ECON 61001: Review of some linear algebra concepts

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- Vector spaces

- Linear (in)dependence
- Rank of a matrix

- Quadratic forms and definiteness of matrices
- Spectral decomposition

Resources: Orme (2009) *Linear Algebra Notes* and sequence of videos, both on BB in folder "Linear Algebra Resources".

The totality of all $n \times 1$ vectors is called the n-dimensional vector space.

- sometimes called the n-dimensional Euclidean space and denoted

$$\mathbb{R}^n = \left\{ \mathbf{x} : \mathbf{x}' = (x_1, x_2, \ldots, x_n); x_i \in \mathbb{R}, i = 1, 2, \ldots, n \right\}.$$

Let $\{\mathbf{e}_i\}_{i=1}^{n}$ denote the n-dimensional unit vectors (i^{th} element of $\mathbf{e}_i$ is one and all others are zero).

Then we have

$$\mathbf{x} \;=\; x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \ldots + x_n\mathbf{e}_n \;=\; \sum_{i=1}^{n} x_i\mathbf{e}_i$$

Note:

(i) any vector $\mathbf{x} \in \mathbb{R}^n$ can be written in this way

(ii) $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are all $n \times 1$ and none of the $\mathbf{e}_i$ can be expressed as a linear combination of the remaining $\mathbf{e}_j$, $j \neq i$.

- (i) $\rightarrow \{\mathbf{e}_i\}_{i=1}^n$ spans $\mathbb{R}^n$.
- (ii) $\rightarrow \{\mathbf{e}_i\}_{i=1}^n$ is a linearly independent set.
- (i) & (ii) $\rightarrow \{\mathbf{e}_i\}_{i=1}^n$ forms a basis for $\mathbb{R}^n$.

A vector space, V, is a non-empty set of vectors satisfying, for $\mathbf{a}, \mathbf{b} \in V$:

- $\mathbf{a} + \mathbf{b} \in V$;
- $\lambda\mathbf{a} \in V$ for any scalar λ.

A sub-space of $\mathbb{R}^n$ is a non-empty subset of $\mathbb{R}^n$ which is also a vector space.

Linearly (in)dependent sets, Orme Chap 2

Let $\{\mathbf{x}_j\}_{j=1}^{m}$ be a collection of $n \times 1$ vectors.

If there exist scalars $\{\lambda_j\}_{j=1}^{m}$ with at least one $\lambda_j \neq 0$ such that $\sum_{j=1}^{m} \lambda_j \mathbf{x}_j = 0$ then $\{\mathbf{x}_j\}_{j=1}^{m}$ is said to form a linearly dependent set.

Conversely, if $\sum_{j=1}^{m} \lambda_j \mathbf{x}_j = 0$ only holds for $\lambda_j = 0$, $j = 1, 2, \ldots, m$ then $\{\mathbf{x}_j\}_{j=1}^{m}$ is said to form a linearly independent set.

Define

$$\underset{n\times m}{\mathbf{X}} = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,m} \end{bmatrix} = \left[\underset{(n\times 1)}{\mathbf{x}_{\bullet,1}}, \underset{(n\times 1)}{\mathbf{x}_{\bullet,2}}, \ldots, \underset{(n\times 1)}{\mathbf{x}_{\bullet,m}} \right]$$

If $\{\mathbf{x}_{\bullet,j}\}_{j=1}^{m}$ form a linearly independent set then $\mathbf{X}$ has full column rank that is, the column rank = # of columns.

If $\{\mathbf{x}_{\bullet,j}\}_{j=1}^{m}$ form a linearly dependent set then $\mathbf{X}$ does not have full column rank and:

column rank of $\mathbf{X}$ = maximum # of columns of $\mathbf{X}$ that can form a linearly independent set.

Similarly

$$\underset{n\times m}{\mathbf{X}} = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,m} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \mathbf{x}_{1,\bullet} \\ \mathbf{x}_{2,\bullet} \\ \vdots \\ \mathbf{x}_{n,\bullet} \end{bmatrix} \end{array} \begin{array}{c} (1\times m) \\ (1\times m) \\ \vdots \\ (1\times m) \end{array}$$

If $\{\mathbf{x}_{i,\bullet}\}_{i=1}^{n}$ form a linearly independent set then $\mathbf{X}$ has full row rank that is, the row rank $= \#$ of rows.

If $\{\mathbf{x}_{i,\bullet}\}_{i=1}^{n}$ form a linearly dependent set then $\mathbf{X}$ does not have full row rank and:

row rank of $\mathbf{X}$ $=$ maximum $\#$ of rows of $\mathbf{X}$ that can form a linearly independent set.

Key result: row rank of $\mathbf{X}$ = column rank of $\mathbf{X}$.

So define rank of $\mathbf{X}$ = row/column rank of $\mathbf{X}$, and denote by $rank(\mathbf{X})$.

Important results involving rank:

- for $\mathbf{X}$ $(n \times m)$: $rank(\mathbf{X}) \leq min[n, m]$.
- $rank(AB) \leq min[rank(A), rank(B)]$.
- if $m = n$ then $\mathbf{X}$ is nonsingular ($det(\mathbf{X}) \neq 0$ and $\mathbf{X}^{-1}$ exists) if and only if $rank(\mathbf{X}) = m(= n)$.

Let $\mathbf{A}$ be a $n \times n$ symmetric matrix, and $\mathbf{x}$ be a $n \times 1$ vector.

A quadratic form in $\mathbf{A}$ takes the form $\mathbf{x}'\mathbf{A}\mathbf{x}$ and is a scalar.

- $\mathbf{A}$ is positive definite (pd) iff $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$.

- $\mathbf{A}$ is positive semi-definite (psd) iff $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \neq 0$.

The definitions of negative definiteness and negative semi-definiteness are analogous only with direction of inequalities reversed.

Note $\mathbf{A}$ can also be indefinite in which case quadratic forms in $\mathbf{A}$ can be either positive or negative depending on $\mathbf{x}$.

Let $\mathbf{A}$ be a (real) symmetric $n \times n$ matrix then there exists

- $\mathbf{\Lambda} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where λ_i are (real) scalars,
- $\mathbf{X} = [\mathbf{x}_{\bullet,1}, \ldots, \mathbf{x}_{\bullet,n}]$ an orthogonal matrix (that is, $\mathbf{X}^{-1} = \mathbf{X}'$),

such that

$$\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}' \quad \sim \text{spectral decomposition of } A$$

$\{\lambda_i, \mathbf{x}_{\bullet,i}\}_{i=1}^{n}$ known as eigenvalues and eigenvectors of $\mathbf{A}$.

As a result:

- $det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$.
- $tr(\mathbf{A}) \stackrel{d}{=} \sum_{i=1}^{n} a_{i,i} = \sum_{i=1}^{n} \lambda_i$

Connection to positive definiteness of $\mathbf{A}$:

- $\mathbf{A}$ is positive (negative) definite iff $\lambda_i > 0$ (< 0) for all $i = 1, 2, \ldots, n$.
- $\mathbf{A}$ is positive (negative) semi- definite iff $\lambda_i \geq 0$ (≤ 0) for all $i = 1, 2, \ldots, n$.

Time to test your understanding

- I have posted some questions on Blackboard that test your understanding. Please try to do these.

- The solutions are also posted for your convenience but please do contact me if you have any questions about this material.