Static Program Analysis

Part 5 – widening and narrowing

http://cs.au.dk/~amoeller/spa/

Anders Møller & Michael I. Schwartzbach

Computer Science, Aarhus University

Interval analysis

- Compute upper and lower bounds for integers
- Possible applications:
  - array bounds checking
  - integer representation
  - ...

- Lattice of intervals:

$$\mathit{Interval} = \mathit{lift}(\{\ [l, h]\ \mid l,h \in N \wedge l \leq h\ \})$$

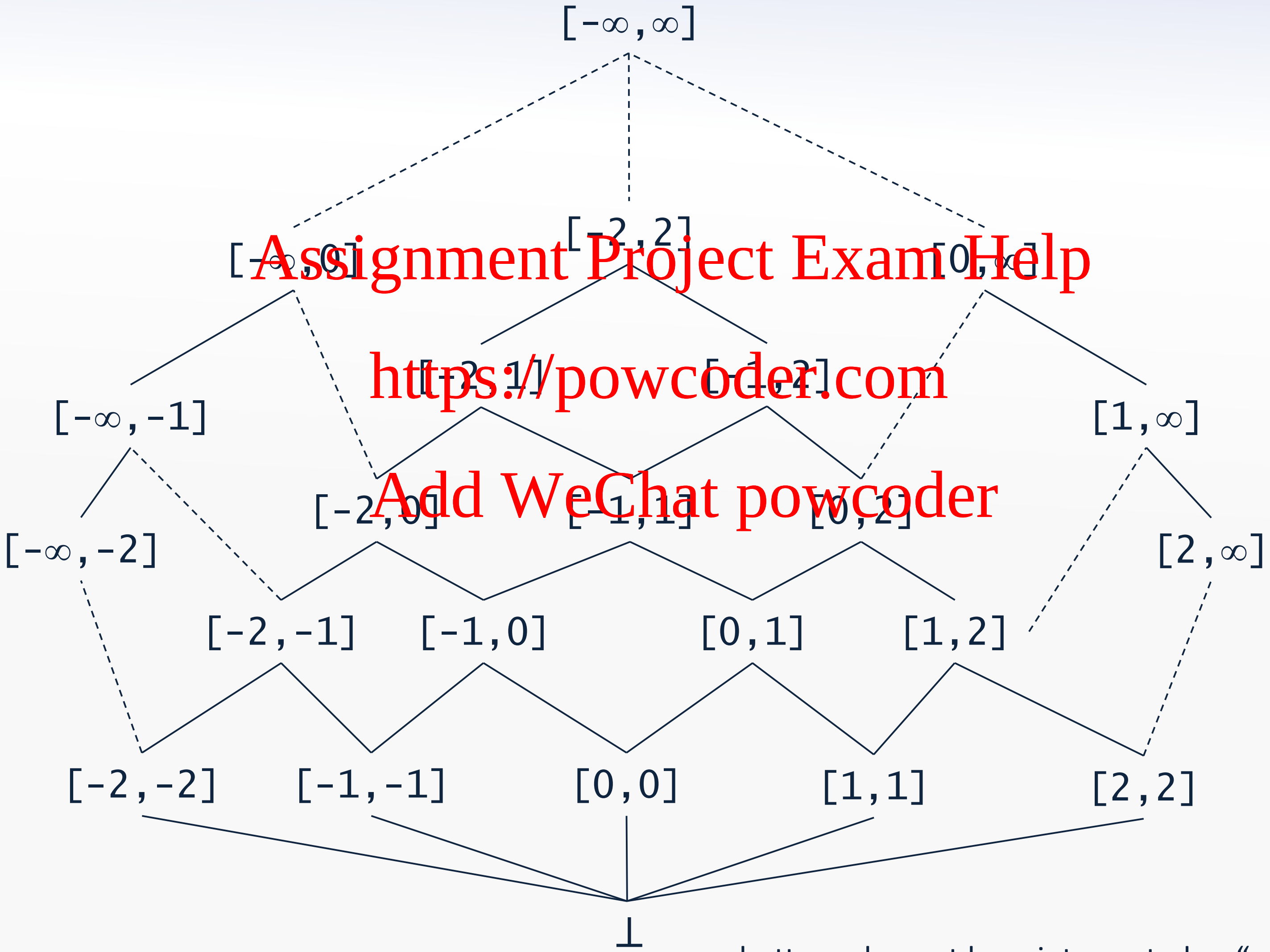
where

$$N = \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$$

and intervals are ordered by inclusion:

$$[l_1, h_1] \sqsubseteq [l_2, h_2] \text{ iff } l_2 \leq l_1 \wedge h_1 \leq h_2$$

