Solutions to Problem Set for Tutorial 3

1.(a) Recall from Lecture 2 that under Assumptions CA1-CA6, we have:

$$\frac{\hat{\beta}_{T,i} - \beta_{0,i}}{\hat{\sigma}_T \sqrt{m_{i,i}}} \sim \text{Student's t distribution with T-k df}$$

where $\hat{\sigma}_T^2 = e'e/(T-k)$ and $e = y - X\hat{\beta}_T$. As discussed in the lecture, we base our decision rule on this statistic with $\beta_{0,i}$ replaced by parameter value that satisfies H_0 but is closest to H_1 which in this case is $\beta_{*,i}$. Our decision rule is then as follows: reject H_0 in favour of H_1 at the $100\alpha\%$ significance level if

$$\frac{\hat{\beta}_{T,i} - \beta_{*,i}}{\hat{\sigma}_T \sqrt{m_{i,i}}} > \tau_{T-k}(1-\alpha)$$

where $\tau_{T-k}(1-\alpha)$ is the $100(1-\alpha)^{th}$ percentile of the Student's t distribution with $T-k$ degrees of freedom.

1.(b) The test statistic is

$$\frac{\hat{\beta}_{T,i}}{\hat{\sigma}_T \sqrt{m_{i,i}}} = \frac{0.067}{0.021} = 3.190$$

Since $T - k = 91$, the critical value is $\tau_{91}(0.95) = 1.662$ to 3dp, and so we reject H_0 at the 5% significance level. This evidence supports the view that increasing the speed limit lead to a higher percentage of traffic accidents involving fatalities.

2.(a) Put $R = [0, 1, 0, -1, 0]$, $r = 0$.

2.(b) Put

$$R = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & -1 \end{bmatrix}$$

$$r = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The easiest way to check $rank(R) = n_r$ is to calculate $det(RR')$. If this determinant is non–zero then R is full rank. For (a): $det(RR') = 2$; for (b) $det(RR') = 10/3$.

3.(a) The i^{th} row of I_k is a $1 \times k$ vector whose i^{th} element is 1 and whose remaining elements are all zero. So in this case, $R\beta_0 = \beta_{0,i}$. It then follows from the definition of r that for the stated choices of R, r, $R\beta_0 = r$ is equivalent to $\beta_{0,i} = \beta_{*,i}$.

3.(b) Recall from Lecture 3 that the generic structure of the F-statistic for testing $H_0 : R\beta_0 = r$ is

$$F \;=\; \frac{(R\hat{\beta}_T \,-\, r)'[R(X'X)^{-1}R']^{-1}(R\hat{\beta}_T \,-\, r)}{n_r\,\hat{\sigma}_T^2}.$$

From part (a) it follows immediately that: $R\hat{\beta}_T - r = \hat{\beta}_{T,i} - \beta_{*,i}$. Now consider $R(X'X)^{-1}R'$. Given the definition of R in part (a), it follows that

$$R(X'X)^{-1}R' \;=\; [0, 0, \ldots 1, 0, \ldots 0](X'X)^{-1}[0, 0, \ldots 1, 0, \ldots 0]'$$

Now $[0, 0, \ldots 1, 0, \ldots 0](X'X)^{-1} = m_i'$ where m_i' is the i^{th} row of $(X'X)^{-1}$, and $m_i'[0, 0, \ldots 1, 0, \ldots 0]' = m_{i,i}$. Therefore in this case, we have (using $n_r = 1$):

$$F \;=\; \frac{(\hat{\beta}_{T,i} - \beta_{*,i})'[m_{i,i}]^{-1}(\hat{\beta}_{T,i} - \beta_{*,i})}{\hat{\sigma}_T^2} \;=\; \left\{ \frac{\hat{\beta}_{T,i} \,-\, \beta_{*,i}}{\hat{\sigma}_T\sqrt{m_{i,i}}} \right\}^2.$$

4.(a) First consider $E[\hat{\beta}_{R,T}]$. Using the formula for RLS estimator, we have:

$$E[\hat{\beta}_{R,T}] \;=\; E\left[\hat{\beta}_T \,-\, (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}(R\hat{\beta}_T \,-\, r)\right].$$

Since $E[\,\cdot\,]$ is a linear operator and X, R and r are constants (the former from Assumption *CA2*), it follows that

$$E[\hat{\beta}_{R,T}] \;=\; E[\hat{\beta}_T] \,-\, (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}(RE[\hat{\beta}_T] \,-\, r). \tag{1}$$

