ECE5884 Wireless Communications

Week 6: Performance of Digital Modulation over Wireless Channels

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Course outline

This week: **Ref. Ch. 6 of [Goldsmith, 2005]**

- Week 1: Overview of Wireless Communications
- Week 2: Wireless Channel (Path Loss and Shadowing)
- Week 3: Wireless Channel Models
- Week 4: Capacity of Wireless Channels
- Week 5: Digital Modulation and Detection
- Week 6 : Performance Analysis
- Week 7: Equalization
- Week 8: Multicarrier Modulation (OFDM)
- Week 9: Diversity Techniques
- Week 10: Multiple-Antenna Systems (MIMO Communications)
- Week 11: Multiuser Systems
- Week 12: Guest Lecture (Emerging 5G/6G Technologies)

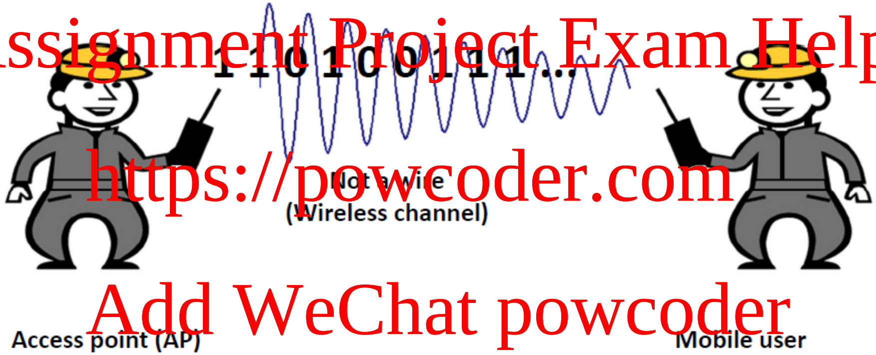


Figure 1: A simple point-to-point wireless communications system.

$$\textit{Gaussian}: \quad X \sim \mathcal{N}(\mu, \sigma^2)$$

$$\textit{PDF}: \quad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}; -\infty < x < \infty \tag{1}$$

$$\textit{CDF}: \quad F_X(x) = \int_{-\infty}^{x} f_x(t)\, dt = 1 - Q\left(\frac{x-\mu}{\sigma}\right) \tag{2}$$

$$Q(x) = 1 - Q(-x) \tag{3}$$

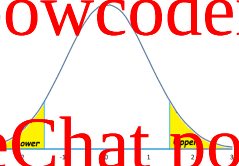


Figure 2: Gaussian lower and upper tails.

$$\textit{For Rayleigh } |h|: \quad f_{|h|^2}(t) = \frac{1}{\Omega_p} e^{-\frac{t}{\Omega_p}} \quad \text{and} \quad F_{|h|^2}(t) = 1 - e^{-\frac{t}{\Omega_p}} \tag{4}$$

$$\textit{Error fun.}: Q(x) = \frac{1}{2}\mathrm{Erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{5}$$

System model

$$\textit{Received signal}: \quad r(t) = h\,s(t) + n(t)$$

$$\textit{SNR (fading)}: \quad \gamma = \frac{|h|^2 P_s}{N_0 B} = \frac{|h|^2 E_s}{N_0 B T_s} = \frac{|h|^2 E_s}{N_0 B (1/B)} = \frac{|h|^2 E_s}{N_0} \tag{6}$$

$$\textit{SNR (AWGN)}: \quad \gamma = \frac{E_s}{N_0} \tag{7}$$

$$\textit{SINR}: \quad \gamma = \frac{\textit{Signal}}{\textit{Interference} + \textit{Noise}} = \begin{cases} \frac{|h|^2 P_s}{P_I + N_0}, & \text{Constant} \\ \frac{|h|^2 P_s}{|f|^2 P_I + N_0}, & \text{Fading} \end{cases} \tag{8}$$

- The AWGN noise, $n = n_r + jn_i$, follows a circularly symmetric complex Gaussian distribution, i.e., $n \sim \mathcal{CN}(0, N_0)$ and $n_r, n_i \sim \mathcal{CN}(0, N_0/2)$.
- In systems with interference, we often use the received signal-to-interference-plus-noise power ratio (SINR). P_I is the average power of the interference ($B = 1$).

$$r(t) = h\,s(t) + I + n(t) \quad \text{or} \quad r(t) = h\,s(t) + f\,i(t) + n(t)$$

