ECE5884 Wireless Communications

Week 4: Capacity of Wireless Channels

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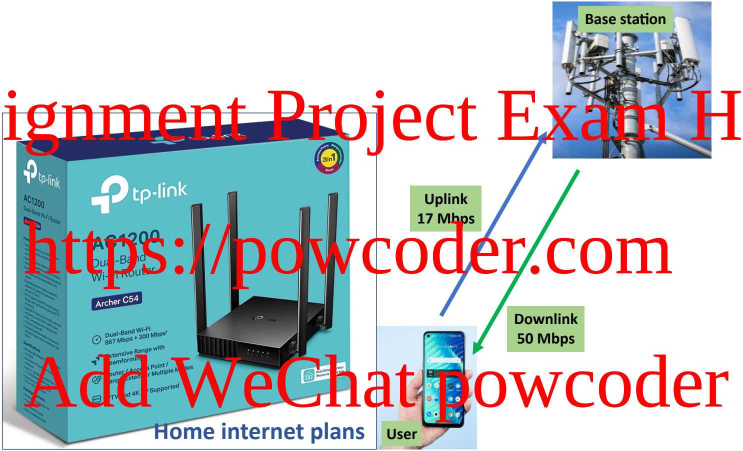
Course outline

This week: Ref. Ch. 4 of [Goldsmith, 2005]

- Week 1: Overview of Wireless Communications
- Week 2: Wireless Channel (Path Loss and Shadowing)
- Week 3: Wireless Channel Models
- Week 4: Capacity of Wireless Channels
- Week 5: Digital Modulation and Detection
- Week 6: Performance Analysis
- Week 7: Equalization
- Week 8: Multicarrier Modulation (OFDM)
- Week 9: Diversity Techniques
- Week 10: Multiple-Antenna Systems (MIMO Communications)
- Week 11: Multiuser Systems
- Week 12: Guest Lecture (Emerging 5G/6G Technologies)

Data rate



typical evening speed 7pm-11pm:	typical evening speed 7pm-11pm:	typical evening speed 7pm-11pm:	typical evening speed 7pm-11pm:
25 Mbps Download / 4 Mbps Upload	50 Mbps Download / 17 Mbps Upload	100 Mbps Download / 17 Mbps Upload	250 Mbps Download / 22 Mbps Upload

Capacity in AWGN

1. For a discrete-time AWGN channel, $h = 1$, input/out relationship at time i:

$$y[i] = x[i] + n[i] \tag{1}$$

where $x[i]$ is the channel input, y[i] is the corresponding channel output, and n[i] is a white Gaussian noise random process.

2. Shannon Capacity:

$$C = B\log_2(1+\gamma)\ bits/sec \quad \text{where} \quad SNR = \gamma = \frac{P_r}{N_0 B} \tag{2}$$

3. Shannon proved that capacity is the maximum error-free data rate a channel can support.

4. So this theoretical limit may not be achievable.
5. Only dependent on channel characteristic, but not dependent on design techniques.
6. C [bits/sec]; P_r received signal power [W]; N_0 Noise power spectral density [W/Hz]; B channel bandwidth [Hz].
7. Any code with rate $R > C$ has a probability of error bounded away from zero.

Capacity at asymptotic regimes

1. Large bandwidth regime:

$$\lim_{B\to\infty} B\log_2\left(1+\frac{P_r}{BN_0}\right) = \log_2(e)\frac{P_r}{N_0}; \text{ as } \lim_{x\to\infty} x\log_2\left(1+\frac{a}{x}\right) = a\log_2(e) \quad (3)$$

   - There is not sufficient power to spread over the large amount of bandwidth available.
   - Capacity no longer depends on the channel bandwidth.

2. Low power regime:

$$B\log_2\left(1+\frac{P_r}{BN_0}\right) \approx \log_2(e)\frac{P_r}{N_0}; \text{ as } \log_e(1+x) \approx x \text{ for small } x \quad (4)$$

   - Capacity no longer depends on the channel bandwidth.

