Problem Set 1 (Solow-Swan, Ramsey, Diamond Models)
ECON 6002/6702
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Due date: Wed, 6 September, 6pm online

NOTE: To receive full marks, show your workings for algebraic manipulations.

1. Suppose the aggregate production function in the Solow-Swan model is Cobb-Douglas, $y = k^\alpha$, with $\alpha = 0.3$. Assume population growth $n = 2\%$, technology growth $g = 3\%$, and depreciation $\delta = 10\%$.

   (a) Derive k^*, y^*, and c^* as functions of the model parameters and determine their values when the saving rate $s = 15\%$.

   (b) Assume both labour and capital are paid their marginal products and the economy is on a balanced growth path at time $t = 0$:

      i) What is the real wage $w(0)$ if $A(0) = 1$?

      ii) What is the growth rate of wages $\dot{w}/w$?

      iii) What is the return to "working" capital r? (*working refers to capital minus depreciation*)

      iv) What are the shares of income going to (both "working" and "dead/depreciated") capital and to labour?

   (c) How would your answers to part (b) change if the saving rate were $s' = 30\%$ and the economy was on the balanced growth path?

   (d) Draw the transition given a change in the saving rate from $s = 15\%$ to $s' = 30\%$ in the basic diagram for the Solow model (You may use MATLAB to make this diagram if you like).

   (e) What are the growth rates of real wages, $\dot{w}/w$, and the return on working capital, $\dot{r}/r$ at the beginning of the transition when $k = 1$? What do these results predict about real wage growth and the return on working capital as an economy with a high saving rate such as China gets closer to the new steady state?

   (f) Can the economy achieve a higher c^* than for $s = 30\%$? Why or why not?

2. Now consider the Ramsey model with a Cobb-Douglas aggregate production function, $y = k^\alpha$ and $\alpha = 0.3$. Assume the discount rate $\rho = 5\%$, population growth $n = 2\%$, technology growth $g = 3\%$, and there is no capital depreciation.

   (a) Derive k^*, y^*, and c^* as functions of the model parameters and determine their values when the coefficient of relative risk aversion $\theta = 5$.

   (b) How do your answers to part (a) change if the coefficient of relative risk aversion changes to $\theta = 2$ (i.e., the intertemporal elasticity of substitution rises from 0.2 to 0.5)?

   (c) Draw the transition given the change in the coefficient of relative risk aversion in the phase diagram for the Ramsey model.

   (d) For this economy, what is the impact of a permanent fall in the growth rate of technology on the saving rate along the balanced growth path? How does your answer depend on the intertemporal elasticity of substitution? Hint: calculate $\partial s^*/\partial g$, where $s^* = 1 - c^*/y^*$.

(e) What happens to s^* given a permanent fall in the growth rate of technology to $g = 2\%$ under both scenarios of $\theta = 5$ and $\theta = 2$?

3. Consider the Diamond model with logarithmic utility ($\ln(C_t)$) and Cobb-Douglas production. Describe in words and using equations how the following affects k_{t+1} as a function of k_t:

   (a) A fall in n.

   (b) A downward shift in the production function (that is, $f(k)$ takes the form Bk^{α}, and B falls).

   (c) A rise in α

4. *The Covid Economic Slump:* Let's consider the economic impact of the pandemic on the economy. Assume that the pandemic evolves according to the basic SIR model:

$$\dot{S} = -\beta S(t)I(t) \qquad (1)$$

$$\dot{I} = \beta S(t)I(t) - \gamma I(t) \qquad (2)$$

$$\dot{R} = \gamma I(t) \qquad (3)$$

with initial conditions $S(0)$, $I(0)$, and $R(0)$, satisfying

$$S(0) + I(0) + R(0) = 1$$

and

$$\dot{S} + \dot{I} + \dot{R} = 0 \Rightarrow S(t) + I(t) + R(t) = 1$$

for all $t \geq 0$. Assume that some fraction $0 < \phi < 1$ of infected people are too sick to work. This implies that during the pandemic the effective labour force $E(t)$ is given by

$$E(t) = (1 - \phi I(t))\bar{L},$$

where we assume the labour ($\bar{L}$) is fixed and does change over time. Finally, assume that production is Cobb-Douglas such that

$$Y(t) = \bar{K}^{\alpha}(\bar{A}E(t))^{1-\alpha},$$

where capital is fixed ($\bar{K}$) and technology ($\bar{A}$) are fixed.

   (a) Find an expression for the growth rate of output in terms of the growth rate of infections $g_I = \dot{I}/I(t)$. Is output growth positively or negatively related to infection growth?

   (b) Let $\bar{K} = \bar{A} = \bar{L} = 1$, $\alpha = 0.33$, $\phi = 0.5$, $S(0) = 1$, and the solution path for infections be

$$I(t) = 1 - S(t) + \frac{1}{\mathcal{R}_0}\ln(S(t)/S(0)).$$

   Create a MATLAB graph that plots output against the percentage of the population that is susceptible. Compare $\mathcal{R}_0 = 2.5$ to $\mathcal{R}_0 = 1.4$, where the latter is the average $\mathcal{R}_0$ of seasonal influenza. Assuming ϕ is the same for Covid and influenza, is Covid's impact on the economy similar to influenza in this model? (*Tip: set the axes to go no higher than 1, e.g.* axis([0 1 0.5 1]) *so that only values of* $I(t) \geq 0$ *are shown.*)

(c) Using your MATLAB program, consider whether it is possible to save lives and the economy by practicing social distancing. Assume that you can lower $\mathcal{R}_0$ (lower β) by lowering $\bar{L}$. Create some graphs that show that it is possible to actually lose less output to the pandemic by forcing some people to stay home. For example, show that if we reduce $\bar{L}$ to 0.95, i.e., 5% of the labour force is made to stay home, but that by eliminating those jobs the $\mathcal{R}_0$ falls to 1.4, it is possible that the recession is less and we save lives.

(d) Critique your answer to (c). What assumptions does it rely on?