Advanced Network Technologies

Queueing Theory

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Outline

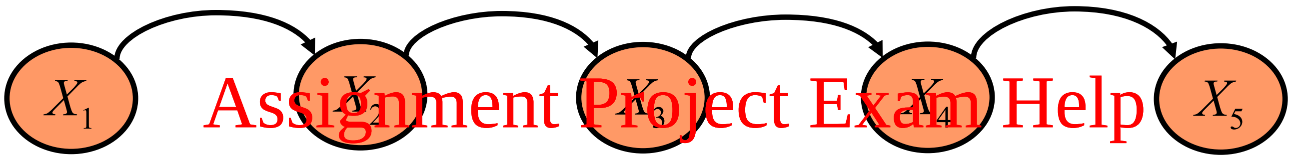
- Markov Chain
- Queueing System and Little's Theorem
- M/M/1 Queue foundations
- M/M/1 Queue

- Markov Chain

- A stochastic process
  - $X_1, X_2, X_3, X_4 \ldots$
  - $\{X_n, n = 1, 2, \ldots\}$
  - X_n takes on a finite or countable number of possible values.
  - $X_n \in \{1,2, \ldots, S\}$
  - i: ith state
  - Markov Property: The state of the system at time $n+1$ depends only on the state of the system at time n

$$\Pr\left[X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_2 = x_2, X_1 = x_1\right] = \Pr\left[X_{n+1} = x_{n+1} \mid X_n = x_n\right]$$

Markov Chain



- Stationary Assumption: Transition probabilities are independent of time (n)

$$\boxed{\Pr\left[X_{n+1} = b \mid X_n = a\right] = p_{ab}}$$

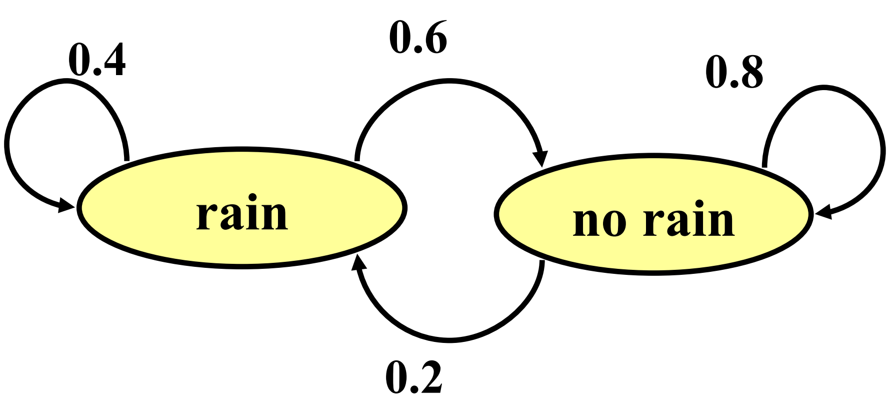
An example

Weather:

- **raining today** → **40% rain tomorrow**
  → **60% no rain tomorrow**

- **not raining today** → **20% rain tomorrow**
  → **80% no rain tomorrow**

Stochastic FSM:



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An example

Weather:

- **raining today** → **40% rain tomorrow**
  → **60% no rain tomorrow**

- **not raining today** → **20% rain tomorrow**
  → **80% no rain tomorrow**

Matrix:

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}$$

- Stochastic matrix:
  Rows sum up to 1

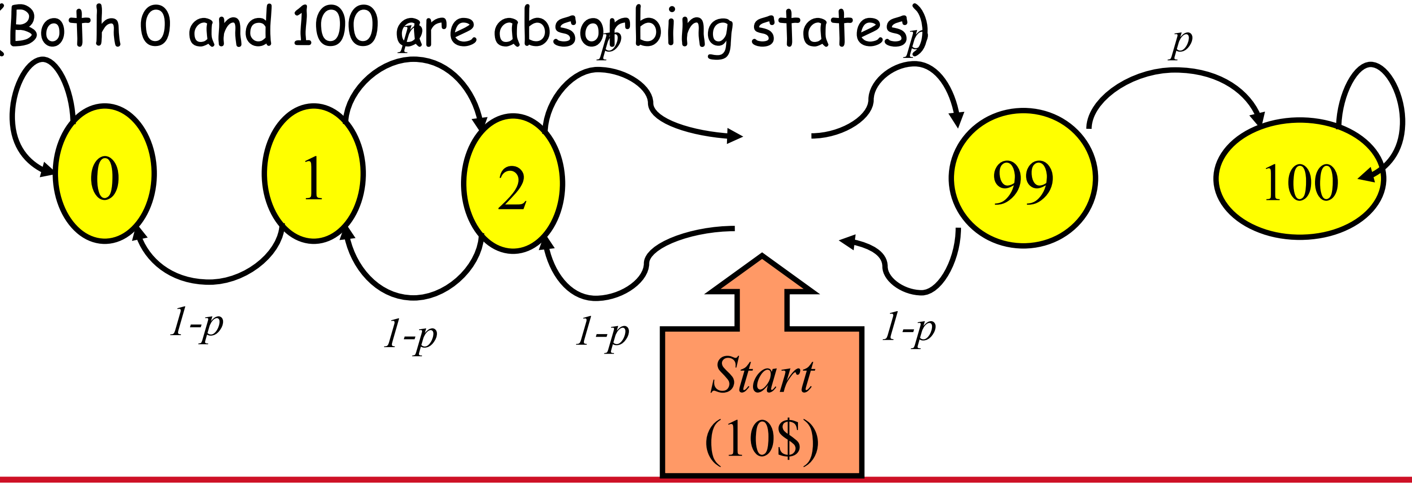
Transition matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1S} \\ p_{21} & p_{22} & \cdots & p_{2S} \\ \vdots & & \ddots & \vdots \\ p_{S1} & p_{S2} & \cdots & p_{SS} \end{pmatrix}$$

Gambler's Example

- Gambler starts with \$10
- At each play we have one of the following:
  - Gambler wins \$1 with probability p
  - Gambler looses \$1 with probability $1\text{-}p$
- Game ends when gambler goes broke, or gains a fortune of \$100

(Both 0 and 100 are absorbing states)



Gambler's Example

- **transient state**

if, given that we start in state *i*, there is a non-zero probability that we will never return to *i*

- **recurrent state**

Non-transient

- **absorbing** state

impossible to leave this state.

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Coke vs. Pepsi Example

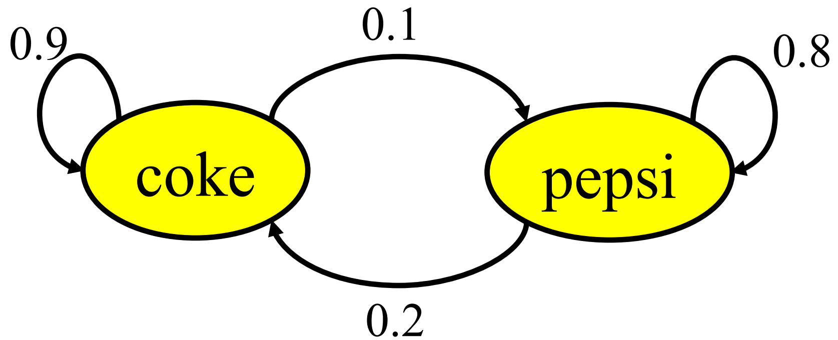
- Given that a person's last cola purchase was Coke, there is a 90% chance that his next cola purchase will also be Coke.

- If a person's last cola purchase was Pepsi, there is an 80% chance that his next cola purchase will also be Pepsi.

transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

state transition diagram:



Coke vs. Pepsi Example

Given that a person is currently a Pepsi purchaser, what is the probability that he will purchase Coke two purchases from now?

Pr[Pepsi→?→Coke] =

Pr[Pepsi→Coke→Coke] + Pr[Pepsi→ Pepsi →Coke] =

0.2 * 0.9 + 0.8 * 0.2 = 0.34

$$P^2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

Pepsi → ?   ? → Coke

Coke vs. Pepsi Example

Given that a person is currently a Coke purchaser, what is the probability that he will purchase Pepsi **three** purchases from now?

$$P^3 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}$$

Coke vs. Pepsi Example

- Assume each person makes one cola purchase per week
- Suppose 60% of all people now drink Coke, and 40% drink Pepsi
- What fraction of people will be drinking Coke three weeks from now?

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \qquad P^3 = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}$$

$$\Pr[X_3 = \text{Coke}] = 0.6 * 0.781 + 0.4 * 0.438 = 0.6438$$

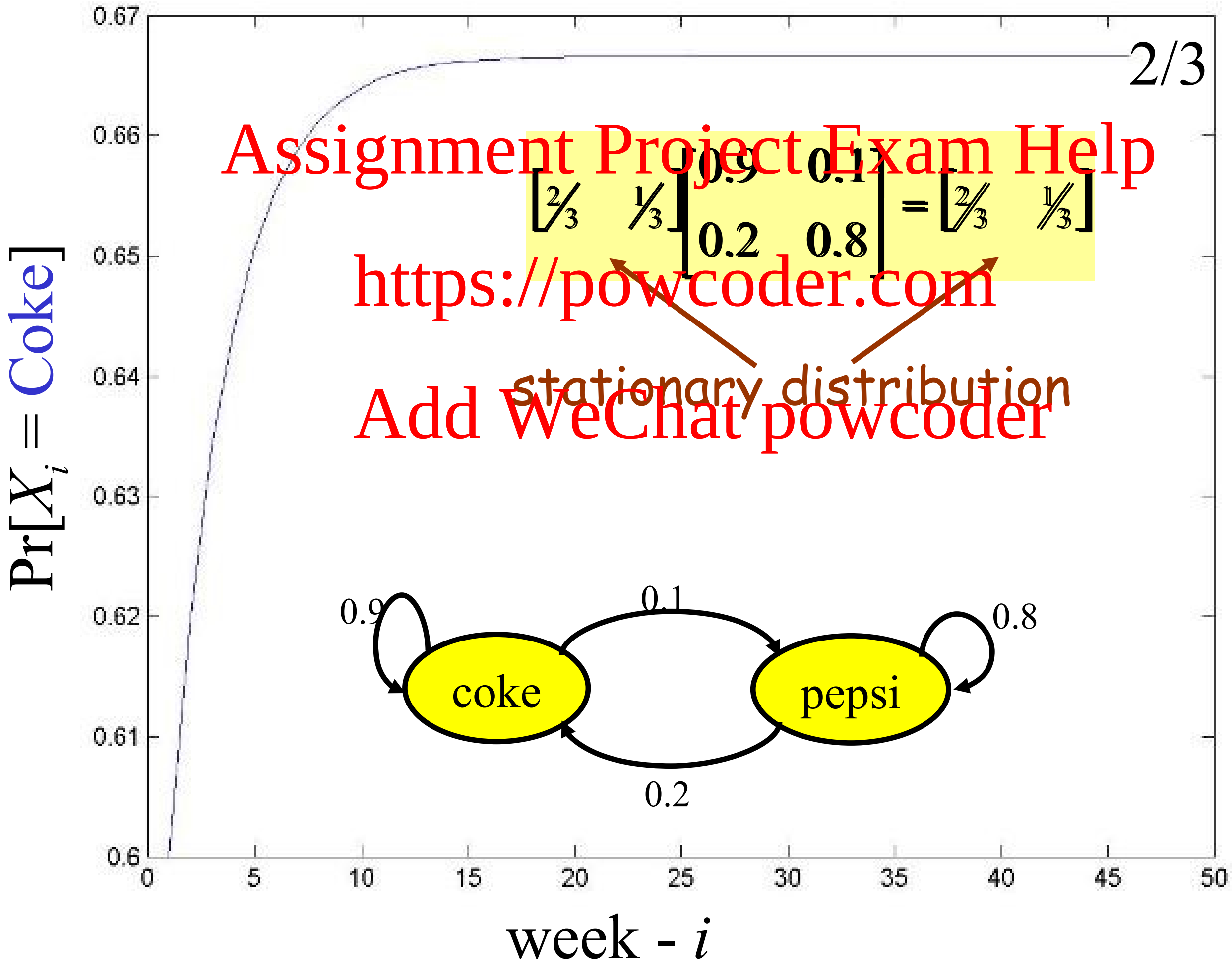
Q_i - the distribution in week i

$Q_0 = (0.6, 0.4)$ - initial distribution

$Q_3 = Q_0 * P^3 = (0.6438, 0.3562)$

Coke vs. Pepsi Example

Simulation:



Steady State and Stationary distribution

$$\lim_{n\to\infty} P(X_n = i) = \pi_i$$

$$\lim_{n\to\infty} P^{n} = \mathbf{1}\boldsymbol{\pi}$$

$$\boldsymbol{\pi} = \boldsymbol{\pi}\cdot P$$

Steady State and Stationary distribution

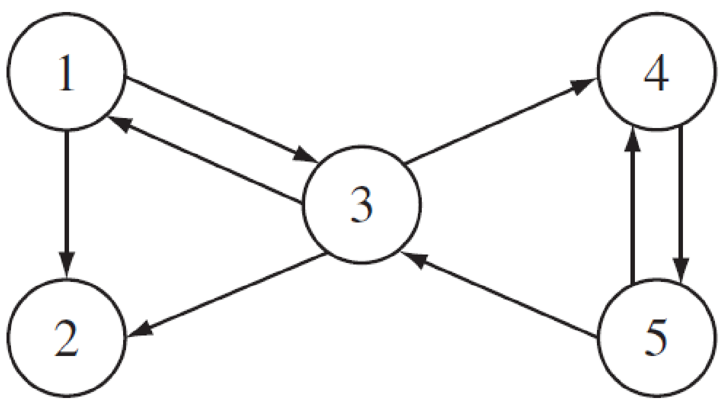
$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \qquad \boldsymbol{\pi} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.6761 & 0.3239 \\ 0.6478 & 0.3522 \end{bmatrix} \qquad P^{100} = \begin{bmatrix} 0.6667 & 0.3333 \\ 0.6667 & 0.3333 \end{bmatrix}$$

$$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

Steady State and Stationary distribution

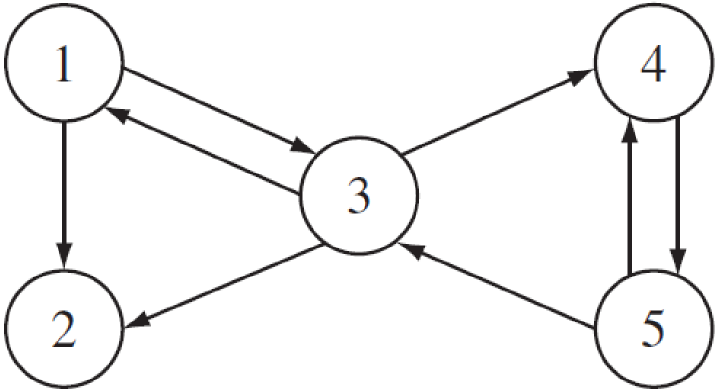
PageRank: A Web surfer browses pages in a five-page Web universe shown in figure. The surfer selects the next page to view by selecting with equal probability from the pages pointed to by the current page. If a page has no outgoing link (e.g., page 2), then the surfer selects any of the pages in the universe with equal probability. Find the probability that the surfer views page i.



Steady State and Stationary distribution

Transition matrix P

$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$



Steady State and Stationary distribution

Stationary Distribution:
Solve the following equations:

$$\boldsymbol{\pi} = \boldsymbol{\pi} \cdot P$$

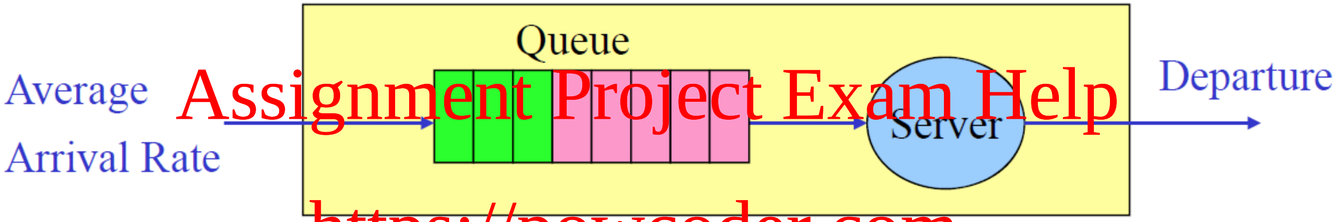
$$\sum_{i=1}^{5} \pi_i = 1$$

$\boldsymbol{\pi}$ = (0.12195, 0.18293, 0.25610, 0.12195, 0.317072)

Search engineer. page rank: 5, 3, 2, 1, 4

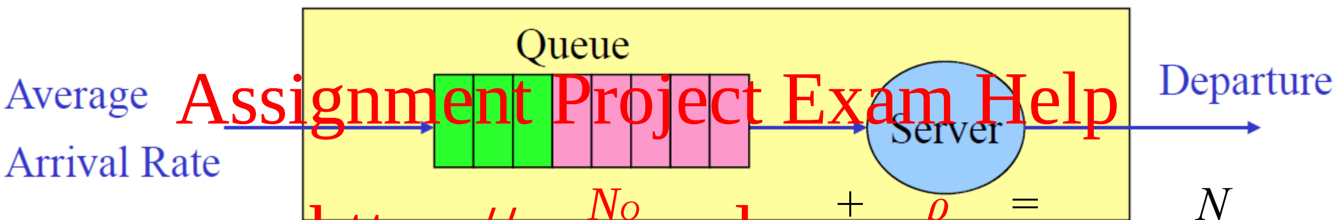
› Queueing System and Little’s Theorem

Queueing System



- Customers = Data packets
- Service Time = Packet Transmission Time (function of packet length and transmission speed)
- Queueing delay = time spent in buffer before transmission
- Average number of customers in systems
  – Typical number of customers either waiting in queue or undergoing service
- Average delay per customer
  – Typical time a customer spends waiting in queue + service time

Queueing System

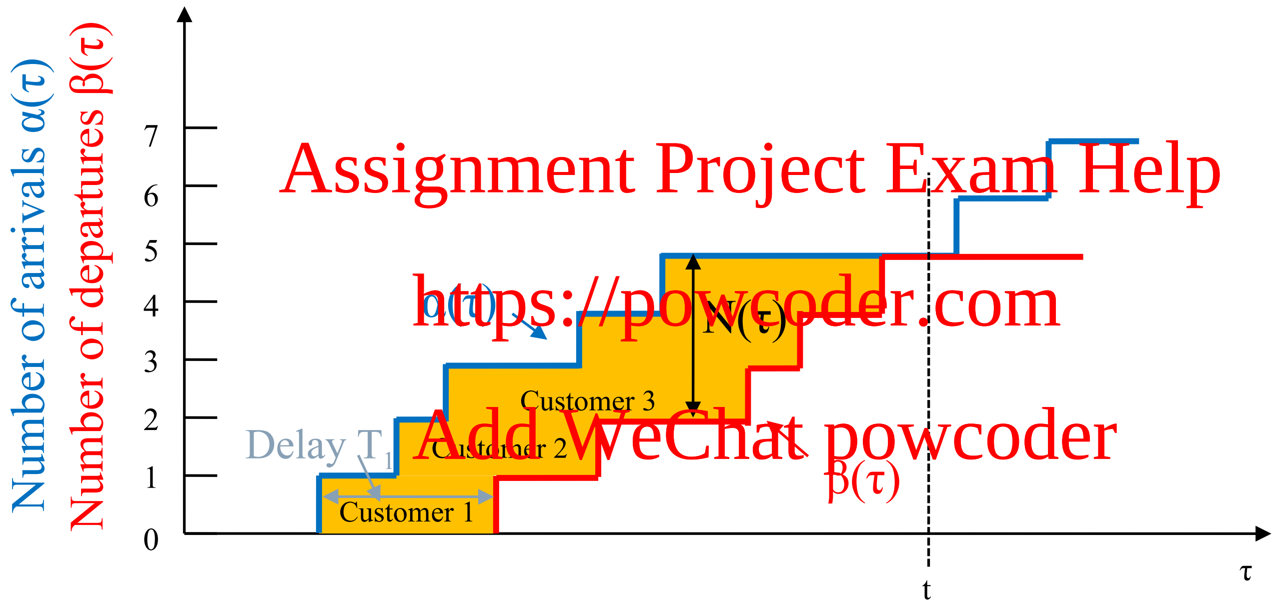


$N_Q \quad + \quad \rho \quad = \quad N$ *Number*

$W \quad + \quad X \quad = \quad T$ *Time*

- W: average waiting time in queue
- X: average service time
- T: average time spent in system ($T = W + X$)
- N_Q = average number of customers in queue
- ρ = utilization = average number of customers in service
- N = average number of customer in system ($N = N_Q + \rho$)
- Want to show later: $N = \lambda T$ (Little's theorem)
- λ Average arrival rate

Little's Theorem



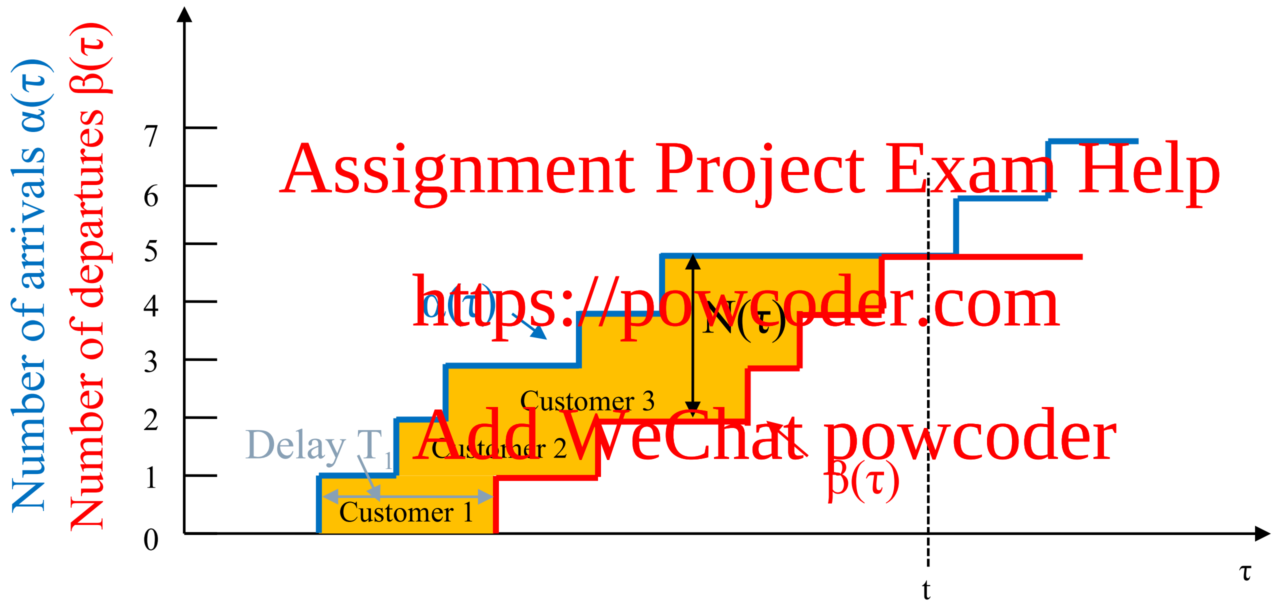
$\alpha(t)$ = Number of customers who arrived in the interval $[0, t]$

