ECE 5884/6884
Wireless Communications
Week 7 Lecture

Pulseshaping and Matched Filtering
Synchronization
Channel Estimation

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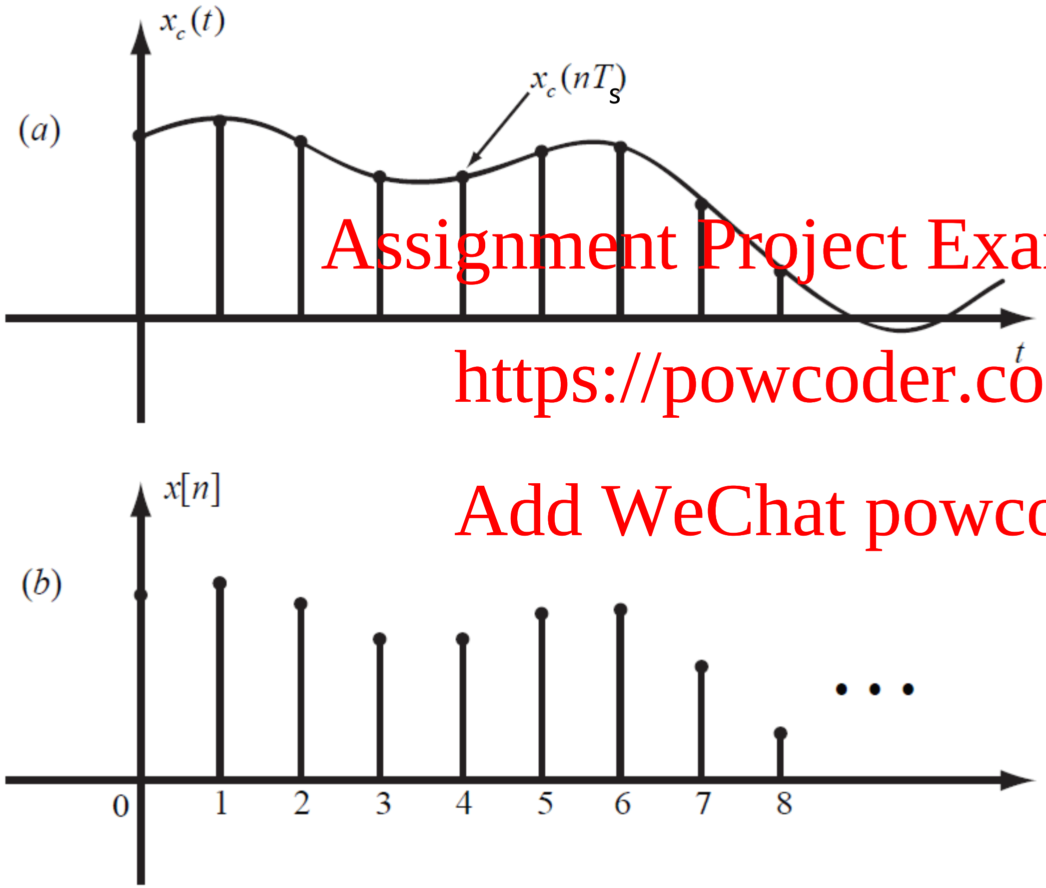
Sessional Lecturer

Discrete-time processing of continuous-time signals



Objective: Determine equivalent combination of sampling, digital filtering, and reconstruction to process a bandlimited continuous-time signal with discrete-time signal processing

Nyquist sampling theorem

Let $x_c(t)$ be a bandlimited signal, which means that $X_c(f) = 0$ for $f \geq f_N$. Then $x_c(t)$ is uniquely determined by its samples $\{x[n] = x_c(nT_s)\}$ $\quad n \in [-\infty, \infty]$

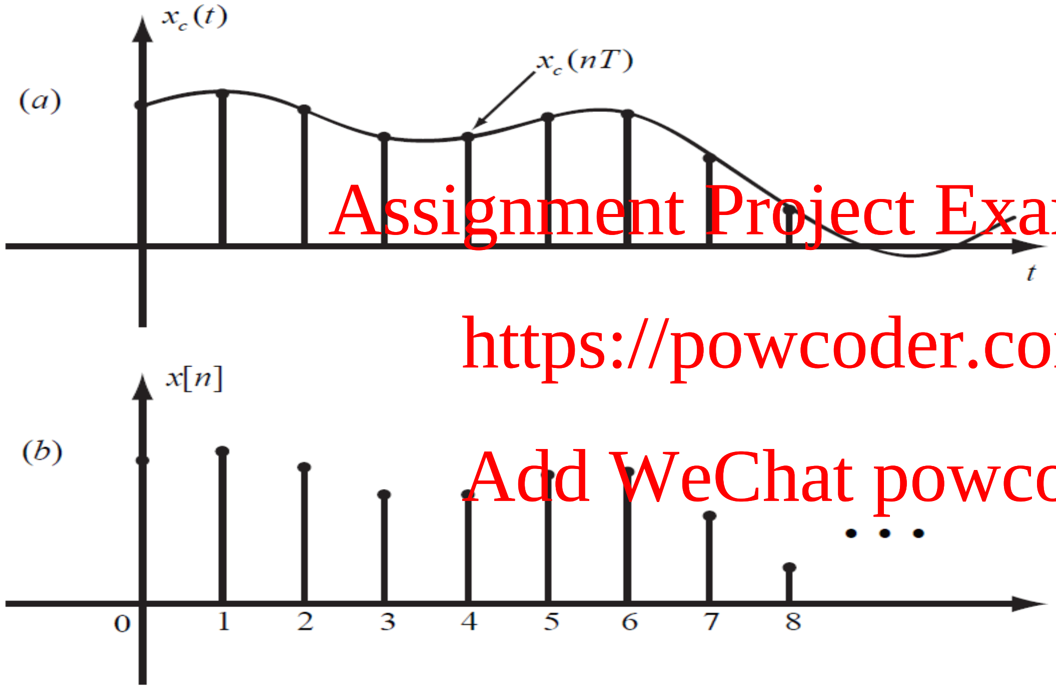
If the sampling frequency satisfies

$$f_s := \frac{1}{T_s} \geq 2f_N$$

where f_N is the Nyquist frequency and $2f_N$ is generally known as the Nyquist rate.

$$x_c(t) = \sum_n \frac{x[n] \sin\left(\pi \frac{(t - nT_s)}{T_s}\right)}{\pi\left(\frac{t - nT_s}{T_s}\right)}$$

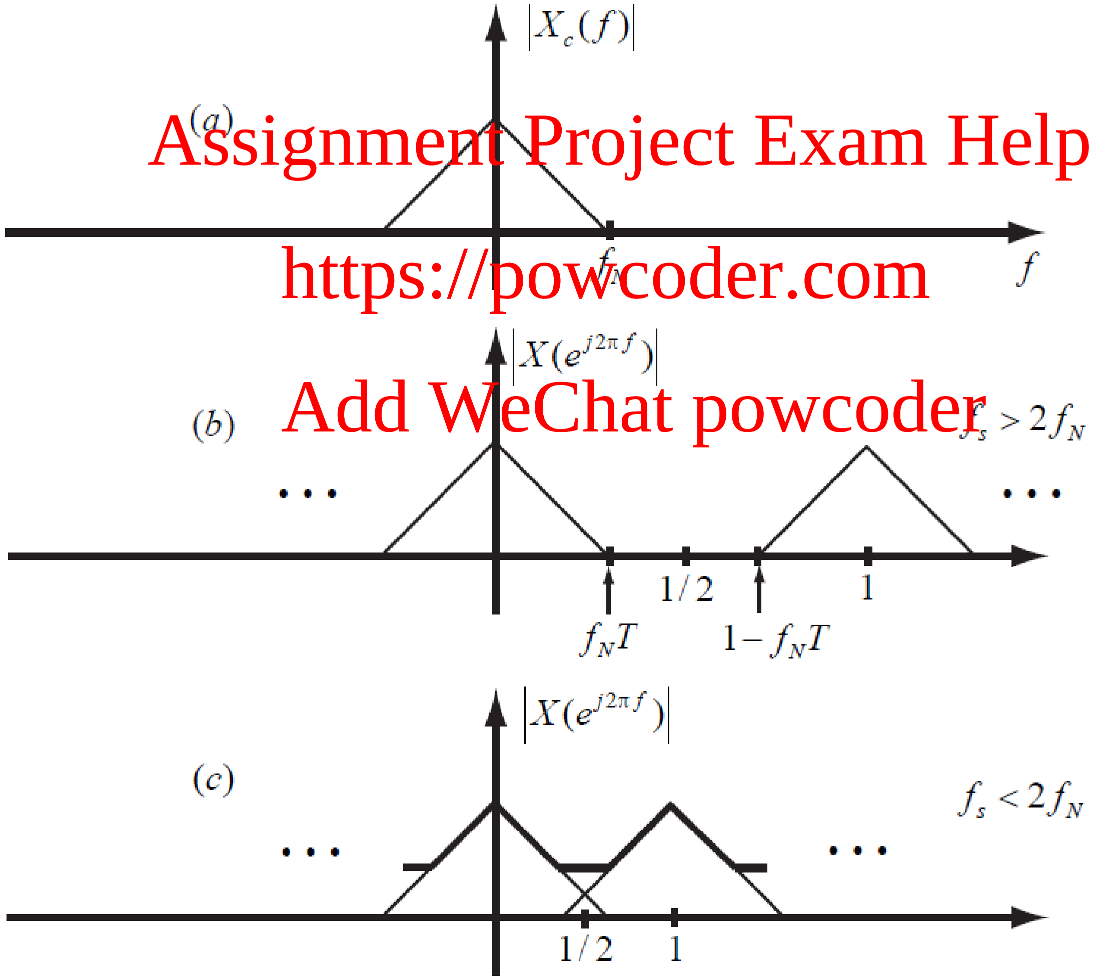
visualize the sampling operation



$$\mathrm{X}_s(f) = \mathcal{F}\left\{x(t)\sum_k \delta(t-kT)\right\} = \mathrm{X}(f) * \frac{1}{T}\sum_n \delta\left(f-\frac{n}{T}\right) = \sum_n \frac{1}{T}\mathrm{X}\left(f-\frac{n}{T}\right)$$

