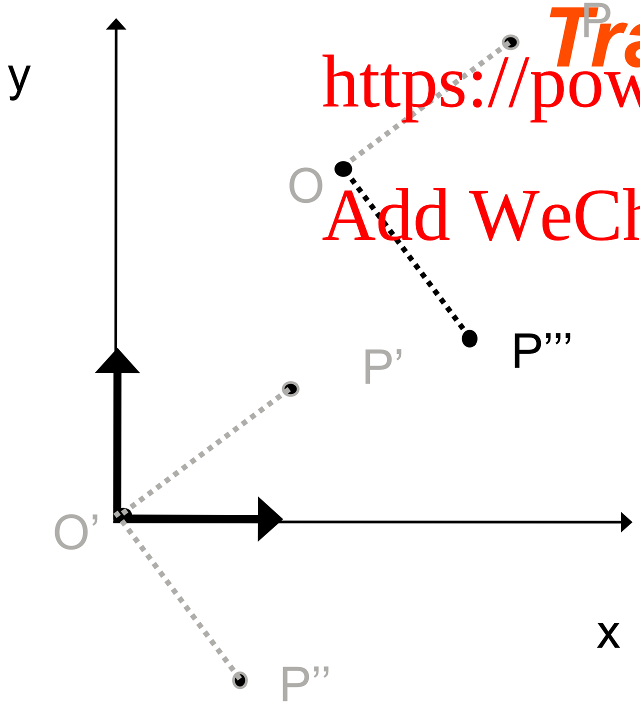
# Final: Put everything back

***Translate(-Ox,-Oy)***

***Rotate(-90)***

***Translate(Ox,Oy)***



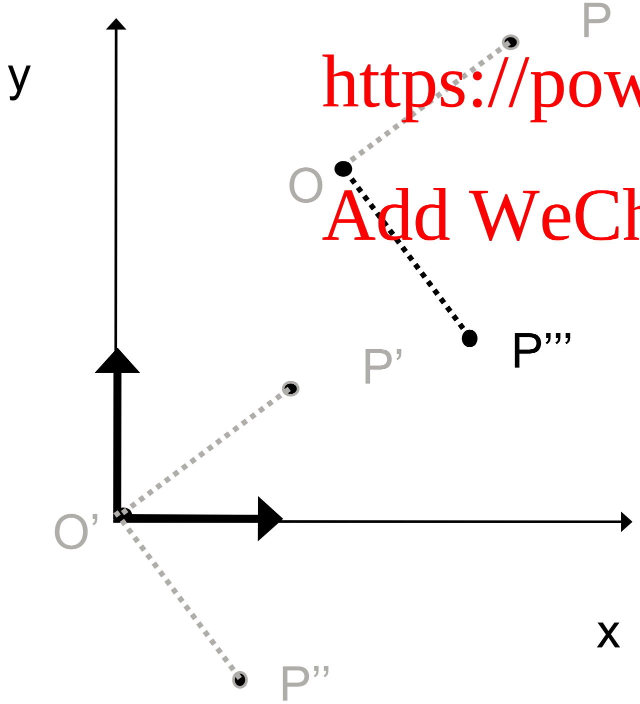


# Rotation about arbitrary point

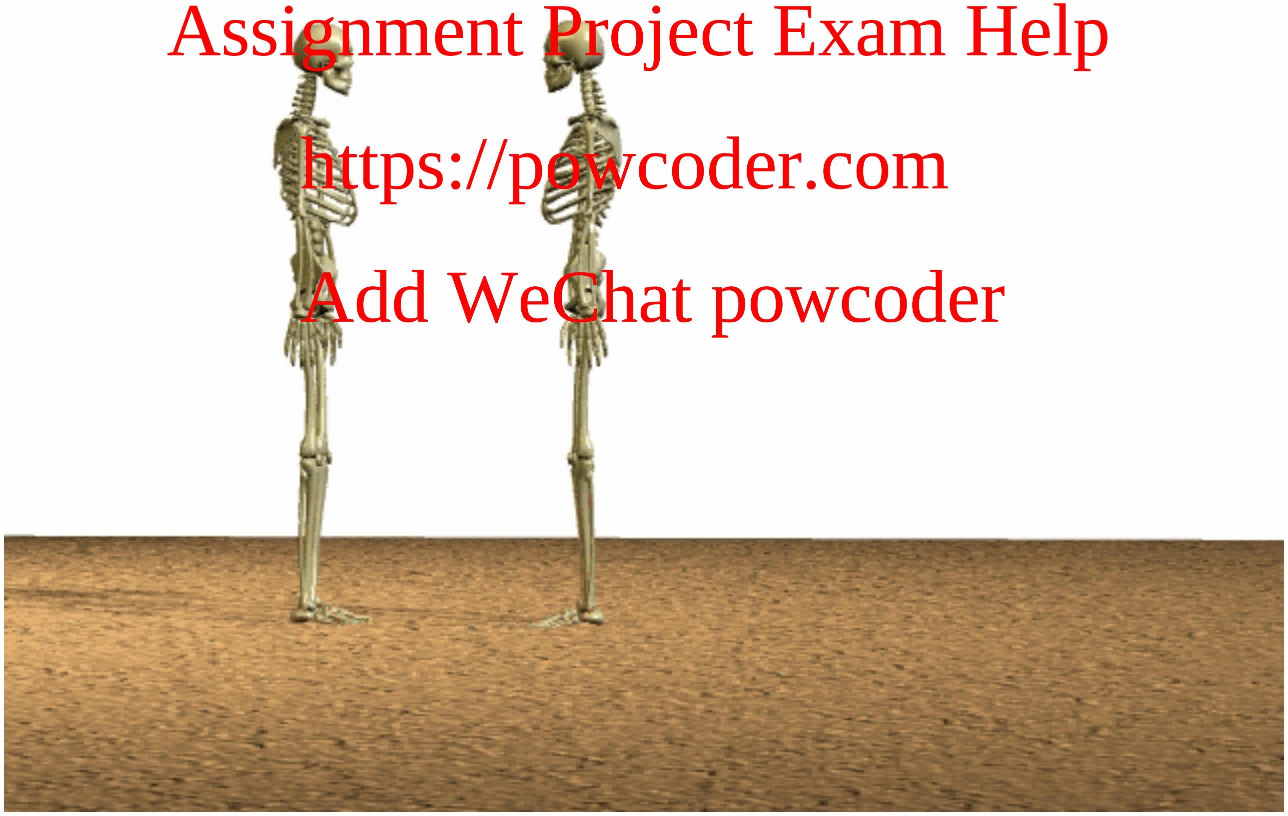
***IMPORTANT!: Order***

$$M = T(Ox,Oy)R(-90)T(-Ox,-Oy)$$





# Affine Transformations in 3D



# Affine Transformations in 3D

## *General form*







$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

# Elementary 3D Affine Transformations

### *Translation*



$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

# Scaling Around the Origin



$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

- Uniform scaling: $s_x = s_y = s_z$

# Shear around the origin

*Along x-axis*



$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

# 3 DRotation

***Various representations***



***Decomposition into axis rotations (x-roll, y-roll, z-roll)***





***CCW positive assumption***

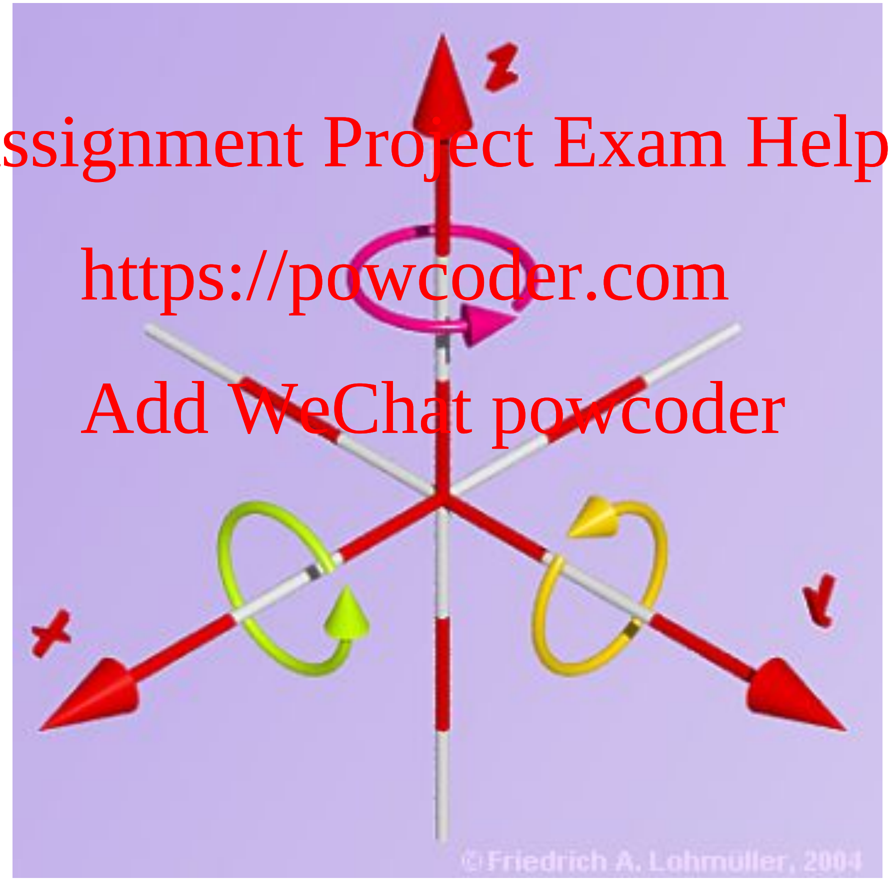
# Reminder 2D z-rotation

$$Q_x = cos\theta P_x - sin\theta P_y$$
$$Q_y = sin\theta P_x + cos\theta P_y$$



$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

# Three axis to rotate around



# Z-roll

$$Q_x = cos\theta P_x - sin\theta P_y$$

$$Q_y = sin\theta P_x + cos\theta P_y$$



$$Q_z = P_z$$





$$R_z(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# X-roll

***Cyclic indexing***

$$x \rightarrow \boxed{y \rightarrow z \rightarrow x} \rightarrow y$$



