ECON 61001: *Lecture 6*

Alastair R. Hall

The University of Manchester

- More on GLS
- Cross-section data with heteroscedasticity
  - OLS-based inference
  - GLS/WLS
  - testing for heteroscedasticity

Recall that our model is:

$$y = X\beta_0 + u$$

where

- *CA1*: true model is: $y = X\beta_0 + u$.
- *CA2*: X is *fixed in repeated samples*.
- *CA3*: X is rank k.
- *CA4*: $E[u] = 0$.
- *CA5-NS* $Var[u] = \Sigma$ where Σ is a $T \times T$ positive definite matrix.
- *CA6*: $u \sim$ Normal.

The Generalized Least Squares estimator of β_0 is:

$$\hat{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$$

If Assumptions *CA1-CA4*, *CA5-NS* and *CA6* hold then:

$$\hat{\beta}_{GLS} \sim N\left(\beta_0, (X'\Sigma^{-1}X)^{-1}\right).$$

But need Σ in order to calculate $\hat{\beta}_{GLS}$.

If Σ unknown then GLS is an infeasible estimator.

Solution assume $t - s^{th}$ element of Σ is given by:

$$\sigma_{t,s} = h_{t,s}(z_t, z_s; \alpha)$$

where

- $h_{t,s}$ is some specified function,
- z_t is a vector of observable variables,
- α is a $p \times 1$ vector of parameters.

So $\Sigma = \Sigma(\alpha)$.

Estimate α from the sample $\{y_t, x_t', z_t';\ t = 1, 2, \ldots, T\} \rightarrow \hat{\alpha}$.

Then set $\hat{\Sigma} = \Sigma(\hat{\alpha})$.

$\rightarrow$ Feasible Generalized Least Squares (FGLS) estimator,

$$\hat{\beta}_{FGLS} = (X'\hat{\Sigma}^{-1}X)^{-1}X'\hat{\Sigma}^{-1}y.$$

Does FGLS have finite sample properties of GLS? Maybe: consider

$$E[\hat{\beta}_{FGLS}] = \beta_0 + E\left[(X'\hat{\Sigma}^{-1}X)^{-1}X'\hat{\Sigma}^{-1}u\right] = \beta_0?$$

FGLS does inherit large sample properties of GLS.

Assumptions:

- *CS1:* $y_i = x_i'\beta_0 + u_i$
- *CS2:* $\{(u_i, x_i'), i = 1, 2, \ldots N\}$ forms an independent and identically distributed sequence.
- *CS3:* $E[x_i x_i'] = Q$, finite, p.d.
- *CS4:* $E[u_i|x_i] = 0$
- *CS5-H:* $Var[u_i|x_i] = h(x_i)$, positive, and $h(x_i) \neq h(x_j)$ for some $i \neq j$.

Note:

- cross-section data with random sample from homogenous population
- have conditional heteroscedasticity but not unconditional heteroscedasticity

Recall:

$$N^{1/2}(\hat{\beta}_N - \beta_0) = \left(N^{-1} \sum_{i=1}^{N} x_i x_i' \right)^{-1} N^{-1/2} \sum_{i=1}^{N} x_i u_i .$$

As in Lecture 4: WLLN & Slutsky's Theorem $\rightarrow$

$$\left(N^{-1} \sum_{i=1}^{N} x_i x_i' \right)^{-1} \xrightarrow{p} Q^{-1},$$

and CLT $\rightarrow$

$$N^{-1/2} \sum_{i=1}^{N} x_i u_i \xrightarrow{d} N(0, \Omega), \text{ where } \Omega = lim_{N\rightarrow\infty} \Omega_N$$

and

$$\Omega_N = Var\left[N^{-1/2} \sum_{i=1}^{N} x_i u_i \right].$$

What is Ω here?

Assumption $CS2 \Rightarrow \{x_i u_i;\ i = 1, 2, \ldots N\}$ are i.i.d. and so $Cov[x_i u_i, x_j u_j] = 0$ $(i \neq j)$.

$\Rightarrow \Omega_N = Var[x_i u_i]$.