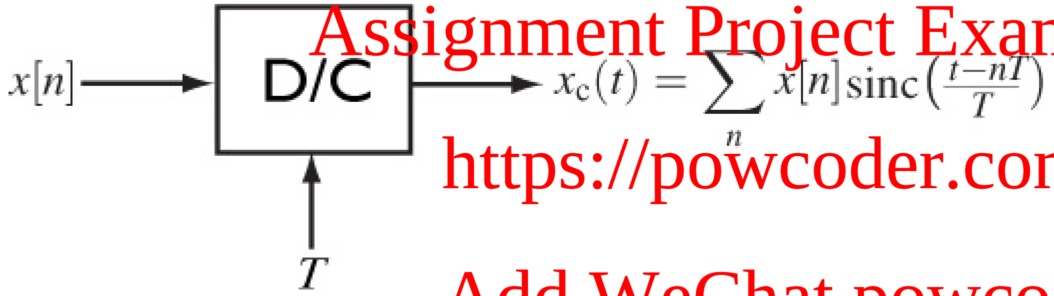
Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

The effect of sampling on the signal bandwidth

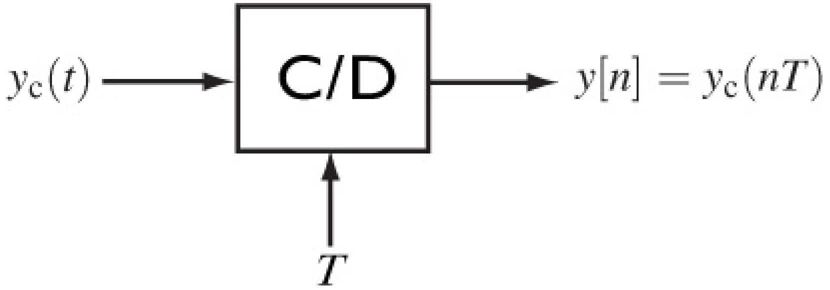


DAC and ADC

Generation of discrete to continuous waveform at the transmitter-practically implemented using the digital to analog converter (DAC)

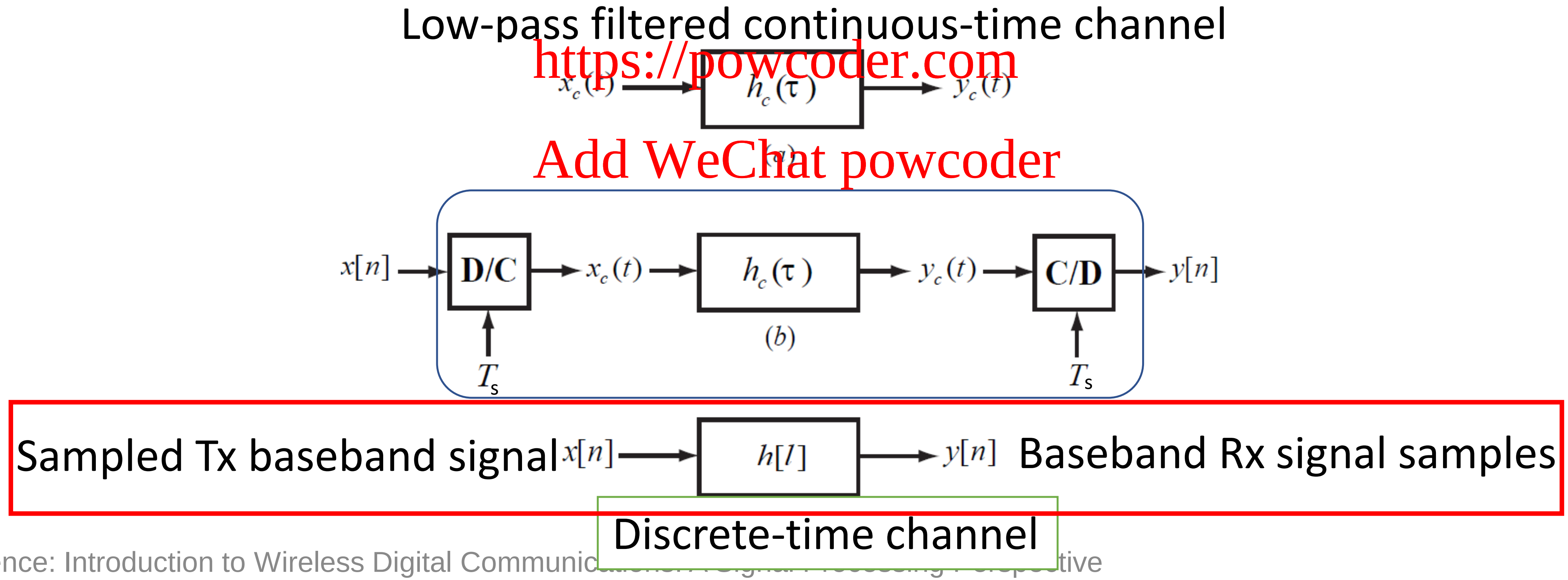


Generation of continuous to discrete samples at the receiver -practically implemented using the analog to digital converter (ADC)



Discrete-time equivalent channel

A continuous time domain signal at the receiver $y_c(t)$ is sampled to obtain its discrete-time equivalent for carrying out digital signal processing



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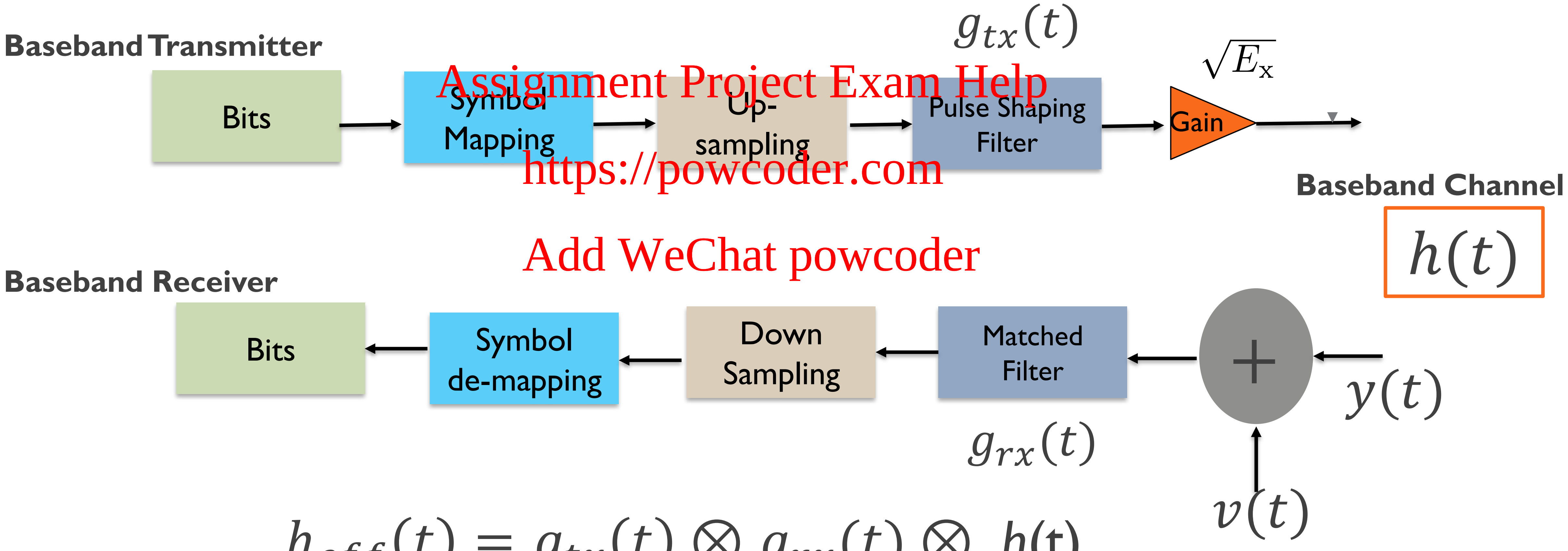
Pulse shaping

What is the best pulse shape for transmission over Wireless channel?



$$h_{eff}(t) = g_{tx}(t) \otimes g_{rx}(t) \otimes h(t)$$

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

Pulse shaping filter design criteria

Ideally in the continuous time domain,

the effective pulse shape $g(t) = g_{tx}(t) \otimes g_{rx}(t) = \delta(t)$

In the discrete-time domain $t = nT$,

$$g(nT) = g_{tx}(nT) \otimes g_{rx}(nT) = \delta(nT)$$

So that the Fourier transform of the sampled $g(nT)$ is

$$\sum G\left(f + \frac{k}{T}\right) = T$$

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

Nyquist criterion for Pulse shaping

The continuous-time received signal at the baseband corresponding to transmit signal $s(m)$

$$y(t) = \sqrt{E_x}\ \ h_{eff}(t) \otimes \sum_m s(m)\delta(t - mT) + g_{rx}(t) \otimes v(t)$$

The discrete-time received signal at the baseband corresponding to transmit signal $s(m)$

$$y(nT) = \sqrt{E_x}\ \ h_{eff}(nT) \otimes \sum_m s(m)\delta(nT - mT) + g_{rx}(nT) \otimes v(nT)$$

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

Discrete-time Received Signal

Assume the baseband channel h(t) = 1

Then, $h_{eff}(t) = g_{tx}(t) \otimes g_{rx}(t)$

$$y(nT) = \sqrt{E_x} \sum_m s(m)g(nT - mT) + g_{rx}(nT) \otimes v(nT)$$

$$y[n] = \boxed{\sqrt{E_x} \sum_m s[m]g[n-m]} + \boxed{g_{rx}[n] \otimes v[n]}$$

Signal component Noise component

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

Zero-ISI Criterion

Signal Energy is calculated at the sampling instant i:e $m = n$

$$\mathrm{E}\left|\sqrt{E_x}\, s[n]\, g[0]\right|^2$$

Energy at all other sampling instants i:e $m \neq n$ interferes with the detection of other symbols and is termed as Inter-Symbol-Interference (ISI)

$$\mathrm{E}\left|\sum_{m\neq n}\sqrt{E_x}\, s[m]\, g[n-m]\right|^2$$

$$\sum E_x g(mT)^2 \quad m = \cdots -1, 0, 1 \ldots.$$

Design Goal is to satisfy the Zero-ISI Criterion for the pulse

$\boxed{\sum E_x g[mT]^2 = 0}$ $\longrightarrow$ $g(nT) = c\delta(n)$ $\longrightarrow$ $\boxed{g(t) = g_{tx}(t) \otimes g_{rx}(-t)}$

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

What are the pulses that satisfy Nyquist criterion?