3. How capacity scales with bandwidth (B to kB for $k > 1$) at high power:

$$\lim_{P_r\to\infty} \frac{kB\log_2\left(1+\frac{P_r}{kBN_0}\right)}{B\log_2\left(1+\frac{P_r}{BN_0}\right)} \approx \lim_{P_r\to\infty} \frac{kB\log_2\left(\frac{P_r}{kBN_0}\right)}{B\log_2\left(\frac{P_r}{BN_0}\right)} = k \quad (5)$$

4. C is also defined as channel's maximum mutual information (Not discussing here!) - Information theoretic perspective ...

Capacity of flat fading channels

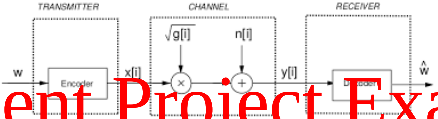


Figure 1: Flat fading channel and system model.

1. Time varying channel gain: $|h| = \sqrt{g}$ and $|h|^2 = g$.
2. The channel gain $g[i]$, called as channel state information or channel side information (CSI), can change at each time i, e.g., an i.i.d. process.
3. Block fading channel: $g[i]$ is constant over some blocklength T, after which time $g[i]$ changes to a new independent value based on the distribution $f_g(t)$.
4. The instantaneous received SNR:

$$\gamma[i] = \frac{\bar{P}g[i]}{N_0 B}; \text{ Expected value: } \bar{\gamma} = \frac{\bar{P}\bar{g}}{N_0 B}; \quad \bar{P} \text{ is the average Tx power} \tag{6}$$

5. The CSI $g[i]$ changes during the transmission of the codeword.

CSI Knowledge

The capacity depends on what is known about $g[i]$ at the transmitter (Tx) and receiver (Rx) [Goldsmith, 1997, Goldsmith, 1999, Caire, 1999].

1. Channel distribution information (CDI): The distribution of $g[i]$ is known to the Tx and Rx.
2. Rx CSI: The value of $g[i]$ is known to the receiver at time i, and both the TX and Rx know the distribution of $g[i]$.
3. Tx and Rx CSI: The value of $g[i]$ is known to the Tx and Rx at time i, and both the Tx and Rx know the distribution of $g[i]$.

Channel estimation for CSI

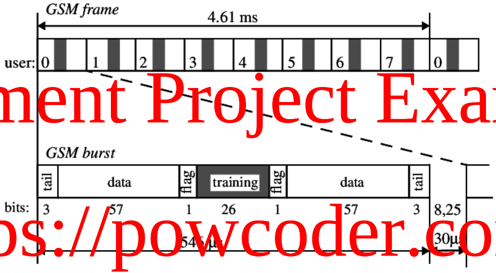


Figure 2: GSM (2G) frame structure.

With Least Squares (LS) estimator:

$$\text{For a single bit training: } y = hs + n \Rightarrow \hat{h} = \frac{y}{s} \tag{7}$$

$$\text{For } m \text{ bits training: } \mathbf{y} = h\mathbf{s} + \mathbf{n} \Rightarrow \hat{h} = \left(\mathbf{s}^{\mathrm{T}}\mathbf{s}\right)\mathbf{s}^{\mathrm{T}}\mathbf{y} \tag{8}$$

$$\text{where} \quad \mathbf{y} = [y_1, \cdots, y_m]^{\mathrm{T}}; \mathbf{s} = [s_1, \cdots, s_m]^{\mathrm{T}}; \mathbf{n} = [n_1, \cdots, n_m]^{\mathrm{T}}$$

Channel estimation error/Imperfect CSI

Due to pilot contamination from neighboring cells, channel frequency offset, Doppler effect, time synchronization mismatch, etc.

$$h = \hat{h} + \epsilon \text{ where } \epsilon \text{ is the channel estimation error} \tag{9}$$

If $h \sim \mathcal{CN}(0, \sigma_h^2)$ and $\epsilon \sim \mathcal{CN}(0, \sigma_\epsilon^2)$, then (Gaussian still Gaussian!!!)