In lecture 2, it is shown that under Assumptions *CA1* - *CA4* we have $E[\hat{\beta}_T] = \beta_0$ and substituting this result into (1),

$$E[\hat{\beta}_{R,T}] \;=\; \beta_0 \,-\, (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}(R\beta_0 \,-\, r). \tag{2}$$

Finally, using the fact that the restrictions hold, $R\beta_0 = r$, it follows form (2) that $E[\hat{\beta}_{R,T}] = \beta_0$. Now consider $Var[\hat{\beta}_{R,T}]$. Recall that since $\hat{\beta}_{R,T} = \beta_0$, we have $Var[\hat{\beta}_{R,T}] = E[(\hat{\beta}_{R,T} - \beta_0)(\hat{\beta}_{R,T} - \beta_0)']$, and so we first obtain an expression for $\hat{\beta}_{R,T} - \beta_0$. Using the formula for the RLS estimator, we have:

$$\hat{\beta}_{R,T} \,-\, \beta_0 \;=\; \hat{\beta}_T \,-\, \beta_0 \,-\, (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}(R\hat{\beta}_T \,-\, r). \tag{3}$$

Notice that if the restrictions hold, $R\beta_0 = r$, then (3) can be re-written as:

$$\hat{\beta}_{R,T} \,-\, \beta_0 \;=\; \hat{\beta}_T \,-\, \beta_0 \,-\, (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}R(\hat{\beta}_T \,-\, \beta_0), \tag{4}$$

and so we have:

$$\hat{\beta}_{R,T} \,-\, \beta_0 \;=\; C(\hat{\beta}_T \,-\, \beta_0), \tag{5}$$

where

$$C \;=\; I_k \,-\, (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}R \;=\; I_k \,-\, C_1, \;\text{say}.$$

Therefore, using (5), C is a constant (as both X and R are), and $E[\hat{\beta}_T] = \beta_0$, it follows that

$$\begin{aligned} Var[\hat{\beta}_{R,T}] &= E[(\hat{\beta}_{R,T} - \beta_0)(\hat{\beta}_{R,T} - \beta_0)'] \;=\; E[C(\hat{\beta}_T - \beta_0)(\hat{\beta}_T - \beta_0)'C'], \\ &= CE[(\hat{\beta}_T - \beta_0)(\hat{\beta}_T - \beta_0)']C' \;=\; CE[(\hat{\beta}_T - E[\hat{\beta}_T])(\hat{\beta}_T - E[\hat{\beta}_T])']C', \\ &= CVar[\hat{\beta}_T]C'. \end{aligned} \tag{6}$$

In lectures, it is shown that if Assumptions *CA1* - *CA5* hold then $Var[\hat{\beta}_T] = \sigma_0^2(X'X)^{-1}$. Using this result in (6), we obtain:

$$Var[\hat{\beta}_{R,T}] \;=\; \sigma_0^2 C(X'X)^{-1}C'. \tag{7}$$

Multiplying out and using $C = I_k - C_1$, we have:

$$\begin{aligned} C(X'X)^{-1}C' &= (X'X)^{-1} - (X'X)^{-1}C_1' - C_1(X'X)^{-1} + C_1(X'X)^{-1}C_1' \\ &= (X'X)^{-1} - D_1 - D_1 + D_1 \\ &= D, \end{aligned} \tag{8}$$

where

$$D_1 \;=\; (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}R(X'X)^{-1}.$$

Combining (7) and (8), we obtain $Var[\hat{\beta}_{R,T}] \;=\; \sigma_0^2 D$.

(b) Using (1) and $E[\hat{\beta}_T] = \beta_0$, we have:

$$E[\hat{\beta}_{R,T}] \;=\; \beta_0 - (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}(R\beta_0 - r) \;=\; \beta_0 + (X'X)^{-1}R'Aa,$$

where $A = \{R(X'X)^{-1}R'\}^{-1}$ and $a = R\beta_0 - r$. The bias of RLS is thus:

$$E[\hat{\beta}_{R,T}] - \beta_0 \;=\; -(X'X)^{-1}R'Aa \;=\; \mu, \text{ say.}$$

We now show that $R\beta_0 \neq r$ implies $\mu \neq 0$. Clearly, $R\beta_0 \neq r$ implies that $a \neq 0$. Now consider $b = Aa$: since A is nonsingular, it follows that $a \neq 0$ implies $b \neq 0$. Under the assumptions here R' is a $k \times n_r$ matrix with full column rank and so if $b \neq 0$ then we have $c = R'b \neq 0$. Finally, as $(X'X)^{-1}$ is nonsingular, it follows that $c \neq 0$ implies $\mu = -(X'X)^{-1}c \neq 0$.