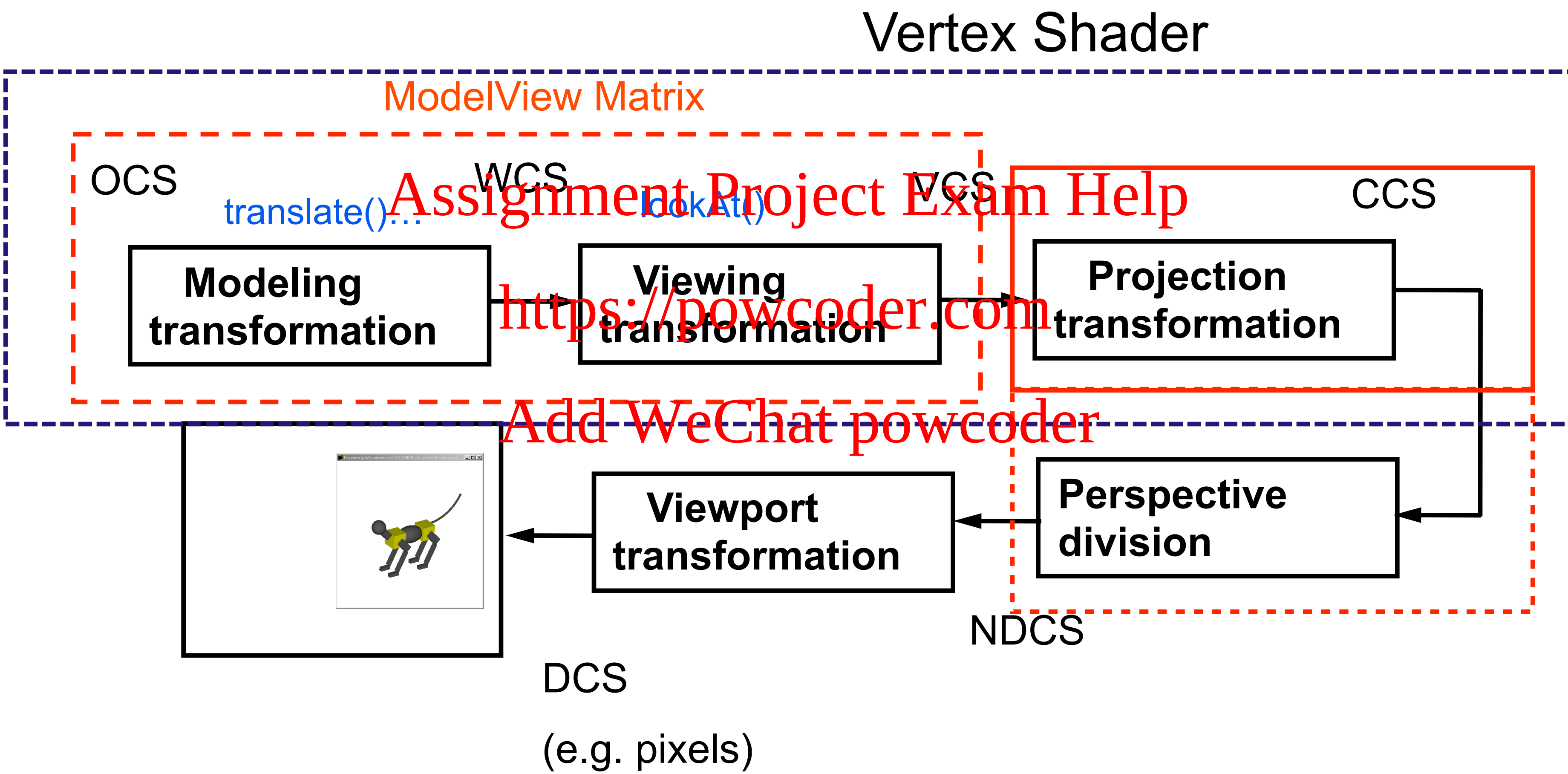
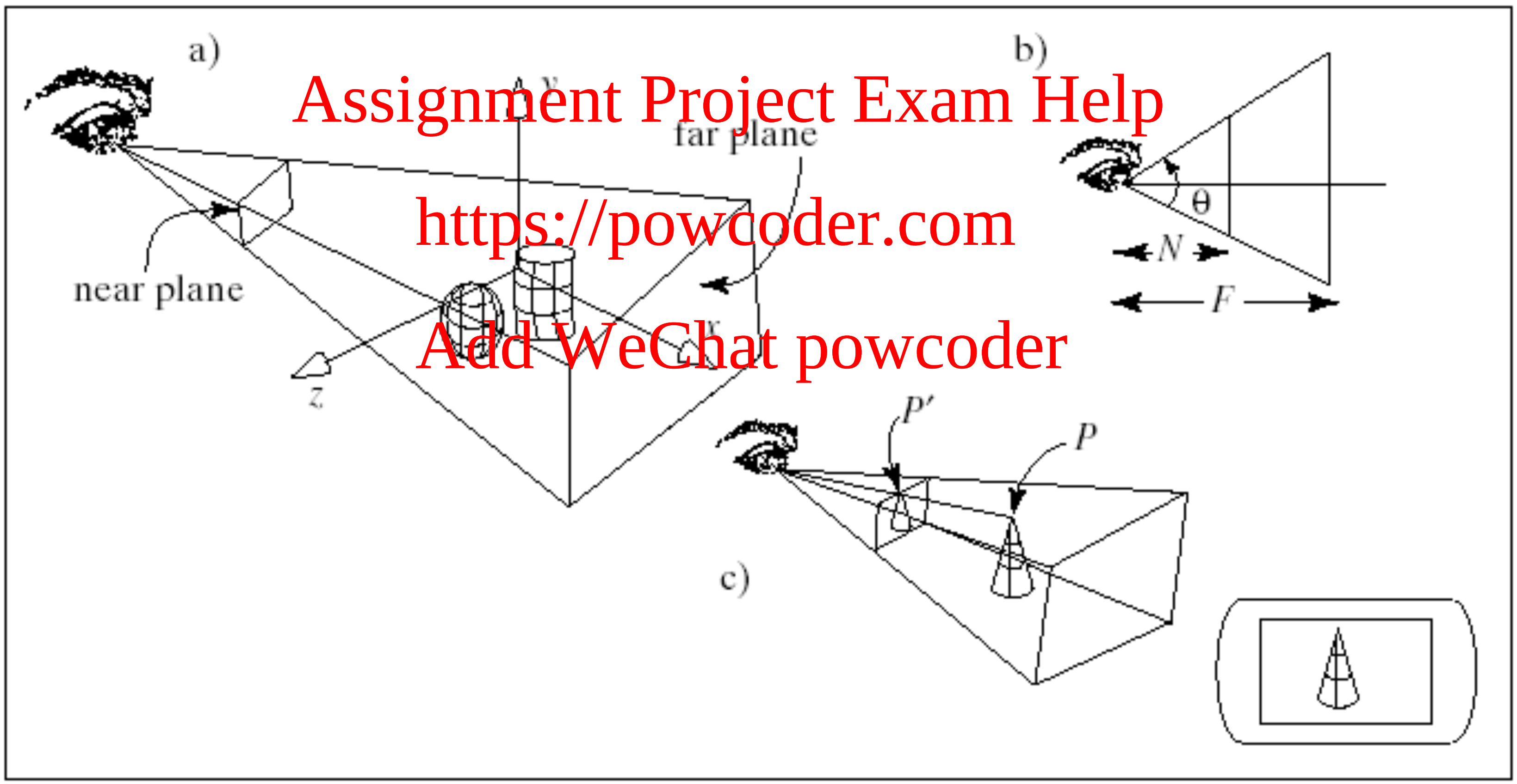
Transformations in the pipeline



