Problem Set for Tutorial 9

In this question, you examine the properties of the error in the linear probability model.

1. Let $\{y_i, x_i'\}_{i=1}^N$ be a sequence of iid random vectors. Suppose that y_i is a dummy variable and so has a sample space of $\{0, 1\}$ with $P(y_i = 1|x_i) = x_i'\beta_0$. Now consider the Linear Probability Model (LPM),

$$y_i = x_i'\beta_0 + u_i.$$

Show that

(a) $E[u_i|x_i] = 0$.

(b) $Var[u_i|x_i] = x_i'\beta_0(1 - x_i'\beta_0)$.

(c) Can u_i have a normal distribution conditional on x_i?

In this question you consider the logit model.

2. Consider the logit model in which $P(y_i = 1|x_i; \beta) = \Lambda(x_i'\beta)$ where x_i is a $(k \times 1)$ vector with ℓ^{th} element $x_{i,\ell}$ and

$$\Lambda(z) = \frac{exp(z)}{1 + exp(z)}.$$

Assume we have a sample of size N and that $\{y_i, x_i'\}_{i=1}^N$ are independently and identically distributed.

(a) Derive the (conditional) likelihood function.

(b) Write down the (conditional) log likelihood function.

(c) Assuming $x_{i,\ell}$ is a continuous variable, derive $\partial\Lambda(x_i'\beta_0)/\partial x_{i,\ell}$.

In this question you consider ML estimation of the linear regression model under Assumptions CA1-CA6*. Recall that under these assumptions, $y \sim N(X\beta_0, \sigma_0^2 I_T)$. Recall from Lecture 9 that in the case of continuous random variables the likelihood function is the joint pdf of the random variables in question.*

3. Consider the linear regression model

$$y = X\beta_0 + u$$

where y and u are $T \times 1$ vectors, and X is $T \times k$ matrix. Suppose that Assumptions *CA1-CA6* (from Lecture 1) hold. Let θ be the $(k + 1) \times 1$ vector consisting of the parameters of this model that is, $\theta = (\beta', \sigma^2)'$.

(a) Show that the log likelihood function is given by

$$LLF_T(\theta) = -\frac{T}{2}ln[2\pi] - \frac{T}{2}ln[\sigma^2] - \frac{(y - X\beta)'(y - X\beta)}{2\sigma^2}.$$

(b) Show that the score equations imply:

$$\begin{bmatrix} X'(y - X\hat{\beta}_T) \\ -T\hat{\sigma}_T^2 + (y - X\hat{\beta}_T)'(y - X\hat{\beta}_T) \end{bmatrix} = 0$$

(c) Show that the MLE's are:

$$\begin{aligned} \hat{\beta}_T &= (X'X)^{-1}X'y \\ \hat{\sigma}_T^2 &= (y - X\hat{\beta}_T)'(y - X\hat{\beta}_T)/T \end{aligned}$$

(d) Compare the MLE's to the OLS estimators of θ_0.