Alastair Hall

ECON61001:
Econometric Methods

Semester 1 2020-21

Practice Questions

1. A researcher is interested in testing the null hypothesis $H_0 : \beta_{0,2} = \beta_{0,3}$ versus the alternative $H_1 : \beta_{0,2} \neq \beta_{0,3}$ in the model

$$y_i \;=\; x_i'\beta_0 \,+\, u_i, \qquad i = 1, 2, \ldots, N$$

where x_i is 3×1 and $\beta_0 = (\beta_{0,1}, \beta_{0,2}, \beta_{0,3})'$. To do so, the researcher estimates the model via OLS and plans to test the hypothesis using the decision rule: reject H_0 if $S_N > \tau_{N-k}(1-\alpha/2)$ where $\tau_{N-k}(1-\alpha/2)$ is the $100(1-\alpha/2)^{th}$ percentile of the Student's t distribution,

$$S_N \;=\; \left| \frac{\hat{\beta}_{N,2} \,-\, \hat{\beta}_{N,3}}{d_N} \right|$$

and $\hat{\beta}_N = (\hat{\beta}_{N,1}, \hat{\beta}_{N,2}, \hat{\beta}_{N,3})'$ is the OLS estimator of β_0. Let y be the $N \times 1$ vector with i^{th} element y_i, X be the $N \times 3$ matrix with i^{th} row x_i', and assume that the regression model satisfies Assumptions SR1-SR6 given on pp.54-5 of the Lecture Notes (with N replacing T). Let $\hat{\sigma}_N^2$ be the OLS estimator of σ_0^2 given in Assumption SR6.

   (a) In order for the test above to have the probability of a Type I error implied by the significance level, d_N must equal $f(y, X, N)$, a particular function of $\{y, X, N\}$: state $f(y, X, N)$.

   (b) Given that $\hat{\beta}_N = (1.136, 0.913, 0.952)'$, $\hat{\sigma}_N^2 = 0.539$, $N = 30$ and

$$(X'X)^{-1} \;=\; \begin{bmatrix} 0.058 & -0.015 & 0.007 \\ -0.015 & 0.010 & -0.001 \\ 0.007 & -0.001 & 0.009 \end{bmatrix}$$

   perform the test using a 5% significance level. Be sure to explain your calculations clearly.

2. Consider the linear regression model

$$y \;=\; X\beta_0 \,+\, u \tag{1}$$

where y is 25×1 vector of observations on the dependent variable, X is 25×5 data matrix of observations on the explanatory variables, β_0 is 5×1 vector of unknown regression coefficients and u is the 25×1 vector containing the error term. Suppose that $X = (\iota_{25}, X_2)$ where ι_{25} is a 25×1 vector of ones and X_2 is a 25×4 matrix, and $\beta_0 = (\beta_{0,1}, \beta_{0,2}')'$ where $\beta_{0,1}$ is a scalar and $\beta_{0,2}$ is a 4×1 vector. Let F be the F-statistic for testing $H_0 : \beta_{0,2} = 0_4$ versus $H_0 : \beta_{0,2} \neq 0_4$ where 0_4 is the 4×1 null vector. If the p-value for this test is 0.05 then what is the adjusted R^2, $\bar{R}^2$, for the estimated version of (1)? Be sure to justify your answer carefully.

3. A researcher estimates the unknown parameter vector γ_0 in the model

$$x \;=\; Z\gamma_0 \,+\, e$$

where x is a $N \times 1$ vector with i^{th} element x_i, Z is $N \times q$ matrix of observable explanatory variables with i^{th} row z_i' based on the population moment condition,

$$E[y_i(x_i \,-\, z_i'\gamma_0)] \;=\; 0, \tag{2}$$

where y_i is $k \times 1$ vector. Assume that $\{z_i, y_i, e_i\}_{i=1}^N$ form a sequence of independently and identically distributed random variables with: (i) $E[z_i y_i'] = M_{zy}$, a matrix of constants with full row rank; (ii) $E[y_i y_i'] = M_{y,y}$, a nonsingular matrix of constants; (iii) $E[e_i|y_i] = 0$; (iv) $Var[e_i|y_i] = a(y_i)$.

(a) Derive the condition for γ_0 to be identified by the population moment condition in (2).

(b) Derive the formula for the IV estimator for γ_0 based on (2).

(c) Show that the IV estimator of γ_0 based on (2) is consistent under the conditions above.

(d) The researcher wishes to test the hypothesis $H_0 : \beta(\gamma_0) = 0$ versus $H_1 : \beta(\gamma_0) \neq 0$ where $\beta(\cdot)$ is a $c \times 1$ vector of continuous differentiable functions. Propose a test based on the IV estimator of γ_0 in part (b), being sure to: explain clearly how to calculate the test statistic from the data; to provide the decision rule; to state any additional conditions (beyond those above) that must be satisfied in order for your decision rule to be valid.