Dense wireless network

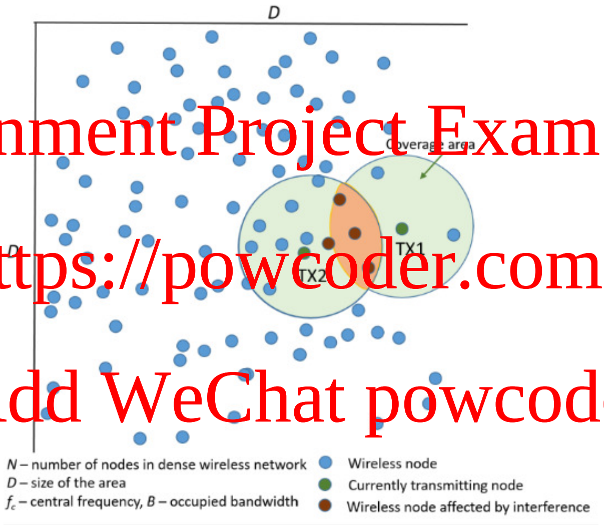
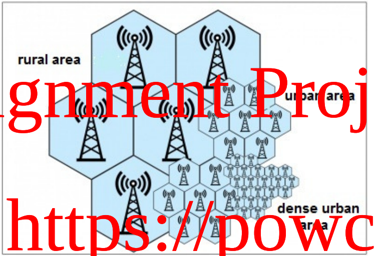


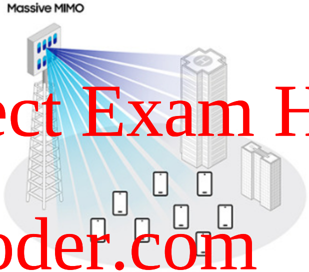
Figure 3: Dense wireless network with interference nodes.

https://www.mdpi.com/1424-8220/21/2/576/htm

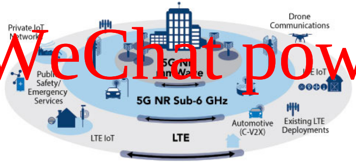
Network (SNR) outage



(a) Cell density.



(b) Massive MIMO.



(c) mmWave.

Figure 4: Different 3G/4G/5G technologies.

SNR outage probability with interference

- The SNR outage probability without interference:

$$P_{out} = \Pr\left[\gamma < \gamma_{th}\right] = \Pr\left[\frac{|h|^2 P_s}{N_0} < \gamma_{th}\right] = \Pr\left[|h|^2 < \frac{N_0\gamma_{th}}{P_s}\right] = F_{|h|^2}\left(\frac{N_0\gamma_{th}}{P_s}\right) \quad (9)$$

- The SNR outage probability with constant interference:

$$P_{out} = \Pr\left[\frac{|h|^2 P_s}{P_I + N_0} < \gamma_{th}\right] = \Pr\left[|h|^2 < \frac{(P_I + N_0)\gamma_{th}}{P_s}\right] = F_{|h|^2}\left(\frac{(P_I + N_0)\gamma_{th}}{P_s}\right) \quad (10)$$

- The SNR outage probability with fading interference:

$$P_{out} = \Pr\left[\frac{|h|^2 P_s}{|f|^2 P_I + N_0} < \gamma_{th}\right] = \Pr\left[|h|^2 < \frac{(|f|^2 P_I + N_0)\gamma_{th}}{P_s}\right]$$

$$= F_{|h|^2 \big| |f|^2 = t}\left(\frac{(tP_I + N_0)\gamma_{th}}{P_s}\right) \quad \text{conditional probability} \quad (11)$$

$$= \int_0^\infty F_{|h|^2 \big| |f|^2 = t}\left(\frac{(tP_I + N_0)\gamma_{th}}{P_s}\right) f_{|f|^2}(t)\, dt \quad \text{unconditional probability} \quad (12)$$

