University of Manchester

ECON61001: Econometric Methods

Mid-Term Exam

Release date/time: 27/11/20, 9.00hrs Greenwich Mean Time (GMT)

Submission deadline: 30/11/20, 9.00hrs GMT

Instructions:

- You must answer **all three** questions.
- Your answers could be typed or hand-written (and scanned to a single pdf file that can be submitted) or a combination of a typed answer with included images of algebra or figures.

- Where relevant, questions include word limits. These are limits, not targets. Excellent answers can be shorter than the word limit. If you go beyond the word limit the additional text will be ignored. Where a question includes a word limit you HAVE to include a word count for your answer (excluding formulae). You could use https://wordcounter.net to obtain word counts.

- Candidates are advised that the examiners attach considerable importance to the clarity with which answers are expressed.

- **You must correctly enter your registration number and the course code on your answer.**

1. Two researchers are interested in whether or not the mean household income is the same in both the North and the South of England. Suppose they obtain a random sample of N observations on the income of households in England at a particular moment in time. Let y_i be the income of the i^{th} household and R_i be a dummy variable that indicates the region in which the household lives as follows:

$$R_i = 1, \text{ if } i^{th} \text{ household lives in the North,}$$
$$\quad = 0, \text{ if } i^{th} \text{ household lives in the South.}$$

Researcher A considers inference based on the regression model

$$y_i = x_i'\beta_0 + u_i, \tag{1}$$

where $x_i' = (1, R_i)$, $\beta_0 = (\beta_{0,1}, \beta_{0,2})'$. Let $\hat{\beta}_N$ denote the OLS estimator of β_0 based on (1).

Researcher B considers inference based on the regression model

$$y_i = w_i'\gamma_0 + u_i, \tag{2}$$

where $w_i' = (1 - R_i, R_i)$, $\gamma_0 = (\gamma_{0,1}, \gamma_{0,2})'$. Let $\hat{\gamma}_N$ denote the OLS estimator of γ_0 based on (2).

Assume that u_i is independent of R_i and that $\{u_i;\ i = 1, 2, \ldots, N\}$ is a sequence of independently and identically distributed random variables with a normal distribution with mean equal to zero and variance equal to σ_0^2, an unknown positive constant.

(a) Show that:

$$\hat{\beta}_N = \begin{bmatrix} \bar{y}_s \\ \bar{y}_n - \bar{y}_s \end{bmatrix},$$

where $\bar{y}_n$, $\bar{y}_s$ are the sample mean incomes for households living, respectively, in the North and the South of England. **[10 marks]**

(b) Under the conditions above, it can be shown that $N^{1/2}(\hat{\beta}_N - \beta_0) \xrightarrow{d} N(0, V_\beta)$ and $N^{1/2}(\hat{\gamma}_N - \gamma_0) \xrightarrow{d} N(0, V_\gamma)$. What is the relationship between V_γ and V_β? Be sure to justify your answer. **[10 marks]**

Continued over

2. Consider the linear regression model

$$y \;=\; X\beta_0 \;+\; u,$$

where X is 20×4 matrix that is fixed in repeated samples with full column rank, and $u \sim N(\,0\,\sigma_0^2 I_{20})$ where σ_0^2 is an unknown positive constant.

(a) Let $\hat{\beta}_{T,3}$ be the OLS estimator of $\beta_{0,3}$, the third element of β_0. Consider the following two inference procedures relating to $\beta_{0,3}$ based on $\hat{\beta}_{T,3}$:

- $100(1-\alpha)\%$ confidence interval for $\beta_{0,3}$;
- the two-sided hypotheses test of $H_0 : \beta_{0,3} = 1$.

Show that the null hypothesis of this two-sided test is rejected at the $100\alpha\%$ significance level if and only if one is not in the $100(1-\alpha)\%$ confidence interval for $\beta_{0,3}$. **[5 marks]**

(b) Propose a 95% confidence interval for σ_0^2 and show that it possesses the stated coverage rate. *Note: you may quote any relevant distributional result from lecture notes without proof.* **[10 marks]**

3. Consider the linear regression model

$$y_i \;=\; x_i'\beta_0 + u_i, \qquad i = 1, 2, \ldots, N$$

where β_0 is the $k \times 1$ vector of unknown regression coefficients, $\{(x_i', u_i)\}_{i=1}^N$ is a sequence of independent and identically distributed random vectors with $E[u_i|x_i] = 0$, $Var[u_i|x_i] = \sigma_0^2$, an unknown positive constant, and $E[x_i x_i'] = Q$, a finite positive definite matrix of constants. Let $\hat{\beta}_{R,N}$ denote the RLS estimator based on the linear restrictions $R\beta = r$ where where R is a $n_r \times k$ matrix of pre-specified constants with rank equal to n_r and r is a $n_r \times 1$ vector of pre-specified constants that is,

$$\hat{\beta}_{R,N} \;=\; \hat{\beta}_N \;-\; (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}(R\hat{\beta}_N \;-\; r),$$

where $\hat{\beta}_N$ is the OLS estimator of β_0 and X is the $N \times k$ matrix with i^{th} row x_i'. Show that $\hat{\beta}_{R,N}$ is a consistent estimator for β_0. **[15 marks]**

