ECON61001: 2020-21
Econometric Methods

Solutions to Mock Exam

1.(a) $x'Ax > 0$ for all $x \neq 0$.

1.(b) $rank(A) = m$ - else there exists $x \neq 0$ such that $Ax = 0$ which violates A pd.

1.(c) For 2×2 matrix A, if A satisfies $tr(A) > 0$ and $|A| > 0$ then A is p.d. Here: $tr(A_1) = 4$, $|A_1| = 3$ so A_1 pd; $tr(A_2) = 10$, $|A_2| = -4$ so A_2 not pd.

2.(a) Given this specification the conditional distribution function of y given x, is:

$$\begin{aligned} P(y=1|x) &= \Lambda(x'\beta_0) \\ P(y=0|x) &= 1 - \Lambda(x'\beta_0). \end{aligned}$$

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Therefore, we have:

$$E[y|x] = 1\Lambda(x'\beta_0) + 0 * (1 - \Lambda(x'\beta_0) = \Lambda(x'\beta_0).$$

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Using the Law of Iterated Expectations $E[y_i] = E[E[y|x]] = E[\Lambda(x'\beta_0)]$.
Finally, we have

$$Var[y|x] = E[(y - E[y|x])^2|x] = (1 - \Lambda(x'\beta_0))^2\Lambda(x'\beta_0) + \Lambda(x'\beta_0)^2(1 - \Lambda(x'\beta_0)\,) = \Lambda(x'\beta_0)(1 - \Lambda(x'\beta_0).$$

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2.(b)(i) Using $\Lambda(z) = e^z/(1+e^z)$ and $\partial e^{x_i'\beta_0}/\partial x_{i,j} = \beta_{0,j}e^{x_i'\beta_0}$, we have:

$$\begin{aligned} \frac{\partial \Lambda(x'\beta_0)}{\partial x_{i,j}} &= \left(\frac{\partial e^{x_i'\beta_0}}{\partial x_{i,j}}\right)\left(\frac{1}{1+e^{x_i'\beta_0}}\right) + (-1)\left(\frac{\partial e^{x_i'\beta_0}}{\partial x_{i,j}}\right)\left(\frac{e^{x_i'\beta_0}}{(1+e^{x_i'\beta_0})^2}\right) \\ &= \Lambda(x'\beta_0)(1 - \Lambda(x'\beta_0))\beta_{0,j}. \end{aligned}$$

2.(b)(ii) $lim_{x'\beta_0 \to \infty} = \Lambda(x'\beta_0)(1 - \Lambda(x'\beta_0))\beta_{0,j} = 0$ - since the probability depends monotonically on the index, as the index gets large the event happens with probability tending to one and so a small change in $x_{i,j}$ has no impact on the probability of the event.

3. This is not true. Unbiasedness is a statement about $E[\hat{\theta}_T]$, the mean of the sampling distribution of $\hat{\theta}_T$. Consistency means $lim_{T\to\infty} P(\|\hat{\theta}_T - \theta_0\| < \epsilon) = 1$ for all $\epsilon > 0$ and so implies the sampling distribution of $\hat{\theta}_T$ collapses to a spike at θ_0. A sufficient condition for consistency is that both $bias(\hat{\theta}_T) \to 0$ and $Var[\hat{\theta}_T] \to 0$ as $T \to \infty$; $bias(\hat{\theta}_T) \to 0$ alone does imply the collapse of the sampling distribution onto a single point.

4.(a) The conditions are: (i) $E[z_iu_i] = 0$; (ii) $E[z_ix_i] \neq 0$. Condition (i) cannot be tested directly as it depends on u_i which is unobservable. Condition (ii) can be tested via the first-stage regression that is, regressing x_i on z_i and testing the joint significance of the coefficients on z_i via, say, a $F-$ statistic, $e.g$ estimate $x_i = z_i'\gamma +$error and test $H_0 : \gamma = 0$ (z_i not relevant) versus $H_1 : \gamma \neq 0$ (z_i relevant) using a F-statistic.