Projection transformations



Introduction to Projection Transformations

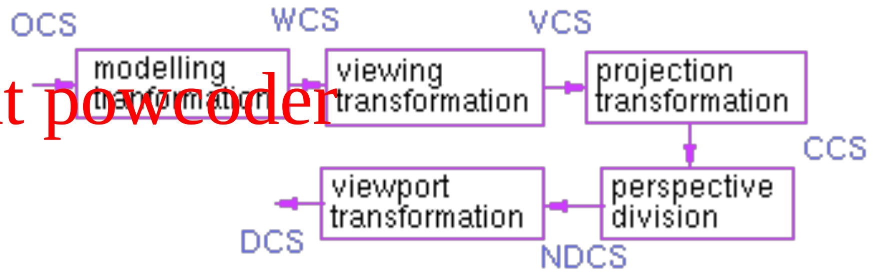
Mapping: $f : R^n \rightarrow R^m$

Projection: $n > m$

Planar Projection: Projection on a plane.

$R^3 \rightarrow R^2$ or

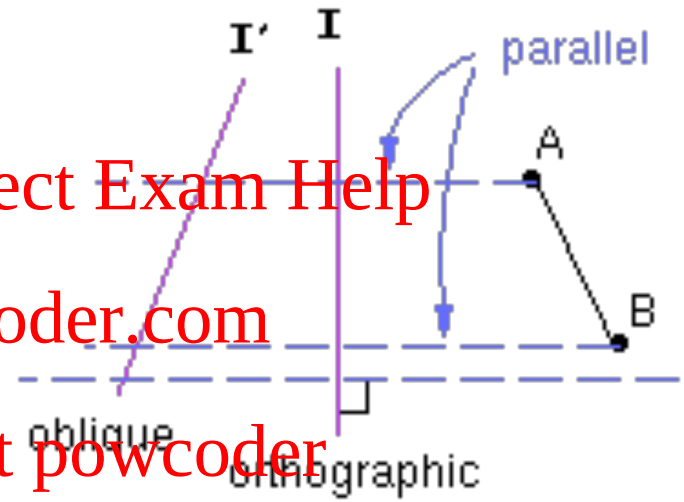
$R^4 \rightarrow R^3$ homogenous coordinates.



