Alastair Hall

ECON61001:
Econometric Methods

Semester 1 2020-21

Problem Set for Tutorial 8

In this question, you investigate the condition for instrument relevance in a simple regression model where β_0 is just-identified.

1. Consider the regression model

$$y_i = \beta_{0,1} + x_{2,i}\beta_{0,2} + u_i = x_i'\beta_0 + u_i$$

where $x_i = (1, x_{2,i})'$, $\beta_0 = (\beta_{0,1}, \beta_{0,2})'$. Define $z_i = (1, z_{2,i})'$, $u_i = y_i - x_i'\beta$, and assume that $E[z_i u_i(\beta_0)] = 0$. Suppose a researcher estimates β_0 using the population moment condition

$$E[z_i u_i(\beta_0)] = 0. \quad (1)$$

(a) Assuming (1) holds, show that $E[z_i u_i(\beta)] \neq 0$ for all $\beta \neq \beta_0$ if and only if $rank\{E[z_i x_i']\} = 2$.

(b) Show that $rank\{E[z_i x_i']\} = 2$ if and only if $Corr(z_{2,i}, x_{2,i}) \neq 0$ where $Corr(a, b)$ denotes the population correlation between r.v.'s a and b. *Hint: Since $E[z_i x_i']$ is square, it is full rank if and only if it is nonsingular.*

(c) Interpret the condition for instrument relevance in this case.

In this question, you derive the formula for the IV estimator in the over-identified case. You may find it useful to refer back to the matrix differentiation results in Tutorial 1 Question 3 and refer back to how we derived the OLS estimator in lectures.

2. Consider the linear regression model

$$y = X\beta_0 + u \quad (2)$$

where y is $T \times 1$ with t^{th} element y_t, X is $T \times k$ with t^{th} row x_t', u is $T \times 1$ with t^{th} element u_t, β_0 is a $k \times 1$ vector of unknown parameters. Let Z be a $T \times q$ matrix with t^{th} row z_t', and define $u(\beta) = y - X\beta$. Assume $rank\{X'Z\} = k$ and $rank\{Z'Z\} = q$. Consider the Instrumental Variables (IV) estimator of β_0, $\hat{\beta}_{IV}$, defined by

$$\hat{\beta}_{IV} = argmin_{\beta} Q_{IV}(\beta), \quad (3)$$

where

$$Q_{IV}(\beta) = u(\beta)'Z(Z'Z)^{-1}Z'u(\beta)$$

(a) By considering the first order conditions for the minimization in (3), show that

$$\hat{\beta}_{IV} \;=\; \left(X'Z(Z'Z)^{-1}Z'X \right)^{-1} X'Z(Z'Z)^{-1}Z'y.$$

(b) Show that if $q = k$ then: $\hat{\beta}_{IV} \;=\; (Z'X)^{-1} Z'y$.

In this question you explore methods for inference in models estimated via IV in cross-sectional data.

3. Consider the linear regression model

$$y_i \;=\; x_i'\beta_0 \,+\, u_i,$$

where: (i) $\{\,(u_i, x_i', z_i'),\;\; i \,=\, 1, 2, \ldots N\}$ forms an independent and identically distributed sequence; (ii) $E[z_i z_i'] \,=\, Q_{zz}$, finite, p.d.; (iii)) $E[z_i x_i'] \,=\, Q_{zx}$, with $rank\{Q_{zx}\} \,=\, k$; (iv) $E[u_i|z_i] = 0$; (v) $Var[u_i|z_i] = h(z_i)$, positive, finite constant. Let $\hat{\beta}_{IV}$ be the IV estimator of β_0 based on $E[z_i u_i] \,=\, 0$. Suppose it is desired to test $H_0 \,:\, R\beta_0 \,=\, r$ versus $H_A \,:\, R\beta_0 \neq r$ where R is a $n_r \times k$ matrix of specified constants and r is a $n_r \times 1$ vector of specified constants. Suggest a suitable decision rule for the test based on $\hat{\beta}_{IV}$, being sure to carefully specify how your test statistics is calculated from the data.

This question consider the regression model in Tutorial 7 Question 3. In that question, you showed that if the regressors include the lagged dependent variable and the errors follow a $MA(1)$ *process then OLS is an inconsistent estimator of the regression parameters. Here you consider an IV approach to estimation of this model.*

4. Consider now the regression model:

$$y_t \;=\; \beta_{0,1} \,+\, \beta_{0,2} y_{t-1} \,+\, u_t, \tag{4}$$

where $|\beta_{0,2}| < 1$ and

$$u_t \;=\; \varepsilon_t \,+\, \phi\varepsilon_{t-1}, \tag{5}$$

where $\phi \neq 0$, $|\phi| < 1$, $\phi \neq -\beta_{0,2}$, and ε_t is white noise. Let $\hat{\beta}_T$ be the OLS estimator of $\beta_0 = (\beta_{0,1}, \beta_{0,2})'$ based on (4). Suppose that this model is estimated via IV using instrument vector $z_t = (1, y_{t-2})'$. Show that z_t satisfies the orthogonality and relevance conditions. *Hint: if* y_t *is generated by (4)-(5) then: (i)* y_t *has the representation* $y_t = \beta_{0,1}/(1-\beta_{0,2}) + \sum_{i=0}^{\infty} \beta_{0,2}^i u_{t-i}$*; (ii)* y_t *is generated by a stationary ARMA(1,1) process and its first order autocorrelation is:*

$$Corr(y_t, y_{t-1}) \;=\; \frac{(\phi + \beta_{0,2})(1 + \phi\beta_{0,2})}{1 + 2\phi\beta_{0,2} + \phi^2}.$$

In this question you consider a simple model in which the instrument relevance condition is not satisfied and the implications of this failure for the large sample behaviour of the IV estimator.

5. Consider the regression model

$$y_i \;=\; x_i\beta_0 \;+\; u_i,$$

where x_i is a scalar and consider estimation of (scalar) β_0 by IV based on the moment condition $E[z_i u_i(\beta_0)] = 0$ where z_i is a scalar and $u_i(\beta) = y_i - x_i\beta$. Let $\hat{\beta}_{IV}$ be the resulting IV estimator and suppose that $E[z_i x_i] = 0$.

(a) Verify that the instrument relevance condition is not satisfied in this model.

(b) What is $E[z_i u_i(\beta)]$ for $\beta \neq \beta_0$? (You may assume that $E[z_i u_i] = 0$.)

(c) Would you expect $\hat{\beta}_{IV}$ to be a consistent estimator for β_0 in this case?