Alastair Hall | **ECON61001: Econometric Methods** | Semester 1, 2020-21

Solutions to Problem Set for Tutorial 5

1. We have:

$$X'X \;=\; \begin{bmatrix} T & T(T+1)/2 \\ T(T+1)/2 & T(T+1)(2T+1)/6 \end{bmatrix}.$$

1.(a) Therefore, it follows that:

$$T^{-1}X'X \;=\; \begin{bmatrix} 1 & (T+1)/2 \\ (T+1)/2 & (T+1)(2T+1)/6 \end{bmatrix}.$$

and so as $T \to \infty$ $T^{-1}X'X$ diverges, and so does not converge to a matrix of finite constants.

1.(b) In this case, we have

$$T^{-2}X'X \;=\; \begin{bmatrix} T^{-1} & (T+1)/2T \\ (T+1)/2T & (T+1)(2T+1)/6T \end{bmatrix}.$$

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Since $lim_{T\to\infty}T^{-1} = 0$ and $lim_{T\to\infty}(T+1)/2T = 0.5$ the $(1,1)$, $(1,2)$ and $(2,1)$ elements of $T^{-2}X'X$ converge to finite constants. However, the $(2,2)$ element of $T^{-2}X'X$ diverges and so the matrix also diverges.

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1.(c) In this case, we have

$$T^{-3}X'X \;=\; \begin{bmatrix} T^{-2} & (T+1)/2T^2 \\ (T+1)/2T^2 & (T+1)(2T+1)/6T^2 \end{bmatrix}.$$

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Since $lim_{T\to\infty}T^{-2} = 0$ and $lim_{T\to\infty}(T+1)/2T^2 = 0$ the $(1,1)$, $(1,2)$ and $(2,1)$ elements of $T^{-3}X'X$ converge to zero. Further, $lim_{T\to\infty}(T+1)(2T+1)/6T^2 = 1/3$ and so $T^{-3}X'X$ converges to a matrix of constants but this limit matrix is singular.

(To develop a limiting distribution theory for the OLS estimator in this case, it is necessary to scale the elements of $\hat{\beta}_T - \beta_0$ by different functions of T, and this leads to different elements of $X'X$ being scaled by different functions of T; see Lecture Notes Section 3.5.)

2.(a) To assess whether u_t is weakly stationary, it is necessary to derive its first two moments. Since we are given that $u_t = \varepsilon_t$, a white noise process, it follows immediately that $E[u_t] = 0$ for all t, $Var[u_t] = \sigma^2$ for all t and $Cov[u_t, u_s] = 0$ for all $t \neq s$. Using Definition 3.7 in the Lecture Notes, it can re recognized that u_t is a weakly stationary process.

2.(b) The first two moments of v_t can be derived as follows.

- $E[v_t]$: Substituting for v_t and using $E[\varepsilon_t] = 0$ we have,

$$E[v_t] \;=\; E[(-1)^t \varepsilon_t] \;=\; (-1)^t E[\varepsilon_t] \;=\; 0.$$

- $Var[v_t]$: Using the previous result it follows that $Var[v_t] = E[v_t^2]$. So substituting for v_t, it follows that

$$Var[v_t] \;=\; E[v_t^2] \;=\; E\left[\{(-1)^t \varepsilon_t\}^2\right] \;=\; (-1)^{2t} E[\varepsilon_t^2] = \sigma^2,$$

where the last equality uses the fact that ε_t is white noise.

- $Cov[v_t, v_s]$: Using $E[v_t] = 0$ for all t, we have $Cov[v_t, v_s] = E[v_t v_s]$. Substituting for v_t, it follows that

$$Cov[v_t, v_s] \;=\; E[v_t v_s] \;= E[(-1)^t (-1)^s \varepsilon_t \varepsilon_s] \;=\; (-1)^{t+s} E[\varepsilon_t \varepsilon_s] \;=\; 0,$$

where the last identity uses the fact that ε_t is white noise and so $Cov[\varepsilon_t \varepsilon_s] = E[\varepsilon_t \varepsilon_s] = 0$.

Therefore, v_t is a weakly stationary process because $E[v_t] = 0$ and $Var[v_t]$ are independent of t, and $Cov[v_t, v_s] = 0$ depends on $|t - s|$ but not t or s individually ($t \neq s$).

2.(c) Note that the event $t = 10$ is not random and so $\mathcal{I}(t = 10)$ can be treated as a constant. Substituting for w_t, we have:

$$E[w_t] \;=\; E[\mathcal{I}(t = 10) \,+\, \varepsilon_t] \;=\; E[\mathcal{I}(t = 10)] \,+\, E[\varepsilon_t] \;=\; \mathcal{I}(t = 10) \,+\, E[\varepsilon_t].$$

Therefore, using $E[\varepsilon_t] = 0$, it follows that $E[w_t] = 0$ for all $t \neq 10$ and $E[w_{10}] = 1$. So the mean of w_t depends on t and w_t is not weakly stationary.

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2.(d) All we know about v_t are the properties of its first two moments for which we can deduce that it is weakly stationary. But in the absence of additional information about the underlying probability distribution, there is no way to assess whether the process is strongly stationary. So the answer is that it may or may not be strongly stationary we lack the information to tell.