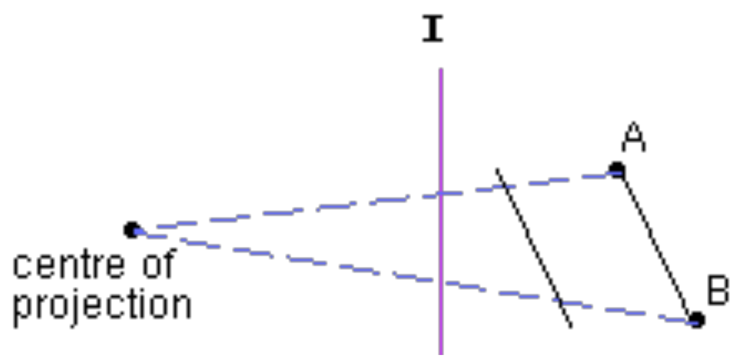
Transformation: $n = m$

Basic projections

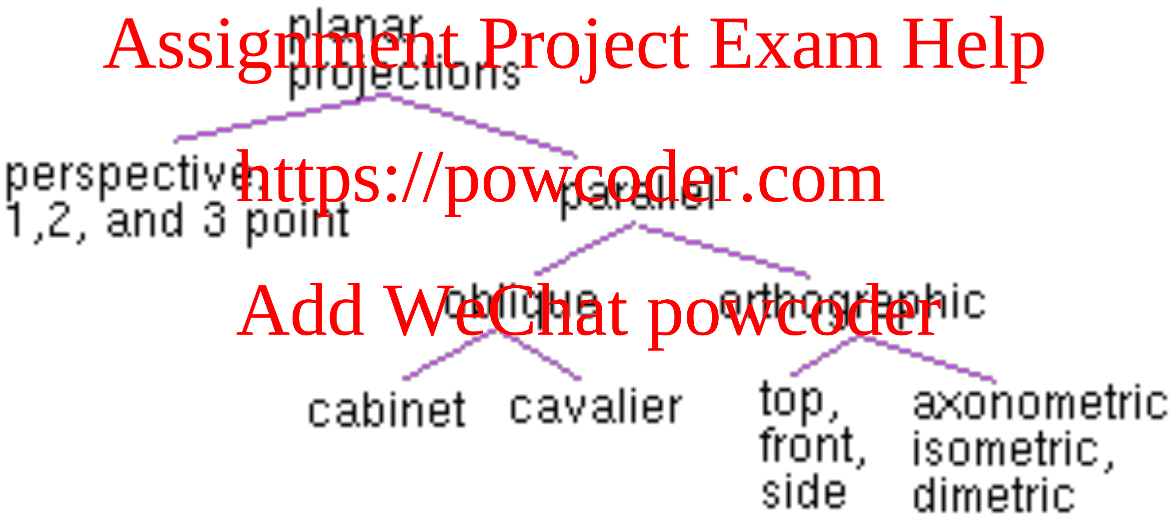
Parallel



Perspective

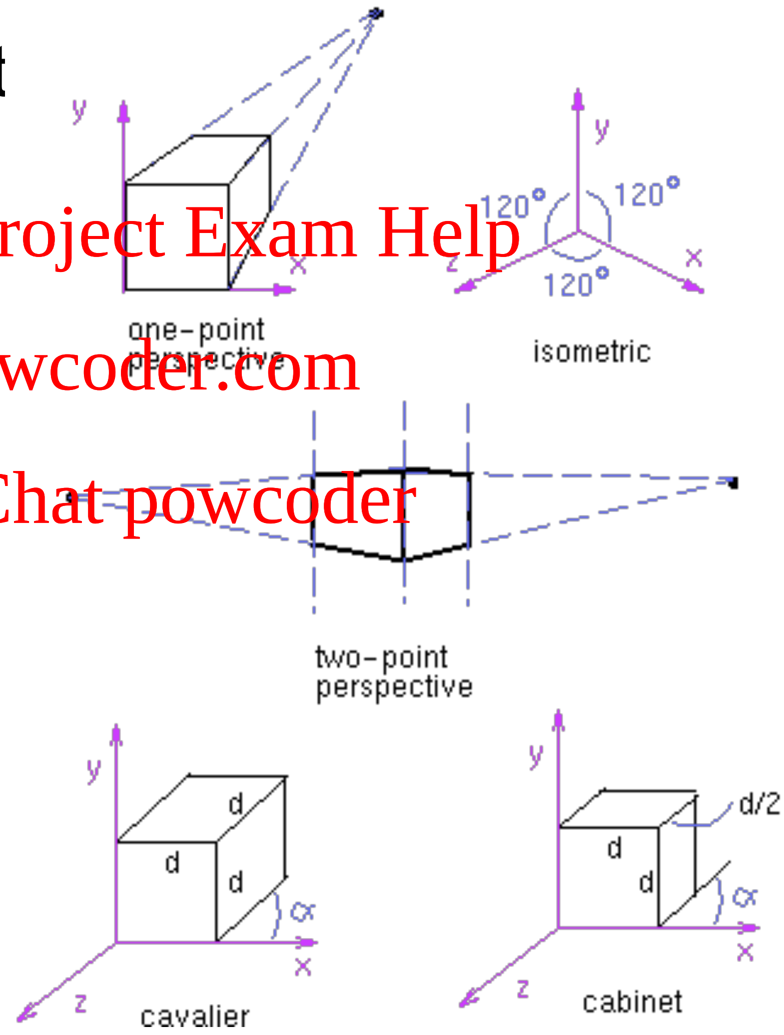


Taxonomy


planar
projections
perspective:
1,2, and 3 point
parallel
oblique
orthographic
cabinet
cavalier
top,
front,
side
axonometric:
isometric,
dimetric

Examples

- All defined with respect to a unit cube

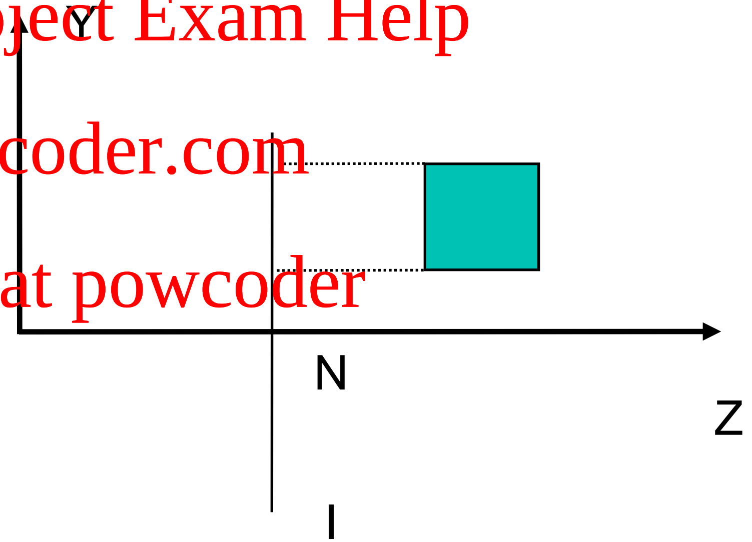


A basic orthographic projection

$$x' = x$$

$$y' = y$$

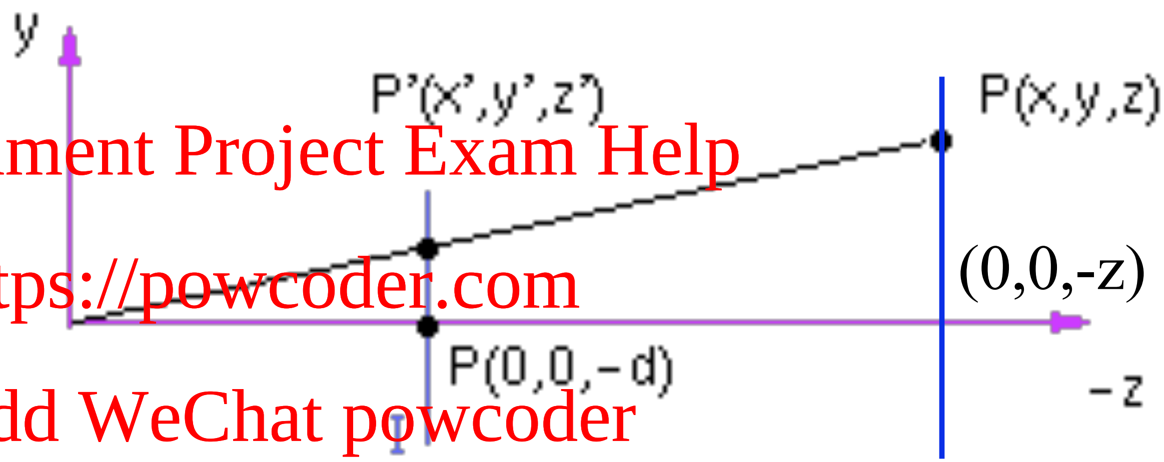
$$z' = N$$



Matrix Form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & N \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ N \\ 1 \end{bmatrix}$$

A basic perspective projection



Note that d > 0

Similar triangles

$x'/d = x/(-z)$ --> $x' = x\,d/(-z)$

$y'/d = y/(-z)$ --> $y' = y\,d/(-z)$

$z' = -d$

Matrix form?

Reminder: Homogeneous Coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\times w} \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

$$\begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix} \xrightarrow{/w} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Canonical matrix form

Matrix form of

$x' = x\, d/(-z)$

$y' = y\, d/(-z)$

$z' = -d$

$d > 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -z/d \end{bmatrix}$$

or

$$\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} xd \\ yd \\ zd \\ -z \end{bmatrix}$$

Moving from 4D to 3D

$$\begin{bmatrix} x \\ y \\ z \\ -z/d \end{bmatrix} \xrightarrow{h=-z/d} \begin{bmatrix} x/h \\ y/h \\ z/h \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} xd/(-z) \\ yd/(-z) \\ -d \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Things to notice

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{-d} & 0 \end{bmatrix} \quad \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Projections in OpenGL

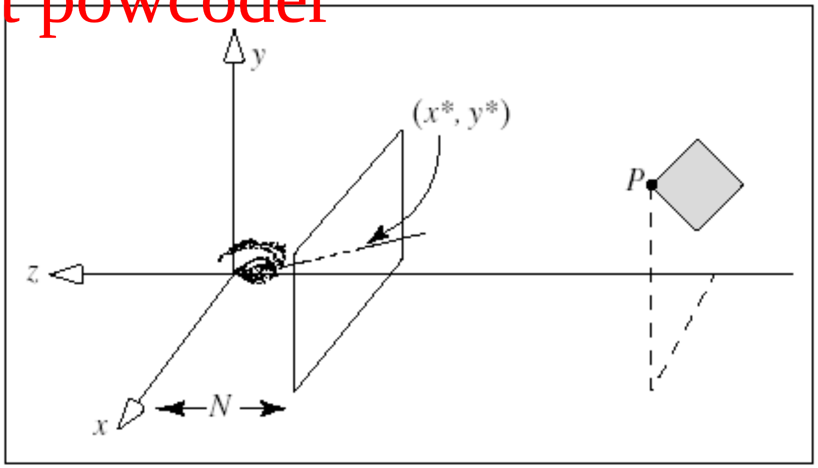
Projections in OpenGL are defined in the camera coordinate system