$\beta(t)$ = Number of customers who departed in the interval $[0, t]$

$N(t)$ = Number of customers in the system at time t, $N(t) = \alpha(t) - \beta(t)$

T_i = Time spent in the system by the i-th arriving customer

Little's Theorem



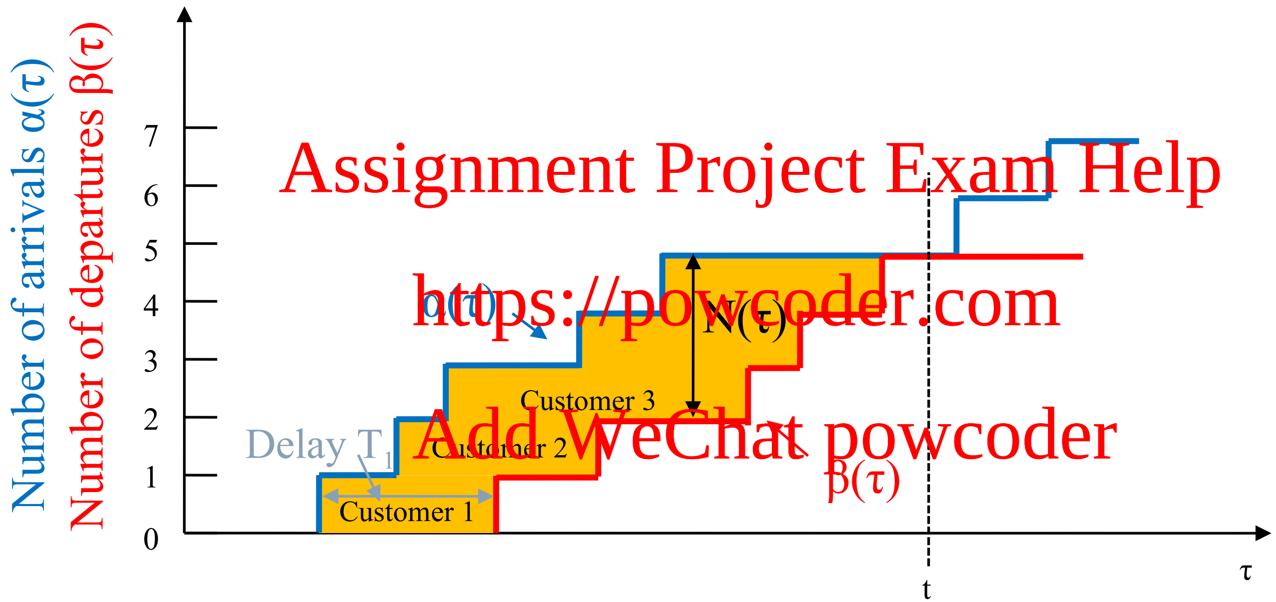
Average # of customers until t

$$N_t = \frac{1}{t}\int_0^t N(\tau)d\tau$$

Average # of customers in long-term

$$N = \lim_{t\to\infty}\frac{1}{t}\int_0^t N(\tau)d\tau$$

Little's Theorem



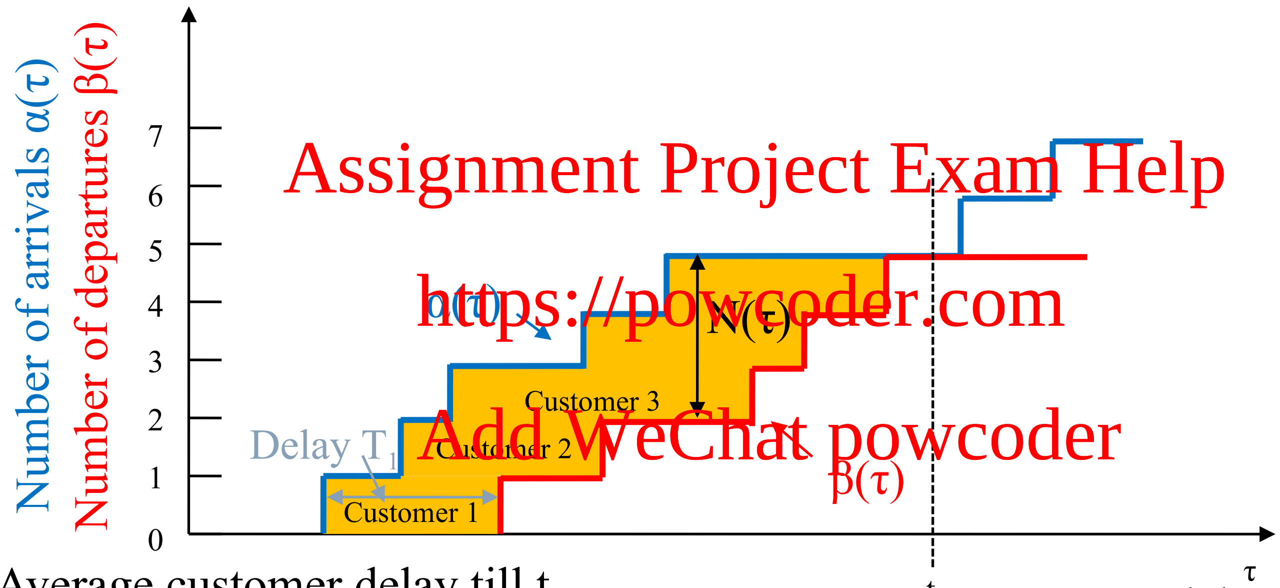
Average # arrival rate until t

$$\lambda_t = \frac{\alpha(t)}{t}$$

Average # arrival rate in long-term

$$\lambda = \lim_{t \to \infty} \frac{\alpha(t)}{t}$$

Little's Theorem



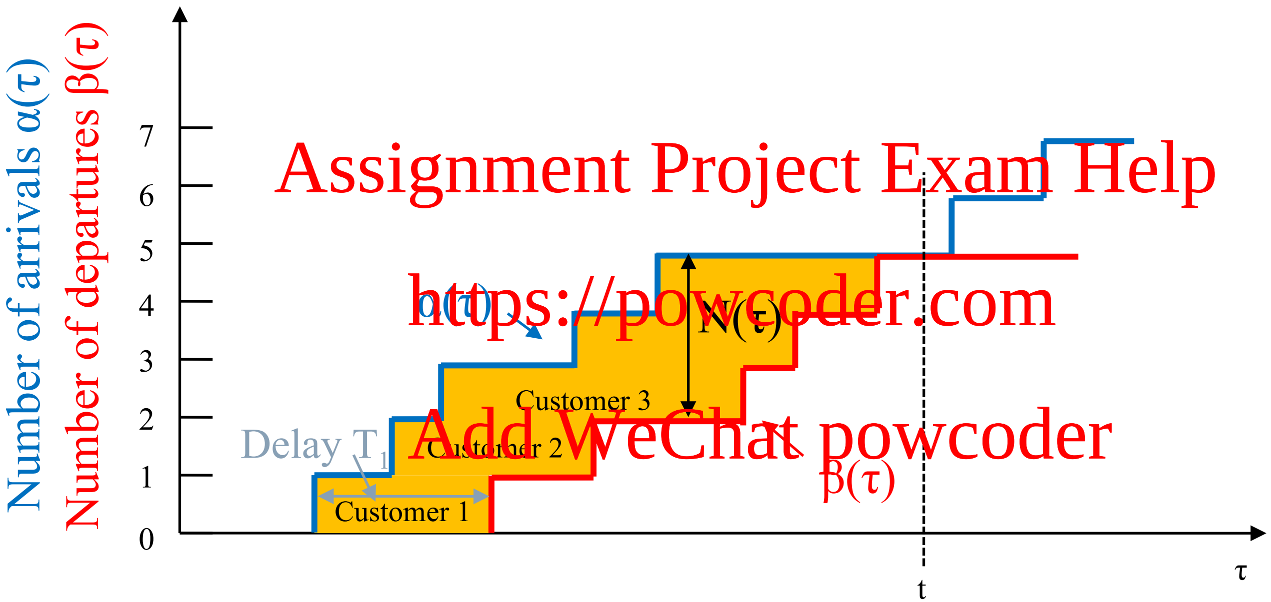
Average customer delay till t

$$T_t = \frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)}$$

Average customer delay in long-term

$$T = \lim_{t\to\infty} \frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)}$$