Note: (i) You may quote the formula for the OLS estimator without proof; (ii) you may quote the generic form of both the Weak Law of Large Numbers, $N^{-1}\sum_{i=1}^N z_i \xrightarrow{p} \mu_z$*, but must verify* μ_z *for the specific choices of* z_i *relevant to your answer; you may also quote the generic form of the Central Limit Theorem,* $N^{-1/2}\sum_{i=1}^N (z_i - \mu_z) \xrightarrow{d} N(0, \Omega)$ *but must verify* μ_z *and* Ω *for the specific choices of* z_i *relevant to your answer.*

END OF EXAMINATION

1 Table 1: Percentage Points for the t distribution

Student's t Distribution Function for Selected Probabilities
The table provides values of $t_{\alpha,v}$ where $\Pr(T \le t_{\alpha,v}) = \alpha$ and $T \sim t_v$

α	0.750	0.800	0.900	0.950	0.975	0.990	0.995	0.9975	0.999	0.9995
ν	Values of $t_{\alpha,v}$									
1	1.000	1.376	3.078	6.314	12.706	31.821	63.657			
2	0.816	1.061	1.886	2.920	4.303	6.965	9.925			
3	0.765	0.978	1.638	2.353	3.182	4.541	5.841			
4	0.741	0.941	1.533	2.132	2.776	3.747	4.604			
5	0.727	0.920	1.476	2.015	2.571	3.365	4.032	4.773		
6	0.718	0.906	1.440	1.943	2.447	3.143	3.707	4.317	5.208	
7	0.711	0.896	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
70	0.678	0.847	1.294	1.667	1.994	2.381	2.648	2.899	3.211	3.435
80	0.678	0.846	1.292	1.664	1.990	2.374	2.639	2.887	3.195	3.416
90	0.677	0.846	1.291	1.662	1.987	2.368	2.632	2.878	3.183	3.402
100	0.677	0.845	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390
110	0.677	0.845	1.289	1.659	1.982	2.361	2.621	2.865	3.166	3.381
120	0.677	0.845	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	0.842	1.282	1.645	1.960	2.326	2.576	2.808	3.090	3.297