4.(b)

- x_i is an endogenous regressor if $E[x_iu_i] \neq 0$. Given the model for x_i, we have:

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$$E[x_iu_i] = E[(z_i'\gamma_0 + v_i)u_i] = \gamma_0'E[z_iu_i] + E[u_iv_i].$$

  Using the Law of Iterated Expectations (LIE) and $E[u_i|z_i] = 0$ (given), it follows that $E[z_iu_i] = E[E[u_i|z_i]z_i] = 0$.

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  Using the Law of Iterated Expectations, $E[u_i|z_i] = 0$ (given) and $Cov[u_i, v_i|z_i] = \Omega_{1,2}$, the $(1,2)$ element of Ω_0, it follows that

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$$E[u_iv_i] = E[E[u_i, v_i|z_i]] = E[Cov[u_i, v_i|z_i]] = E[\Omega_{1,2}] = \Omega_{1,2}.$$

  So condition for x_i to be endogenous is $\Omega_{1,2} \neq 0$.

- Orthogonality condition: Using LIE $E[u_i|z_i]$ (given), $E[z_iu_i] = E[E[u_i|z_i]z_i] = 0$.
- Relevance condition: Using LIE and $E[v_i|z_i]$ (given), we have

$$E[z_ix_i] = E[z_i(z_i'\gamma_0 + v_i)] = M_{zz}\gamma_0 + E[E[v_i|z_i]z_i] = M_{zz}\gamma_0$$

  Since M_{zz} is pd (given), the condition for relevance is $\gamma_0 \neq 0$.

5.(a) A scalar time series $\{v_t\}$ is covariance stationary if its first two moments do not depend on t that is, $E[v_t] = \mu$, $Var[v_t] = \gamma_0$ and $Cov[v_t, v_{t-j}] = \gamma_{|j|}$. We are given that e_t is white noise and so: $E[e_t] = 0$, for all t; $Var[e_t] = E[e_t^2] = \sigma^2$ for all t; $Cov[e_t, e_s] = E[e_te_s] = 0$ for all $t \neq s$.

(i) $u_t = e_t$ and so using the properties stated above $E[u_t] = 0$, $Var[u_t] = \sigma^2$, $Cov[u_t, u_s] = 0$ $\Rightarrow u_t$ is cov stat.

(ii) $E[v_t] = (-1)^t = E[e_t] = (-1)^t$ as $E[e_t] = 0$ and so the mean depends on time: for example, $E[v_1] = -1$ & $E[v_2] = 1$. Therefore, v_t is not covariance stationary.

(iii) Using the properties of white noise stated above, we have: $E[w_t] = (-1)^t E[e_t] = 0$; $Var[w_t] = (-1)^{2t} E[e_t^2] = \sigma^2$; $Cov[w_t, w_s] = E[w_t w_s] = (-1)^{t+s} E[e_t e_s] = 0$. Therefore, u_t is covariance stationary.

5.(b) v_t is not strictly stationary because it is not covariance stationary; u_t and w_t may or may not be strictly stationary but we have insufficient information about $\{e_t\}$ in order to assess this.

6.(a) Substituting for y in the formula for the estimator, we have

$$\tilde{\beta}_T \;=\; (Z'X)^{-1}Z'y \;=\; \beta_0 \,+\, (Z'X)^{-1}Z'u, \tag{1}$$

and so

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$$E[\tilde{\beta}_T] = E\left[\beta_0 + (Z'X)^{-1}Z'u\right]$$

Using X, Z constant and $E[u] = 0$, it follows that

$$E[\tilde{\beta}_T] = \beta_0 + (Z'X)^{-1}Z'E[u] = \beta_0.$$

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6.(b) We have $Var[\tilde{\beta}_T] = E\left[\tilde{\beta}_T - E[\tilde{\beta}_T]\right]\left[\tilde{\beta}_T - E[\tilde{\beta}_T]\right]'$. Using part (a) and (1), it follows that

$$Var[\tilde{\beta}_T] \;=\; E\left[(Z'X)^{-1}Z'uu'Z(X'Z)^{-1}\right]$$

Since Z, X are fixed in repeated samples (given), we can bring the expectation operator through