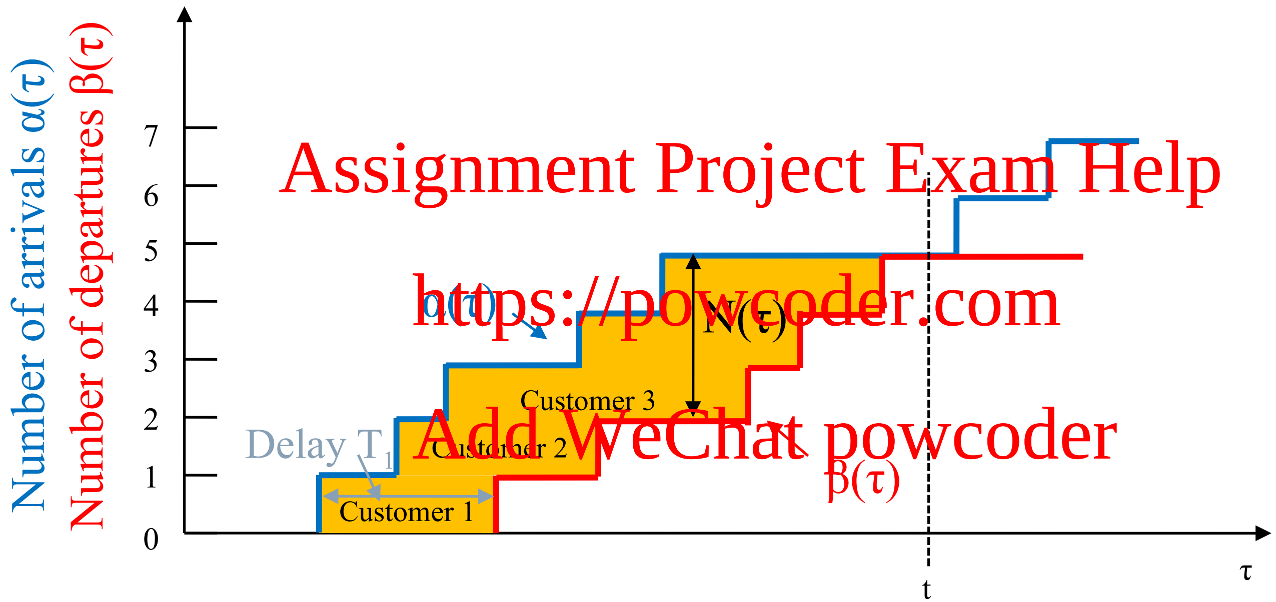
Little's Theorem



Shaded area when the queue is empty: two ways to compute

$$\int_0^t N(\tau)d\tau = \sum_{i=1}^{\alpha(t)} T_i$$

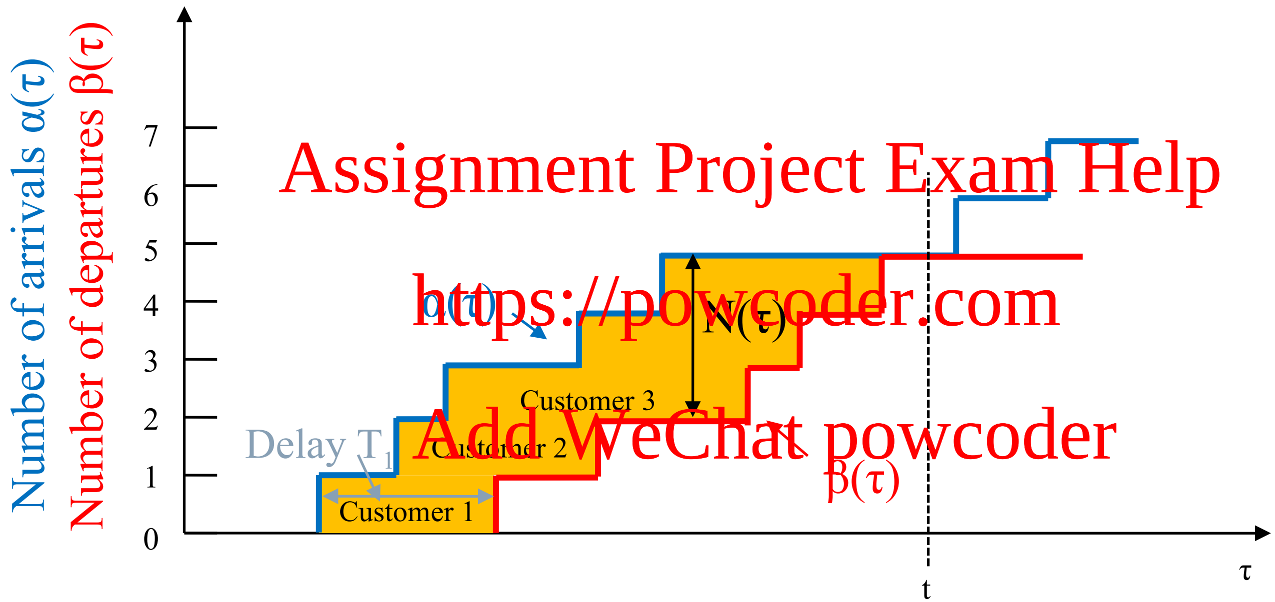
Little's Theorem



Shaded area when the queue is empty: two ways to compute

$$\frac{1}{t}\int_0^t N(\tau)d\tau = \frac{1}{t}\sum_{i=1}^{\alpha(t)} T_i$$

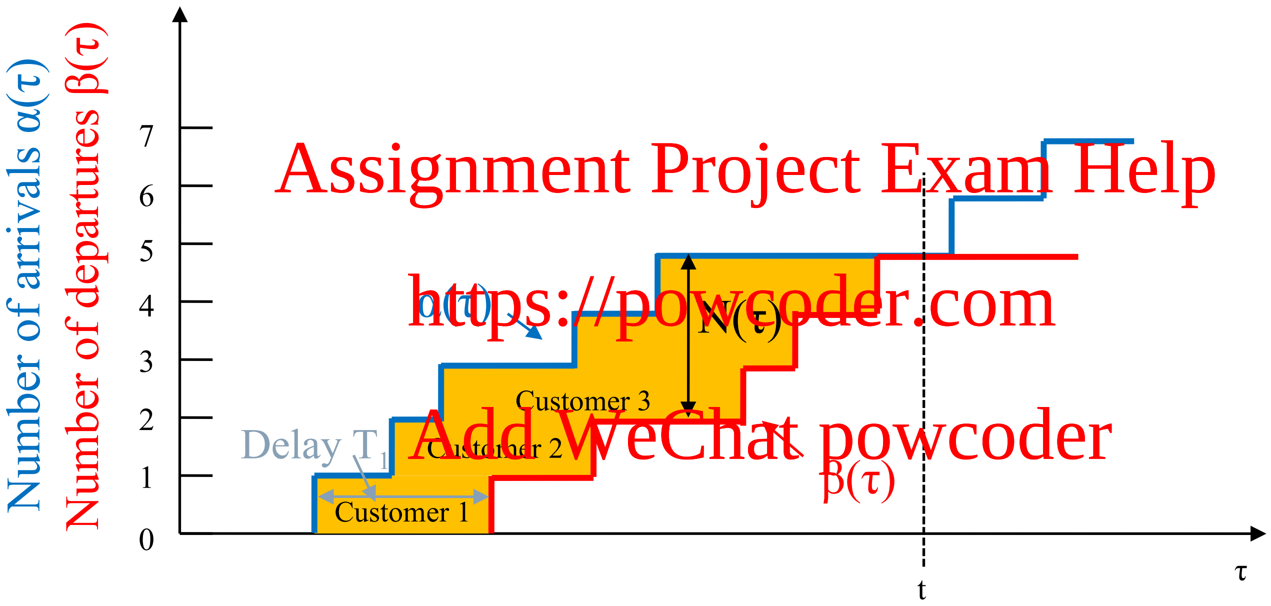
Little's Theorem



Shaded area when the queue is empty: two ways to compute

$$\underbrace{\frac{1}{t}\int_0^t N(\tau)d\tau}_{N_t} = \underbrace{\frac{\alpha(t)}{t}}_{\lambda_t}\underbrace{\frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)}}_{T_t}$$

Little's Theorem



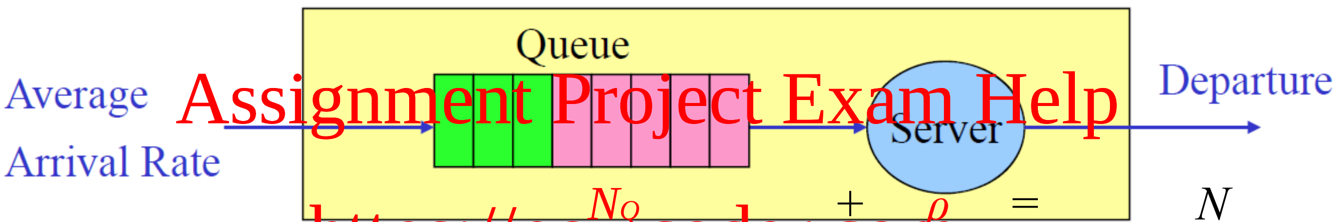
Shaded area when the queue is empty: two ways to compute

$$N_t = \lambda_t T_t$$

$$N = \lambda T$$

Note that the above Little’s Theorem is valid for any service disciplines (e.g., first-in-first-out, last-in-first-out), interarrival time distributions, and service time distributions.

Little’s Theorem



N_Q	$+$	ρ	$=$	N		*Number*
W	$+$	X	$=$	T		*Time*
λ		λ		λ		*Rate*

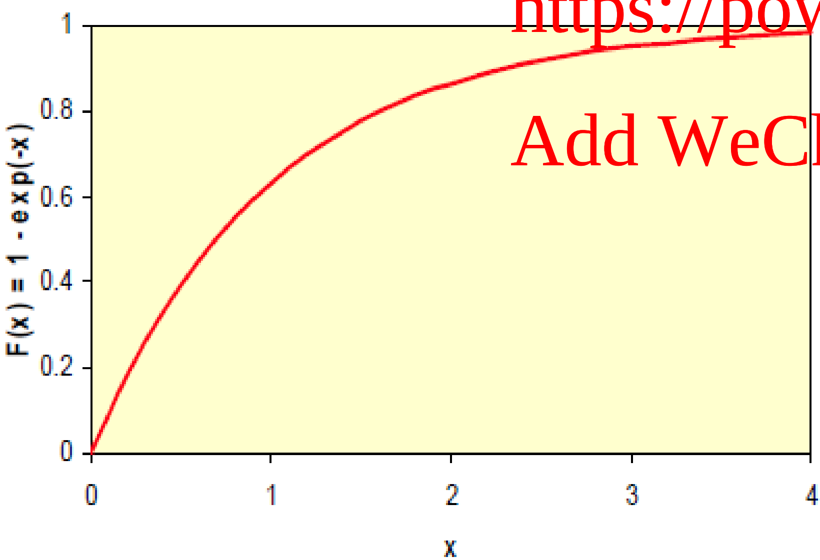
Rate × *Time* = *Number*

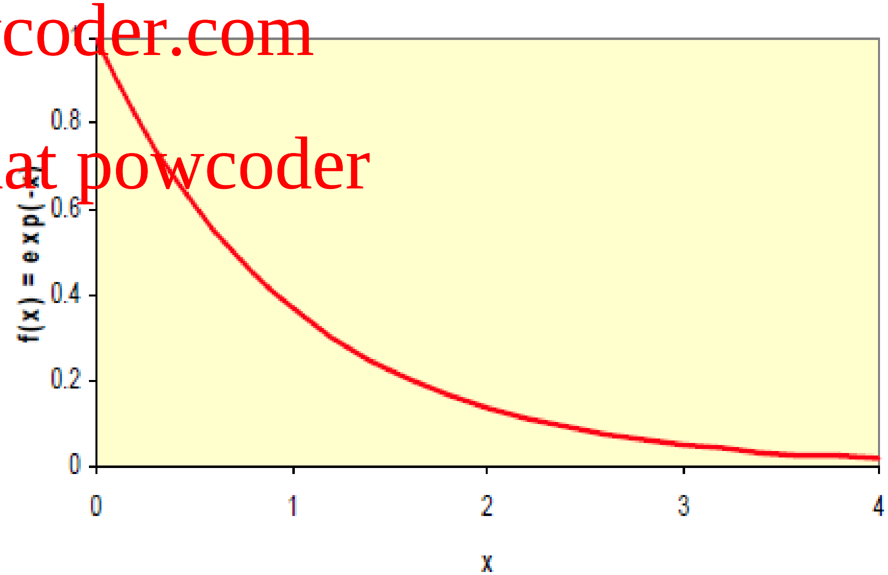
- $N = \lambda T$
- $N_Q = \lambda W$
- ρ = proportion of time that the server is busy = λX
- $T = W + X$
- $N = N_Q + \rho$

- M/M/1 Queue foundations

Exponential Distribution

- Exponential Distribution
- • The cumulative distribution function $F(x)$ and probability
- density function $f(x)$ are:
- $F(x) = 1 - e^{-\lambda x}$ $f(x) = \lambda e^{-\lambda x}$ $x \geq 0,\ \lambda > 0$





The mean is equal to its standard deviation: $E[X] = \sigma_X = 1/\lambda$

Memoryless Property

- $P(X > s + t \mid X > t) = P(X > s)$ for all $s, t \geq 0$
- The only continuous distribution with this property
- Practice Q2 in Tutorial Week 4

Other Properties of Exponential Distribution

- Let $X_1, \ldots, X_n$ be i.i.d. exponential r.v.s with mean $1/\lambda$,
- then $X_1+X_2+\ldots+X_n$ (Practice Q1 in Tutorial Week 4)

$$f_{X_1+\cdots+X_{n-1}+X_n}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

- gamma distributio
- Suppose X_1 and X_2 are ... h means
- $1/\lambda_1$ and $1/\lambda_2$, respecti

$$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

Counting Process

- A stochastic process $\{N(t), t \geq 0\}$ is a counting process if $N(t)$ represents the total # of events that have occurred up to time t.
- 1. $N(t) \geq 0$ and $N(t)$ is integer valued.
- 2. If $s < t$, then $N(s) \leq N(t)$

- 3. For $s < t$, $N(t) - N(s)$ = # of events that have occurred in (s, t)

- • Examples:

- – # of people who have entered a particular store by time t
- – # of packets sent by a mobile phone
- • A counting process is said to be independent increment if # of events which occur in disjoint time intervals are independent.
- • A counting process is said to be stationary increment if the distribution of # of events which occur in any interval of time depends only on the length of the time interval.