$$\hat{h} \sim \mathcal{CN}(0, \sigma_h^2 - \sigma_\epsilon^2) \tag{10}$$

SNR with channel estimation error:

$$y = hs + n \tag{11}$$

$$\hat{h}^* y = \hat{h}^*(\hat{h} + \epsilon)s + \hat{h}^* n \tag{12}$$

$$= \underbrace{|\hat{h}|^2 s}_{\text{signal}} + \underbrace{(\hat{h}^* s)\epsilon}_{\text{channel estimation error}} + \underbrace{\hat{h}^* n}_{\text{noise}} \tag{13}$$

$$SNR\ \gamma = \frac{\text{Received signal power}}{\text{Est. error power+ Noise power}} = \frac{|\hat{h}|^2 \bar{P}}{\sigma_\epsilon^2 \bar{P} + N_0} \tag{14}$$

CSI at receiver

- Shannon (ergodic) capacity: Shannon capacity for an AWGN channel with SNR γ, and then averaged over the distribution of γ.

$$
\begin{aligned}
C &= \int_0^\infty B\log_2(1+\gamma) f_\gamma(\gamma)\, d\gamma && (15)\\
&= \mathbf{E}_\gamma\left[B\log_2(1+\gamma)\right] && (16)\\
&\leq B\log_2\left(1+\mathbf{E}_\gamma[\gamma]\right); \quad \text{using Jensen's inequality} && (17)\\
&= B\log_2(1+\bar{\gamma}) \quad \bar{\gamma} \text{ is the average SNR on the channel} && (18)
\end{aligned}
$$

- $C_{AWGN} = B\log_2(1+\bar{\gamma})$ is equivalent to the capacity of AWGN channel with the average SNR $\bar{\gamma}$.
- Ergodic capacity over Rayleigh fading channel: From Eq. (15),

$$
C_{Rayleigh} = \int_0^\infty B\log_2(1+\gamma)\frac{1}{\bar{\gamma}}e^{-\frac{\gamma}{\bar{\gamma}}}\, d\gamma = -\frac{B}{\log(2)}e^{\frac{1}{\bar{\gamma}}}\,\mathrm{Ei}\left(-\frac{1}{\bar{\gamma}}\right) \qquad (19)
$$

where $\mathrm{Ei}(x)$ is the Exponential Integral, $\mathrm{Ei}(x) = -\int_{-x}^\infty \frac{e^{-t}}{t}\, dt$.

Numerical example

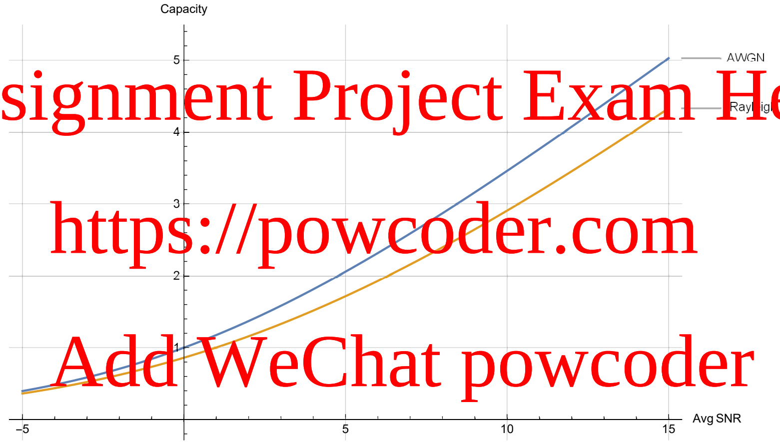


Figure 3: Capacity (bits/sec/Hz) vs average SNR (dB) over AWGN and Rayleigh channels for $B = 1$ Hz.

Capacity outage probability

- The capacity outage probability is the probability that the (instantaneous) capacity C falls below a certain predetermined capacity threshold C_{th}

$$C_{out} = \Pr\left[C < C_{th}\right] \tag{20}$$

$$= \Pr\left[B\log_2(1+\gamma) < C_{th}\right] = \Pr\left[\gamma < 2^{\frac{C_{th}}{B}} - 1\right] = F_\gamma\left(2^{\frac{C_{th}}{B}} - 1\right) \tag{21}$$

- For Rayleigh fading:

$$F_\gamma(t) = 1 - e^{-\frac{t}{\bar{\gamma}}} \tag{22}$$

$$C_{out} = 1 - e^{-\frac{2^{\frac{C_{th}}{B}}-1}{\bar{\gamma}}} \tag{23}$$

- Similarly, you can evaluate the capacity outage probabilities for Rician and Nakagami-m fading channels!