$$Var[\tilde{\beta}_T] \;=\; (Z'X)^{-1}Z'\,E[uu']\,Z(X'Z)^{-1}$$

Given that $E[u] = 0$ and $Var[u] = \sigma_0^2 I_T$, it follows that $E[uu'] = \sigma_0^2 I_T$ and thus that

$$\begin{aligned} Var[\tilde{\beta}_T] \;&=\; (Z'X)^{-1}Z'\sigma_0^2 I_T Z(X'Z)^{-1} \;=\; \sigma_0^2 (Z'X)^{-1}Z' I_T Z(X'Z)^{-1} \\ &=\; \sigma_0^2 (X'Z(Z'Z)^{-1}Z'X)^{-1}, \end{aligned}$$

where have also used $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

6.(c) $Cov[\hat{\beta}_{T,i}, \hat{\beta}_{T,j}] = \sigma_0^2\{(X'Z(Z'Z)^{-1}Z'X)^{-1}\}_{i,j}$ where $\{A\}_{i,j}$ denotes the $i-j^{th}$ element of any matrix A.

6.(d) From (1), $u \sim Normal$ and X, Z, β_0 constants, $\tilde{\beta}_T$ is a linear combination of normal random variables and so has a normal distribution.

6.(e) No, $\tilde{\beta}_T$ is a linear in y unbiased estimator of β_0 and so we know that under the conditions in the question that the OLS estimator is the efficient estimator within the class of linear unbiased estimators.

6.(f) It is not unbiased because $E[(Z'X)^{-1}Z'u]$ depends on both X and Z and so cannot use the Law of Iterated Expectations and $E[u|Z] = 0$ to deduce $E[(Z'X)^{-1}Z'u]$ is zero. However, estimator is consistent because consistency rests on $E[z_t u_t] = 0$ which is implied by $E[u|Z] = 0$. So the estimator is consistent subject to additional regularity conditions for WLLN.

7.(a) First need $E[u_t]$. We have $E[u_t] = E[w_t] + \phi E[w_{t-2}] = 0$ as $E[w_t] = 0$ (given). Let $\Omega_{t,s}$ denote the (t,s) element of Ω. By definition $\Omega_{t,t} = Var[u_t] = E[u_t^2]$ (as $E[u_t] = 0$) and $\Omega_{t,s} = Cov[u_t, u_s] = E[u_t u_s]$ (as $E[u_t] = 0$). Noting that $w_t \sim iid(0, \sigma^2)$ (given) we have: $E[w_t^2] = Var[w_t] = \sigma^2$ and $E[w_t w_s] = Cov[w_t, w_s] = 0$ for all $s \neq t$. Using these properties we have:

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$$
\begin{aligned}
Var[u_t] &= E\left[(w_t + \phi w_{t-2})^2\right], \\
&= E\left[w_t^2 + \phi^2 w_{t-2}^2 + 2\phi w_t w_{t-2}\right], \\
&= E[w_t^2] + \phi^2 E[w_{t-2}^2] + 2\phi E[w_t w_{t-2}], \\
&= \sigma^2 + \phi^2\sigma^2, \qquad \text{b/c } E[w_t w_{t-2}] = 0, \\
&= (1 + \phi^2)\sigma^2 = \gamma_0,
\end{aligned}
$$

and

$$
\begin{aligned}
Cov[u_t, u_{t-s}] &= E\left[(w_t + \phi w_{t-2})(w_{t-s} + \phi w_{t-s-2})\right], \\
&= E\left[w_t w_{t-s} + \phi w_{t-2} w_{t-s} + \phi w_t w_{t-s-2} + \phi^2 w_{t-2} w_{t-s-2}\right] = \gamma_s, \text{ say.}
\end{aligned}
$$

From the previous equation we obtain: $\gamma_1 = 0$; $\gamma_2 = \phi E[w_{t-2}^2] = \phi\sigma^2$, and for $s > 2$, $\gamma_s = 0$, because $Cov[w_t, w_{t-j}] = 0$ for all $j \neq 0$. So Ω is a matrix whose elements are zero apart from: $\Omega_{t,t} = \gamma_0$, for $t = 1, 2, \ldots, T$, and $\Omega_{t,s} = \gamma_2$ for all (t,s) such that $t = j$, $s = j+2$ for $j = 1, 2, \ldots T-2$, and $t = j$, $s = j-2$, $j = 3, 4, \ldots, T$.