- The standard sync pulse satisfies the Nyquist criterion, however, the impulse response

$$g_{\text{sync}}(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

1. Is a non-causal system (with impulse response non-zero for t<0) and hence difficult to approximate
2. The slope of the sync waveform is 1/t at time instants other than zero crossings which is very slow.
3. Due to this it is very sensitive to sample timing errors causing significant interference to adjacent symbols.
4. faster decay such as $\frac{1}{t^2}$ or even $\frac{1}{t^3}$ are desirable to minimize the ISI due to timing jitter in adjacent samples

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

Design of desired pulse shapes using Nyquist criterion

- In the frequency domain,

$$G(f) = \prod\left(\frac{f}{f_s}\right) \otimes Z(f)$$

where, $Z(f) = Z(-f)$ → Even Function

and, $Z(f) = 0, \ |f| \geq f_s \geq \frac{1}{2T}$ → Band limited filter

- In the time domain,

$$g_{rc} = \frac{\sin(\pi t/T)}{(\pi t/T)} z(t)$$

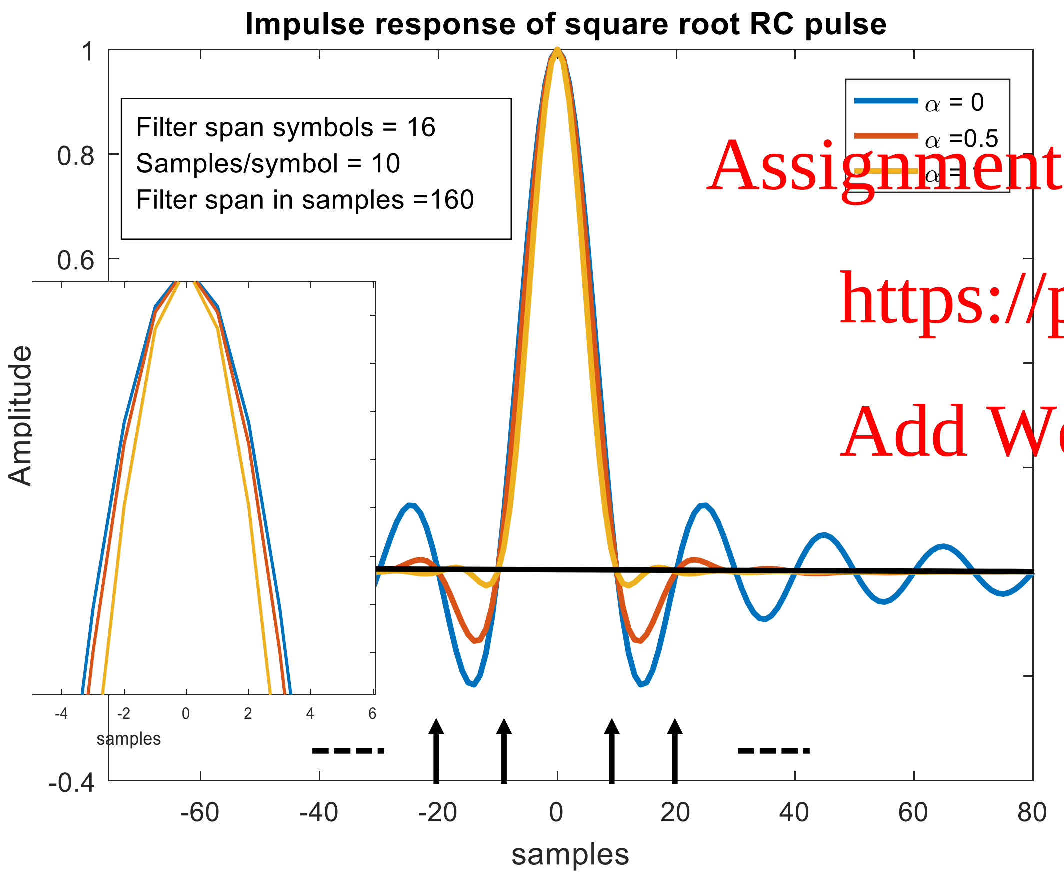
Raised Cosine Pulse

$$g_{rc}(t) = \frac{\sin(\pi\, t/T)}{(\pi\, t/T)} \left(\frac{\cos(\pi \alpha t)}{1 - (2\pi\alpha\, t/T)^2} \right)$$

$$G_{rc}(f) = \begin{cases} T, & 0 \le |f| \le \dfrac{1-\alpha}{2T} \text{ (passband)} \\ \dfrac{T}{2}\left[1 + cos\left(\dfrac{\pi T}{\alpha}\left[|f| - \dfrac{1-\alpha}{2T}\right]\right)\right], & \dfrac{1-\alpha}{2T} \le |f| \le \dfrac{1+\alpha}{2T} \text{ (transition band)} \\ 0 & |f| > \dfrac{1+\alpha}{2T} \text{ (out of band)} \end{cases}$$

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

Impulse response of the Raised Cosine filter

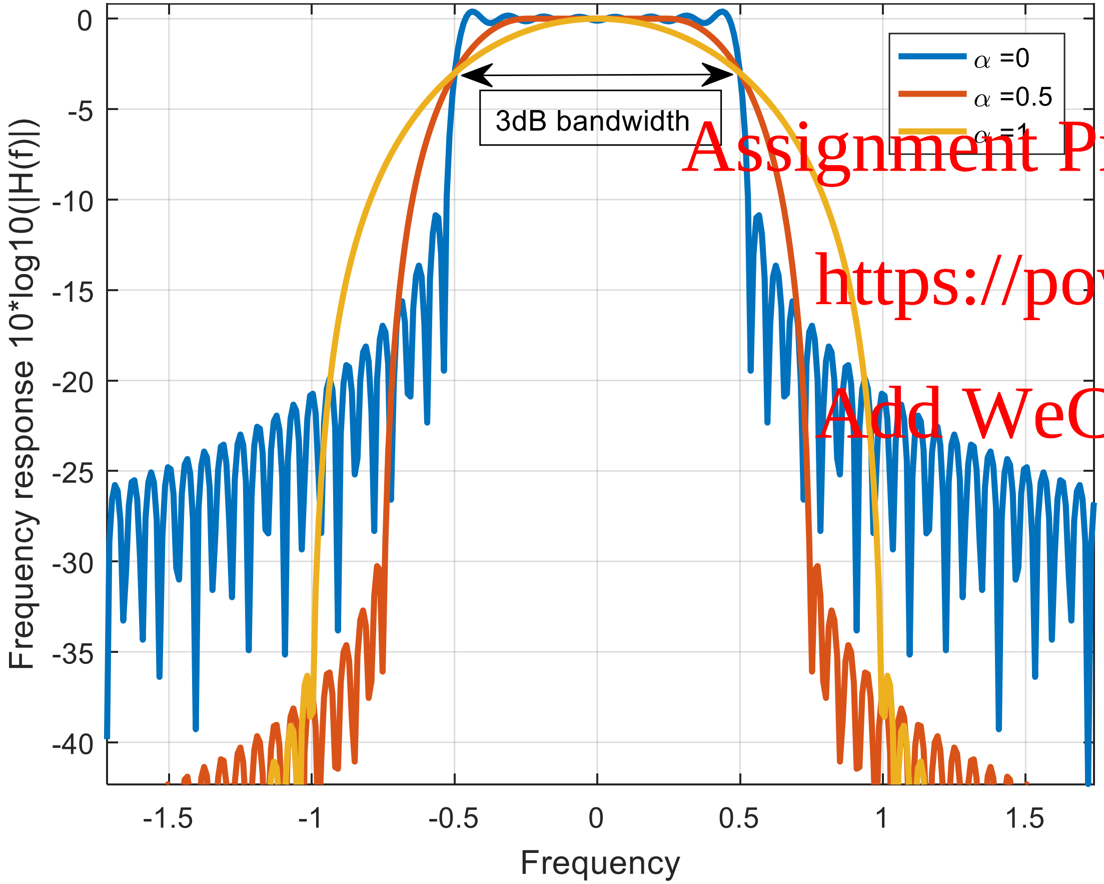


Questions on impulse response of the RC pulse shape

1. What is the effect of filter span (sometimes referred to as filter order) on the transmitted signal?
2. What are the implications of choosing a long filter span?
3. What does roll-off factor α in pulse shaping filters control?
4. What are the implications of designing pulse shaping filters with 0<α<1.

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

Frequency response of the Raised Cosine filter



Questions on frequency response of the RC pulse:

1. What is the effect of filter span (sometimes referred t o as filter order) on the spectrum of the transmitted si gnal?

2. Is there an ideal choice for the filter span?

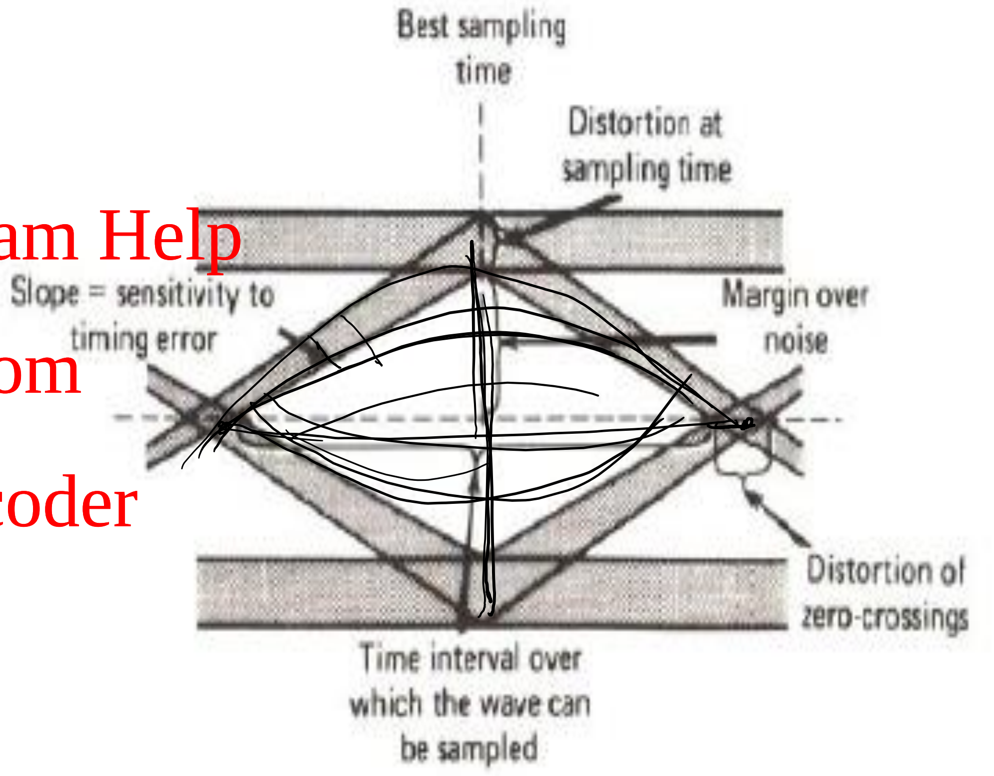
3. How does roll-off factor α in pulse shaping filters affec t the pass band and out of band of the transmit signal ?