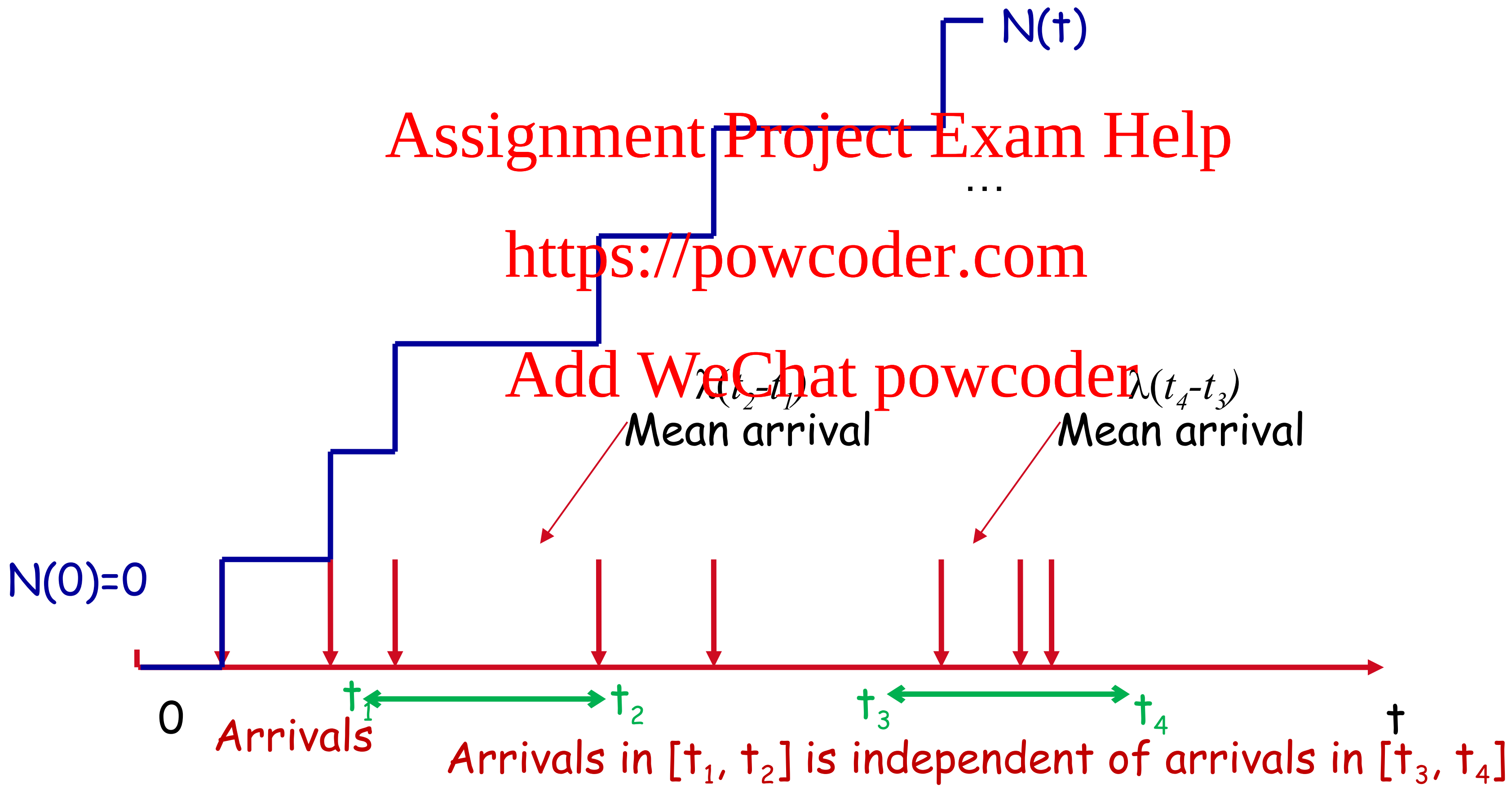
Poisson Process

- The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate $\lambda > 0$, if
- 1. $N(0) = 0$
- 2. The process has independent increments (i.e., # of events which occur in disjoint time intervals are independent)
- – for $0 < t_1 < t_2 < t_3 < t_4$, for all n , $j \geq 0$
- – $P\{ N(t_4) - N(t_3) = n \mid N(t_2) - N(t_1) = j \} = P\{ N(t_4) - N(t_3) = n \}$
- 3. Number of events in any interval of length t is Poisson distributed with mean λt. That is, for all s, $t \geq 0$

$$E(N(t+s) - N(s)) = \lambda t$$

$$P(N(t+s) - N(s) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Poisson Process

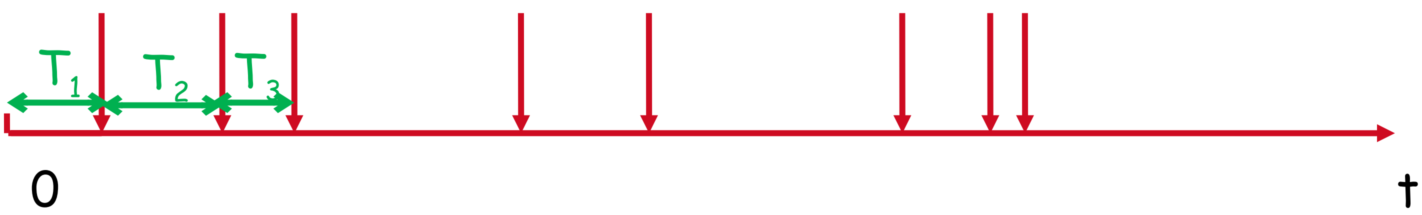


Poisson Process: Inter arrival time distribution

Exponential distribution with parameter λ
(mean $1/\lambda$)

$$P(T_2 > t) = P\{T_2 > t \mid T_1 = t_1\} = e^{-\lambda t}$$

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$



Number of arrivals in a short period of time

Number of arrival events in a very short period

$P\{N(t+h) - N(t) = 1\} = \lambda h + o(h)$; and

$P\{N(t+h) - N(t) \geq 2\} = o(h)$.

$o()$. Small o notation. The function $f(.)$ is said to be $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

Poisson process:

Independent increments

# of arrivals: Poisson distributed

# of arrivals in a small period of time h. 1 arrival, probability λh

Inter-arrival time distribution: exponential distribution

- M/M/1 Queue

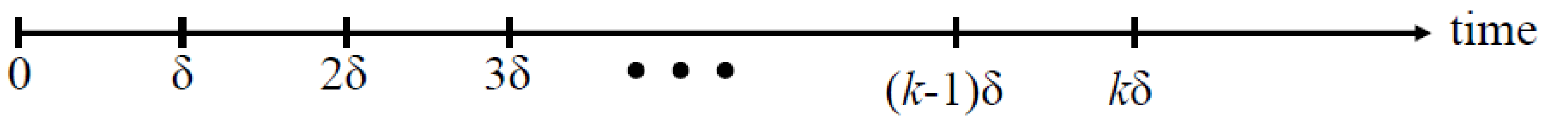
Queues: Kendall's notation

- Notations Used in Queueing Systems
- *X*/*Y*/*Z*
- – *X* refers to the distribution of the interarrival times
- – *Y* refers to the distribution of service times
- – *Z* refers to the number of servers
- Common distributions:
- – *M* = Memoryless = exponential distribution
- – *D* = Deterministic arrivals or fixed-length service
- – *G* = General distribution of interarrival times or service times
- *M*/*M*/*1* refers to a single-server queuing model with exponential interarrival times (i.e., Poisson arrivals) and exponential service times.
- In all cases, successive interarrival times and service times are assumed to be statistically independent of each other.

M/M/1 Queue

- Arrival:
- Poisson arrival with rate λ
- Service:
- Service time: exponential distribution with mean $1/\mu$
- μ: service rate,
- $\lambda < \mu$: Incoming rate < outgoing rate

Markov Chain Formulation



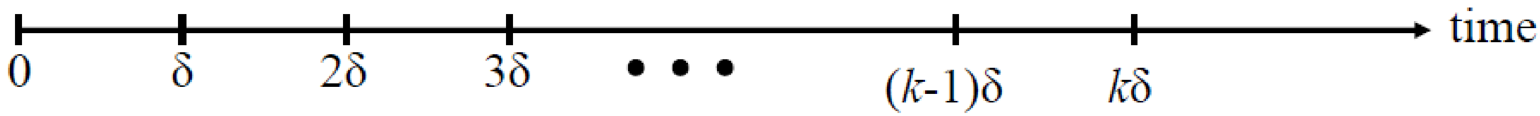
δ: a small value

N_k = Number of customers in the system at time $k\delta$

N_0 N_1 N_2... is a Markov Chain!

Q: How to compute the transition probability?

Markov Chain Formulation

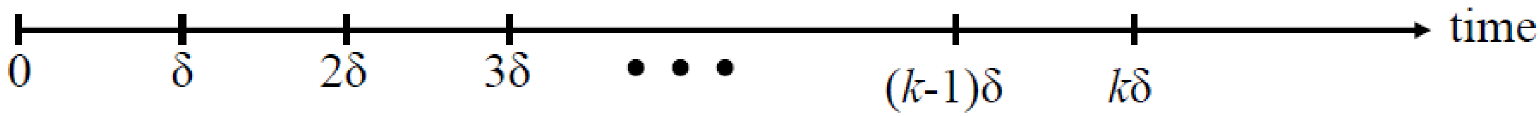


$$P(0 \quad \text{customer} \quad \text{arrives}) = 1 - \lambda\delta + o(\delta)$$

$$P(1 \quad \text{customer} \quad \text{arrives}) = \lambda\delta + o(\delta)$$

$$P(\geq 2 \quad \text{customer} \quad \text{arrives}) = o(\delta)$$

Markov Chain Formulation



$$P(0 \quad \text{customer} \quad \text{leaves}) = 1 - \mu\delta + o(\delta)$$

$$P(1 \quad \text{customer} \quad \text{leaves}) = \begin{cases} \mu\delta + o(\delta) & i \quad 1 \\ 0 & i = 0 \end{cases}$$

$$P(\quad 2 \quad \text{customer} \quad \text{leaves}) = o(\delta)$$

No one in the system

Markov Chain Formulation

$$Aim\ to\ compute\ P_{ij} = P\{N_{k+1} = j \mid N_k = i\}$$

$$For\ example,\ P\{N_{k+1} = i \mid N_k = i\},\ i>0$$

$$\begin{aligned} & \boldsymbol{P}(\mathbf{0\ \ customer\ \ arrives})\boldsymbol{P}(\mathbf{0\ \ customer\ \ departs}) \\ & + \boldsymbol{P}(\mathbf{1\ \ customer\ \ arrives})\boldsymbol{P}(\mathbf{1\ \ customer\ \ departs}) \\ & + \boldsymbol{P}(\mathbf{other}) \end{aligned}$$