- Although not advisable, with shaders you can actually change that if you wish

- That means they are also applied in the camera coordinate system, i.e. they are applied to a point or vector given in **camera coordinates**

Camera coordinate system

- Camera at (0,0,0)
- Looking at $-z$
- Image plane is the near plane
  $z = -d,\ d > 0$



Perspective projection of a point

Point or vector in eye coordinates

$P_{eye} = (x,y,z)$

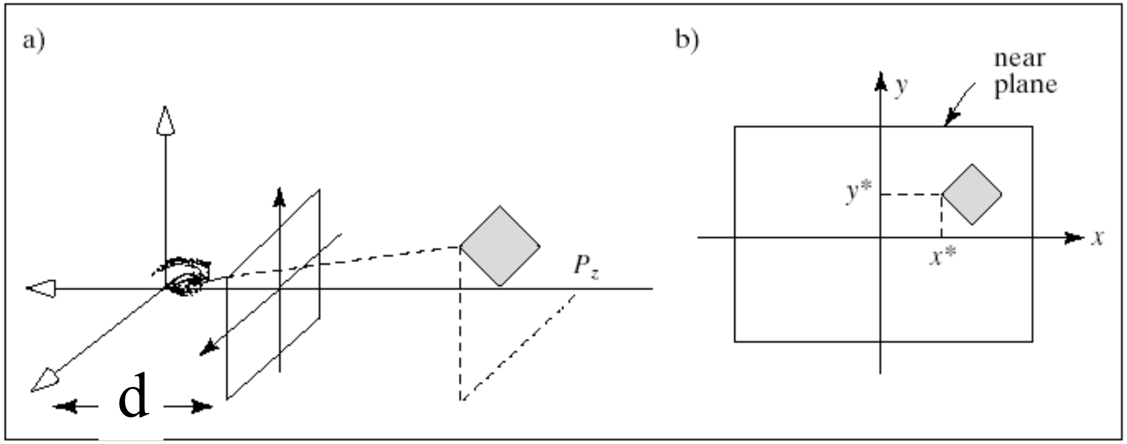
Projected coordinates:

$x' = x\, d/(-z)$

$y' = y\, d/(-z)$

$z' = -d$

$d > 0$



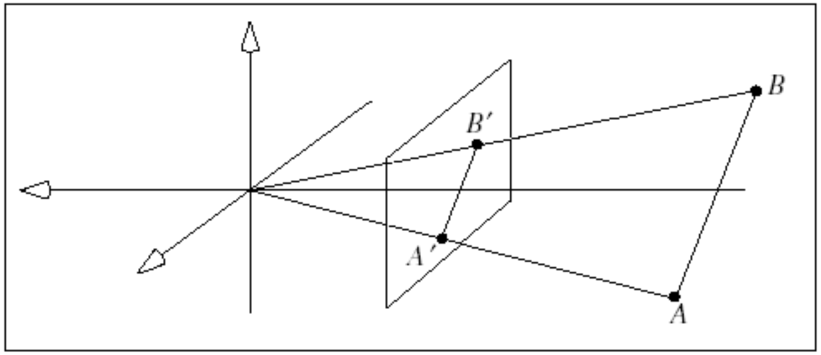
Observations

- Perspective foreshortening
- Denominator becomes undefined for z = 0
- If P is behind the eye z changes sign
- Near plane just scales the picture
- Straight line -> straight line

$$x' = -d\frac{x}{z}$$

$$y' = -d\frac{y}{z}$$

$$z' = -d$$



Perspective projection of a line

$$L(t) = \mathbf{A} + \vec{\mathbf{c}}t = \begin{bmatrix} A_x \\ A_y \\ A_z \\ 1 \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \\ c_z \\ 0 \end{bmatrix} t$$

$$\widetilde{L}(t) = \mathbf{M}L(t) = \mathbf{M}(\mathbf{A} + \vec{\mathbf{c}}t) = \mathbf{M}\begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \\ 1 \end{bmatrix} = \begin{bmatrix} N(A_x + c_x t) \\ N(A_y + c_y t) \\ N(A_z + c_z t) \\ -(A_z + c_z t) \end{bmatrix}$$

Perspective Division, drop fourth coordinate

$$L'(t) = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

Is it a line?

$$\text{Original}: L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected}: L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

$$x' = -N(A_x + c_x t)/(A_z + c_z t) \Rightarrow x'(A_z + c_z t) = -N(A_x + c_x t) \Rightarrow$$

$$x' A_z + x' c_z t = -NA_x - Nc_x t \Rightarrow \begin{cases} x' A_z + NA_x = -(x' c_z + Nc_x)t \\ \quad \text{and similarly for y} \\ y' A_z + NA_y = -(y' c_z + Nc_y)t \end{cases}$$

Cont'd next slide

Is it a line? (cont'd)

$$\left.\begin{array}{l} x'A_z + NA_x = -(x'c_z + Nc_x)t \\ y'A_z + NA_y = -(y'c_z + Nc_y)t \end{array}\right| \Rightarrow \left.\begin{array}{l} x'A_z + NA_x = -(x'c_z + Nc_x)t \\ -(y'c_z + Nc_y)t = y'A_z + NA_y \end{array}\right| \Rightarrow$$

$$(x'A_z + NA_x)(y'c_z + Nc_y) = (x'c_z + Nc_x)(y'A_z + NA_y) \Rightarrow$$

$$x'A_z y'c_z + x'A_z Nc_y + NA_x y'c_z + N^2 A_x c_y = x'c_z y'A_z + x'c_z NA_y + Nc_x y'A_z + N^2 A_y c_x \Rightarrow$$

$$(A_z Nc_y - c_z NA_y)x' + (NA_x c_z + Nc_x A_z)y' + N^2(A_x c_y + A_y c_x) = 0 \Rightarrow$$

$\Rightarrow$ $\boxed{ax' + by' + c = 0}$ which is the equation of a line.

So is there a difference?