4. What are the implications of designing a pulse shapin g filters with 0<α<1.

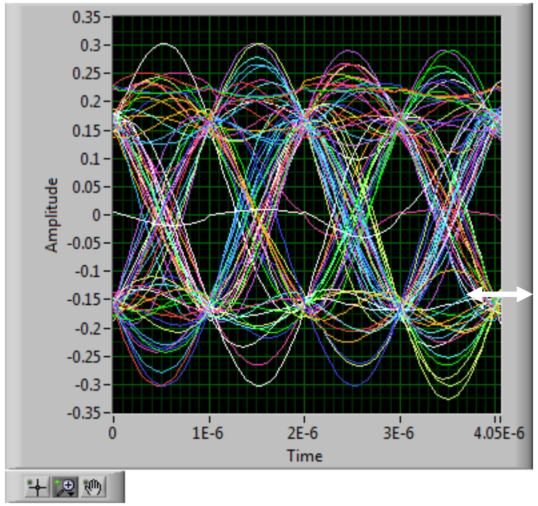
Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

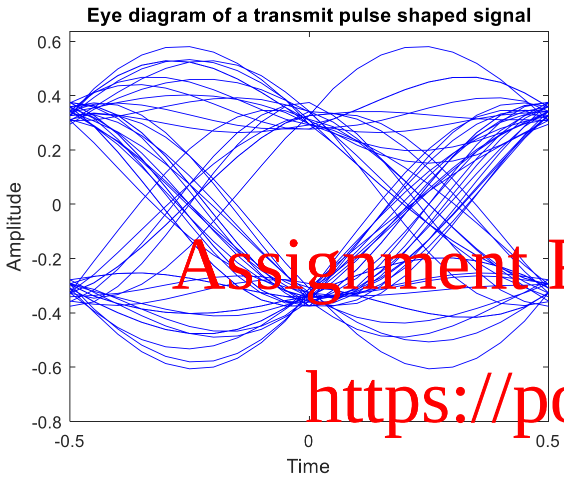
Eye Diagrams

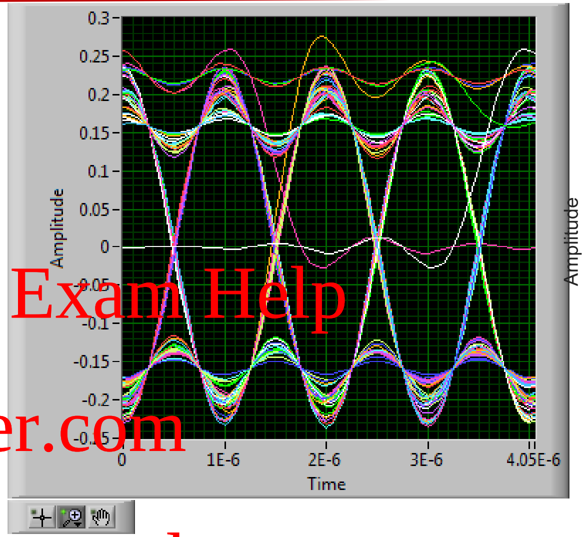
- The width of the eye opening defines the time interval over which the received signal can be sampled without error from ISI. It is intuitive that the preferred time for sampling is the instant of time at which the eye is open the widest.
- Sensitivity of the system to timing errors is determined by the slope of the eye as the sampling time is varied.
- The height of the eye opening specifies the *noise margin* of the system
- Pulses with more distortion of zero-crossings imply susceptibility to synchronisation errors.

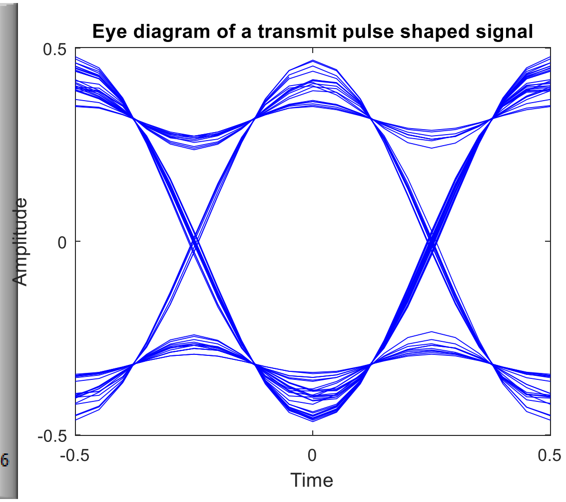


Comparison of filters with different alphas









RRC pulse shape with 10% roll-off

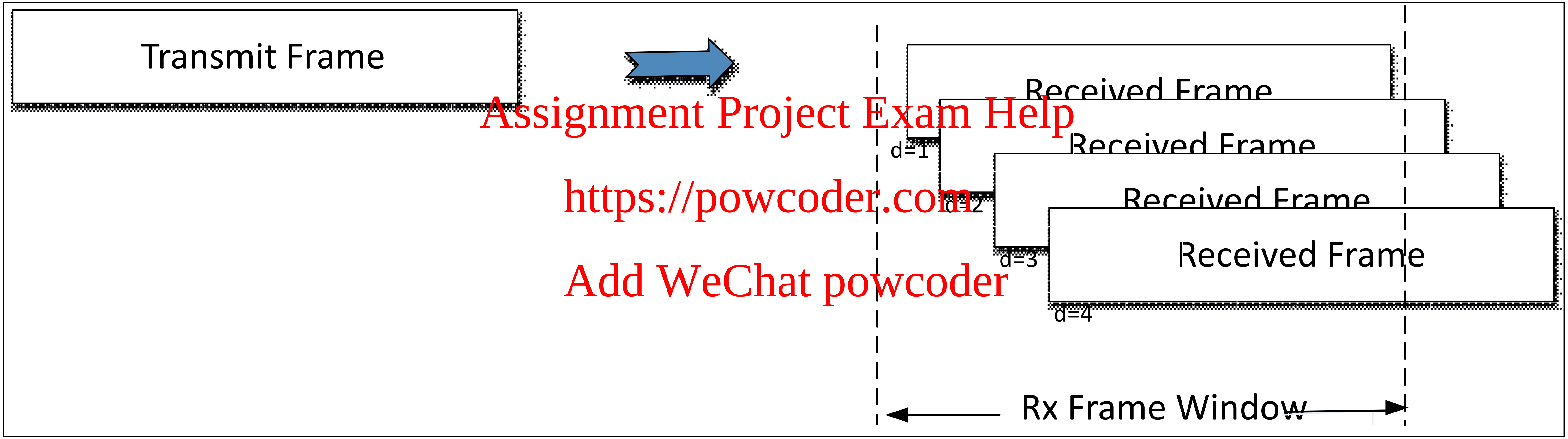
RRC pulse shape with 100% roll-off

Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

Channel impairments

- Time offset
- Frequency offset
- Multipath channel distortions

Timing offset



Reference: Introduction to Wireless Digital Communications: A Signal Processing Perspective

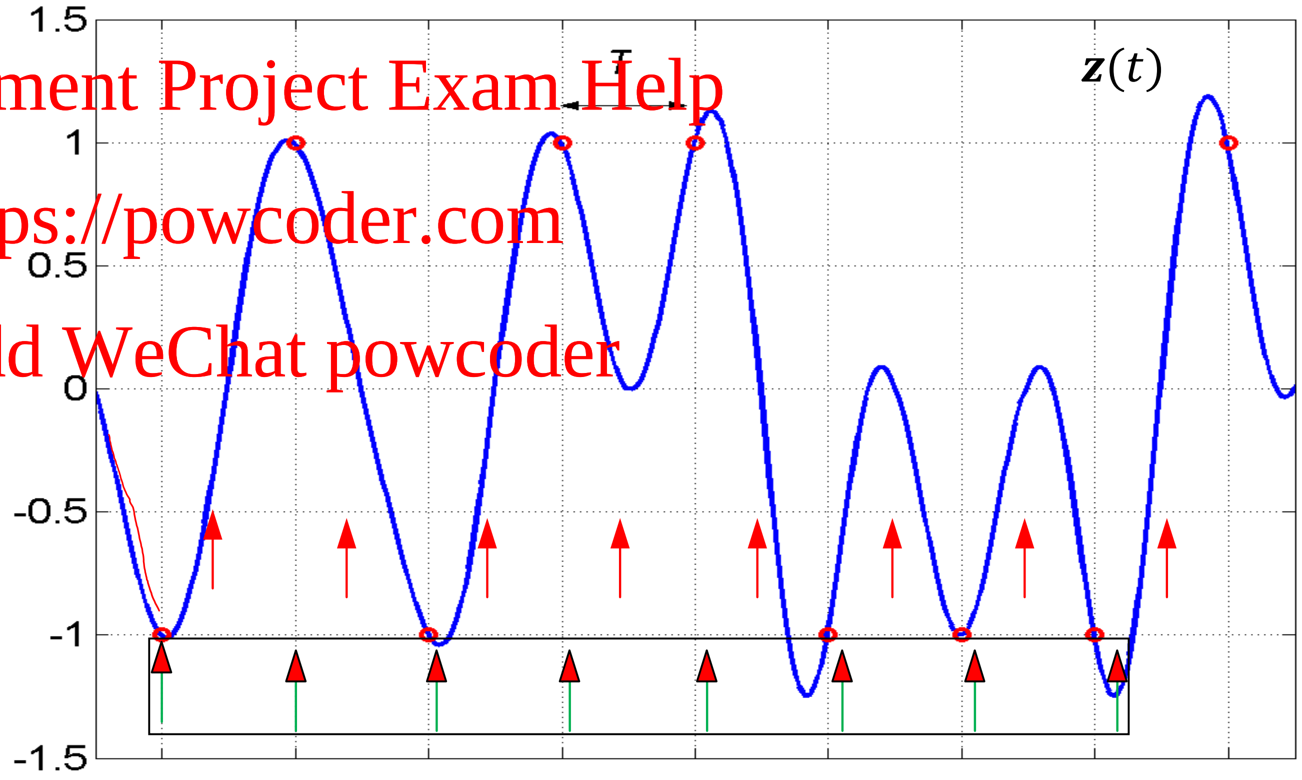
Time Synchronization

1. How to know the start of the Frame?

✓ Frame Synchronization

2. How to get the optimal sampling instants from the oversampled received signal?

✓ Symbol synchronization



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Timing synchronization algorithm

