Alastair Hall | **ECON61001: Econometric Methods** | Semester 1, 2020-21

Solutions to Problem Set for Tutorial 7

1.(a) We have

$$T^{1/2}\hat{\rho}_T \;=\; \frac{T^{-1/2}\sum_{t=2}^{T} u_t u_{t-1}}{T^{-1}\sum_{t=1}^{T} u_t^2}. \tag{1}$$

Using the WLLN, we have

$$T^{-1}\sum_{t=1}^{T} u_t^2 \;\xrightarrow{p}\; E[u_t^2] \;=\; \sigma_\varepsilon^2. \tag{2}$$

Now consider $T^{-1/2}\sum_{t=2}^{T} u_t u_{t-1}$. Since we are given that u_t is an independent sequence with mean zero, it follows that $E[u_t u_{t-1}] = Cov[u_t, u_{t-1}] = 0$. Therefore, using the CLT for time series, we have

$$T^{-1/2}\sum_{t=2}^{T} u_t u_{t-1} \;\xrightarrow{d}\; N(\,0, \omega) \tag{3}$$

where $\omega = \gamma_0 + 2\sum_{i=1}^{\infty}\gamma_i$ and $\gamma_i = E[u_t u_{t-1} u_{t-i} u_{t-i-1}]$. Using the fact that u_t is an independent sequence it follows that: (i) $\gamma_0 = E[u_t^2]E[u_{t-1}^2] = \sigma_\varepsilon^4$; (ii) for $i > 0$, $\gamma_i = E[u_t]E[u_{t-1}u_{t-i}u_{t-i-1}] = 0$ as $E[u_t] = 0$. Therefore $\omega = \sigma_\varepsilon^4$.

Using Lemma 3.5 in the Lecture Notes (in Section 3.1), it follows from (1)-(3) that:

$$T^{1/2}\hat{\rho}_T \xrightarrow{d} N(0,1).$$

1.(b) We have

$$\hat{\rho}_T \;=\; \frac{T^{-1}\sum_{t=2}^{T} u_t u_{t-1}}{T^{-1}\sum_{t=1}^{T} u_t^2}. \tag{4}$$

From Tutorial 5 Question 4 parts (b) and (c) respectively it follows that, it follows from the WLLN that

$$T^{-1}\sum_{t=1}^{T} u_t^2 \;\xrightarrow{p}\; E[u_t^2] \;=\; \frac{\sigma_\varepsilon^2}{1\,-\,\theta^2}, \tag{5}$$

$$T^{-1}\sum_{t=2}^{T} u_t u_{t-1} \;\xrightarrow{p}\; E[u_t u_{t-1}] \;=\; \frac{\theta\sigma_\varepsilon^2}{1\,-\,\theta^2}. \tag{6}$$

Combining (4-(6) and using Slutsky's theorem, it follows that: $\hat{\rho}_T \xrightarrow{p} \theta$.

1.(c) We have

$$\hat{\rho}_T \;=\; \frac{T^{-1}\sum_{t=2}^{T} u_t u_{t-1}}{T^{-1}\sum_{t=1}^{T} u_t^2}. \tag{7}$$

From Tutorial 5 Question 3 parts (b) and (c) respectively it follows that, it follows from the WLLN that

$$T^{-1}\sum_{t=1}^{T} u_t^2 \xrightarrow{p} E[u_t^2] = \sigma_\varepsilon^2(1+\phi^2), \tag{8}$$

$$T^{-1}\sum_{t=2}^{T} u_t u_{t-1} \xrightarrow{p} E[u_t u_{t-1}] = \sigma_\varepsilon^2\phi. \tag{9}$$

Combining (4-(6) and using Slutsky's theorem, it follows that:

$$\hat{\rho}_T \xrightarrow{p} \frac{\phi}{1+\phi^2}.$$

1.(d) We have

$$\hat{\rho}_T = \frac{T^{-1}\sum_{t=2}^{T} u_t u_{t-1}}{T^{-1}\sum_{t=1}^{T} u_t^2}. \tag{10}$$