The interval lattice



bottom element here interpreted as “not an integer”

Interval analysis lattice

- The total lattice for a program point is

  $$L = \mathit{Vars} \rightarrow \mathit{Interval}$$

  that provides bounds for each (integer) variable

- If using the worklist solver that initializes the worklist with only the *entry* node, use the lattice *lift*(L)
  - bottom value of *lift*(L) represents "unreachable program point"
  - bottom value of L represents "maybe reachable, but all variables are non-integers"

- This lattice has *infinite height*, since the chain

  $$[0,0] \sqsubseteq [0,1] \sqsubseteq [0,2] \sqsubseteq [0,3] \sqsubseteq [0,4] \ldots$$

  occurs in *Interval*

Interval constraints

- For assignments:

$$[\![x = E]\!] = JOIN(v)[x \rightarrow eval(JOIN(v), E)]$$

- For all other nodes:

$$[\![v]\!] = JOIN(v)$$

where $JOIN(v) = \bigsqcup_{w \in pred(v)} [\![w]\!]$

Evaluating intervals

- The *eval* function is an *abstract evaluation*:
  - $eval(\sigma, x) = \sigma(x)$
  - $eval(\sigma, intconst) = [intconst, intconst]$
  - $eval(\sigma, E_1 \text{ op } E_2) = \overline{\text{op}}(eval(\sigma,E_1),eval(\sigma,E_2))$

- Abstract arithmetic operators:
  - $\overline{\text{op}}([l_1, h_1], [l_2, h_2]) =$

$$[\min_{x\in[l_1,h_1],\, y\in[l_2,h_2]} x \text{ op } y\,,\ \max_{x\in[l_1,h_1],\, y\in[l_2,h_2]} x \text{ op } y]$$

not trivial to implement!

- Abstract comparison operators (could be improved):
  - $\overline{\text{op}}([l_1, h_1], [l_2, h_2]) = [0, 1]$

Fixed-point problems

- The lattice has infinite height, so the fixed-point algorithm does not work ☹

- In L^n, the sequence of approximants

  $f^i(\bot, \bot, \ldots, \bot)$

  is not guaranteed to converge

- (Exercise: give an example of a program where this happens)

- Restricting to 32 bit integers is not a practical solution
- *Widening* gives a useful solution…

Widening

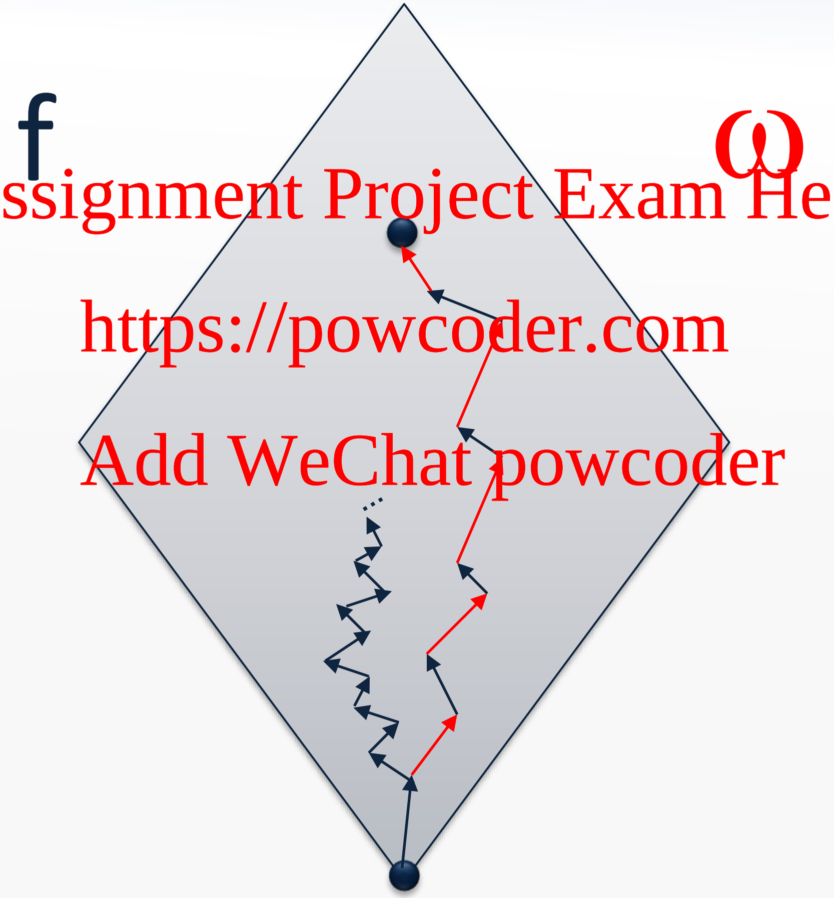
- Introduce a *widening* function ω: $L^n \rightarrow L^n$ so that