Two-stage Timing synchronization algorithm

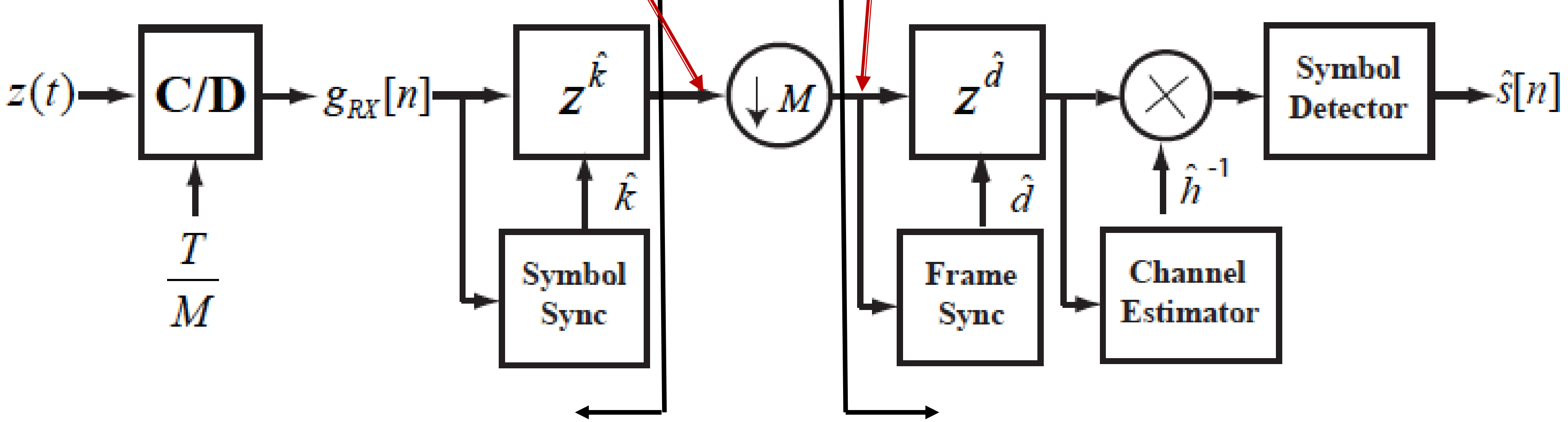
Let, the propagation time in sec be denoted by

$$\tau_d = \tau_d^{integer} + \tau_d^{fractional} \longrightarrow \tau_d = dT + k^* \frac{T}{M}$$

d is delay in number of symbol periods, and k^* is the optimal fractional delay sample periods

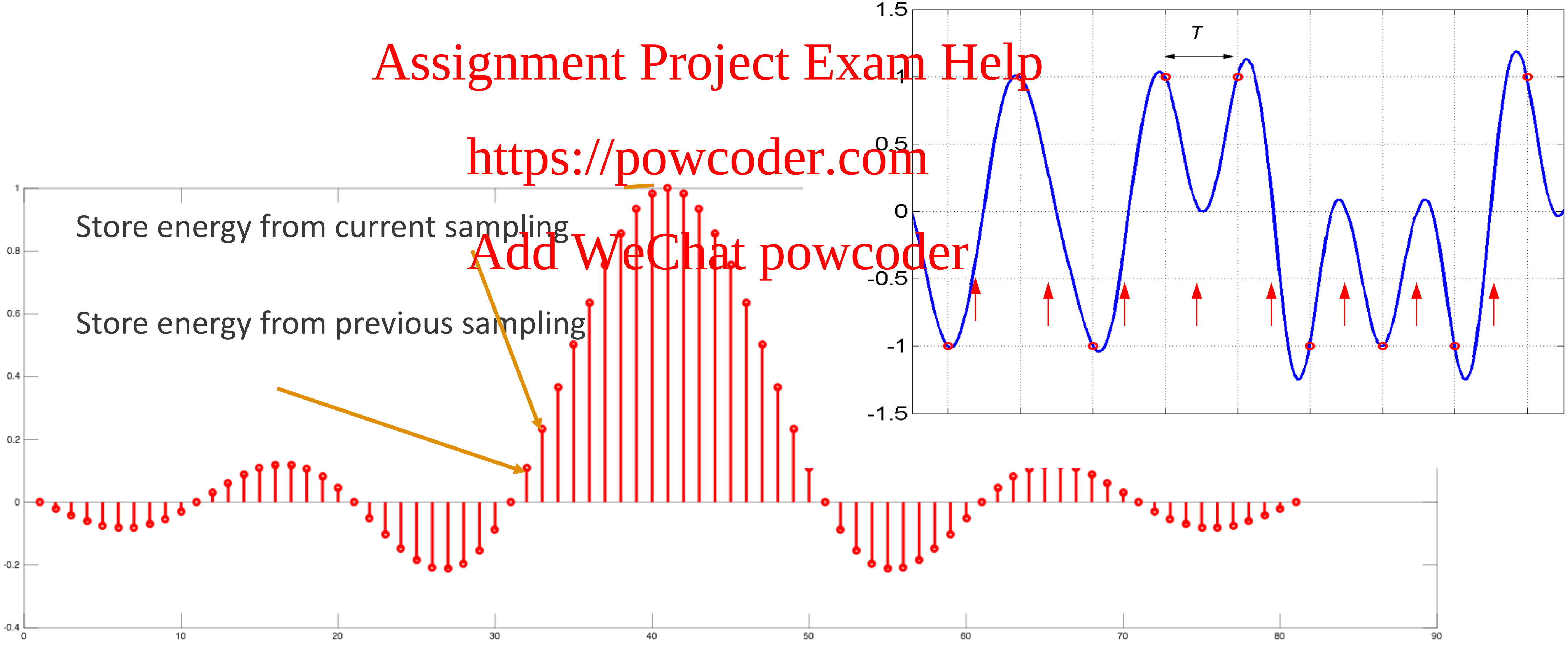
Energy based symbol synchronization
Performed on over-sampled signal

Correlation based Peak detection for
frame synchronization.



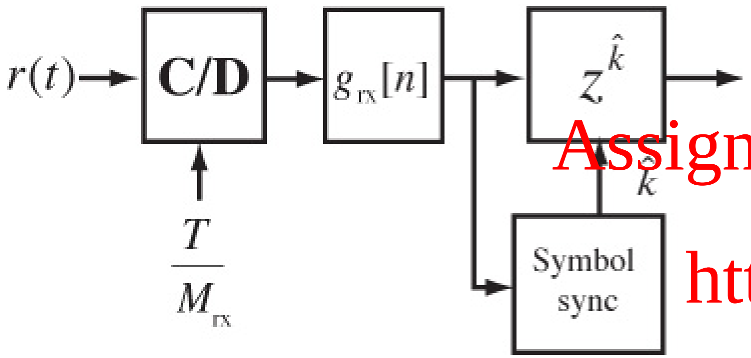
Output energy tracking

Consider transmission of a Nyquist pulse shaped signal with a waveform as shown in the figure below



Symbol Synchronization

Output Energy Maximization



$$\tilde{y}[n] = \sum_{m=-\infty}^{\infty} r[m] g_{rx}[n-m]$$

We use this sampled signal to compute a discrete-time version of $J_{MOE}(\tau)$ given by

$$J_{MOE}[k] = \mathrm{E}[|\tilde{y}[nM+k]|^2]$$

where k is the sample offset between 0, 1, . . . , $M-1$ corresponding to an estimate of the fractional part of the timing offset given by kT/M

replace the expectation with a time average over P symbols

$$J_{MOE}[k] = \frac{1}{P}\left|\sum_{p=0}^{P-1} \tilde{y}[pM+k]\right|^2$$

$\hat{k}$ is the is the sample delay offset that maximizes the J_{MOE}

Frame Synchronization

The objective is to determine the transmission delay d $\quad y[n] = \sqrt{E_x}\,\alpha e^{j\phi} s[n-d] + v[n]$

Transmission frame with a training sequence N_t appended at the start

$s_{\text{sync}}[n]$

L Train	P-L data	L Train	......

