Alastair Hall

ECON61001:
Econometric Methods

Semester 1, 2020-21

Solutions to Problem Set for Tutorial 4

1.(a) We have $T^{1/2}\bar{v}_T = T^{-1/2}\iota_T' v$. Using Lemma 2.1 with $w = T^{1/2}\bar{v}_T$, $c = 0$, $C = T^{-1/2}\iota_T$, $\mu = 0$ and $\Omega = I_T$, we obtain $T^{1/2}\bar{v}_T \sim N(0, T^{-1}\iota_T'\iota_T)$. Since $\iota_T'\iota_T = T$, we have $T^{1/2}\bar{v}_T \sim N(0,1)$.

1.(b) The event $|\bar{v}_T| < n$ is equivalent to the event $|T^{1/2}\bar{v}_T| < T^{1/2}n$ and so $P(|\bar{v}_T| < n) = P(|z| < T^{1/2}n)$. As $n > 0$, we have $T^{1/2}n \to \infty$ as $T \to \infty$; so $lim_{T\to\infty}P(|\bar{v}_T| < n) = 1$ because $lim_{T\to\infty}P(|z| < T^{1/2}n) = 1$. Since this holds for any n, it follows from Definition 3.1 (in the lecture notes) that $\bar{v}_T \xrightarrow{p} 0 = E[v_t]$ which is the WLLN.

1.(c) The event $|T\bar{v}_T| < n$ is equivalent to the event $|T^{1/2}\bar{v}_T| < T^{-1/2}n$, and so $P(|T\bar{v}_T| < n) = P(|z| < T^{-1/2}n)$. Therefore $lim_{T\to\infty}P(|T\bar{v}_T| < n) = 0$ for any n. As a result $T\bar{v}_T$ is said to *diverge* as $T \to \infty$ and does not have a well-defined limiting distribution.

Parts (a)-(c) illustrate the crucial role of the scaling in our limit theorems for the sample mean. Unscaled the sample mean converges in probability to the population mean, a constant; this is the WLLN. If we scale the difference between the sample and population means by $T^{1/2}$ then we obtain a well defined rv whose limiting distribution is normal; this is the CLT. If we scale the difference between the sample and population means by T^a for $a > 1/2$ then the resulting statistic diverges.

2.(i) u_i and x_i cannot be independent because of the discrete sample space for y_i. Specifically, since the sample space of y_i is $\{0,1\}$, u_i can only take one of two values given x_i: $1 - x_i'\beta_0$ or $-x_i'\beta_0$. Therefore, the sample space for u_i depends on x_i.

2.(ii) Since there only two possible outcomes for u_i conditional on x_i, the conditional distribution for u_i given x_i cannot be normal.

3. Recall that $\hat{\sigma}_N^2 = e'e/(N-k) = u'(I_N - P)u/(N-k)$ for $P = X(X'X)^{-1}X'$. Multiplying out and for convenience multiplying and dividing by N, we obtain

$$\hat{\sigma}_N^2 = \left(\frac{N}{N-k}\right)\left\{N^{-1}u'u - N^{-1}u'X(N^{-1}X'X)^{-1}N^{-1}X'u\right\}$$

$$= \left(\frac{N}{N-k}\right)\left\{N^{-1}\sum_{i=1}^{N}u_i^2 - N^{-1}\sum_{i=1}^{N}u_i x_i'(N^{-1}\sum_{i=1}^{N}x_i x_i')^{-1}N^{-1}\sum_{i=1}^{N}x_i u_i\right\} \quad (1)$$

We now analyze the large sample behaviour of the terms of the right hand side of the previous equation. First note that as k is finite, we have $lim_{N\to\infty}N/(N-k) = 1$. For the remaining terms, we note they all involve sums of i.i.d random variables. As in Lecture 4, we can apply the WLLN to deduce:

$$
\begin{aligned}
N^{-1}\sum_{i=1}^{N} x_i x_i' &\xrightarrow{p} Q, \\
N^{-1}\sum_{t=1}^{N} x_i u_i &\xrightarrow{p} 0.
\end{aligned}
$$