$$\text{Result}: 1 - \lambda\delta - \mu\delta + o(\delta)$$

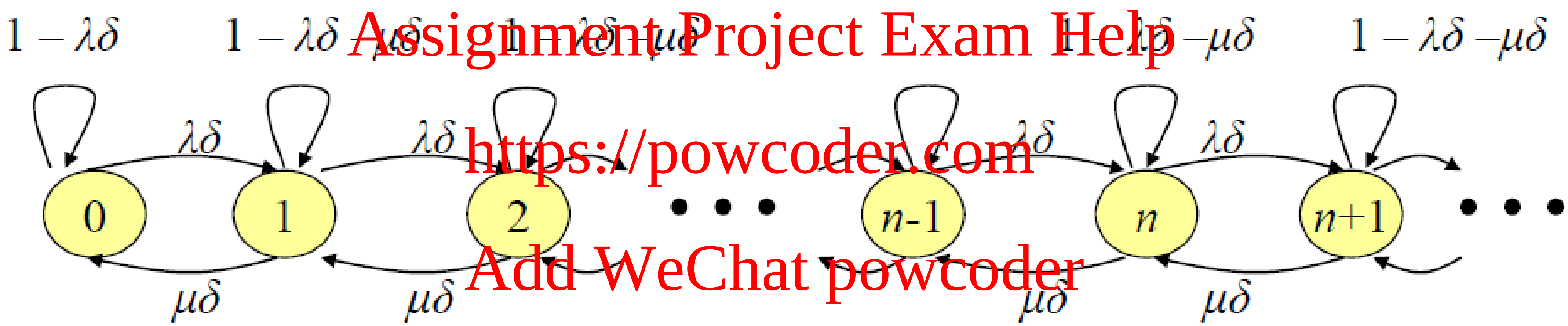
$$[1 - \lambda\delta + o(\delta)][1 - \mu\delta + o(\delta)] = 1 - \lambda\delta - \mu\delta + o(\delta)$$

$$[\lambda\delta + o(\delta)][\mu\delta + o(\delta)] = o(\delta)$$

$$o(\delta)o(\delta) = o(\delta)$$

Markov Chain Formulation

Result:

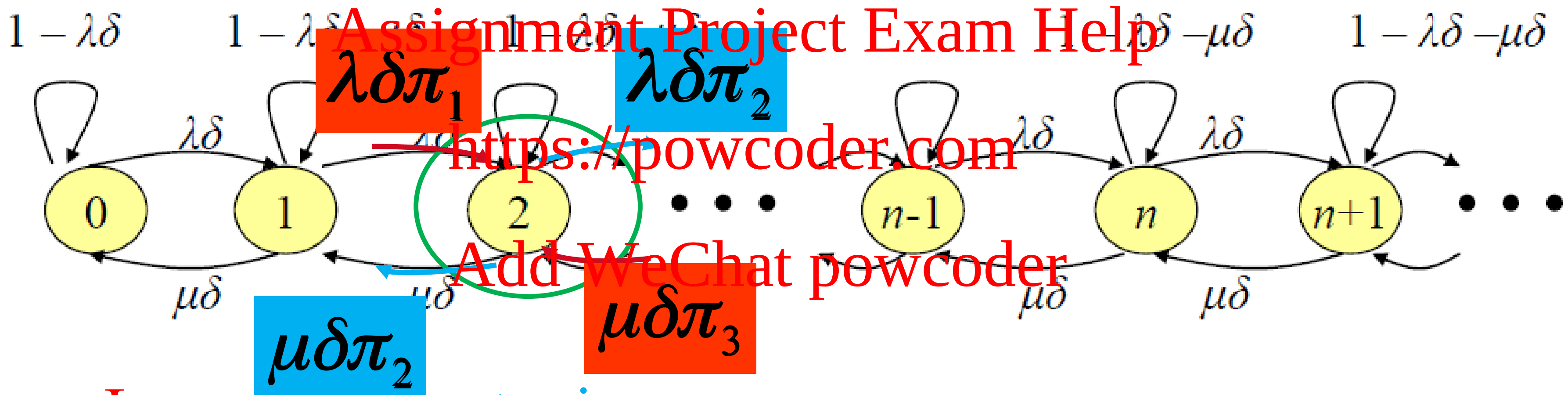


$\boldsymbol{\pi}_{\mathrm{i}}$ Stationary distribution of state *i*

The probability that there are *i* units in the system

Stationary Distribution Derivation

How to derive $\boldsymbol{\pi_i}$ *balance equation satisfied*

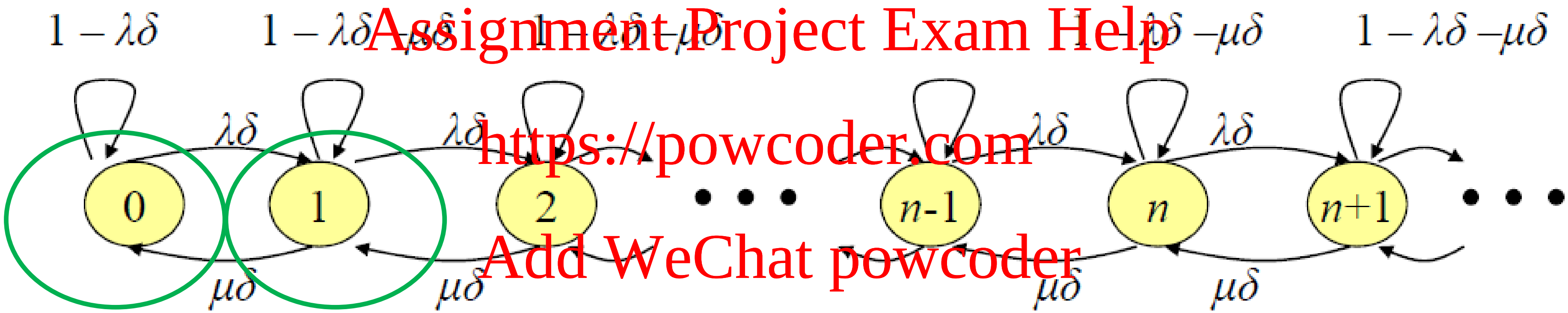


Incoming = outgoing

$$\lambda\delta\pi_2 + \mu\delta\pi_2 = \lambda\delta\pi_1 + \mu\delta\pi_3$$

Stationary Distribution Derivation

How to derive π_i



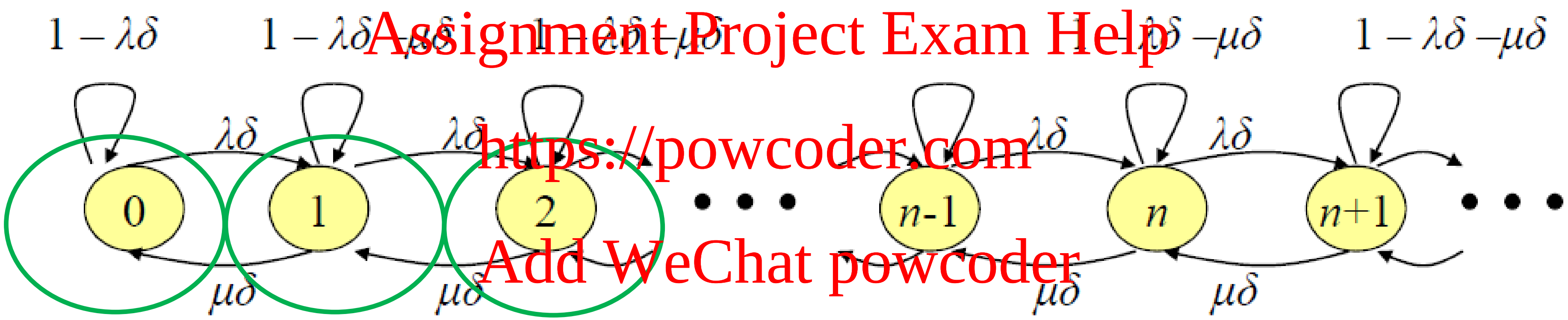
balance equation is performed at each state

$$\lambda\delta\pi_0 = \mu\delta\pi_1$$

$$\lambda\delta\pi_1 + \mu\delta\pi_1 = \lambda\delta\pi_0 + \mu\delta\pi_2 \Rightarrow \lambda\delta\pi_1 = \mu\delta\pi_2$$

Stationary Distribution Derivation

How to derive π_i



balance equation is performed at each state

$$\lambda\delta\pi_0 = \mu\delta\pi_1$$

$$\lambda\delta\pi_1 = \mu\delta\pi_2$$

$$\lambda\delta\pi_2 + \mu\delta\pi_2 = \lambda\delta\pi_1 + \mu\delta\pi_3 \Rightarrow \lambda\delta\pi_2 = \mu\delta\pi_3$$

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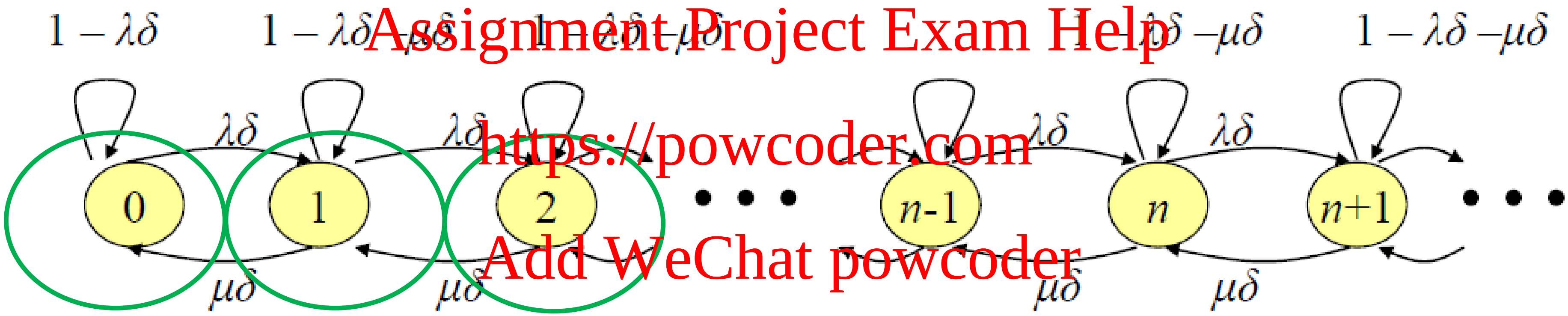