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3.(a) We are given that ε_t is white noise and so $E[\varepsilon_t] = 0$ for all t. Thus, it follows that:

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$$\begin{aligned} E[y_t] &= E[\varepsilon_t \,+\, \phi\varepsilon_{t-1}] \\ &= E[\varepsilon_t] \,+\, \phi E[\varepsilon_{t-1}] \\ &= 0, \qquad \text{b/c } E[\varepsilon_t] = 0. \end{aligned}$$

3.(b) Since ε_t is white noise, we have $E[\varepsilon_t^2] = Var[\varepsilon_t] = \sigma^2$ and $E[\varepsilon_t \varepsilon_s] = Cov[\varepsilon_t, \varepsilon_s] = 0$ for all $s \neq t$. Using these properties and part(a), it follows that:

$$\begin{aligned} Var[y_t] &= E\left[(\varepsilon_t \,+\, \phi\varepsilon_{t-1})^2\right], \qquad \text{b/c } E[y_t] = 0 \\ &= E\left[\varepsilon_t^2 \,+\, \phi^2 \varepsilon_{t-1}^2 \,+\, 2\phi\varepsilon_t \varepsilon_{t-1}\right] \\ &= E[\varepsilon_t^2] \,+\, \phi^2 E[\varepsilon_{t-1}^2] \,+\, 2\phi E[\varepsilon_t \varepsilon_{t-1}] \\ &= \sigma^2 \,+\, \phi^2 \sigma^2, \qquad \text{b/c } E[\varepsilon_t \varepsilon_{t-1}] = 0, \\ &= (1 \,+\, \phi^2)\sigma^2. \end{aligned}$$

3.(c) Using part(a), we have:

$$\begin{aligned}
Cov[y_t, y_{t-s}] &= E\left[(\varepsilon_t + \phi\varepsilon_{t-1})(\varepsilon_{t-s} + \phi\varepsilon_{t-s-1})\right] \\
&= E\left[\varepsilon_t\varepsilon_{t-s} + \phi\varepsilon_{t-1}\varepsilon_{t-s} + \phi\varepsilon_t\varepsilon_{t-s-1} + \phi^2\varepsilon_{t-1}\varepsilon_{t-s-1}\right] \\
&= E[\varepsilon_t\varepsilon_{t-s}] + \phi E[\varepsilon_{t-1}\varepsilon_{t-s}] + \phi E[\varepsilon_t\varepsilon_{t-s-1}] + \phi^2 E[\varepsilon_{t-1}\varepsilon_{t-s-1}].
\end{aligned}$$

From the previous equation we obtain (using $E[\varepsilon_t\varepsilon_n] = 0$ for all $t \neq n$): for $s = 1$,

$$Cov[y_t, y_{t-s}] = \phi E[\varepsilon_{t-1}^2] = \phi\sigma^2,$$

and for $s > 1$,

$$Cov[y_t, y_{t-s}] = 0.$$

3.(d) By inspection of the results in parts (a)-(c), it can be seen that y_t is a weakly stationary process because $E[y_t]$ and $Var[y_t]$ do not depends on t and $Cov[y_t, y_{t-s}]$ only depends on s.

3.(e) The long run variance is defined in Section 3.3.1 of the lecture notes. By convention, we use γ_i to denote the i^{th} autocovariance of a scalar and so the the long run covariance of y_t is given by: $\gamma_0 + \sum_{i=1}^{\infty}(\gamma_i + \gamma_i')$ where $\gamma_i = Cov[y_t, y_{t-i}]$. Since the transpose of a scalar is itself, the formula can be equivalently written as:

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$$\Omega_y = \gamma_0 + 2\sum_{i=1}^{\infty}\gamma_i \tag{1}$$

Using the results from parts (b)-(c), we have:

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$$\Omega_y = \sigma^2(1 + \phi^2 + 2\phi) = \sigma^2(1 + \phi)^2.$$

4.(a) By similar arguments to the answer to Question 3.(a), we have:

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$$\begin{aligned}
E[y_t] &= E\left[\sum_{i=0}^{\infty}\theta^i\varepsilon_{t-i}\right] \\
&= \sum_{i=0}^{\infty}\theta^i E[\varepsilon_{t-i}] \\
&= 0, \quad \text{b/c } E[\varepsilon_t] = 0.
\end{aligned}$$

4.(b) By similar arguments to the answer to Question 3.(b):

$$\begin{aligned}
Var[y_t] &= Var[\sum_{i=0}^{\infty}\theta^i\varepsilon_{t-i}] \\
&= E[(\sum_{i=0}^{\infty}\theta^i\varepsilon_{t-i})^2] \qquad \text{since } E[\sum_{i=0}^{\infty}\theta^i\varepsilon_{t-i}] = 0
\end{aligned}$$

$$
\begin{aligned}
&= E[\sum_{i=0}^{\infty} \theta^{2i} \varepsilon_{t-i}^2] \qquad \text{since } E[\varepsilon_t \varepsilon_s] = 0 \ \forall t \neq s \\
&= \sigma^2 \sum_{i=0}^{\infty} \theta^{2i} \\
&= \sigma^2/(1-\theta^2)
\end{aligned}
$$

where the last equality uses the hint (note that $|\theta| < 1$ implies $0 \leq \theta^2 < 1$.)