Numerical example

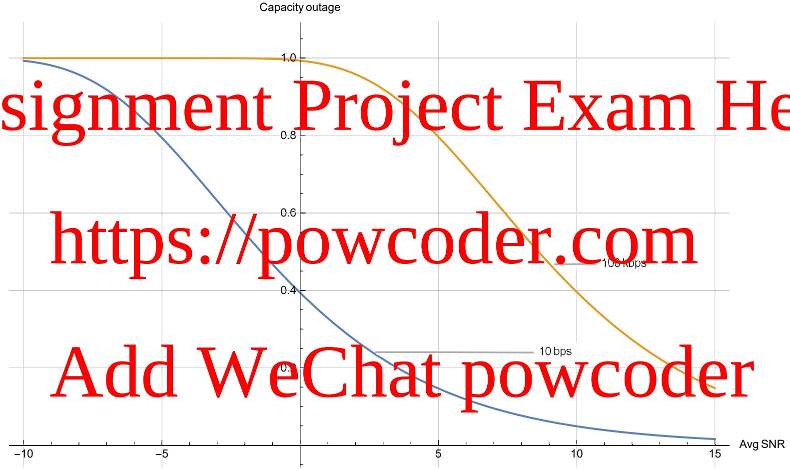


Figure 4: Capacity outage vs average SNR over Rayleigh channel for $C_{th} = 10$ bits/sec and $C_{th} = 100$ kbits/sec at $B = 30$ kHz.

CSI at transmitter and receiver

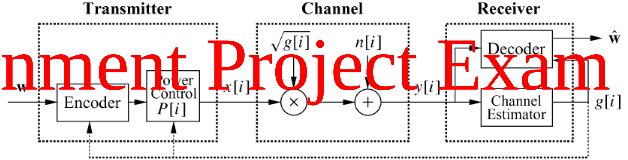


Figure 5: System model with transmitter and receiver CSI [Goldsmith, 2005].

- How we can improve following C (which is valid when CSI at Rx only):

$$C = \int_0^\infty B \log_2(1+\gamma) f_\gamma(\gamma)\, d\gamma \tag{24}$$

- If both the transmitter and receiver have CSI then the transmitter can adapt its transmission strategy (e.g., Tx power and code) relative to this CSI.

CSI at transmitter and receiver

- Let us allow the transmit power $P(\gamma)$ to vary with instantaneous SNR γ subject to an average power constraint $\bar{P}$.

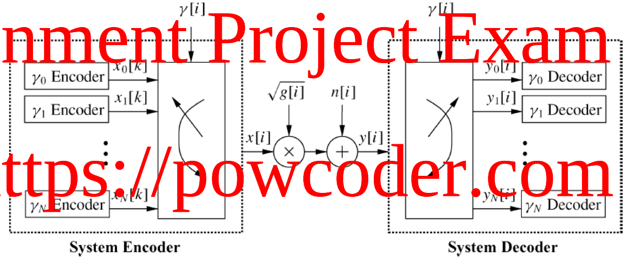


Figure 6: Multiplexed coding and decoding [Goldsmith, 2005].

1. The range of fading values is quantized to a finite set $\{\gamma_j, 1 \le j \le N\}$.
2. For each γ_j, an encoder-decoder pair for an AWGN channel with SNR γ_j.
3. The input x_j for encoder has average power $P(\gamma_j)$ and data rate C_j where C_j is the capacity of time-invariant AWGN channel with received SNR $P(\gamma_j)\gamma_j/\bar{P}$.