From the WLLN, it follows that

$$T^{-1}\sum_{t=1}^{T} u_t^2 \xrightarrow{p} E[u_t^2], \tag{11}$$

$$T^{-1}\sum_{t=2}^{T} u_t u_{t-1} \xrightarrow{p} E[u_t u_{t-1}] \tag{12}$$

To evaluate these expectations, we substitute in for u_t and use the given properties of ε_t as follows.

$$\begin{aligned} E[u_t^2] &= E[(\varepsilon_t + \phi\varepsilon_{t-2})^2] \\ &= E[\varepsilon_t^2 + 2\phi\varepsilon_t\varepsilon_{t-2} + \phi^2\varepsilon_{t-2}^2] \\ &= E[\varepsilon_t^2] + 2\phi E[\varepsilon_t\varepsilon_{t-2}] + \phi^2 E[\varepsilon_{t-2}^2] \\ &= \sigma_\varepsilon^2(1+\phi^2), \end{aligned} \tag{13}$$

using the given information that $E[\varepsilon_t^2] = \sigma_\varepsilon^2$, and that ε_t is independent of ε_{t-2} so that $E[\varepsilon_t\varepsilon_{t-2}] = E[\varepsilon_t]E[\varepsilon_{t-2}] = 0$ as $E[\varepsilon_t] = 0$.

$$\begin{aligned} E[u_t u_{t-1}] &= E[(\varepsilon_t + \phi\varepsilon_{t-2})(\varepsilon_{t-1} + \phi\varepsilon_{t-3})] \\ &= E[\varepsilon_t\varepsilon_{t-1} + \phi\varepsilon_{t-1}\varepsilon_{t-2} + \phi\varepsilon_t\varepsilon_{t-3} + \phi^2\varepsilon_{t-2}\varepsilon_{t-3}] \\ &= E[\varepsilon_t\varepsilon_{t-1}] + \phi E[\varepsilon_{t-1}\varepsilon_{t-2}] + \phi E[\varepsilon_t\varepsilon_{t-3}] + \phi^2 E[\varepsilon_{t-2}\varepsilon_{t-3}] \\ &= 0 \end{aligned} \tag{14}$$

using the given information that ε_t is independent of ε_s for all $t \neq s$ so that $E[\varepsilon_t\varepsilon_s] = E[\varepsilon_t]E[\varepsilon_s] = 0$ as $E[\varepsilon_t] = 0$. From (10)-(14) and using Slutsky's Theorem, it follows that $\hat{\rho}_T \xrightarrow{p} 0$.

2. The assumed model specification is:

$$
\begin{aligned}
y_t &= x_t'\beta_0 + u_t \\
u_t &= \rho u_{t-1} + \varepsilon_t
\end{aligned} \tag{15}
$$

If the assumed specification is correct then the strategy makes sense.

- if $\rho = 0$ then both OLS and FGLS inferences are valid, but those based on OLS are likely more reliable as FGLS is based on a model that is overfit (*i.e.* we estimate ρ but its true value is zero and so this parameter could have been omitted.)
- if $\rho \neq 0$ then FGLS inferences are valid, but those based on OLS are not as we have the wrong variance estimator. Note that from Question 1 part (b) and the hint, the large sample behaviour of the test statistic is determined by $|T^{1/2}\theta|$. Therefore, as $T \to \infty$, $|T^{1/2}\hat{\rho}_T| \to \infty$ with probability one and so we reject with probability one as $T \to \infty$. Therefore, inferences are based on the FGLS estimator in this case with probability one as $T \to \infty$. (If a test rejects with probability one as $T \to \infty$ then it is said to be *consistent*. This is, perhaps, an unfortunate choice of words and is not to be confused with the concept of consistency of an estimator.)

However, using the results from Question 1 part (c) and the hint, it is clear that if the errors are generated by a MA(1) model - instead of an AR(1) model - then the test also rejects with probability one as $T \to \infty$. In this case, the FGLS inferences would be invalid.