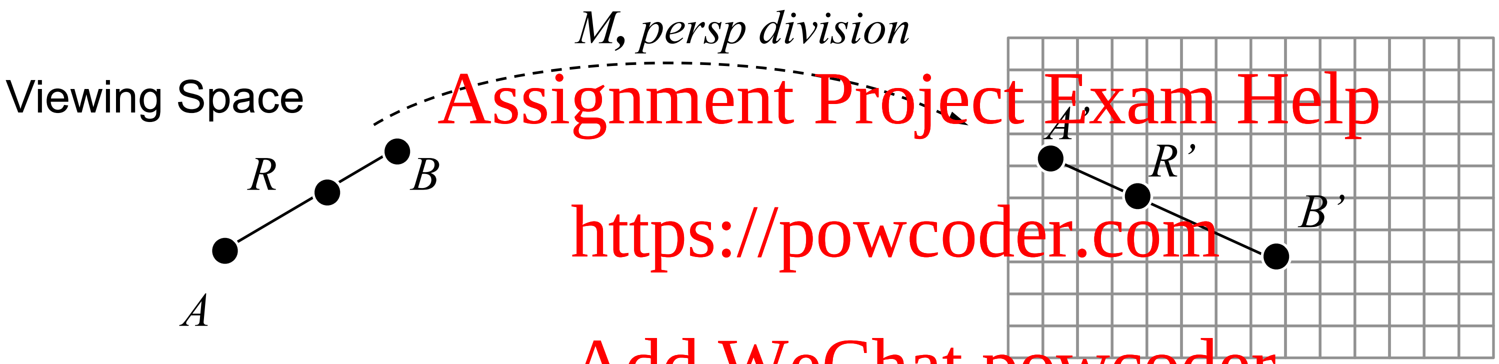
$$\text{Original}: L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected}: L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

Non-linearity of perspective projection

How do points on lines project ?

NDCS and eventually
Screen Space



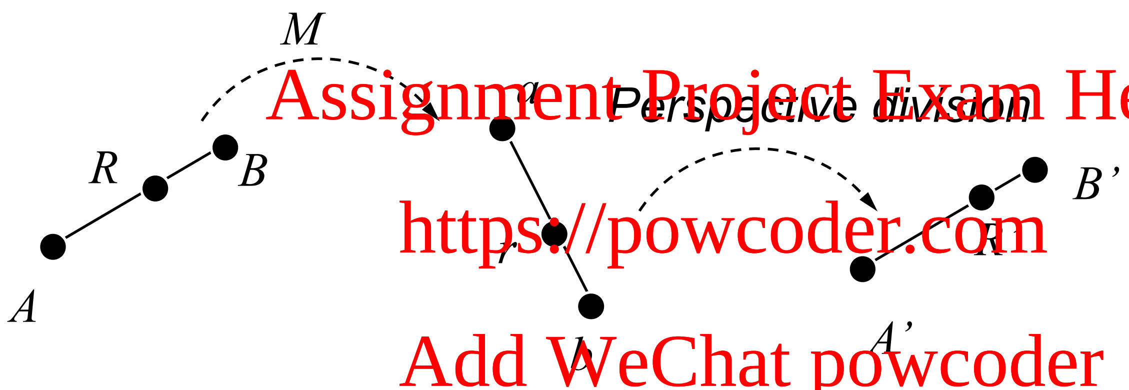
Viewing space: $R(g) = (1-g)A + gB$

NDCS Coordinates: $R'(f) = (1-f)A' + fB'$

What is the relationship between g and f?

Non-linearity of perspective projection

Point goes through two stages



Viewing space: $R(g) = (1-g)A + gB$

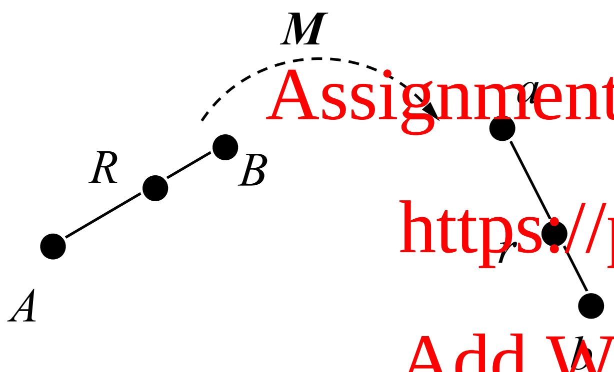
Projected (4D) : $r = MR$

Projected cartesian: $R'(f) = (1-f)A' + fB'$

What is the relationship between g and f?

First step

Viewing to homogeneous space (4D)