balance equation is performed at each state

$$\lambda\delta\pi_0 = \mu\delta\pi_1$$

$$\lambda\delta\pi_1 = \mu\delta\pi_2$$

$$\lambda\delta\pi_2 = \mu\delta\pi_3$$

$$\Rightarrow \quad \lambda\delta\pi_i = \mu\delta\pi_{i+1}$$

For any i

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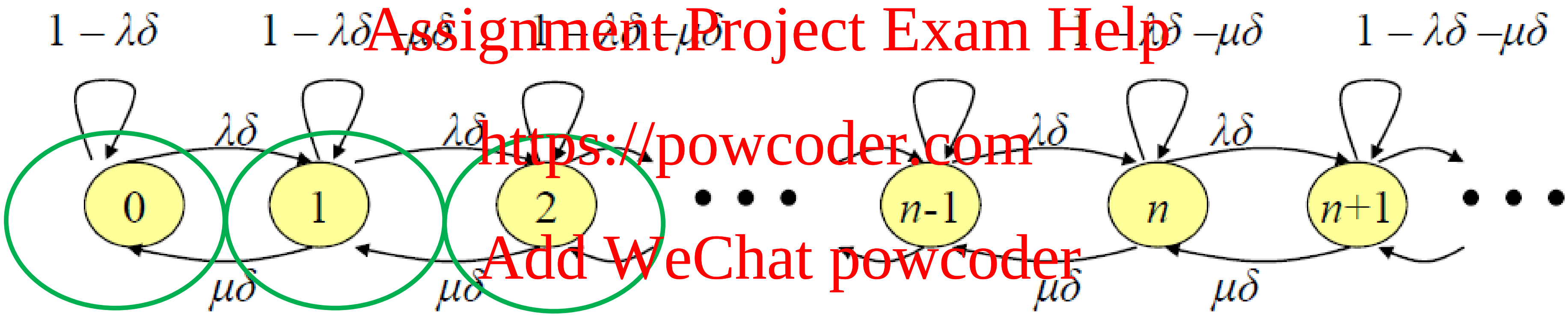

balance equation is performed at each state

$$\pi_1 = \frac{\lambda}{\mu}\pi_0 \qquad \pi_2 = \left(\frac{\lambda}{\mu}\right)^2 \pi_0 \qquad \cdots \qquad \pi_i = \left(\frac{\lambda}{\mu}\right)^i \pi_0$$

$$\sum_{i=0}^{\infty} \pi_i = 1$$

Sum of geometric sequence

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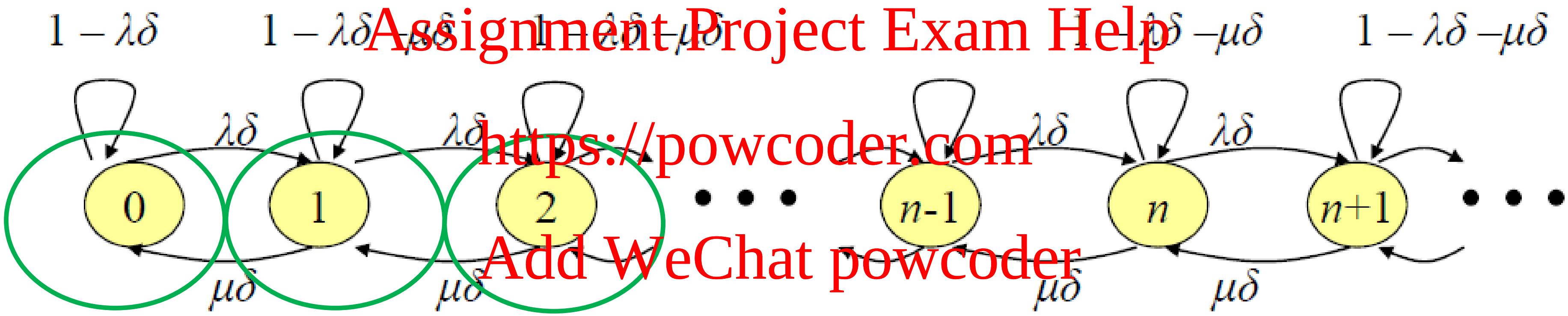
How to derive π_i



balance equation is performed at each state

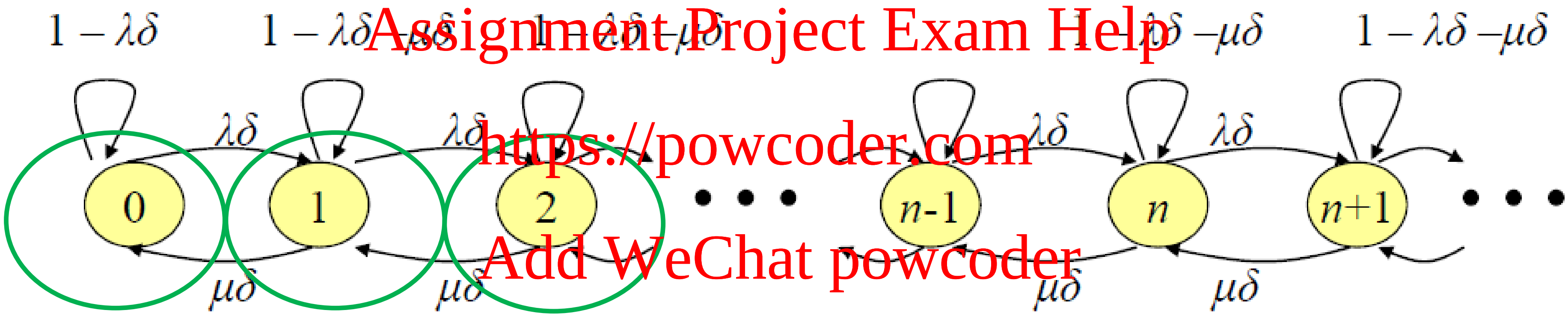
$$\pi_1 = \rho\pi_0 \qquad \pi_2 = (\rho)^2\pi_0 \qquad \cdots \qquad \pi_i = (\rho)^i\pi_0 \qquad \rho = \frac{\lambda}{\mu} < 1$$

$$\sum_{i=0}^{\infty}\pi_i = 1$$

Sum of geometric sequence

Stationary Distribution Derivation

How to derive π_i



balance equation is performed at each state

$$\lim_{N\to\infty} \frac{\pi_0(1-\rho^N)}{1-\rho} = \frac{\pi_0}{1-\rho} = 1$$

$$\pi_0 = 1 - \rho$$

$$\pi_i = (1-\rho)\rho^i$$

Sum of geometric sequence

Average number of user

Average number of users in the system

$$E(N) = \sum_{n=0}^{\infty} n(1-\rho)\rho^{n}$$

$$= \rho(1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1}$$

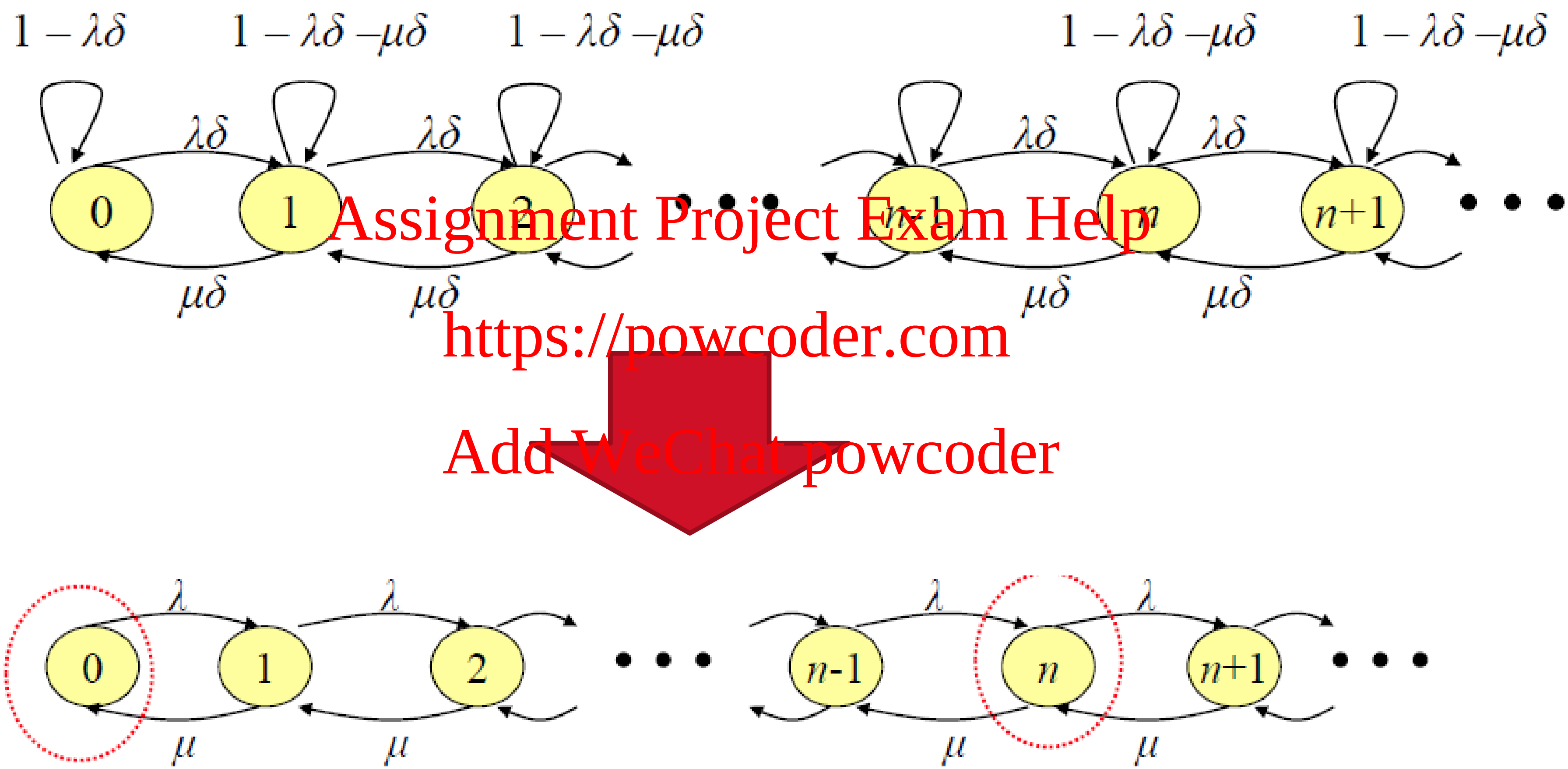
$$= \rho(1-\rho)\frac{\partial\left[\sum_{n=0}^{\infty}\rho^{n}\right]}{\partial\rho}$$

$$= \rho(1-\rho)\frac{\partial\left[\frac{\rho}{1-\rho}\right]}{\partial\rho} = \frac{\rho}{1-\rho}$$