2 Table 2: Percentage Points for the χ^2 distribution

The χ^2 Distribution Function for Selected Probabilities											
The table provides values of $\chi^2_{\alpha,v}$ where $\Pr(\chi^2 \leq \chi^2_{\alpha,v}) = \alpha$ and $\chi^2 \sim \chi^2_v$											
α	0.005	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99	0.995
v	Values of $\chi^2_{\alpha,v}$										
1	0.000	0.000	0.001	0.004	0.016	0.455	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	1.386	4.605	5.991	7.378	9.210	10.60
3	0.072	0.115	0.216	0.352	0.584	2.366	6.251	7.815	9.348	11.34	12.84
4	0.207	0.297	0.484	0.711	1.064	3.357	7.779	9.488	11.14	13.28	14.86
5	0.412	0.554	0.831	1.145	1.610	4.351	9.236	11.07	12.83	15.09	16.75
6	0.676	0.872	1.237	1.635	2.204	5.348	10.64	12.59	14.45	16.81	18.55
7	0.989	1.239	1.690	2.167	2.833	6.346	12.02	14.07	16.01	18.48	20.28
8	1.344	1.646	2.180	2.733	3.490	7.344	13.36	15.51	17.53	20.09	21.95
9	1.735	2.088	2.700	3.325	4.168	8.343	14.68	16.92	19.02	21.67	23.59
10	2.156	2.558	3.247	3.940	4.865	9.342	15.99	18.31	20.48	23.21	25.19
11	2.603	3.053	3.816	4.575	5.578	10.34	17.28	19.68	21.92	24.72	26.76
12	3.074	3.571	4.404	5.226	6.304	11.34	18.55	21.03	23.34	26.22	28.30
13	3.565	4.107	5.009	5.892	7.042	12.34	19.81	22.36	24.74	27.69	29.82
14	4.075	4.660	5.629	6.571	7.790	13.34	21.06	23.68	26.12	29.14	31.32
15	4.601	5.229	6.262	7.261	8.547	14.34	22.31	25.00	27.49	30.58	32.80
16	5.142	5.812	6.908	7.962	9.312	15.34	23.54	26.30	28.85	32.00	34.27
17	5.697	6.408	7.564	8.672	10.09	16.34	24.77	27.59	30.19	33.41	35.72
18	6.265	7.015	8.231	9.390	10.86	17.34	25.99	28.87	31.53	34.81	37.16
19	6.844	7.633	8.907	10.12	11.65	18.34	27.20	30.14	32.85	36.19	38.58
20	7.434	8.260	9.591	10.85	12.44	19.34	28.41	31.41	34.17	37.57	40.00
21	8.034	8.897	10.28	11.59	13.24	20.34	29.62	32.67	35.48	38.93	41.40
22	8.643	9.542	10.98	12.34	14.04	21.34	30.81	33.92	36.78	40.29	42.80
23	9.260	10.20	11.69	13.09	14.85	22.34	32.01	35.17	38.08	41.64	44.18
24	9.886	10.86	12.40	13.85	15.66	23.34	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	24.34	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	25.34	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	26.34	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	27.34	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	28.34	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	29.34	40.26	43.77	46.98	50.89	53.67
35	17.19	18.51	20.57	22.47	24.80	34.34	46.06	49.80	53.20	57.34	60.27
40	20.71	22.16	24.43	26.51	29.05	39.34	51.81	55.76	59.34	63.69	66.77
45	24.31	25.90	28.37	30.61	33.35	44.34	57.51	61.66	65.41	69.96	73.17
50	27.99	29.71	32.36	34.76	37.69	49.33	63.17	67.50	71.42	76.15	79.49
50	27.99	29.71	32.36	34.76	37.69	49.33	63.17	67.50	71.42	76.15	79.49
70	43.28	45.44	48.76	51.74	55.33	69.33	85.53	90.53	95.02	100.4	104.2
80	51.17	53.54	57.15	60.39	64.28	79.33	96.58	101.9	106.6	112.3	116.3
90	59.20	61.75	65.65	69.13	73.29	89.33	107.6	113.1	118.1	124.1	128.3
100	67.33	70.06	74.22	77.93	82.36	99.33	118.5	124.3	129.6	135.8	140.2
150	109.1	112.7	118.0	122.7	128.3	149.3	172.6	179.6	185.8	193.2	198.4
200	152.2	156.4	162.7	168.3	174.8	199.3	226.0	234.0	241.1	249.4	255.3

3 Table 3: Upper 5% percentage points for the F distribution

The F Distribution Function for $\alpha = 0.05$

The table provides values of F_{α,v_1,v_2} where $\Pr(F \geq F_{\alpha,v_1,v_2}) = 0.05$ and $F \sim F(v_1, v_2)$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01
35	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11	2.04	1.96
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92
45	4.06	3.20	2.81	2.58	2.42	2.31	2.22	2.15	2.10	2.05	1.97	1.89
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.87
55	4.02	3.16	2.77	2.54	2.38	2.27	2.18	2.11	2.06	2.01	1.93	1.85
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.89	1.81
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.88	1.79
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.86	1.78
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.85	1.77
110	3.93	3.08	2.69	2.45	2.30	2.18	2.09	2.02	1.97	1.92	1.84	1.76
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.82	1.73

4 Table 4: Upper 1% percentage points for the F distribution

The F Distribution Function for $\alpha = 0.01$												
The table provides values of F_{α,v_1,v_2} where $\Pr(F \geq F_{\alpha,v_1,v_2}) = 0.01$ and $F \sim F(v_1, v_2)$												
	$v_1 \rightarrow$											
$v_2 \downarrow$	1	2	3	4	5	6	7	8	9	10	12	15
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70
35	7.42	5.27	4.40	3.91	3.59	3.37	3.20	3.07	2.96	2.88	2.74	2.60
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52
45	7.23	5.11	4.25	3.77	3.45	3.23	3.07	2.94	2.83	2.74	2.61	2.46
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.56	2.42
55	7.12	5.01	4.16	3.68	3.37	3.15	2.98	2.85	2.75	2.66	2.53	2.38
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35
70	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67	2.59	2.45	2.31
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.42	2.27
90	6.93	4.85	4.01	3.53	3.23	3.01	2.84	2.72	2.61	2.52	2.39	2.24
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.37	2.22
110	6.87	4.80	3.96	3.49	3.19	2.97	2.81	2.68	2.57	2.49	2.35	2.21
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19
150	6.81	4.75	3.91	3.45	3.14	2.92	2.76	2.63	2.53	2.44	2.31	2.16