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ \boxed{\begin{matrix} y \\ z \\ x \end{matrix}} \\ y \end{bmatrix}$$

$$Q_y = cos\theta P_y - sin\theta P_z$$

$$Q_z = sin\theta P_y + cos\theta P_z$$

$$Q_x = P_x$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Y-roll

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ x \\ y \end{bmatrix}$$



$$Q_z = cos\theta P_z - sin\theta P_x$$

$$Q_x = sin\theta P_z + cos\theta P_x$$

$$Q_y = P_y$$

$$R_y(\theta) = \begin{pmatrix} cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Inversion of transformations

*Translation:* $T^{-1}(a,b,c) = T(-a,-b,-c)$

*Rotation:* $R^{-1}_{axis}(b) = R_{axis}(-b)$



*Scaling:* $S^{-1}(sx,sy,sz) = S(1/sx,1/sy,1/sz)$



*Shearing:* $Sh^{-1}(a) = Sh(-a)$

# Inverse of Rotations

***Pure rotation only, no scaling or shear.***







$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

$$M^{-1} = M^T$$

# Composition of 3D Affine Transformations

***The composition of affine transformations is an affine transformation.***





***Any 3D affine transformation can be performed as a series of elementary affine transformations.***

# Composite 3D Rotation around origin



$$R = R_z(\theta_3)R_y(\theta_2)R_x(\theta_1)$$



The order is important !!

It is often convenient to use other representations for 3D rotations….

# Gimbal lock

$$R(\theta_1, \theta_2, \theta_3) = R_z(\theta_3) R_y(\theta_2) R_x(\theta_1)$$