Average waiting time Little's Theorem

$$E(T) = \frac{E(N)}{\lambda} = \frac{1}{\mu - \lambda}$$

Stationary Distribution Derivation



Stationary Distribution Derivation

balance equation is performed at each state

$$\lambda\pi_0 = \mu\pi_1$$

$$\lambda\pi_1 + \mu\pi_1 = \lambda\pi_0 + \mu\pi_2$$

$$\cdots$$

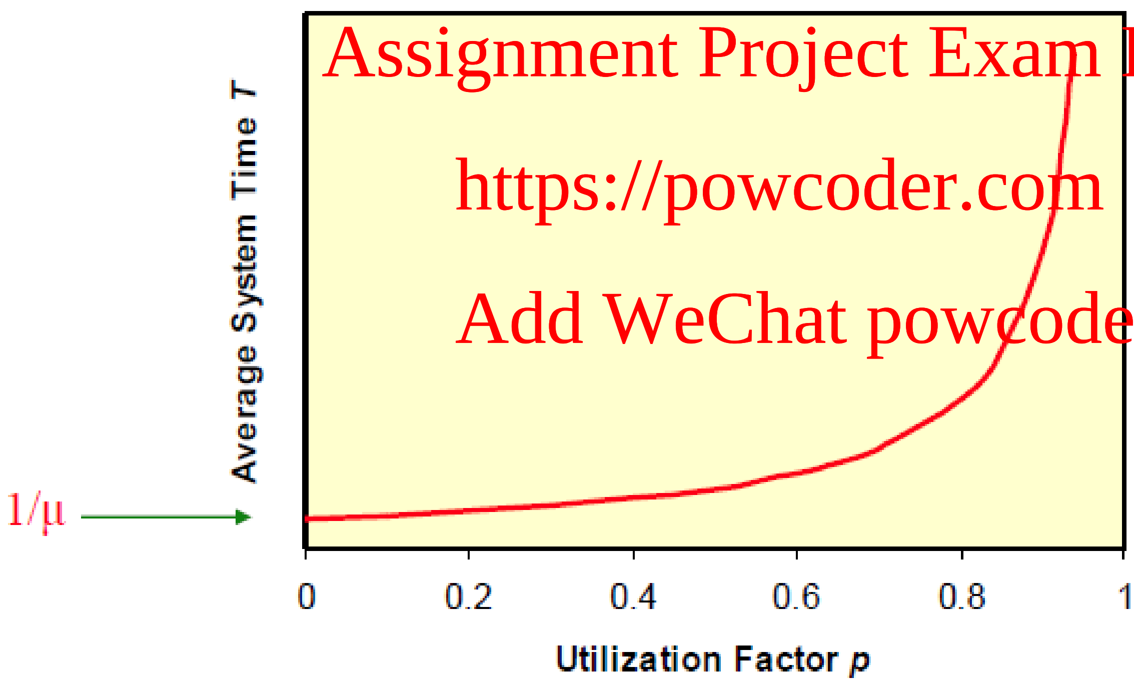
Stationary Distribution Derivation

balance equation is performed at each state

Following the same step, derive the same result

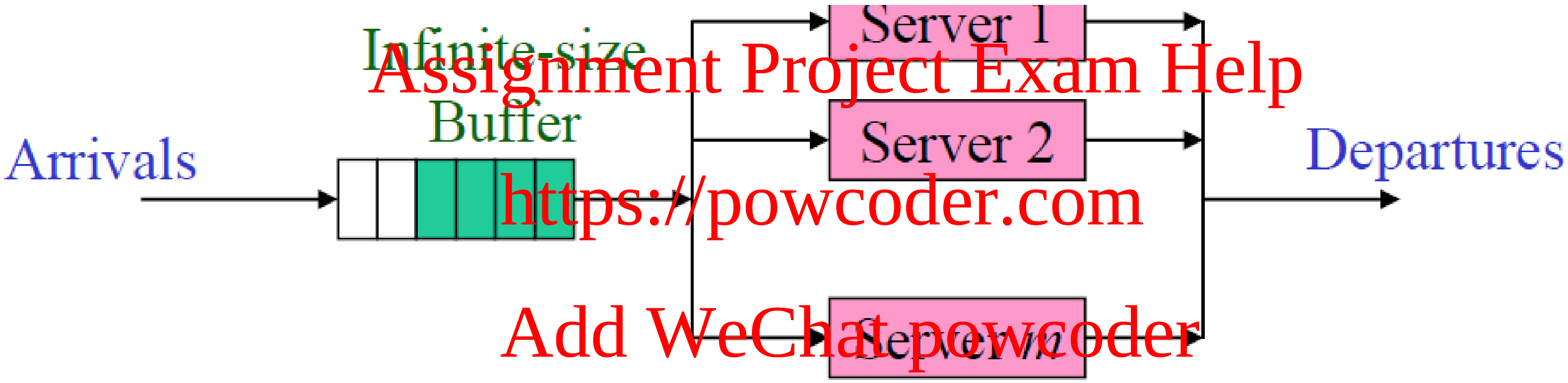
Stationary Distribution Derivation

Queueing delay goes to infinity when arrival rate approaches service rate!



when ρ approaches 1, T goes to infinity.

M/M/m Queue


Infinite-size
Buffer
Arrivals
Server 1
Server 2
Server m
Departures

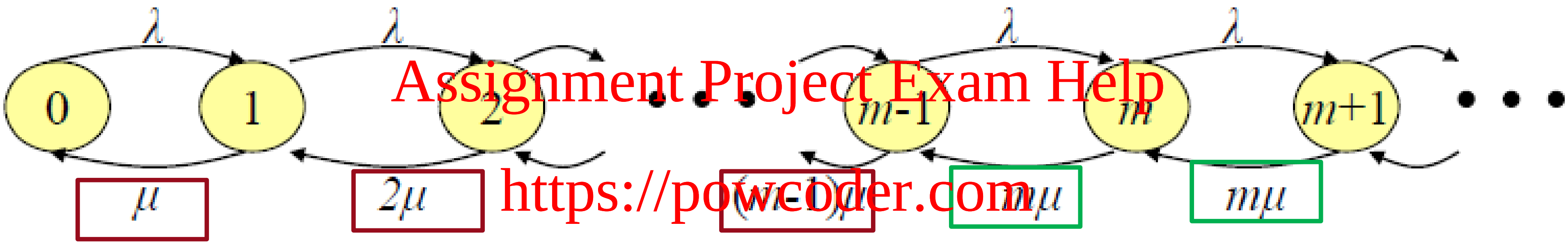
M/M/m Queue

- Arrival:
- Poisson arrival with rate λ
- Service:

- Service time for one server: exponential distribution with mean $1/\mu$

- service rate is i μ, if there are i<m users in the system
- service rate is mμ, if there are i>=m users in the system

M/M/m Queue



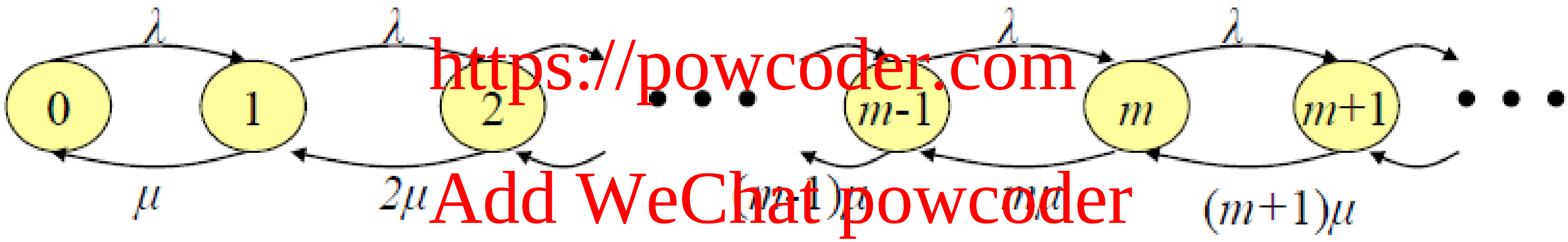
$$\lambda \pi_{i-1} = i\mu\pi_i \qquad i \le m$$

$$\lambda \pi_{i-1} = m\mu\pi_i \qquad i > m$$

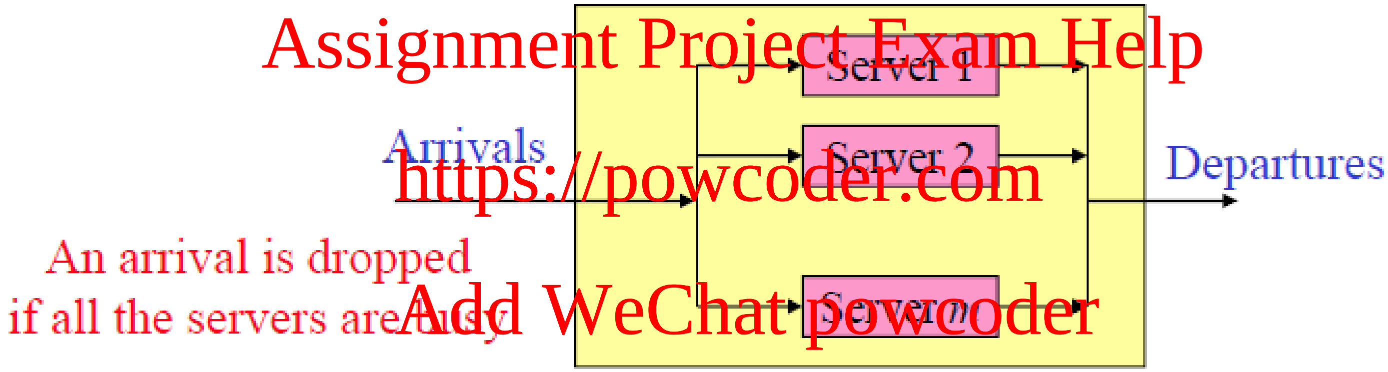
$$\pi_n = \begin{cases} \pi_0 \dfrac{(m\rho)^n}{n!} & n \le m \\ \pi_0 \dfrac{m^m \rho^n}{m!} & n > m \end{cases} \qquad \rho = \frac{\lambda}{m\mu} < 1$$

Then, π_0 can be solved

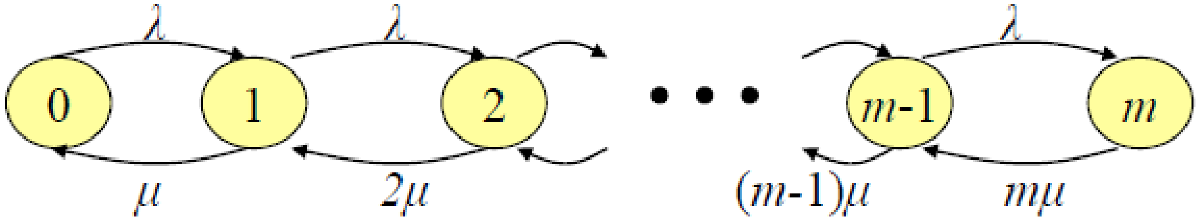
M/M/∞ Queue



M/M/m/m Queue



Server 1
Server 2
Server m
Arrivals
Departures
An arrival is dropped
if all the servers are busy

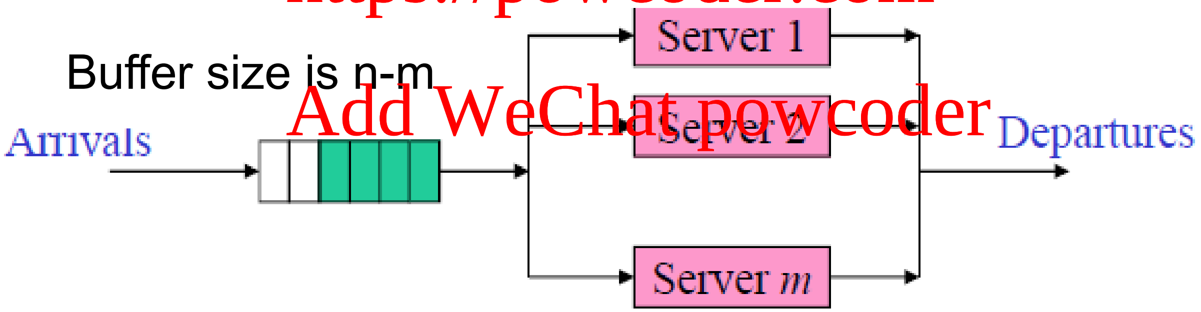


λ
λ
λ
0
1
2
m-1
m
μ
2μ
(m-1)μ
mμ

M/M/m/n Queue

Arrivals will dropped if there are n users in the system.

Buffer size is n-m



How do you derive its stationary distribution?

Practice

- Analyze M/M/ ∞, M/M/m/n queues
  - Draw the state transition diagrams
  - Derive their stationary distributions
  - For M/M/m/n queue, calculate the probability that an incoming user is dropped. Calculate the probability that the queue is empty (i.e., all users are served in the servers or there are no users at all.)