Similarly, if the errors are generated by the process in part (d) then this type of dependence may not be detected by the test and so the failure to reject does not necessarily mean that the errors are serially uncorrelated. Using similar arguments to Tutorial 5 Question 3 part (e), we can use the CLT to deduce that

$$
T^{-1/2}\sum_{t=2}^{T} u_t u_{t-1} \xrightarrow{d} N(0, \omega_1),
$$

where the exact form of ω_1 need not concern us, and from Question 1(d) we have

$$
T^{-1}\sum_{t=1}^{T} u_t^2 \xrightarrow{p} E[u_t^2] = \sigma_\varepsilon^2(1 + \phi^2);
$$

and so using Lemma 2.11, we have

$$
T^{1/2}\hat{\rho}_T \xrightarrow{d} N(0, \omega_\rho),
$$

where $\omega_\rho = \omega_1/\{\sigma_\varepsilon^2(1+\phi^2)\}^2$. Therefore, if the errors are generated by the MA(2) process in Question 1 part (d) then the probability H_0 is not rejected in the limit as $T \to \infty$ is $P(|\xi| \le 1.96)$ where $\xi \sim N(0, \omega_\rho)$; this probability is non-zero.

The following terminology is used to underscore the differences between the null and alternative hypotheses of the test and the process for which the test does not reject. Here the *nominal null hypothesis* is that u_t is generated via (15) with $\rho = 0$ and the *nominal alternative hypothesis* is that u_t is generated via (15)with $\rho \neq 0$. The *implicit alternative hypothesis* consists of all process for u_t for which the rejects with probability one in the limit (in our context here), and the *implicit nominal null hypothesis* consists of processes for which there is non-zero probability of failing to reject H_0 as $T \to \infty$.[1]

The only protection against the issues highlighted here is to perform a joint test for serial correlation at multiple lags, and then, if this test rejects, to engage in a more general investigation of the model specification.

3. As discussed in lectures, the key condition for the consistency of the OLS estimator is $E[x_t u_t] = 0$. This case, we have $x_t = [1, y_{t-1}]$. Consider $E[y_{t-1}u_t]$. Since

$$y_{t-1} = \beta_{0,1} + \beta_{0,2}y_{t-2} + \varepsilon_{t-1} + \phi\varepsilon_{t-2}$$

and

$$u_t = \varepsilon_t + \phi\varepsilon_{t-1},$$

so $E[y_{t-1}u_t] = \phi Var[\epsilon_{t-1}] \neq 0$. Therefore $E[x_t u_t] \neq 0$ and so OLS is not consistent.

4. Define:

$$\hat{\tau}_2 = \frac{\hat{\beta}_{T,2}}{\sqrt{\hat{V}_{sc,2,2}/T}}$$

where $\hat{V}_{sc,2,2}$ is the $(2,2)^{th}$ element of $\hat{V}_{sc}$,

$$\hat{V}_{sc}/T = T(X'X)^{-1}\hat{\Omega}_{HAC}(X'X)^{-1},$$

$$\hat{\Omega}_{HAC} = \hat{\Gamma}_0 + \sum_{i=1}^{T-1} \omega(i,T)\left\{\hat{\Gamma}_i + \hat{\Gamma}_i'\right\},$$

$$\hat{\Gamma}_j = T^{-1}\sum_{t=j+1}^{T} x_t x_{t-j}' e_t e_{t-j},$$

where $\{e_t\}_{t=1}^T$ are the OLS residuals,

$$\omega(i,T) = \begin{cases} 1 - a_i \text{ for } a_i \leq 1 \\ 0 \text{ for } a_i > 1 \end{cases}$$

for $a_i = i/(b_T + 1)$ and $b_T \propto T^{1/3}$.

[1] We note that the precise definition of the implicit null and alternative can differ in the statistics literature although the spirit is always the same. We have adapted the concepts to fit our setting here in which we deal with large sample properties against what are known as "fixed alternatives".

A suitable decision rule is to reject $H_0 : \beta_{0,2} = 0$ at the (approximate) $100\alpha\%$ significance level in favour of $H_A : \beta_{0,2} \neq 0$ if $|\hat{\tau}_2| > z_{1-\alpha/2}$ where $z_{1-\alpha/2}$ is the $1001 - \alpha/2^{th}$ percentile of the standard normal distribution.