  $$(\omega \circ f)^i(\bot, \bot, \dots, \bot)$$

  converges on a fixed-point that is a safe approximation of each $f^i(\bot, \bot, \dots, \bot)$

- i.e. the function ω coarsens the information

Turbo charging the iterations



Widening for intervals

- The function ω is defined pointwise on L^n
- Parameterized with a fixed finite subset $B \subset \mathcal{N}$
  - must contain $-\infty$ and ∞ (to retain the $\top$ element)
  - typically seeded with all integer constants occurring in the given program
- Idea: Find the nearest enclosing allowed interval
- On single elements from *Interval* :

$$\omega([a\,,b]) = [\ \mathit{max}\{i \in B \mid i \leq a\},\ \mathit{min}\{i \in B \mid b \leq i\}\ ]$$

$$\omega(\bot) = \bot$$

Divergence in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \rightarrow \perp, y \rightarrow \perp]$

[x → [8,8], y → [0,1]]

[x → [8,8], y → [0,2]]

[x → [8,8], y → [0,3]]

…

Widening in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
   x = 7;
   x = x+1;
   y = y+1;
}
```

$[x \to \bot, y \to \bot]$

$[x \to [7,\infty], y \to [0,1]]$

$[x \to [7,\infty], y \to [0,7]]$

$[x \to [7,\infty], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

Correctness of widening

- Widening works when:
  - ω is an *extensive* and *monotone* function, and
  - $\omega(L)$ is a *finite-height* lattice

- Safety: $\forall i$: $f^i(\bot, \bot, \ldots, \bot) \sqsubseteq (\omega \circ f)^i(\bot, \bot, \ldots, \bot)$
  since f is monotone and ω is extensive
- $\omega \circ f$ is a monotone function $\omega(L) \to \omega(L)$
  so the fixed-point exists

- Almost “correct by definition”!
- When used in the worklist algorithm, it suffices to apply widening on back-edges in the CFG

Narrowing

- Widening generally shoots over the target
- *Narrowing* may improve the result by applying f
- Define:

$$\mathit{fix} = \bigsqcup f^i(\bot, \bot, \ldots, \bot) \qquad \mathit{fix}\omega = \bigsqcup (\omega \circ f)^i(\bot, \bot, \ldots, \bot)$$

then $\mathit{fix} \sqsubseteq \mathit{fix}\omega$

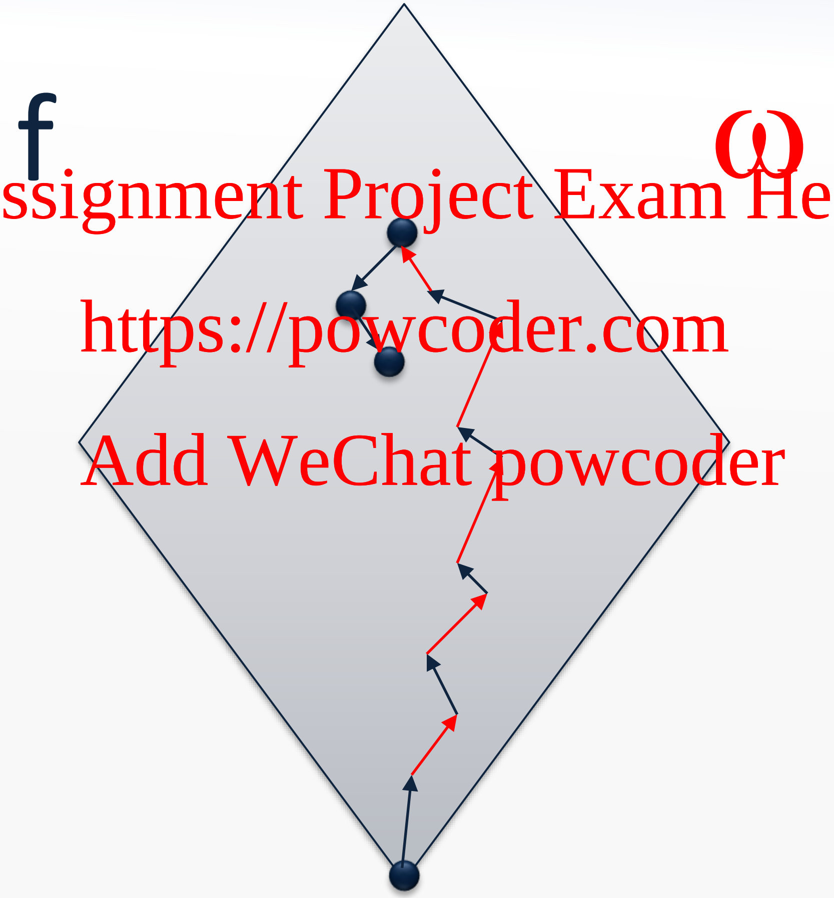
- But we also have that

$$\mathit{fix} \sqsubseteq f(\mathit{fix}\omega) \sqsubseteq \mathit{fix}\omega$$

so applying f again may improve the result and remain sound!

- This can be iterated arbitrarily many times
  - may diverge, but safe to stop anytime

Backing up



Narrowing in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
   x = 7;
   x = x+1;
