Alastair Hall

ECON61001:
Econometric Methods

Semester 1, 2020-21

Problem Set for Tutorial 7

In this question, you investigate the properties the first order autocorrelation of a time series under certain assumptions about how it is generated. This analysis involves the large sample behaviour of terms involving sums that start at $t = 2$ rather than $t = 1$. In the cases below, this difference is negligible in large samples and so the large sample behaviour of terms involving sums that start at $t = 2$ is the same as that of the analogous terms with the sums starting at $t = 1$. You may find it useful to refer back to Tutorial 5.

1. Define $\hat{\rho}_T$ to be the sample first-order autocorrelation of the weakly stationary process u_t with mean zero that is,[1]

$$\hat{\rho}_T \;=\; \frac{\sum_{t=2}^{T} u_t u_{t-1}}{\sum_{t=1}^{T} u_t^2}.$$

Let $\{\varepsilon_t\}_{t=1}^{T}$ denote a sequence of i.i.d. random variables with mean zero and variance σ_ε^2.

(a) Assume that $u_t = \varepsilon_t$. Show that $T^{1/2}\hat{\rho}_T \xrightarrow{d} N(0,1)$.

(b) Assume

$$u_t \;=\; \theta u_{t-1} \;+\; \varepsilon_t,$$

where $|\theta| < 1$. Show that $\hat{\rho}_T \xrightarrow{p} \theta$.

(c) Assume

$$u_t \;=\; \varepsilon_t \;+\; \phi\varepsilon_{t-1}.$$

Show that $\hat{\rho}_T \xrightarrow{p} \phi/(1+\phi^2)$.

(d) Assume

$$u_t \;=\; \varepsilon_t \;+\; \phi\varepsilon_{t-2}.$$

Show that $\hat{\rho}_T \xrightarrow{p} 0$.

[1] We use the knowledge that the process has mean zero to simplify the formula by not explicitly stating the formula in terms of $u_t - E[u_t]$.

In this question you must use the results from Question 1 to evaluate a modeling strategy for estimation of a time series regression model

2. Consider the time series regression model

$$y_t \;=\; x_t'\beta_0 \,+\, u_t$$

where x_t does not contain any lagged values of y_t. Suppose a researcher estimates this model via OLS and calculates the OLS residuals $\{e_t\}_{t=1}^T$. The researcher is concerned that the errors may be serially correlated and hence that the OLS estimators are inefficient. To this end, she adopts the following strategy. She assumes that the errors satisfy

$$u_t \;= \rho u_{t-1} \,+\, \varepsilon_t, \tag{1}$$

where ε_t is defined as in Question 1 and can be assumed independent of x_t, and tests $H_0: \rho = 0$ versus $H_A: \rho \neq 0$ using the following decision rule:

Reject H_0 *at the* 5% *significance level if* $|T^{1/2}r_T| > 1.96$ *where* $r_T \;=\; \dfrac{\sum_{t=2}^T e_t e_{t-1}}{\sum_{t=1}^T e_t^2}$

If the test does not reject then she will base inference on the OLS estimators using the standard formula for variance estimator, "$\hat{\sigma}_T^2(X'X)^{-1}$". If the test does reject then she will base inference on a feasible GLS estimator based on the assumption that the errors are generated by (1), and so the variance is estimator is "$(X'\hat{\Sigma}^{-1}X)^{-1}$".

In light of your answers to Question 1, do you think this is a good strategy? (You may assume that if the errors are generated via (1) then $T^{1/2}(\hat{\beta}_{FGLS} - \beta_0)$ has the same large sample distribution as is asymptotically equivalent to $T^{1/2}(\hat{\beta}_{GLS} - \beta_0)$, and that $T^{-1}X'\hat{\Sigma}^{-1}X \,-\, T^{-1}X'\Sigma^{-1}X \xrightarrow{p} 0$.)

Hint: it can be shown under certain conditions (which you may assume here) that (i) under the conditions of Question 1(a)& (d), $T^{1/2}r_T - T^{1/2}\hat{\rho}_T \xrightarrow{p} 0$*; (ii) under the conditions on* u_t *in parts (b)-(c) the large sample behaviour of* $T^{1/2}r_T$ *is determined by* $T^{1/2}plim\hat{\rho}_T$*.*

3. Consider now the regression model:

$$y_t \;=\; \beta_{0,1} \,+\, \beta_{0,2}y_{t-1} \,+\, u_t, \tag{2}$$

where

$$u_t \;=\; \varepsilon_t \,+\, \phi\varepsilon_{t-1}, \tag{3}$$

where $\phi \neq 0$ and ε_t is white noise. Let $\hat{\beta}_T$ be the OLS estimator of $\beta_0 = (\beta_{0,1}, \beta_{0,2})'$ based on (2). Is $\hat{\beta}_T$ consistent for β_0?

4. Consider the time series regression model

$$y_t \;=\; x_t'\beta_0 \;+\; u_t \tag{4}$$

where $x_t = (1, z_t)'$ and z_t is a scalar time series variable. Let $\hat{\beta}_T$ be the OLS estimator of β_0 based on (4). Assume that: (i) y_t is generated by (4); (ii) (y_t, z_t) is a weakly stationary and weakly dependent time series; (iii) $E[x_t x_t'] = Q$, a positive definite matrix; (iv) $E[x_t u_t] = 0$; (v) $T^{1/2}(\hat{\beta}_T - \beta_0) \xrightarrow{d} N(0, V_{sc})$ where $V_{sc} = Q^{-1}\Omega Q^{-1}$, $\Omega = \Gamma_0 + \sum_{i=1}^{\infty}(\Gamma_i + \Gamma_i')$ and $\Gamma_i = Cov[x_t u_t, x_{t-i}u_{t-i}]$. Propose a decision rule to test $H_0 : \beta_{0,2} = 0$ vs $H_A : \beta_{0,2} \neq 0$, being sure to define your test statistic in terms of the data.