A correlation based detector correlates the received with the training sequence to find peak

$$R[n] = \sum_{p=0}^{N_t-1} |t^*[p]y[n+p]|$$

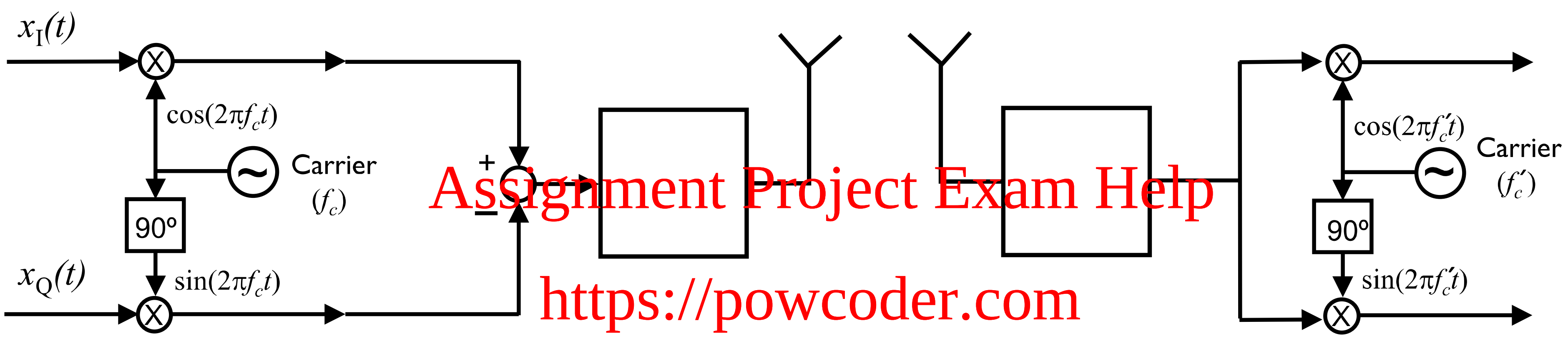
And, an estimate of the delay is obtained as the index n that corresponds to maximum correlator output

$$\hat{d} = \max_n R[n] \qquad \tau_d = \tau_d^{integer} + \tau_d^{fractional}$$

Frequency Offset

- What is the origin of frequency offset?
- Analyze a simple frequency offset estimation algorithm based on sending training sequence

Downconversion

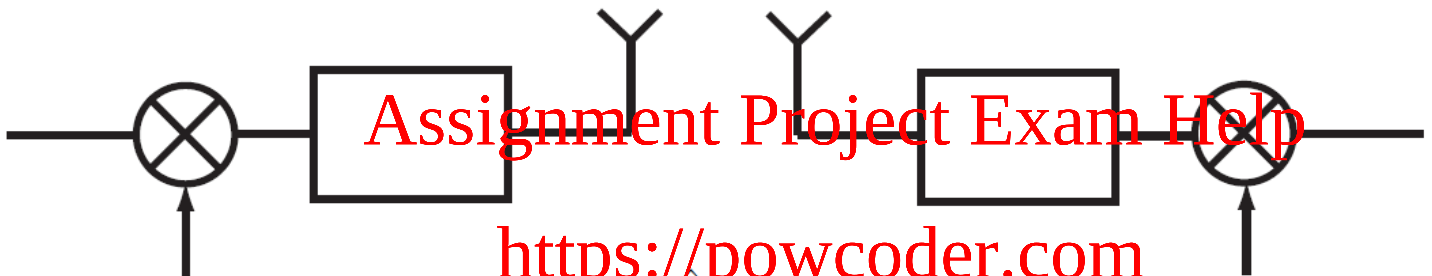


- Consider the received signal after downconversion

- What if only f_c' not f_c is known at the receiver?
  - The result is carrier frequency offset (CFO)

Carrier frequency offset (CFO)

What is it? Frequency offset occurs when $f_c \neq f_c'$ and is *unknown*!



Let $f_e = f_c - \hat{f}_c$ and $\epsilon = T f_e$

CFO in Hz

CFO normalised

Question: A certain digital transmission scheme has 1 M symbols per sec and a CFO of 200Hz. What is the normalized CFO?

The effect of frequency offset on discrete-time signal

- Assume the offset is small, the front-end bandwidth is sufficiently wide,

$$y(t) = e^{(j2\pi f_e t)}(h_{eff}(t) \otimes s(t)) + v(t)$$

- In discrete time($t = nT$), including noise and with $\epsilon = f_e T$

- Assume, a Matched filter implementation so that, $h_{eff}(t) = g_{tx}(t) \otimes g_{rx}(t) \otimes h(t) = h(t)$

$$y[n] = e^{(j2\pi\epsilon n)}(h(n) \otimes s(n)) + v(n)$$

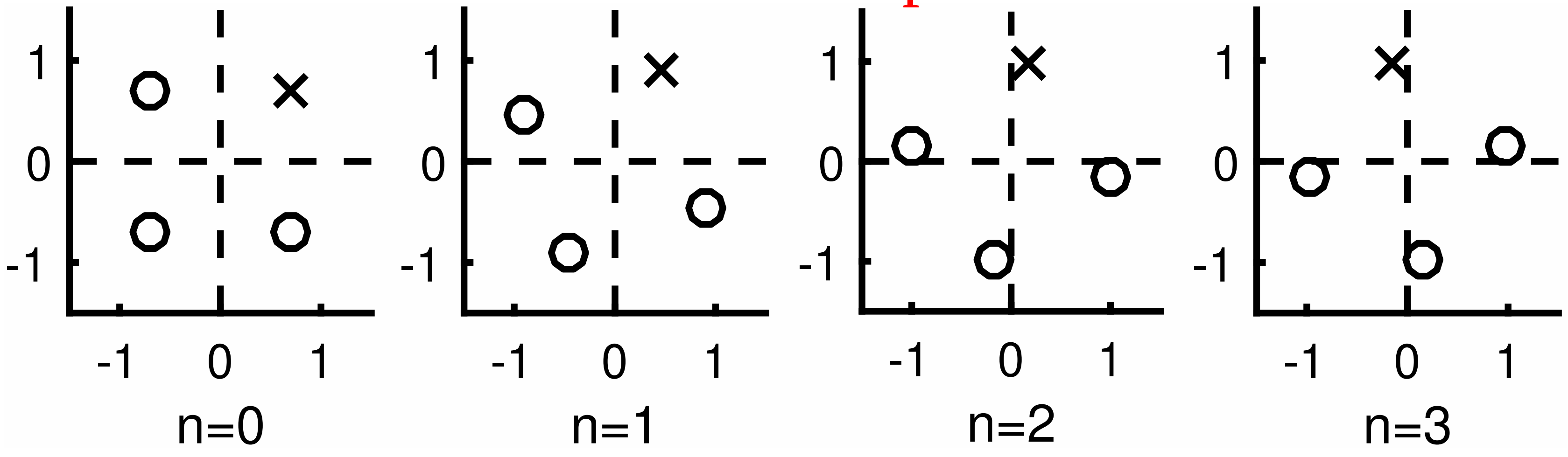
  - Rotation occurs after the convolution
  - Impacts channel estimation and thus equalization

Visualizing the frequency offset effect

- Special case of flat fading channel

$$y[n] = e^{(j2\pi\epsilon n)}h(n)s(n) + v(n)$$

  - ϵ is generally small but unknown
  - Rotates constellation by $e^{(j2\pi\epsilon n)}$



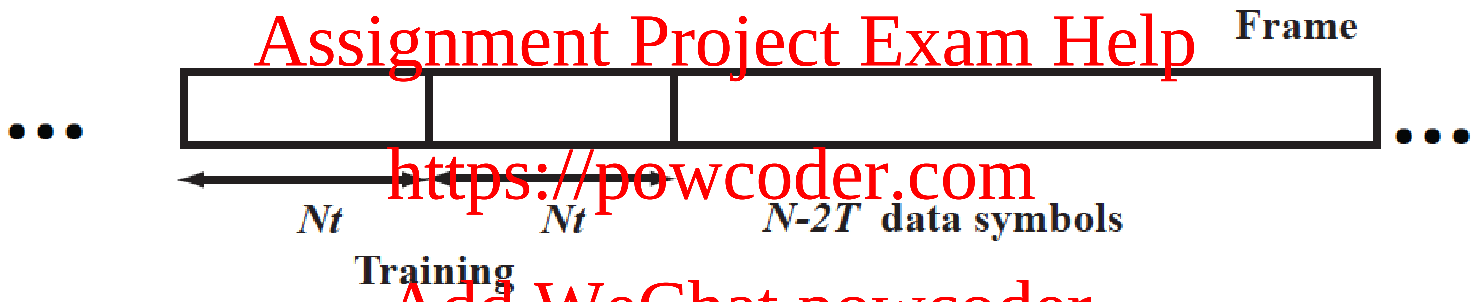
Frequency offset synchronization

- Frequency offset is a severe impairment
  - Even a small offset leads to significant degradation
- Frequency offset synchronization is challenging
  - Offset occurs after the convolution with the unknown channel
  - Impacts channel estimation and frame synchronization
- Methods for offset correction
  - Exploit structure in the received signal
  - Create and exploit structure using a known training signal

Frequency offset estimation and channel estimation

Assume transmission of two blocks of N_t training symbols (Barker Sequence) each

$$s[n] = s[n+N_t] = t(n) \qquad for \qquad n = 0,1,\dots.N_t - 1$$



This transmission structure is exploited at the receiver to estimate

The received sequence at the sampling instant n is

$$y[n] = e^{(j2\pi\epsilon n)}(h[n] \otimes s[n]) + v[n]$$

The received sequence at the sampling instant $n + N_t$ is

$$y[n+N_t] \\ = e^{(j2\pi\epsilon(n+N_t))}(h[n] \otimes s[n+N_t]) + v[n+N_t]$$

Frequency synchronization using training symbols

Exploiting the training structure of the frame format

$$s[n] = s[n+N_t] = t(n)\ n = 0,1,\ldots.N_t - 1$$

The discrete time received signal at n and $n + N_t$ is given by

$$y[n+N_t] \approx e^{(j2\pi\epsilon N_t)} y[n]$$
$$y[n+N_t] \approx a y[n]$$

The goal is to estimate a and apply to the entire transmission frame, hence the objective function is

$$J(a) = \|y[n+N_t] - a y[n]\|^2$$

Frequency synchronization using training symbols

$$\frac{\partial}{\partial a^*}\sum_{l=L}^{N_t-1} |y[n+N_t]-ay[n]|^2=\frac{\partial}{\partial a^*}\sum_{l=L}^{N_t-1}\left(y[n+N_t]-ay[n]\right)^*\left(y[n+N_t]-ay[n]\right)=0$$

Applying orthogonality principle

$$=\sum_{l=0}^{N_t-1}\left(y[n+N_t]-ay[n]\right)^* y[n+N_t]=0$$

$$\hat{a}=\frac{\sum_{l=L}^{N_t-1} y^*[n]y[n+N_t]}{\sum_{l=L}^{N_t-1} y^*[n+N_t]y[n+N_t]}$$