$$R = (1-g)A + gB$$

$$r = MR = M[(1-g)A + gB] = (1-g)MA + gMB \Rightarrow$$

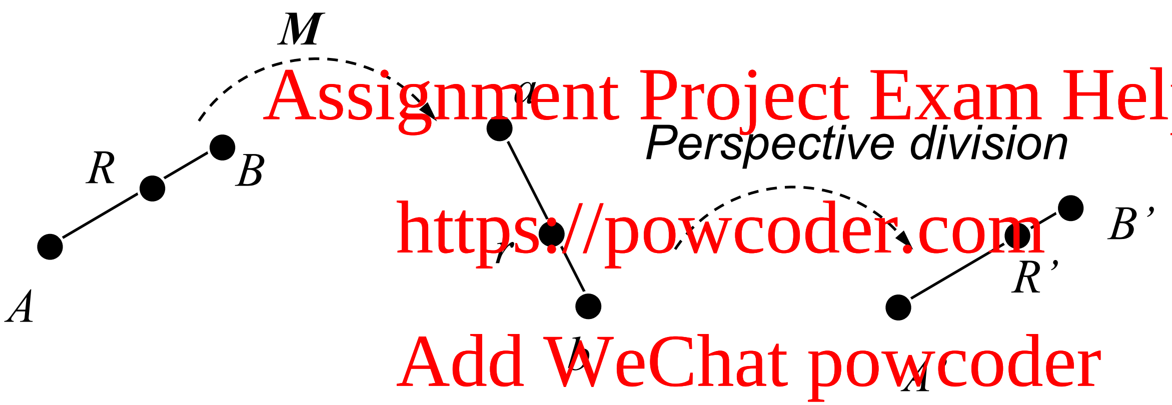
$$r = (1-g)a + gb$$

$$a = MA = (a_1, a_2, a_3, a_4)$$

$$b = MB = (b_1, b_2, b_3, b_4)$$

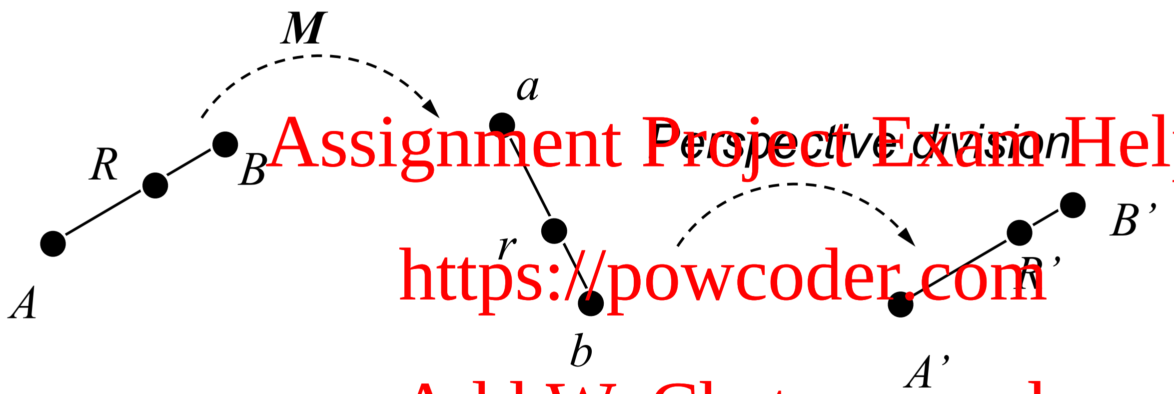
Second step

Perspective division



$$\left.\begin{aligned} r &= (1-g)a + gb \\ \\ r &= (r_1, r_2, r_3, r_4) \\ a &= (a_1, a_2, a_3, a_4) \\ b &= (b_1, b_2, b_3, b_4) \end{aligned}\right\} \rightarrow \boxed{R'_1 = \frac{r_1}{r_4} = \frac{(1-g)a_1 + gb_1}{(1-g)a_4 + gb_4}}$$

Putting all together



$$R_1' = \frac{(1-g)a_1 + gb_1}{(1-g)a_4 + gb_4} = \frac{lerp(a_1, b_1, g)}{lerp(a_4, b_4, g)}$$

$$At \;\; the \;\; same \;\; time:$$

$$R' = (1-f)A' + fB' \Rightarrow R_1' = (1-f)A_1' + fB_1'$$

$$R_1' = (1-f)\frac{a_1}{a_4} + f\frac{b_1}{b_4} = lerp(\frac{a_1}{a_4}, \frac{b_1}{b_4}, f)$$

Relation between the fractions

$$\left.\begin{array}{l} R_1'(g) = \frac{lerp(a_1, b_1, g)}{lerp(a_4, b_4, g)} \\ \\ R_1'(f) = lerp(\frac{a_1}{a_4}, \frac{b_1}{b_4}, f) \end{array}\right\} \rightarrow g = \frac{f}{lerp(\frac{b_4}{a_4}, 1, f)}$$

substituting this in $R(g) = (1 - g)A + gB$ yields

$$R_1 = \frac{lerp(\frac{A_1}{a_4}, \frac{B_1}{b_4}, f)}{lerp(\frac{1}{a_4}, \frac{1}{b_4}, f)}$$

THAT MEANS: For a given f in **screen space** and A,B in **viewing space** we can find the corresponding R (or g) in **viewing space** using the above formula.

“A,B” can be texture coordinates, position, color, normal etc.

Effect of perspective projection on lines

Equations

$$\text{Original}: L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected}: L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

What happens to parallel lines?

Effect of perspective projection on lines

Parallel lines

$$\text{Original}: L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected}: L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

If parallel to view plane then:

$$c_z = 0 \rightarrow L'(t) = -\frac{N}{A_z}(A_x + c_x t, A_y + c_y t)$$

$$\textbf{slope} \quad = \frac{c_y}{c_x}$$

Effect of perspective projection on lines

Parallel lines

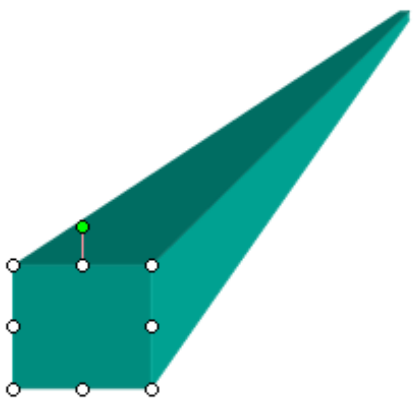
$$\text{Original}: L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected}: L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

If not parallel to view plane then:

$$c_z \neq 0 \rightarrow \lim_{t \to \infty} L'(t) = -\frac{N}{c_z}(c_x, c_y)$$

Vanishing point!



Summary

Forshortening

Non-linear

Lines go to lines

Parallel lines either intersect or remain parallel

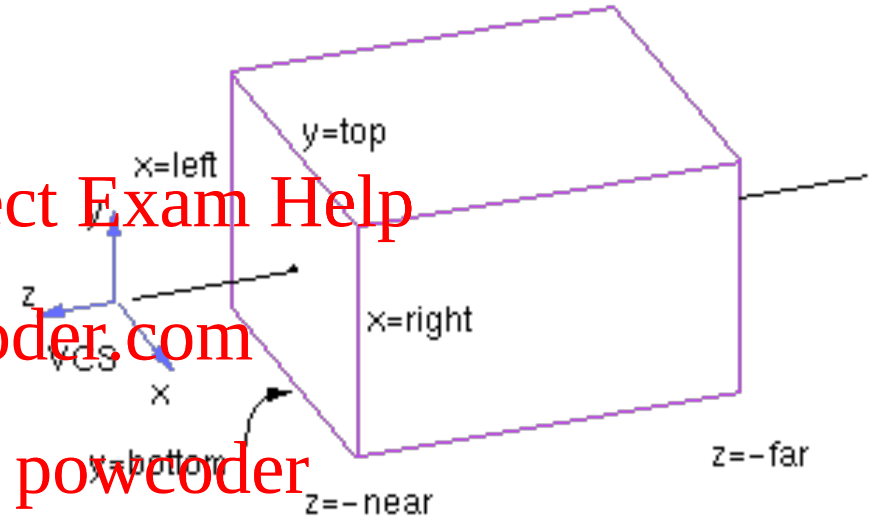
Inbetweeneness (interpolation)

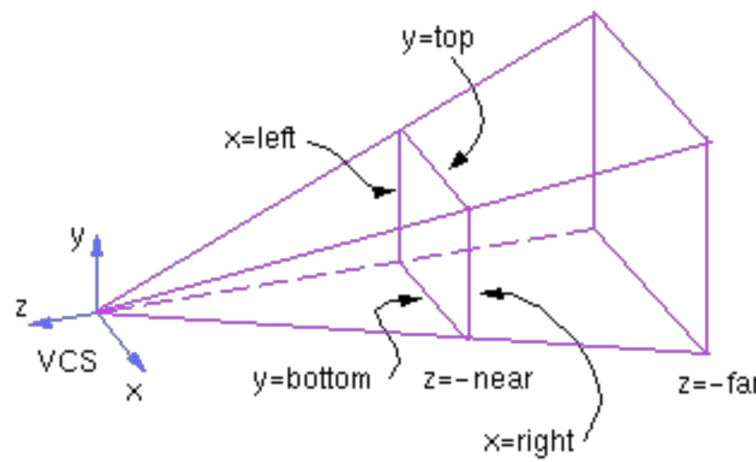
Screen space and viewing space are not linearly related