7.(b)(i) y_{t-1} is contemporaneously exogenous if $E[u_t|y_{t-1}] = 0$ which would imply via LIE that $E[u_t y_{t-1}] = E\left[E[u_t|y_{t-1}]\right] = 0$. So if $E[u_t y_{t-1}] \neq 0$ then y_{t-1} is not contemporaneously exogenous. Using the hint, $y_{t-1} = \psi_0 u_{t-1} + \psi_1 u_{t-2} + f(u_{t-3}, u_{t-4}, \ldots,)$, we have:

$$\begin{aligned} E[u_t y_{t-1}] &= E[u_t\left(\psi_0 u_{t-1} + \psi_1 u_{t-2} + f(u_{t-3}, u_{t-4}, \ldots,)\right)] \\ &= \psi_0 E[u_t u_{t-1}] + \psi_1 E[u_t u_{t-2}] + E[u_t f(u_{t-3}, u_{t-4}, \ldots,)] \end{aligned}$$

Consider the terms on the right-hand side:

- Since $E[u_t] = 0$ and $\gamma_1 = 0$ from part (a), we have $E[u_t u_{t-1}] = \gamma_1 = 0$.
- Since $E[u_t] = 0$ and $\gamma_2 \neq 0$ from part (a), we have $E[u_t u_{t-2}] = \gamma_2 \neq 0$.
- Since u_t is a function of $\{w_t, w_{t-2}\}$, $\{u_{t-3}, u_{t-4}, \ldots\}$ is a function of $\{w_{t-3}, w_{t-4}, \ldots\}$ and $w_t \sim i.i.d.$, it follows that u_t is independent of $\{u_{t-3}, u_{t-4}, \ldots\}$ and so using $E[u_t] = 0$ (from part (a)), we have $E[u_t f(u_{t-3}, u_{t-4}, \ldots,)] = E[u_t]E[f(u_{t-3}, u_{t-4}, \ldots,)] = 0$.

Therefore, given $\psi_{0,1} \neq 0$, it follows that $E[u_t y_{t-1}] = \psi_{0,1}\gamma_2 \neq 0$ and so y_{t-1} cannot be contemporaneously exogenous in this model.

7.(b)(ii) y_{t-1} is strictly exogenous if $E[u_t|y_{T-1}, y_{T-2}, \ldots, y_0] = 0$. This implies that $E[y_s u_t] = 0$ for all $s = 1, 2, \ldots, T-1$. From part (i), we already know this is not the case, and so it is not strictly exogenous.

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8.(a) We can write

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$$\hat{\beta}_N = \left(\sum_{i=1}^{N} x_i x_i'\right)^{-1} \sum_{i=1}^{N} x_i y_i$$

and substituting for y_i this yields:

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$$\hat{\beta}_N = \beta_0 + \left(\sum_{i=1}^{N} x_i x_i'\right)^{-1} \sum_{i=1}^{N} x_i u_i.$$

Therefore, we have:

$$N^{1/2}(\hat{\beta}_N - \beta_0) = \left(N^{-1}\sum_{i=1}^{T} x_i x_i'\right)^{-1} N^{-1/2}\sum_{i=1}^{N} x_i u_i,$$

Using the WLLN, we have $N^{-1}\sum_{i=1}^{N} x_i x_i' \xrightarrow{p} E[x_i x_i'] = Q$. Since Q is pd, we can apply Slutsky's Theorem to deduce:

$$\left(N^{-1}\sum_{i=1}^{N} x_i x_i'\right)^{-1} \xrightarrow{p} Q^{-1}.$$

To use the CLT, need to evaluate: $E[x_i u_i]$ and $\Omega = \lim_{N\to\infty} Var[N^{-1/2}\sum_{i=1}^{N} x_i u_i]$. Via LIE, we have $E[x_i u_i] = E[x_i E[u_i|x_i]] = 0$ as $E[u_i|x_i] = 0$ (given). Multiplying out, we have:

$$Var[N^{-1/2}\sum_{i=1}^{N} x_i u_i] \;=\; N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} Cov[x_i u_i, x_j u_j].$$