Using $E[x_i u_i] = 0$

$Var[x_i u_i] \;=\; E[u_i^2 x_i x_i'] = E\left[\, E[u_i^2 | x_i] x_i x_i' \,\right] = E[h(x_i) x_i x_i'] = \Omega_h,$ say.

Therefore, under our assumptions, we have:

$$N^{-1/2} \sum_{i=1}^{N} x_i u_i \xrightarrow{d} N\,(\,0,\, \Omega_h\,)\,.$$

Under Assumptions *CS1*-*CS4* and *CS5-H*:

$$N^{1/2}(\hat{\beta}_N - \beta_0) \xrightarrow{d} N\,(\,0,\, V_h\,)\,.$$

where $V_h = Q^{-1}\Omega_h Q^{-1}$.

To use this result as basis for inference, need a consistent estimator of V_h and so Ω_h.

Using WLLN we have:

$$N^{-1}\sum_{i=1}^{N} u_i^2 x_i x_i' \xrightarrow{p} E[u_i^2 x_i x_i'] = \Omega_h.$$

Also it can be shown that

$$N^{-1}\sum_{i=1}^{N} e_i^2 x_i x_i' - N^{-1}\sum_{i=1}^{N} u_i^2 x_i x_i' \xrightarrow{p} 0.$$

and so

$$\hat{\Omega}_h = N^{-1}\sum_{i=1}^{N} e_i^2 x_i x_i' \xrightarrow{p} \Omega_h.$$

Set $\hat{Q} = N^{-1}X'X$ then

$$\hat{V}_h = \hat{Q}^{-1}\hat{\Omega}_h\hat{Q}^{-1} \xrightarrow{p} V_h.$$

Let $\hat{V}_{h,\ell,\ell}$ is (ℓ,ℓ) element of $\hat{V}_h$ then:

$$\sqrt{\hat{V}_{h,\ell,\ell}/N} \sim \text{“White standard errors”}$$

Can then perform inference using same techniques as in Lecture 4 provided we use modified variance estimator.

For example, an approximate $100(1-\alpha)\%$ confidence interval for $\beta_{0,\ell}$ is given by,

$$\left(\hat{\beta}_{N,\ell} \pm z_{1-\alpha/2}\sqrt{\hat{V}_{h,\ell,\ell}/N}\right).$$

For other inference procedures see Lecture Notes Section 4.3.1.

Impose Assumptions *CS1-CS4*, & *CS5-H*: $\Rightarrow E[u|X] = 0$ & (with $\sigma_i^2 = h(x_i)$)

$$Var[u|X] = \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & 0 & \sigma_N^2 \end{bmatrix}.$$

$\rightarrow$

$$D = \begin{bmatrix} \sigma_1 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & 0 & \sigma_N \end{bmatrix}, \text{ and } S = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & 0 & \frac{1}{\sigma_N} \end{bmatrix}$$

and transformed regression model is

$$\frac{y_i}{\sigma_i} = \left(\frac{1}{\sigma_i}\right)\beta_{0,1} + \left(\frac{x_{i,2}}{\sigma_i}\right)\beta_{0,2} + \ldots + \left(\frac{x_{i,k}}{\sigma_i}\right)\beta_{0,k} + \left(\frac{1}{\sigma_i}\right)u_i$$

Asymptotic properties of GLS

Theorem: *If Assumptions CS1 - CS4 hold then* $\hat{\beta}_{GLS}$ *is a consistent estimator for* β_0.

Theorem: *If Assumptions CS1 - CS4 and CS5-H hold then:*

$$N^{1/2}(\hat{\beta}_{GLS} - \beta_0) \xrightarrow{d} N(0, V_{GLS}),$$

where $V_{GLS} = \{E[\sigma_i^{-2} x_i x_i']\}^{-1}$.