The normalised frequency offset is obtained from the angle

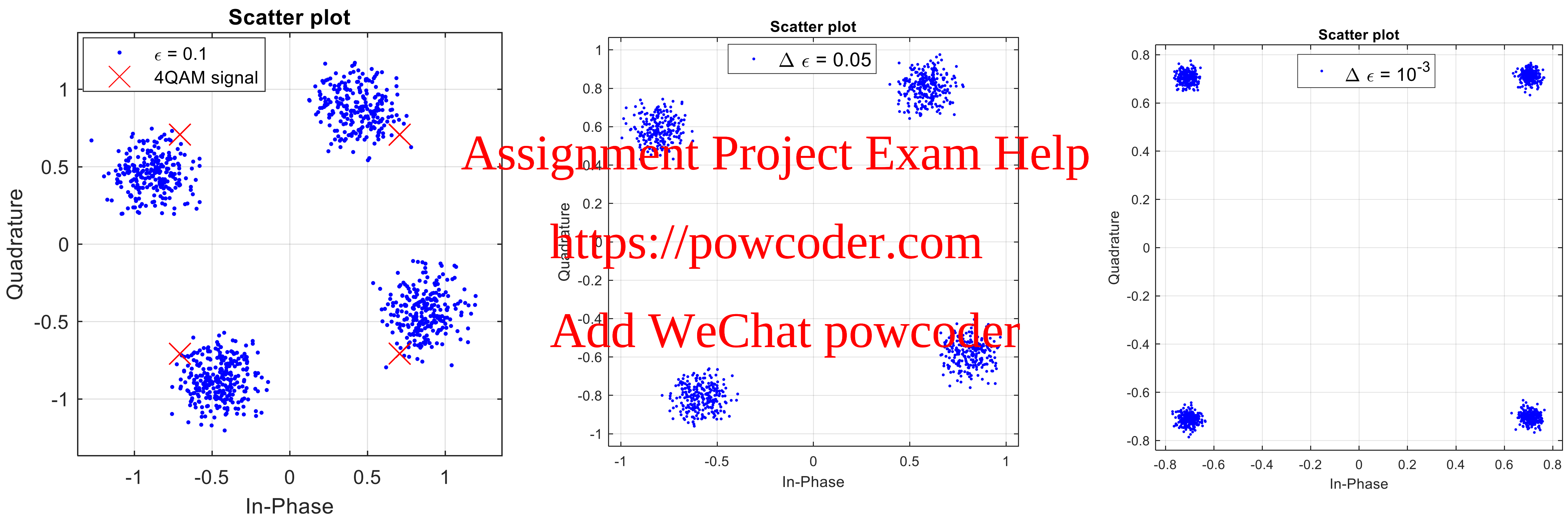
$$\angle\hat{a}=\angle\sum_{l=L}^{N_t-1} y^*[n]y[n+N_t]$$

$$2\pi\hat{\epsilon}N_t=\angle\sum_{l=L}^{N_t-1} y^*[n]y[n+N_t]$$

$$\hat{\epsilon}=\frac{\angle\sum_{l=L}^{N_t-1} y^*[n]y[n+N_t]}{2\pi N_t}$$

Is the estimate of the normalised frequency offset

Effect of frequency offset on digital constellation



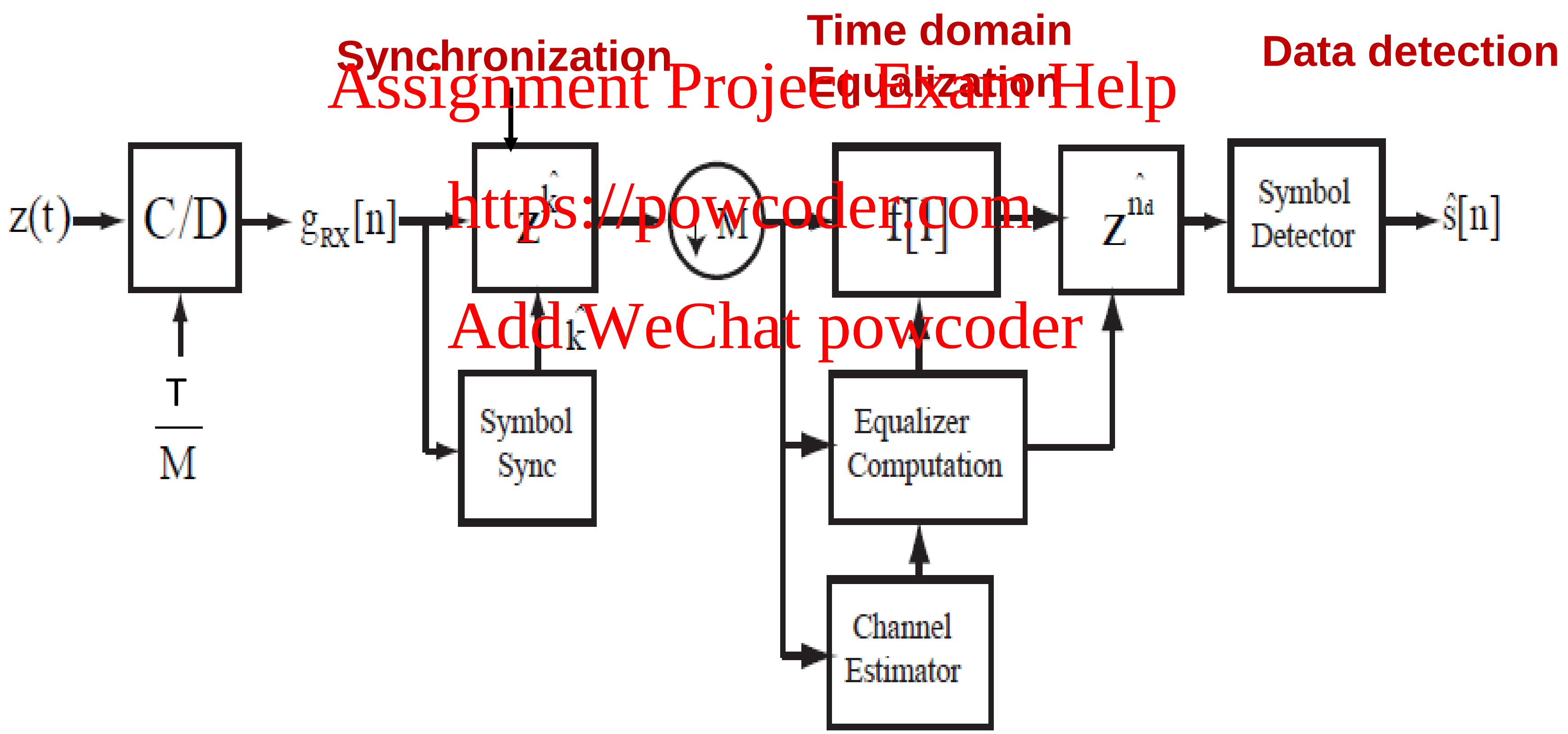
CFO estimation error decrease from 5% to 0.1%

Channel estimation and Equalization

$$y[n] = e^{(j2\pi\epsilon n)}(h(n) \otimes s(n)) + v(n)$$

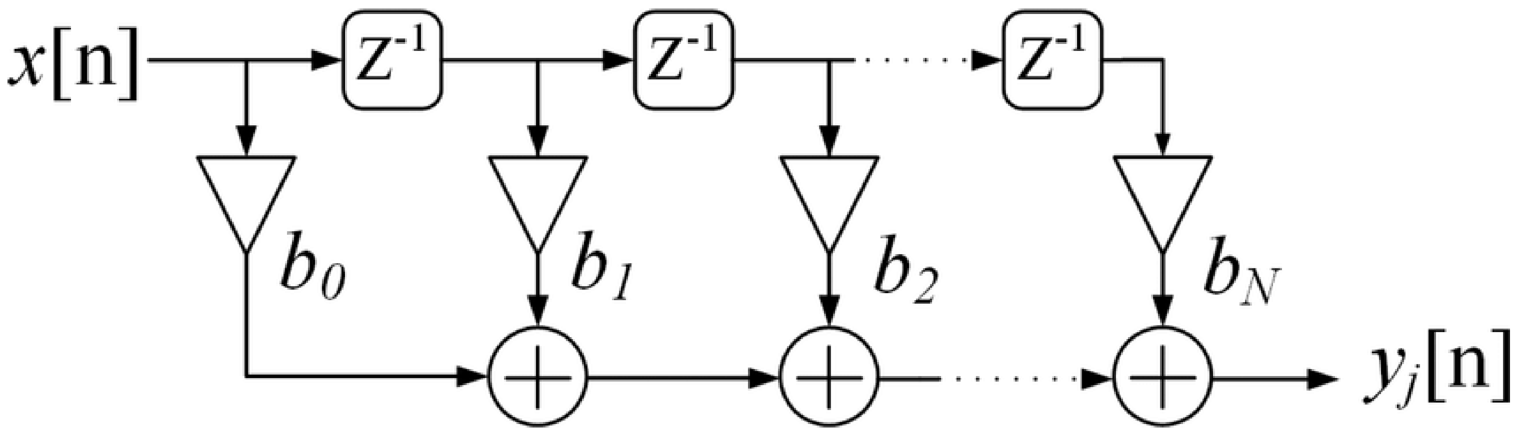
Next task is channel *estimation* and *equalization*

Receiver Processing stages



Training based Channel estimation

1. Channel in general is assumed to be *causal* and *finite impulse response* (FIR).
1. Each multipath component arrives with a different delay and phase shift
2. More channel parameters for estimation in the multipath/frequency selective channels when compared to frequency –flat channels.