Communication system: Symbols/bits in error

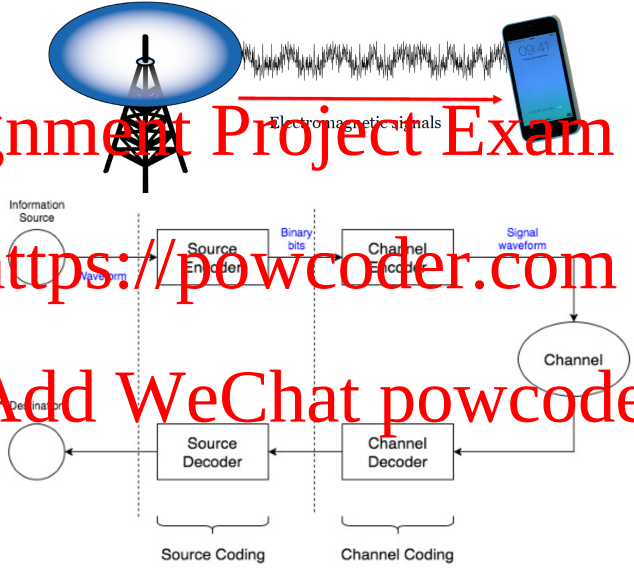
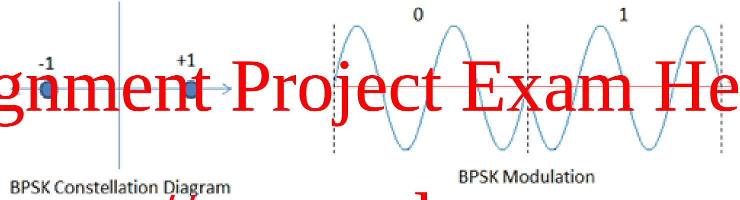


Figure 5: Block diagram of a digital communication system.

Binary Phase Shift Keying (BPSK) modulation



1. The average energy: $E_s = \frac{1}{2}\left((-A)^2 + A^2\right) = A^2 \Rightarrow A = \sqrt{E_s}$.
2. The average power: $P_s = E_s/T_s$ where T_s is the symbol time.
3. For BPSK, symbol energy (E_s) = bit energy (E_b), i.e., $E_s = E_b$.
4. Received signal over AWGN channel:

$$r = \begin{cases} s_0 + n, & \text{when bit 0 is transmitted.} \\ s_1 + n, & \text{when bit 1 is transmitted.} \end{cases} \Rightarrow r = \begin{cases} -\sqrt{E_s} + n, \\ +\sqrt{E_s} + n, \end{cases} \tag{13}$$

Binary Phase Shift Keying (BPSK) modulation

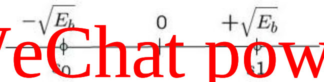
1. Received signal over AWGN channel:

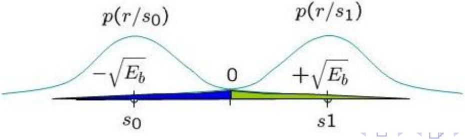
$$r = \begin{cases} -\sqrt{E_s} + n, \\ +\sqrt{E_s} + n, \end{cases} \Rightarrow \Re(r) = \begin{cases} -\sqrt{E_s} + n_r, \\ +\sqrt{E_s} + n_r, \end{cases} \text{ and } \Im(r) = \begin{cases} n_i, \\ n_i \end{cases} \tag{14}$$

2. Probability distribution:

$$\Re(r) \sim \begin{cases} N(-\sqrt{E_s}, N_0/2), \\ N(+\sqrt{E_s}, N_0/2), \end{cases} \text{ and } \Im(r) \sim \begin{cases} N(0, N_0/2), \\ N(0, N_0/2), \end{cases} \tag{15}$$

3. Illustration:





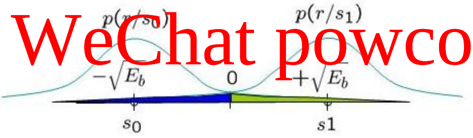
BPSK - Symbol error rate (SER)

1. Possible signal/symbol set: $S \in \{s_1, \cdots, s_M\}$
2. The probability of symbol error:

$$P_s = \sum_{i=1}^{M} p\left(\hat{s} \neq s_i | s_i \text{ sent}\right) p\left(s_i \text{ sent}\right) \quad (16)$$

3. For BPSK:

$$\begin{aligned} P_{s,bpsk} &= \Pr\left(\hat{s} \neq s_0 | s_0 \text{ sent}\right) \Pr\left(s_0 \text{ sent}\right) + \Pr\left(\hat{s} \neq s_1 | s_1 \text{ sent}\right) \Pr\left(s_1 \text{ sent}\right) \\ &= \Pr\left(\hat{s} \neq s_0 | s_0 \text{ sent}\right) \frac{1}{2} + p\left(\hat{s} \neq s_1 | s_1 \text{ sent}\right) \frac{1}{2}; \quad \text{equiprobable} \end{aligned}$$



$$P_{s,bpsk} = \Pr\left(\hat{s} \neq s_0 | s_0\right) = \text{Green} = Q\left(\frac{\left(0 - \left(-\sqrt{E_s}\right)\right)}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \quad (17)$$

BPSK - Bit error rate (BER)