For $i = j$, $Cov[x_i u_i, x_j u_j] = Var[x_i u_i]$. For $i \neq j$, $Cov[x_i u_i, x_j u_j] = 0$ because $\{u_i, x'_{2,i})' \sim$ iid implies that $x_i u_i$ and $x_j u_j$ are independent for $i \neq j$. Therefore, $\Omega_N = N^{-1}\sum_{i=1}^{N} Var[x_i u_i]$. To find $Var[x_i u_i]$, use $E[x_i u_i] = 0$ from above and so via LIE

$$Var[x_i u_i] = E[u_i^2 x_i x_i'] = E[E[u_i^2|x_i]x_i x_i'] = E[h(x_i)x_i x_i'] = \Omega_h.$$

Using Lemma 3.5 from the Lecture Notes (If $M_T \xrightarrow{p} M$, finite pd and $b_T \xrightarrow{d} N(0, V)$ then $M_T b_T \xrightarrow{d} N(0,\, MVM')$), it follows that $T^{1/2}(\hat{\beta}_T - \beta_0) \xrightarrow{d} N(0,\, V_h)$.

8.(b)(i) Errors are heteroskedastic but serially uncorrelated. Hence we can perform a t-test with either White or Newey-West standard errors as both are consistent estimators of the true s.e in this case. Let $s.e._W(\cdot)$ and $s.e._{NW}(\cdot)$ denote the White and Newey-West standard errors of the coefficient estimator in the parentheses. We have

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$$\frac{\hat{\beta}_{0,1} - 0}{s.e_W(\hat{\beta}_{0,1})} \text{ and } \frac{\hat{\beta}_{0,1} - 0}{s.e_{NW}(\hat{\beta}_{0,1})}$$

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are both asymptotically N(0,1) under H_0. Here we perform the test with White standard errors as is a more efficient estimate of the true standard errors under heteroscedasticity. Here $\hat{\beta}_{0,1} = 0.389$ and $s.e(\hat{\beta}_{0,1}) = 1.362$ and $\frac{\hat{\beta}_{0,1}-0}{s.e_W(\hat{\beta}_{0,1})} = 0.389/1.362 = 0.286$ where the critical value of the two-sided 5% test is 1.96 and hence we cannot find evidence to reject the null.

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8(b)(ii) Errors are heteroskedastic and serially correlated. Hence we can only construct a valid test performing a t-test with Newey-West standard errors, and so use test statistic

$$\frac{\hat{\beta}_{0,2} - 0}{s.e_{NW}(\hat{\beta}_{0,2})}$$

Performing the test in this case $\frac{\hat{\beta}_{0,2}-0}{s.e_{NW}(\hat{\beta}_{0,2})} = 0.336/0.321 = 1.05$ where the c.v is 2.58 and hence no evidence to reject the null.

8(b)(iii) Errors are spherical and hence can perform a joint tests based on the F-stat with any s.e's. We'd prefer to use the formula based on OLS s.e's as it is more efficient. Given the info we may only perform this test at any rate.

The test-stat in this case is $(R^2/2)/((1-R^2)/97))$ which has an exact F(2,97) distribution. $(97 * R^2)/(2 * (1-R^2) = 40.5$ where the c.v is 2.15, hence evidence to reject the null.