Model for the received signal

- Consider the received signal after matched filtering and sampling

Channel distorted signal

Transmitter output

$$y(t) = g_{\mathrm{rx}}(t) * h(t) * \sqrt{E_{\mathrm{x}}} \sum_{m=-\infty}^{\infty} s[m] g_{\mathrm{tx}}(t - mT) + g_{\mathrm{rx}}(t) * v(t)$$

- The effective channel $h_{\mathrm{eff}} = g_{\mathrm{rx}}(t) * g_{\mathrm{tx}}(t) * \sqrt{E_{\mathrm{x}}} h(t)$

$$y(t) = \sum_{m=-\infty}^{\infty} s[m] h_{\mathrm{eff}}(t - mT) + g_{\mathrm{rx}}(t) * v(t)$$

Model for the received signal

- Samples obtained at the output of receive matched filter and down sampler

$$y[n] = \sqrt{E_x} \sum_{l=-\infty}^{\infty} h[l]s[n-l] + v[n]$$

- Suppose the channel is FIR and the signal decays with distance, meaning long reflections are very weak. The simplified system with FIR

Order of the FIR channnel, usually assumed known

$$y[n] = \sqrt{E_x} \sum_{l=0}^{L} h[l]s[n-l] + v[n]$$

Example

- Suppose that $h[n] = \sqrt{E_\mathrm{x}}\delta[n] + \sqrt{E_\mathrm{x}}h_1\delta[n-1]$
  - This is a two-tap discrete-time channel

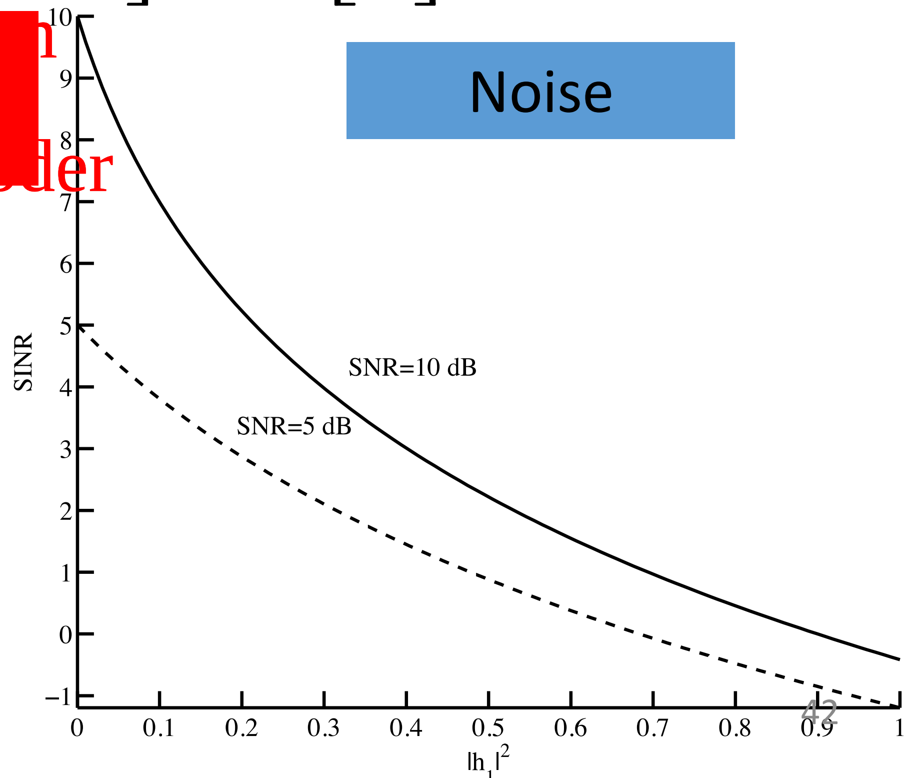
$$y[n] = \sqrt{E_\mathrm{x}}s[n] + \sqrt{E_\mathrm{x}}h_1 s[n-1] + v[n]$$

Signal

Inter Symbol Interference (ISI)

Noise

$$\mathrm{SINR} = \frac{E_\mathrm{x}}{E_\mathrm{x}|h_1|^2 + N_\mathrm{o}}$$



Training based Channel estimation

After symbol timing offset and frame synchronization, the disc rete time received signal

- in *frequency flat-fading* channel is given by

$$y[n] = \sqrt{(E_x)}hs[n] + v[n] \rightarrow \quad h = \alpha e^{j\phi}$$

- in *frequency-selective* fading environments is given by

$$y[n] = \sqrt{(E_x)} \sum_{l=0}^{L} h[l]s[n-l] + v[n]$$

Frequency flat channel estimation

Assume transmission of N_t training symbols so that $s[n] = t[n]$ and $y[n]$ is written as

$$y[n] = \sqrt{E_x}\alpha e^{j\phi} t[n] + v[n]$$

The n^th received signal during training substitute $a = \sqrt{E_x}\alpha e^{j\phi}$

$$y[n] = at[n] + v[m]$$

The objective is to estimate the unknown scalar channel 'a'

Objective function

$$J(\mathrm{a}) = \sum_{n=0}^{N_t-1} \|y[n] - at[n]\|^2$$

$$= \sum_{n=0}^{N_t-1} (y[n] - at[n])^*(y[n] - at[n])$$

Optimization

Taking Partial derivative of the cost function w.r.t a^*, we have,

$$\frac{\partial}{\partial \mathrm{a}^*} J(\mathrm{a}) = \frac{\partial}{\partial \mathrm{a}^*} \sum_{n=0}^{N_t-1} (y[n] - at[n])^*(y[n] - at[n])$$

$$= \frac{\partial}{\partial \mathrm{a}^*} \sum_{n=0}^{N_t-1} (y^*[n]y[n] - a^*t^*[n]y[n] - at[n]y^*[n] + a^*at^*[n]t[n]$$

$$\frac{\partial}{\partial \mathrm{a}^*} J(\mathrm{a}) = \sum_{n=0}^{N_t-1} at^*[n]t[n] - t^*[n]y[n]$$

Sliding Correlator

The optimal Least squares estimate for the channel is obtained by

$$\sum_{n=0}^{N_t-1} at^*[n]t[n] - t^*[n]y[n] = 0$$

$$h_{LS} = \frac{\sum_{n=0}^{N_t-1} t^*[n]y[n]}{\sum_{n=0}^{N_t-1} t^*[n]t[n]}$$

$$h_{LS} = (\boldsymbol{t}^*\boldsymbol{t})^{-1}(\boldsymbol{t}^*\boldsymbol{y})$$

performs both frame sync and channel estimation

Frequency selective channel estimation

Least squares based channel estimation

$$\{\hat{h}[0], \hat{h}[1], \ldots, \hat{h}[L]\} = \arg \min_{a[0],a[1],\ldots,a[L]} \sum_{n=L}^{N_t} \left\| y[n] - \sum_{l=0}^{L} a[l]t[n-l] \right\|^2$$

$$\underbrace{\begin{bmatrix} y[L] \\ y[L+1] \\ \vdots \\ y[N_t-1] \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} t[L] & \cdots & t[0] \\ t[L+1] & \ddots & \vdots \\ \vdots & & \vdots \\ t[N_t-1] & \cdots & t[N_t-1-L] \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} a[0] \\ a[1] \\ \vdots \\ a[L] \end{bmatrix}}_{\mathbf{a}}.$$

$$\hat{\mathbf{h}} = (\mathbf{T}^*\mathbf{T})^{-1}\mathbf{T}^*\mathbf{y}.$$

Channel estimation

- Signal model

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N_{\mathrm{tr}}-1] \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} t[0] \\ t[1] \\ \vdots \\ t[N_{\mathrm{tr}}-1] \end{bmatrix}}_{\mathbf{t}} h + \underbrace{\begin{bmatrix} v[0] \\ v[1] \\ \vdots \\ v[N_{\mathrm{tr}}-1] \end{bmatrix}}_{\mathbf{v}}$$

$$\boxed{\mathbf{y} = \mathbf{t}h + \mathbf{v}}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x}_{LS} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\mathbf{b}$$

- LS solution is

$$\hat{h} = (\mathbf{t}^*\mathbf{t})^{-1}\mathbf{t}^*\mathbf{y}$$

$$= \frac{\sum_{n=0}^{N_{\mathrm{tr}}-1} t^*[n]y[n]}{\sum_{n=0}^{N_{\mathrm{tr}}-1} t^*[n]t[n]}$$

Correlator – can jointly perform frame synchronization and channel estimation

Corrects for scaling

Blind frequency offset estimation for 4-QAM

Exploit symmetry in the 4-QAM constellation to develop a blind frequency offset estimator,

Normalized 4-QAM constellation symbols are points on the unit circle. Assuming a static channel h

Taking the fourth power that $s^4[n] = \left(me^{\left(\frac{j\pi}{4}\right)}\right)^4 = -1$

$$y^4[n] = e^{(j2\pi\epsilon 4n)}(h^4 \otimes s^4[n]) + v^4[n]$$

$$y^4[n] = e^{(j2\pi\epsilon 4n)}(h^4 \otimes -1) + v[n]$$

$$= -e^{(j2\pi\epsilon 4n)}(h^4) + v[n]$$

Calculate the phase from

$$\angle(y^4[n+1]\, y^{*4}[n]) = \angle e^{(j2\pi\epsilon 4(n+1))} e^{(-j2\pi\epsilon 4n)} = 8\pi\epsilon + \tilde{v}\,[n]$$

$$\hat{\epsilon} = \frac{1}{8\,\pi(N-1)} \sum_{n=1}^{N} \angle(y^4[n+1]\, y^{*4}[n])$$

Minimum Mean squared error (MMSE) Channel estimation

$y[n] = ht[n] + v[n]$

Let g be an optimal MMSE chaanel estimate of h

The MMSE estimate of $\mathbf{t} = \left[t[0] \quad t[L+1] \quad \ldots \quad t[N-1] \right]$

$\hat{t}[n] = g^* y[n] = ht[n] + v[n]$

$e[n] = g^* y[n] - \hat{t}[n]$

Taking the partial derivative w.r.t g^* $\mathrm{E}\left[|e[n]|^2 \right]$

$$\mathrm{E}\left[\left(g^* y[n] - \hat{t}[n] \right)^* y[n] \right] = 0$$

$$g_{MMSE} = \left(\mathrm{E}\left[\mathbf{y}^* \mathbf{y} \right] \right)^{-1} \mathrm{E}\left[\mathbf{t}^* \mathbf{y} \right]$$

$$g_{MMSE} = \left(\mathbf{C}_{YY} \right)^{-1} \mathbf{C}_{Yt}$$

Compare LS and MMSE channel Estimators

LS based channel estimate

$$h_{LS} = (t^* t)^{-1} (t^* y)$$

MMSE based channel estimate $\mathbf{y} = h\mathbf{t} + \mathbf{v}$

$$g_{MMSE} = \left(\mathbf{C}_{YY}\right)^{-1} \mathbf{C}_{Yt}$$

$$\mathbf{C}_{yy} = \mathrm{E}\left[\mathbf{y}^* \mathbf{y}\right]$$

$$\mathbf{C}_{yt} = \mathrm{E}\left[\mathbf{y}^* \mathbf{t}\right]$$

$$\hat{\mathbf{t}}_{MMSE} = g^*{}_{MMSE} \mathbf{y}$$

$$g_{MMSE} = \left(h^* h + \frac{\sigma_n^2}{\gamma^2} \right)^{-1} h^* \qquad \mathrm{E}\left[\mathbf{v}^* \mathbf{v}\right] = \sigma_n^2 \quad \text{and} \quad \mathrm{E}\left[\mathbf{t}^* \mathbf{t}\right] = \gamma^2$$