CSI at Tx and RX: Power allocation

1. an average power constraint $\bar{P}$:

$$\int_0^\infty P(\gamma) f_\gamma(\gamma)\, d\gamma \le \bar{P} \tag{25}$$

2. Power is a positive value when Tx happens:

$$P(\gamma) > 0 \tag{26}$$

3. Objective:

$$C = \max_{P(\gamma):\int_0^\infty P(\gamma) f_\gamma(\gamma)\, d\gamma = \bar{P}} \int_0^\infty B \log_2\left(1 + \frac{P(\gamma)\gamma}{\bar{P}}\right) f_\gamma(\gamma)\, d\gamma \tag{27}$$

Optimization problem: power allocation

$$\begin{aligned} \max_{P(\gamma)} \quad & \int_0^\infty B \log_2\left(1 + \frac{P(\gamma)\gamma}{\bar{P}}\right) f_\gamma(\gamma)\, d\gamma \\ \text{subject to} \quad & \int_0^\infty P(\gamma) f_\gamma(\gamma)\, d\gamma \le \bar{P} \\ & P(\gamma) > 0 \end{aligned} \tag{28}$$

Optimal power allocation

1. Lagrangian

$$J\left(P(\gamma),\lambda\right) = \int_0^\infty B\log_2\left(1+\frac{P(\gamma)\gamma}{\bar{P}}\right) f_\gamma(\gamma)\,d\gamma - \lambda\left(\int_0^\infty P(\gamma)f_\gamma(\gamma)\,d\gamma - \bar{P}\right)$$

2. Differentiate the Lagrangian and set the derivative equal to zero:

$$\frac{\partial J\left(P(\gamma),\lambda\right)}{\partial P(\gamma)} = \int_0^\infty \frac{B}{\ln(2)}\frac{\frac{\gamma}{\bar{P}}}{\left(1+\frac{P(\gamma)\gamma}{\bar{P}}\right)} f_\gamma(\gamma)\,d\gamma - \lambda\int_0^\infty f_\gamma(\gamma)\,d\gamma = 0 \quad (29)$$

$$\frac{B}{\ln(2)}\frac{\frac{\gamma}{\bar{P}}}{\left(1+\frac{P(\gamma)\gamma}{\bar{P}}\right)} - \lambda = 0 \Rightarrow \left(1+\frac{P(\gamma)\gamma}{\bar{P}}\right) = \frac{B}{\ln(2)}\frac{\gamma}{\lambda\bar{P}} \quad (30)$$

$$\frac{P(\gamma)}{\bar{P}} = \frac{1}{\gamma_0} - \frac{1}{\gamma} \quad \text{where} \quad \gamma_0 = \frac{B}{\ln(2)\lambda\bar{P}} \quad (31)$$

3. Solution:

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma}; & \gamma \ge \gamma_0 \\ 0; & \gamma < \gamma_0 \end{cases} \quad (32)$$

where γ_0 is a cutoff/threshold value.

Optimal power allocation: water-filling

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma}; & \gamma \geq \gamma_0 \\ 0; & \gamma < \gamma_0 \end{cases} \quad \text{then} \quad C = \int_{\gamma_0}^{\infty} B \log_2\left(\frac{\gamma}{\gamma_0}\right) f_\gamma(\gamma)\, d\gamma \tag{33}$$

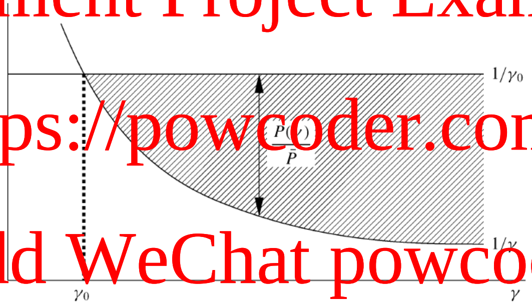


Figure 7: Water-filling technique [Goldsmith, 2005]. The better channel, the more power and the higher data rate!

The line $1/\gamma$ sketches out the bottom of a bowl, and power is poured into the bowl to a constant water level of $1/\gamma_0$.

Calculate γ_0!

- γ_0 is found from the power constraint.
- Since using the maximum available power will always be optimal, we have eq. (25) as

$$\int_0^\infty \frac{P(\gamma)}{\bar{P}} f_\gamma(\gamma)\, d\gamma = 1 \tag{34}$$

$$\int_{\gamma_0}^\infty \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) f_\gamma(\gamma)\, d\gamma = 1 \tag{35}$$

- The value for γ_0 must be found numerical integration and iteration, because no closed-form solutions exist for typical continuous distributions $f_\gamma(\gamma)$ [Goldsmith, 1999].