   y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [7,\infty], y \to [0,1]]$
$[x \to [7,\infty], y \to [0,7]]$
$[x \to [7,\infty], y \to [0,\infty]]$
…
$[x \to [8,8], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

Correctness of (repeated) narrowing

- $f(\mathit{fix}\,\omega) \sqsubseteq \omega(f(\mathit{fix}\,\omega)) = (\omega \circ f)(\mathit{fix}\,\omega) = \mathit{fix}\,\omega$ since ω is extensive
  - by induction we also have, for all i:

    $f^{i+1}(\mathit{fix}\,\omega) \sqsubseteq f^{i}(\mathit{fix}\,\omega) \sqsubseteq \mathit{fix}\,\omega$
  - i.e. $f^{i+1}(\mathit{fix}\,\omega)$ is at least as precise as $f^{i}(\mathit{fix}\,\omega)$

- $\mathit{fix} \sqsubseteq \mathit{fix}\,\omega$ hence $f(\mathit{fix}) = \mathit{fix} \sqsubseteq f(\mathit{fix}\,\omega)$ by monotonicity of f
  - by induction we also have, for all i:

    $\mathit{fix} \sqsubseteq f^{i}(\mathit{fix}\,\omega)$
  - i.e. $f^{i}(\mathit{fix}\,\omega)$ is a sound approximation of fix

More powerful widening

- Defining the widening function based on constants occurring in the given program may not work

```
f(x) { // ”McCarthy’s 91 function”
  var r;
  if (x > 100) {
    r = x - 10;
  } else {
    r = f(f(x + 11));
  }
  return r;
}
```

https://en.wikipedia.org/wiki/McCarthy_91_function

- Note: this example requires interprocedural analysis…

More powerful widening

- A *widening* is a function ∇: $L \times L \rightarrow L$ that is extensive in both arguments and satisfies the following property:

  for all increasing chains $z_0 \sqsubseteq z_1 \sqsubseteq \dots$,

  the sequence $y_0 = z_0, \dots, y_{i+1} = y_i \nabla z_{i+1}, \dots$ converges

  (i.e. stabilizes after a finite number of steps)

- Now replace the basic fixed point solver by computing $x_0 = \bot, \dots, x_{i+1} = x_i \nabla F(x_i), \dots$ until convergence

More powerful widening for interval analysis

Extrapolates unstable bounds to B:

$$\bot \nabla y = y$$

$$x \nabla \bot = x$$

$$[a_1 , b_1] \nabla [a_2 , b_2] =$$

$$[\text{if } a_1 \leq a_2 \text{ then } a_1 \text{ else } \mathit{max}\{i \in B \mid i \leq a_2\},$$

$$\text{if } b_2 \leq b_1 \text{ then } b_1 \text{ else } \mathit{min}\{i \in B \mid b_2 \leq i\}]$$

The ∇ operator on L is then defined pointwise down to individual intervals

For the small example program, we now get the same result as with simple widening plus narrowing (but now without using narrowing)