Projections in the Graphics Pipeline

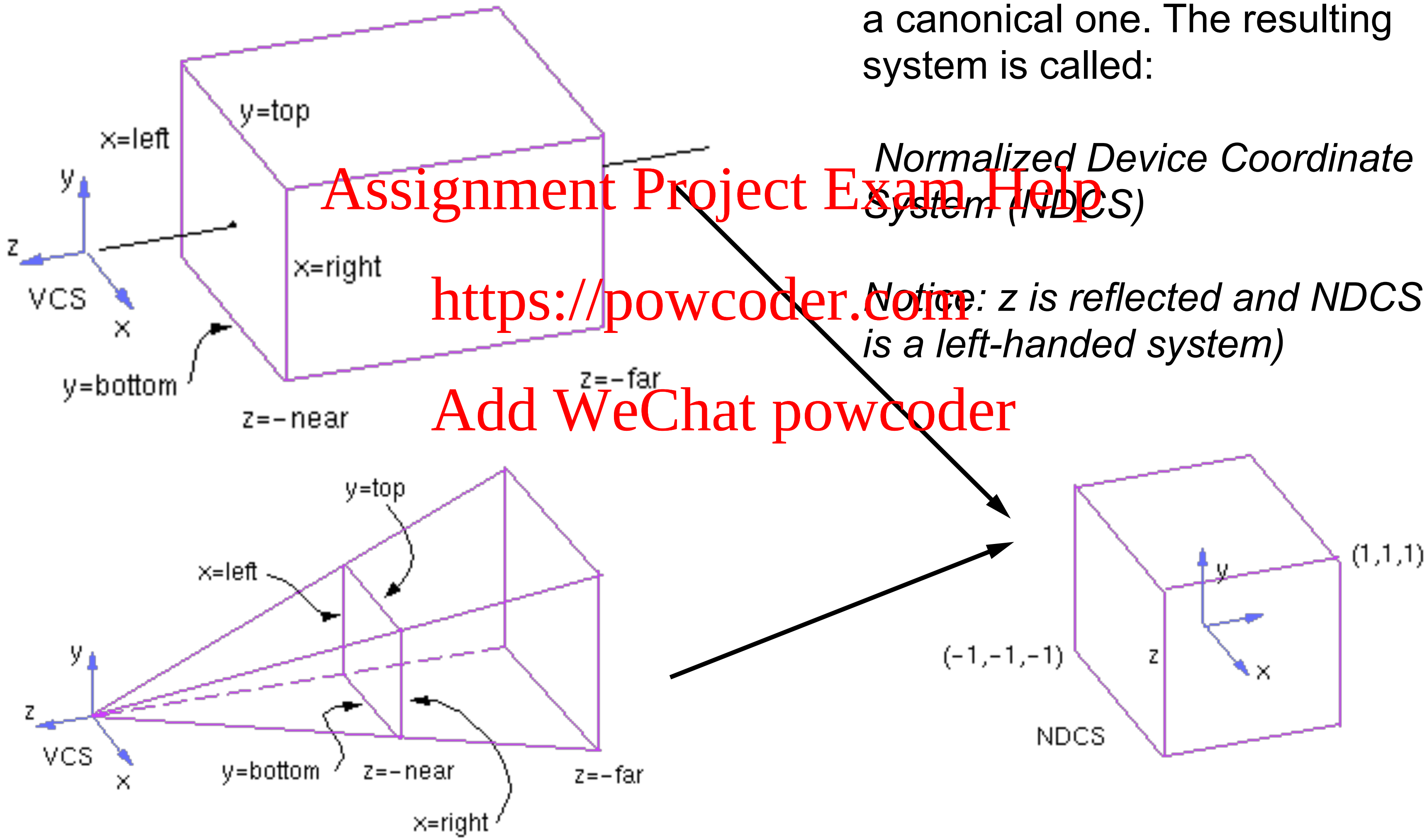
View volumes

- Primarily two:
  - *Orthographic*
  - *Perspective*
- This stage also defines the view window
- What is visible with each projection?
  - *a cube*
  - *a truncated pyramid*





Projection Stage in Graphics Pipeline



Transforms the view volume into a canonical one. The resulting system is called:

Normalized Device Coordinate System (NDCS)

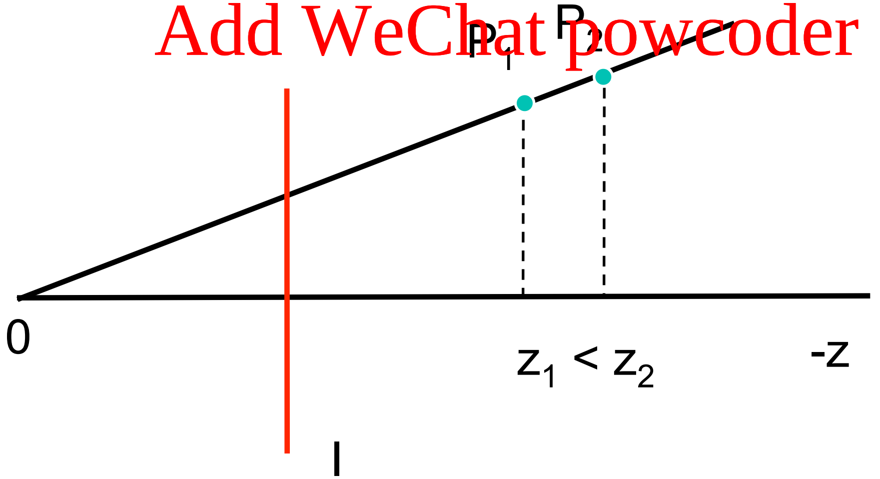
Notice: z is reflected and NDCS is a left-handed system)

Transformation vs Projection

We want to keep z

Why?

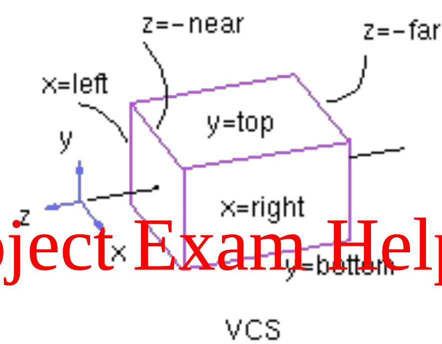
- Pseudodepth

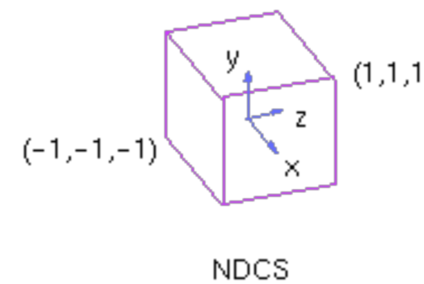


Derivation of the orthographic transformation

Map each axis separately:

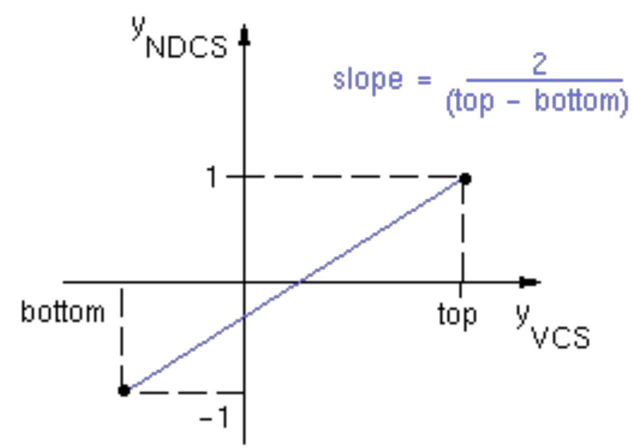
- Scaling and translation





Let's look at y:

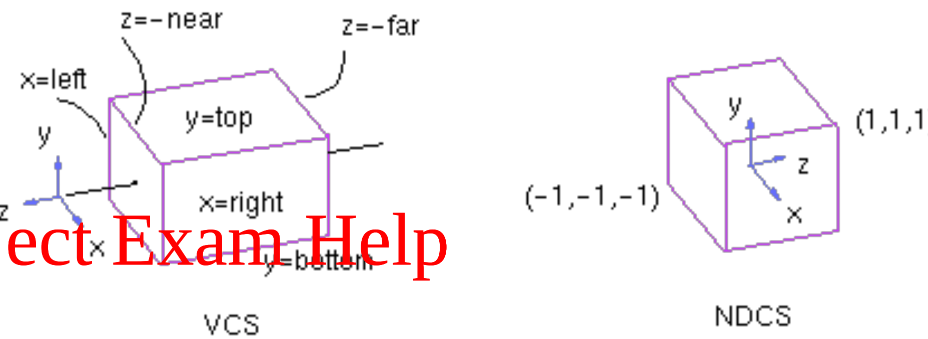
- $y' = ay+b$ such that
  bottom → -1
  top → 1



- Note: left,right,near,far,top,bottom>0

Derivation of the orthographic transformation

Scaling and Translation



$$y_{VCS} \rightarrow y$$

$$y_{NDCS} \rightarrow y'$$

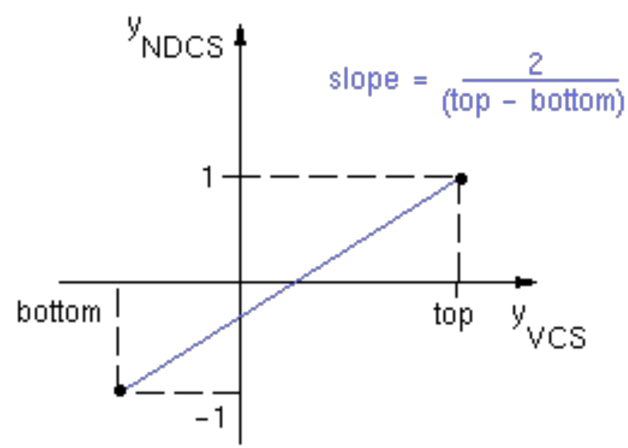
$$(y_b, y_b') = (bottom, -1) \quad and$$

$$(y_t, y_t') = (top, 1)$$

$$Line \quad equation \quad \frac{y' - y_b'}{y - y_b} = \frac{y_t' - y_b'}{y_t - y_b}$$

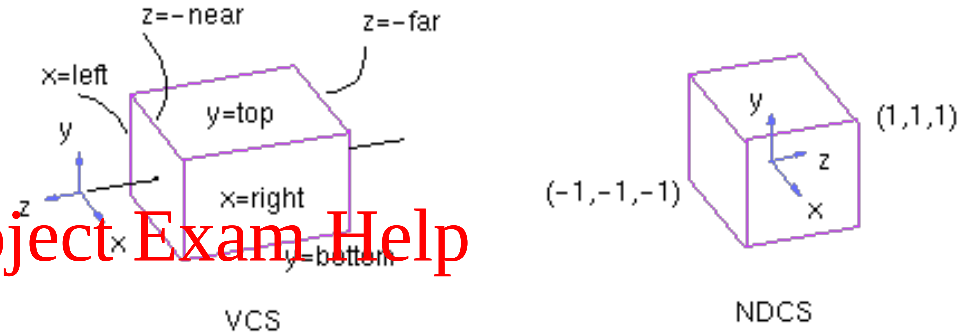
$$\frac{y' - (-1)}{y - bottom} = \frac{1 - (-1)}{top - bottom} \rightarrow$$

$$y' = \frac{2}{top - bottom} y - \frac{top + bottom}{top - bottom}$$



All three coordinates

Scaling and Translation

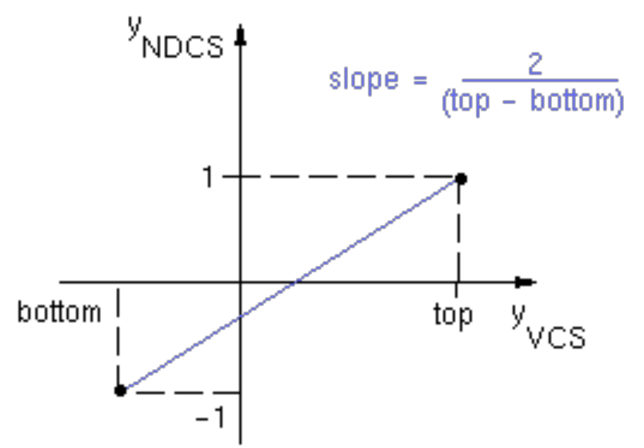


$Similarly,$

$$x' = \frac{2}{right - left}x - \frac{right + left}{right - left}$$

$$y' = \frac{2}{top - bottom}y - \frac{top + bottom}{top - bottom}$$

$$z' = \frac{-2}{far - near}z - \frac{far + near}{far - near}$$



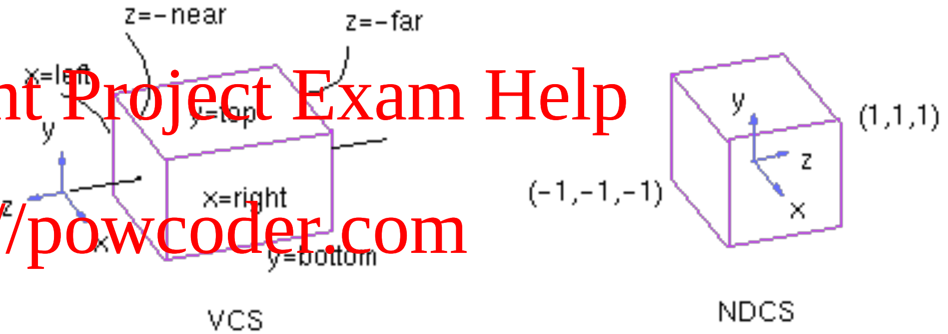
Matrix form

$$P' = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix} P$$

Alternative way

Scaling and translation of a cube

Note: $r, l, b, t, n, f > 0$



$$\mathbf{M}_O = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -\frac{l+r}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & -\frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$