4.(c) Assume $s > 0$. Using the $MA(\infty)$ representation and part(a), we have

$$
\begin{aligned}
Cov[y_t, y_{t-s}] &= Cov[\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}, \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}] \\
&= E\left[\left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}\right)\left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)\right]. \qquad (2)
\end{aligned}
$$

At this point it is convenient to write $\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}$ as the sum of two components,

$$
\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i} = \sum_{i=0}^{s-1} \theta^i \varepsilon_{t-i} + \sum_{i=s}^{\infty} \theta^i \varepsilon_{t-i}.
$$

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Notice further that

$$
\sum_{i=s}^{\infty} \theta^i \varepsilon_{t-i} = \theta^s \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i},
$$

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and so we have:

$$
\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i} = \sum_{i=0}^{s-1} \theta^i \varepsilon_{t-i} + \theta^s \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}. \qquad (3)
$$

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Substituting (3) into (2), it follows that

$$
\begin{aligned}
Cov[y_t, y_{t-s}] &= E\left[\left(\sum_{i=0}^{s-1} \theta^i \varepsilon_{t-i} + \theta^s \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)\left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)\right] \\
&= E\left[\left(\sum_{i=0}^{s-1} \theta^i \varepsilon_{t-i}\right)\left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)\right] + E\left[\theta^s \left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)^2\right] \qquad (4)
\end{aligned}
$$

Consider each of these two terms in turn. Multiplying out

$$
\left(\sum_{i=0}^{s-1} \theta^i \varepsilon_{t-i}\right)\left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)
$$

it can be seen that the result is a linear combination of terms of the form $\varepsilon_t \varepsilon_n$ where $t \neq n$. Since the expectation of a linear combination of rv's is the corresponding linear combination of the expectations and $E[\varepsilon_t \varepsilon_n] = 0$ for all $t \neq n$ (as ε_t is white noise), it follows that

$$E\left[\left(\sum_{i=0}^{s-1} \theta^i \varepsilon_{t-i}\right)\left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)\right] = 0. \tag{5}$$

Now consider the other term. Since θ^s is a constant, it follows that

$$E\left[\theta^s \left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)^2\right] = \theta^s E\left[\left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)^2\right], \tag{6}$$

and so using the same argument as for part (b), we have

$$\theta^s E\left[\left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-s-i}\right)^2\right] = \theta^s \sigma^2/(1-\theta^2). \tag{7}$$

The desired result then follows from (4)-(7).

4.(d) Inspection of parts (a)-(c) reveals that $E[y_t]$ and $Var[y_t]$ do not depend on t, and $Cov[y_t, y_{t-s}]$ depends on s but neither t or $t-s$. Therefore the process is weakly stationary. (The "stationarity" condition in ARMA models ensures that the process is weakly stationary.)

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4.(e) As y_t is a scalar, we evaluate (1). Using parts (b)-(d), it follows that:

$$\Omega_y = \frac{\sigma^2}{1-\theta^2} + 2\frac{\sigma^2}{1-\theta^2}\left\{\theta + \theta^2 + \ldots\right\} = \frac{\sigma^2}{1-\theta^2}\left\{1 + 2\sum_{i=1}^{\infty} \theta^i\right\}. \tag{8}$$

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Now, using the hint, we have:

$$1 + 2\sum_{i=1}^{\infty} \theta^i = 2\sum_{i=0}^{\infty} \theta^i - 1 = \frac{2}{1-\theta} - 1 = \frac{1+\theta}{1-\theta},$$

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and substituting this into (8), we obtain:

$$\Omega_y = \left(\frac{\sigma^2}{1-\theta^2}\right)\left(\frac{1+\theta}{1-\theta}\right) = \frac{\sigma^2}{(1-\theta)^2}.$$

5. As can be seen from Figures 1 and 2, the pdf is still bell-shape but the bell has been elongated along the negative 45-degree line. Most of each contour lies in in the North-West and South-East quadrants of the plot. This reflects that if u_1 and u_2 are negatively correlated then positive values of u_1 are more likely to occur with negative values of u_2 and *vice versa*. The shape of the plots is different from those presented in Figures 4.1-4.6.

Figure 1: Probability density function with non-spherical errors due to negative correlation

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Figure 2: Probability contours with non-spherical errors due to negative correlation

6. A suitable program is contained in the file Tut5timetrend.m where you need to make the comparison on the screen; this program calls the file ols.m used in the solutions to Tutorial 2. It is common practice to include a time trend in time series regression models. This exercise demonstrates a general result: if we regress y_t on an intercept, a time trend and a vector of variables z_t then the estimated coefficients on z_t are the same as those obtained by regressing $\tilde{y}_t$ on $\tilde{z}_t$ where $(\tilde{\cdot})$ denotes the linearly de-trended value of $(\cdot)$.

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