Now consider, $N^{-1}\sum_{i=1}^{N} u_i^2$. Since $E[u_i] = 0$, we have $E[u_i^2] = Var[u_i] = \sigma_0^2$ under our assumptions. Therefore, it follows from the WLLN that

$$
N^{-1}\sum_{i=1}^{N} u_i^2 \xrightarrow{p} \sigma_0^2.
$$

From (1), it can be seen that $\hat{\sigma}_N^2$ is a continuous function of $N^{-1}\sum_{i=1}^{N} u_i^2$, $N^{-1}\sum_{i=1}^{N} x_i x_i'$, $N^{-1}\sum_{i=1}^{N} x_i u_i$ and $N/(N-k)$. From Slutsky's Theorem, it follows that $\hat{\sigma}_N^2$ converges in probability to the corresponding function of the limits of these terms, and so we have:

$$
\hat{\sigma}_N^2 \xrightarrow{p} (1)\left\{\sigma_0^2 - 0 \times Q^{-1} \times 0\right\} = \sigma_0^2.
$$

4. From the solutions to Tutorial 2, Question 2, we have:

$$
\hat{\gamma}_N = \beta_{0,1} + (X_1'X_1)^{-1}X_1'X_2\beta_{0,2} + (X_1'X_1)^{-1}X_1'u,
$$

where the i^{th} row of X_ℓ is $x_{\ell,i}'$ for $\ell = 1, 2$, from which it follows that

$$
\hat{\gamma}_N = \beta_{0,1} + (N^{-1}X_1'X_1)^{-1}N^{-1}X_1'X_2\beta_{0,2} + (N^{-1}X_1'X_1)^{-1}N^{-1}X_1'u. \tag{2}
$$

From (2), it can be seen that $\hat{\gamma}_N$ is a continuous function of $N^{-1}X_1'X_1$, $N^{-1}X_1'X_2$ and $N^{-1}X_1'u$. By similar arguments to Lecture 4, we have from the WLLN that $N^{-1}X_1'u \xrightarrow{p} 0$, and (decomposing X)

$$
N^{-1}X'X = \begin{bmatrix} N^{-1}X_1'X_1 & N^{-1}X_1'X_2 \\ N^{-1}X_2'X_1 & N^{-1}X_2'X_2 \end{bmatrix} \xrightarrow{p} \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix} = Q.
$$

Thus, using Slutsky's Theorem, it follows from (2) and these applications of the WLLN that: $\hat{\gamma}_N \xrightarrow{p} \beta_{0,1} + Q_{1,1}^{-1}Q_{1,2}\beta_{0,2}$.

5. Recall from Lecture 4 that we can test $H_0 : g(\beta_0) = 0$ vs. $H_1 : g(\beta_0) \neq 0$ using the statistic:

$$
W_N^{(g)} = Ng(\hat{\beta}_N)'\left[G(\hat{\beta}_N)(N^{-1}X'X)^{-1}G(\hat{\beta}_N)'\right]^{-1} g(\hat{\beta}_N)/\hat{\sigma}_N^2
$$

where $G(\bar{\beta}) = \partial g(\beta)/\partial\beta'|_{\beta=\bar{\beta}}$, and that under H_0: $W_N^{(g)} \xrightarrow{d} \chi^2_{n_g}$ where n_g is the number of restrictions. For this question, $g(\beta_0) = \beta_{0,2}\beta_{0,3} - 1$. Therefore, we have $G(\beta) = [0, \beta_3, \beta_2, 0, 0]$.

Note that since we are given that $\beta_{0,i} \neq 0$ for $i = 2, 3$, it follows that $rank\{G(\beta_0)\} = 1 = n_g$. Using these results to specialize $W_N^{(g)}$ to this case, we obtain:

$$W_N^{(g)} \;=\; \frac{\left(\hat{\beta}_{N,2}\hat{\beta}_{N,3} - 1\right)^2}{\hat{\sigma}_N^2\left(\hat{\beta}_{N,3}^2 m_{2,2} \;+\; \hat{\beta}_{N,2}^2 m_{3,3} \;+\; 2\hat{\beta}_{N,2}\hat{\beta}_{N,3} m_{2,3}\right)}$$

where $m_{i,j}$ is the $(i,j)^{th}$ element of $(X'X)^{-1}$. The decision rule is to: reject H_0 at the approximate $100\alpha\%$ significance level if $W_N^{(g)} > c_1(1-\alpha)$ where $c_1(1-\alpha)$ is the $100(1-\alpha)^{th}$ percentile of the χ_1^2 distribution.