Need to assume functional form for $\Sigma(\alpha)$:

- $\sigma_i^2 = \sigma_0^2 * v(x_i)$
  - suffices to divide variables by $\sqrt{v(x_i)}$ (see example later)

- $\sigma_i^2 = h(x_i, \alpha)$
  - $E[u_i^2|x_i] = h(x_i, \alpha) \Rightarrow$

$$u_i^2 = h(x_i, \alpha) + a_i, \text{ where } E[a_i|x_i] = 0.$$

  - So estimate α using model via Nonlinear LS (NLS) or OLS

$$e_i^2 \;=\; h(x_i, \alpha) + \text{“error”}.$$

Let $\{w_i;\ i = 1, 2, \ldots, N\}$ be a set of positive constants and use to weight observations in regression model:

$$w_i * y_i = (w_i * x_i)'\beta_0 + w_i * u_i.$$

Weighted Least Squares (WLS) is OLS estimator based on weighted regression model:

$$\hat{\beta}_{WLS} = (X'W_2X)^{-1}X'W_2y.$$

where $W_2 = diag(w_1^2, w_2^2, \ldots, w_N^2)$.

It can be shown that under Assumptions *CS1-CS4*, & *CS5-H* that

$$N^{1/2}(\hat{\beta}_{WLS} - \beta_0) \xrightarrow{d} N(0, V_{WLS})$$

where $V_{WLS} = Q_w^{-1}\Omega_w Q_w^{-1}$, $Q_w = plim_{T\to\infty} N^{-1}X'W_2X$, $\Omega_w = plim_{T\to\infty} N^{-1}X'W_2\Sigma W_2X$.

- $w_i = 1$ for all $i \Rightarrow$ WLS=OLS

- $w_i = 1/\sigma_i$ for all $i \Rightarrow$ WLS=GLS

Re-weighting of observations is source of efficiency gains from GLS over OLS.

What happens if assume incorrect model for σ_i^2?

- GLS is WLS and may or may not be more efficient than OLS;
- inferences based on "GLS" valid but must be performed with heteroscedasticity robust estimator of WLS variance formula (see Lecture Notes).

Assume:

$$\sigma_i^2 \;=\; h(\delta \,+\, z_i'\alpha)$$

where

- $h(\,\cdot\,)$ is (a twice continuously differentiable) function, independent of i, and $h(\,\cdot\,) > 0$,
- z_i is a $(p \times 1)$ vector of observable variables,

- $(\delta, \alpha')'$ is a vector of $(p+1) \times 1$ of parameters.

Test:

- $H_0 : \alpha = 0$ vs $H_A : \alpha_\ell \neq 0$ for at least one $\ell = 1, 2, \ldots, p$
- Test stat is $BP_N = NR^2$ where R^2 is from regression of e_i^2 on $(1, z_i')$.
- Under H_0: $BP_N \xrightarrow{d} \chi^2_p$ - see Lecture Notes for regularity conditions.

Note:

- statistic does not depend on $h(\cdot)$.

- depends on z_i
  - White's (1980) direct test for heteroscedasticity (see Lecture Notes)

Suppose a researcher is interested in studying the savings behaviour of households:

$$y_i = x_i'\beta_0 + u_i = \beta_{0,1} + \beta_{0,2} m_i + u_i$$

where

- y_i is the level of savings of household i
- m_i is household income.
- $Var[u_i|x_i]$ arguably depends on m_i.

OLS estimation results:

$$\begin{array}{cccccc} y_i & = & 124.84 & + & 0.147 & m_i. \\ & & (655.39) & & (0.058) & \\ & & [522.91] & & [0.061] & \end{array}$$

where $(\cdot) =$ conventional OLS s.e.'s, $[\cdot] =$ White s.e.'s

Now suppose $Var[u_i \mid x_i] = \sigma_0^2 m_i$.

WLS estimation results:

$$\begin{array}{cccccc} y_i & = & -124.95 & + & 0.172 & m_i. \\ & & (480.86) & & (0.057) & \\ & & [266.59] & & [0.050] & \end{array}$$

The number in parentheses are the standard errors assuming have correct model for heteroscedasticity; the number in brackets are heteroscedastcity robust standard errors.

Further reading

- Notes: Section 4.3.
- Greene:
  - Large sample properties of GLS, Section 9.3 (general discussion)
  - Heteroscedasticity: OLS, Section 9.4
  - Testing for heteroscedasticity, Section 9.5
  - Heteroscedasticity: GLS/WLS, Section 9.6
  - Application, Section 9.7