Numerical Example (from the text book) [Goldsmith, 2005]

Received SNR γ_i	Probability $p(\gamma_i)$
$\gamma_1 = 0.8333$	0.1
$\gamma_2 = 83.33$	0.5
$\gamma_3 = 333.33$	0.4

Table 1: Parameters

When $B = 30$ kHz, find

1. the capacity for AWGN channel
2. the ergodic capacity assuming that only Rx has CSI
3. the ergodic capacity assuming that both Tx and Rx have CSI.

Average SNR $\bar{\gamma} = \sum_{i=1}^{3} \gamma_i p(\gamma_i) = 175.08$

1. the capacity for AWGN channel: $C = B\log_2(1+\bar{\gamma}) = 223.80$ kbits/sec
2. the ergodic capacity assuming that only Rx has CSI:

$$C = \int_0^\infty B\log_2(1+\gamma)\, f_\gamma(\gamma)\, d\gamma = \sum_{i=1}^{3} B\log_2(1+\gamma_i)\, p(\gamma_i) = 199.26\, kbits/sec.$$

3. the ergodic capacity when both Tx and Rx have CSI (next pages).

Example

- Calculate γ_0: Use average power constraint in eq. (35)

$$\int_{\gamma_0}^{\infty}\left(\frac{1}{\gamma_0}-\frac{1}{\gamma}\right) f_\gamma(\gamma)\, d\gamma = 1 \Rightarrow \sum_i \left(\frac{1}{\gamma_0}-\frac{1}{\gamma_i}\right) p(\gamma_i) = 1$$

$$0.1\left(\frac{1}{\gamma_0}-\frac{1}{0.833}\right)+0.5\left(\frac{1}{\gamma_0}-\frac{1}{83.33}\right)+0.4\left(\frac{1}{\gamma_0}-\frac{1}{333.33}\right)=1$$

$$\gamma_0 = 0.387116$$

- Neglect channel's SNR with $\gamma_i < \gamma_0$, i.e., no data is transmitted over the ith time interval. In this case we avoid γ_1.
- Again calculate γ_0 for the remaining channel's SNRs:

$$0.5\left(\frac{1}{\gamma_0}-\frac{1}{83.33}\right)+0.4\left(\frac{1}{\gamma_0}-\frac{1}{333.33}\right)=1$$

$$\gamma_0 = 0.893566$$

- Again check for $\gamma_i < \gamma_0$. Now all SNRs satisfy $\gamma_i \geq \gamma_0$ condition! So start power allocation ☺

Example

- The ergodic capacity assuming that both Tx and Rx have instantaneous CSI: The capacity of the channel can be evaluated by using Eq. (33) as

$$
\begin{aligned}
C &= \int_{\gamma_0}^{\infty} B \log_2\left(\frac{\gamma}{\gamma_0}\right) f_\gamma(\gamma)\, d\gamma = \sum_{i=2}^{3} B \log_2\left(\frac{\gamma_i}{\gamma_0}\right) p(\gamma_i) \qquad (36)\\
&= 30 \times 10^3 \left[0.5 \log_2\left(\frac{83.33}{0.893566}\right) + 0.4 \log_2\left(\frac{333.33}{0.893566}\right)\right]\\
&= 200.82\ kbits/sec
\end{aligned}
$$

- This rate (200.82 kbits/sec) is only slightly higher than for the case of receiver CSI only (199.26 kbits/sec), and it is still significantly below that of an AWGN channel with the same average SNR (199.26 kbits/sec). However, this may not be the case always!

References

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- A. J. Goldsmith and P. P. Varaiya, “Capacity of fading channels with channel side information,” *IEEE Trans. Inform. Theory*, pp. 1986–92, November 1997.
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- M.-S. Alouini and A. J. Goldsmith, “Capacity of Rayleigh fading channels under different adaptive transmission and diversity combining techniques,” *IEEE Trans. Veh. Tech.* pp. 1165–81, July 1999.

Thank You!

See you again ☺