1. Since each BPSK symbol has only ONE bit: $SER = BER$
2. Definitions:

$$\text{SNR per symbol: } \gamma_s = \frac{E_s}{N_0} \quad (18)$$

$$\text{SNR per bit: } \gamma_b = \frac{E_b}{N_0} \quad (19)$$

3. For BPSK: $d_{min} = 2\sqrt{E_s}$

$$P_s = P_b = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q\left(\sqrt{2\gamma_s}\right) = Q\left(\sqrt{2\gamma_b}\right) = Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \quad (20)$$

4-Quadrature Amplitude Modulation (4-QAM)

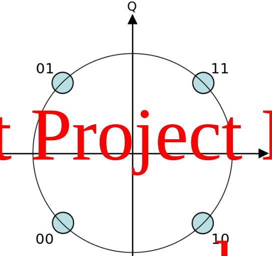


Figure 6: 4-QAM (or 4-PSK).

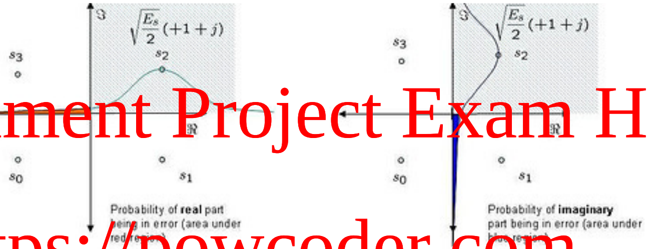
1. The average energy: $E_s = \frac{1}{4}4(2A^2) = 2A^2 \Rightarrow A = \sqrt{\frac{E_s}{2}}$.
2. Received signal over AWGN channel:

$$r = s_i + n;\ s_i \in \{11, 01, 00, 10\}$$

$$r_{11} = s_{11} + n \Rightarrow \begin{cases} \Re(r_{11}) = \sqrt{\frac{E_s}{2}} + n_r \sim N\left(\sqrt{\frac{E_s}{2}}, \frac{N_0}{2}\right) \\ \Im(r_{11}) = \sqrt{\frac{E_s}{2}} + n_i \sim N\left(\sqrt{\frac{E_s}{2}}, \frac{N_0}{2}\right) \end{cases} \quad (21)$$

4-QAM

1. Illustration:



2. Symbol error rate:

$$P_s = 1 - \text{Correct prob.} = 1 - \frac{1}{4} \cdot 4 \left[\Pr\left(\Re(r_{11}) \geq 0 \right) \text{ and } \Pr\left(\Im(r_{11}) \geq 0 \right) \right]$$

$$= 1 - \left[1 - Q\left(\sqrt{\frac{E_s/2}{N_0/2}} \right) \right]^2 = 2Q\left(\sqrt{\frac{E_s}{N_0}} \right) - Q\left(\sqrt{\frac{E_s}{N_0}} \right)^2 \quad (22)$$

$$= 2Q\left(\sqrt{\gamma_s} \right) - Q\left(\sqrt{\gamma_s} \right)^2 = 2Q\left(d_{min}/\sqrt{2N_0} \right) - Q\left(d_{min}/\sqrt{2N_0} \right)^2 \quad (23)$$

For high SNR : $\gamma_s \ggg 0$; $\quad P_s \approx 2Q\left(\sqrt{\gamma_s} \right) = 2Q\left(d_{min}/\sqrt{2N_0} \right) \quad (24)$

General SER and BER

1. Assume:
   - The symbol energy is divided equally among all bits
   - Gray encoding is used (one symbol error corresponds to exactly one bit error)
   - Reasonable SNRs

$$\gamma_b = \frac{\gamma_s}{\log_2(M)} \tag{25}$$

$$P_b = \frac{P_s}{\log_2(M)} \tag{26}$$

2. The nearest neighbor approximation:

$$P_s \approx M_{d_{min}} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \Rightarrow \begin{cases} Q\left(d_{min}/\sqrt{2N_0}\right) & \text{for BPSK} \\ 2Q\left(d_{min}/\sqrt{2N_0}\right) & \text{for 4-QAM} \end{cases} \tag{27}$$

   - $M_{d_{min}}$ is the largest number of nearest neighbors for any constellation point in the constellation;
   - d_{min} is the minimum distance in the constellation.
   - More accurate for for nonrectangular constellations.