$$\begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta_2) & 0 & sin(\theta_2) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta_2) & 0 & cos(\theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta_1) & -sin(\theta_1) & 0 \\ 0 & sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$R(\theta_1, 90^o, \theta_3) = R_z(\theta_3) R_y(90^o) R_x(\theta_1)$$



$$\begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta_1) & -sin(\theta_1) & 0 \\ 0 & sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & cos\theta_1) & -sin(\theta_1) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Loss of degree of freedom

$$R(\theta_1, 90^o, \theta_3) = \begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & cos\theta_1) & -sin(\theta_1) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$



$$\begin{pmatrix} 0 & cos(\theta_3)sin(\theta_1) - sin(\theta_3)cos(\theta_1) & cos(\theta_3)cos(\theta_1) + sin(\theta_3)sin(\theta_1) & 0 \\ 0 & cos(\theta_3)cos(\theta_1) + sin(\theta_3)sin(\theta_1) & -cos(\theta_3)sin(\theta_1) + sin(\theta_3)cos(\theta_1) & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & sin(\theta_1 - \theta_3) & cos(\theta_1 - \theta_3) & 0 \\ 0 & cos\theta_1 - \theta_3) & -sin(\theta_1 - \theta_3) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & cos\theta) & -sin(\theta) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R(\theta) \qquad (\theta_1, \theta_3) \rightarrow \theta = (\theta_1 - \theta_3)$$

# Rotation around an arbitrary axis

***Euler's theorem: Any rotation or sequence of rotations around a point is equivalent to a single rotation around an axis that passes through the point.***



What does the matrix look like?