9.(a) From notes, we have the conditional log likelihood function takes the form:

$$CLLF_N = \sum_{i=1}^{N} \ell_i(\beta),$$

where

$$\ell_i(\beta) = y_i ln[\Phi(x_i'\beta)] + (1 - y_i) ln[1 - \Phi(x_i'\beta)].$$

In this question $x_i = 1$ and so specializing to model here, we have:

$$\begin{aligned} LLF_N(\beta) &= \sum_{i=1}^{N} \{ y_i ln[\Phi(\beta)] + (1-y_i) ln[(1-\Phi(\beta)]\} \\ &= N_1 ln[\Phi(\beta)] + (N - N_1) ln[(1-\Phi(\beta)]. \end{aligned}$$

The score function is:

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$$s(\beta) = \frac{\partial LLF_N(\beta)}{\partial \beta} = N_1 \frac{\phi(\beta)}{\Phi(\beta)} - (N - N_1)\frac{\phi(\beta)}{1 - \Phi(\beta)}$$

The MLE estimator of β_0 is obtained by solving the first order conditions $s(\hat{\beta}) = 0$ which in this case are:

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$$N_1 \frac{\phi(\hat{\beta})}{\Phi(\hat{\beta})} - (N - N_1)\frac{\phi(\hat{\beta})}{1 - \Phi(\hat{\beta})} = 0.$$

Since $\phi(\beta) \neq 0$, the MLE is also the solution to

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$$N_1(1 - \Phi(\hat{\beta})) - (N - N_1)\Phi(\hat{\beta}) = 0,$$

from which it follows that $\Phi(\hat{\beta}) = N_1/N$. Therefore, $\hat{\beta} = \Phi^{-1}(N_1/N)$. Since $y_i \in \{0, 1\}$ it follows that $\bar{y} = \sum_{i=1}^{N} y_i/N = N_1/N$ and so $\hat{\beta} = \Phi^{-1}(\bar{y})$.

9.(b)(i) Let β_p be the coefficient on *ptcon* in the probit model. The decision rule is to reject $H_0 : \beta_p = 0$ in favour of $H_A : \beta_p \neq 0$ at the $100\alpha\%$ significance level if $|\hat{\beta}_p/s.e(\hat{\beta}_p)| > z_{1-\alpha/2}$ where $z_{1-\alpha/2}$ is the $100(1-\alpha/2)^{th}$ percentile of the standard normal distribution. Here the test stat is $|-2.4217/1.0982| = 2.2052$. From Tables $z_{0.975} = 1.96$ and $z_{0.995} = 2.576$ so we can reject the null hypothesis at the 5% but not the 1% level.

9.(b)(ii) Let β_{li} be the coefficient on *loginc* in the Probit model. The marginal response is: $\beta_{li}\phi(x_i'\beta)$ where $\phi(v)$ is pdf of standard normal and so depends on x_i. However sign of estimate gives sign of marginal response and so marginal response is positive here. So same sign for response to income as log is a monotonic increasing transformation. To test if the effect is different from zero, we can test: The decision rule is to reject $H_0 : \beta_{li} = 0$ in favour of $H_A : \beta_{li} \neq 0$ at the $100\alpha\%$ significance level if $|\hat{\beta}_{li}/s.e(\hat{\beta}_{li})| > z_{1-\alpha/2}$ where $z_{1-\alpha/2}$ is the $100(1-\alpha/2)^{th}$ percentile of the standard normal distribution. From the output, the test statistic equals $|2.434/0.821| = 2.96$ and so we can reject at the 1% significance level - using cv's given in part (b)(i) - the null that *loginc* has no effect on the probability of approval.

9.(b)(iii) LR test: $LR = 2\{LLF(\hat{\beta}_U) - LLF(\hat{\beta}_R)\}$ where $\hat{\beta}_U$ is the (unrestricted) MLE, $\hat{\beta}_R$ is the (restricted) MLE s.t. $R\beta_0 = r$, and LLF is the log likelihood function. Here R equals rows 2 to 8 of I_8 and r is 8×1 vector of zeros. The decision rule is to reject at the $100\alpha\%$ significance level if $LR > c_{1-\alpha}(df)$ where $c_{1-\alpha}(df)$ is the $100(1-\alpha)^{th}$ percentile of the χ^2_{df} distribution and df equals the number of restrictions. From output $LLF(\hat{\beta}_U) = -52.84$. Using part (a), $LLF(\hat{\beta}_R) = 59 * ln(59/95) + 36 * ln(36/95) = -63.04$. Therefore $LR = 20.39$ and as $c_{0.99}(8) = 20.09$ can reject H_0 at 1% sig level.

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