General SER and BER

BFSK		$P_b = Q(\sqrt{\gamma_b})$
BPSK		$P_b = Q(\sqrt{2\gamma_b})$
QPSK, 4-QAM	$P_s \approx 2Q(\sqrt{\gamma_s})$	$P_b \approx Q(\sqrt{2\gamma_b})$
MPAM	$P_s = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\overline{\gamma}_s}{M^2-1}}\right)$	$P_b \approx \frac{2(M-1)}{M\log_2 M} Q\left(\sqrt{\frac{6\overline{\gamma}_b \log_2 M}{M^2-1}}\right)$
MPSK	$P_s \approx 2Q\left(\sqrt{2\gamma_s}\sin\left(\frac{\pi}{M}\right)\right)$	$P_b \approx \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M}\sin\left(\frac{\pi}{M}\right)\right)$
Rectangular MQAM	$P_s \approx 4Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\overline{\gamma}_b \log_2 M}{M-1}}\right)$
Nonrectangular MQAM	$P_s \approx 4Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\overline{\gamma}_b \log_2 M}{M-1}}\right)$

Figure 7: Approximate symbol and bit error probabilities for coherent modulations

1. Coherent systems need carrier phase information at the receiver and they use matched filters to detect and decide what data was sent
2. Noncoherent systems do not need carrier phase information and use methods like square law to recover the data.

Over Fading

$$r = \begin{cases} s_i + n, & \text{AWGN.} \\ h \cdot s_i + n, & \text{Fading.} \end{cases} \tag{28}$$

1. The received signal power varies randomly over distance or time as a result of shadowing and/or multipath fading.
2. γ_s is a random variable with distribution $f_{\gamma_s}(t)$, e.g., Rayleigh, Rician, Nakagami-m, etc.

3. P_s is also random, i.e., $P_s(\gamma_s)$.
4. Assume $T_s \approx T_c$, so γ_s is roughly constant over a symbol time.
5. Average SER is computed by integrating the error probability in AWGN over the fading distribution.

$$\bar{P}_s = \int_0^\infty P_s(t) f_{\gamma_s}(t)\, dt \tag{29}$$

1. For AWGN:

$$P_s = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q\left(\sqrt{2\gamma_s}\right) \tag{30}$$

2. For Fading channel h:

$$P_{s||h|} = Q\left(\sqrt{\frac{2|h|^2 E_s}{N_0}}\right) = Q\left(\sqrt{2g\gamma_s}\right) \tag{31}$$

3. For Rayleigh fading channel $|h|$: we have $f_{|h|^2}(t) = \frac{1}{\Omega_p} e^{-\frac{t}{\Omega_p}}$, then

$$\bar{P}_s = \int_0^\infty Q\left(\sqrt{2\gamma_s t}\right) f_g(t)\, dt = \frac{1}{\Omega_p}\int_0^\infty Q\left(\sqrt{2\gamma_s t}\right) e^{-\frac{t}{\Omega_p}}\, dt \tag{32}$$

$$= \frac{1}{2\Omega_p}\int_0^\infty \mathrm{Erfc}\left(\sqrt{\gamma_s t}\right) e^{-\frac{t}{\Omega_p}}\, dt = \frac{1}{2\left(\Omega_p\gamma_s + \sqrt{\Omega_p\gamma_s(\Omega_p\gamma_s + 1)} + 1\right)}$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{\Omega_p\gamma_s}{1 + \Omega_p\gamma_s}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}_s}{1 + \bar{\gamma}_s}}\right) \quad \text{when } \bar{\gamma}_s = \Omega_p\gamma_s \tag{33}$$

Numerical results

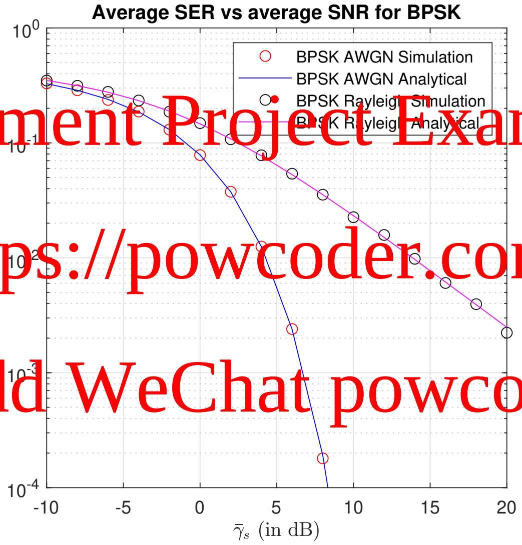


Figure 8: SER vs average SNR for BPSK over AWGN and Rayleigh fading channels.

References

A. Goldsmith, *Wireless Communications*, Cambridge University Press, USA, 2005.

Thank You!

See you again ☺