4. Suppose $\{v_i\}_{i=1}^N$ is a sequence of i.i.d. normal random variables with mean μ_0 and variance σ_0^2.

(a) Derive the Wald, Likelihood Ratio (LR) and Lagrange Multiplier (LM) statistics for testing $H_0 : \mu_0 = 0$ versus $H_1 : \mu_0 \neq 0$. *Hint: you may quote the form of the log likelihood function, score equations and maximum likelihood estimators without derivation.*

(b) Given a sample of size $N = 100$, $\{v_i\}_{i=1}^N$, for which $\sum_{i=1}^N v_i = 12.31$ and $\sum_{i=1}^N v_i^2 = 135.28$, what is the outcome of the Wald, LR and LM tests in part (a). Be sure to justify your conclusions carefully.

5. Consider the case where $y_i \in \{0, 1, 2\}$ and the value of y_i^* is latent variable generated via

$$y_i^* \;=\; x_i'\beta_0 \,+\, u_i$$

where x_i is vector of observable explanatory variables, β_0 is an unknown parameter vector, and $u_i \sim N(0, 1)$. Suppose that the outcome of y_i is determined as follows:

$$\begin{aligned} y_i &= 0, \text{ if } y_i^* < 0, \\ &= 1, \text{ if } 0 \leq y_i^* < 2, \\ &= 2, \text{ if } 2 \geq y_i^*. \end{aligned}$$

Assume that $\{x_i, u_i\}_{i=1}^N$ is an independently and identically distributed sequence, and that y_i is observable.

(a) Derive the probability distribution function for y_i conditional on x_i.

(b) Write down $CLLF_N(\beta)$, the conditional log-likelihood function based on a sample $\{y_i, x_i\}_{i=1}^N$.

6. Suppose that y_t is generated via

$$y_t \;=\; \beta_0 y_{t-1} \,+\, u_t \tag{3}$$

where $\beta_0 = 0$,

$$u_t \;=\; \varepsilon_t \,+\, \phi\varepsilon_{t-1}, \tag{4}$$

$\phi \neq 0$ and $\{\varepsilon_t\}$ is sequence of independently and identically distributed random variables with mean zero and variance σ^2. Let $\hat{\beta}_T$ be the OLS estimator of β_0 based on (3) that is,

$$\hat{\beta}_T \;=\; \frac{\sum_{t=1}^T y_{t-1}y_t}{\sum_{t=1}^T y_{t-1}^2}.$$

(a) Derive the probability limit of $\hat{\beta}_T$ and verify whether or not this estimator is consistent for β_0.

(b) Suppose that (3) is estimated via Instrumental Variables (IV) using y_{t-2} as instrument for y_{t-1} that is, estimation is based on the moment condition $E[y_{t-2}(y_t - \beta_0 y_{t-1})] = 0$. Verify whether or not this choice of instrument satisfies the orthogonality and relevance conditions associated with IV estimation.

(c) Let $\tilde{\beta}_T$ be the IV estimator of β_0 based on the population moment condition in part (b). Show that $T^{1/2}(\tilde{\beta}_T - \beta_0) \xrightarrow{d} N(0, H)$ where H is a matrix that you must specify as part of your answer.

(d) Suppose now that $\phi = 0$. How does $T^{1/2}(\tilde{\beta}_T - \beta_0)$ behave as $T \to \infty$? Justify your answer briefly.

Hint: y_t is a covariance stationary process and you may quote the generic form of: (i) the Weak Law of Large Numbers for covariance stationary and weakly dependent processes, $T^{-1}\sum_{t=1}^T w_t \xrightarrow{p} \mu_w$, but must verify the specific forms of the limits relevant to the quantities analyzed in your answer; (ii) the Central Limit Theorem, $T^{-1/2}\sum_{t=1}^T (w_t - \mu_w) \xrightarrow{d} N(0, V_w)$ but must verify the specific form of μ_w and V_w relevant to the quantities in your answer.

7. Consider the model

$$y \;=\; X\beta_0 \,+\, u \tag{5}$$

where y is a $T \times 1$ observable random vector, X is a $T \times k$ observable matrix that is fixed in repeated samples and u is a $T \times 1$ vector of unobservable errors. It is assumed that the model satisfies Classical Assumptions CA1-CA6 given in the Lecture Notes pp.10-11. In addition, it is assumed that $R\beta_0 = r$ where R is a $n_r \times k$ matrix of constants with $rank(R) = n_r$, and r is a $n_r \times 1$ vector of constants. If (5) is estimated by least squares subject to $R\beta_0 = r$ then the resulting RLS estimator of β_0 is denoted $\hat{\beta}_{R,T}$, and the RLS estimator of σ_0^2 is denoted $\hat{\sigma}^2_{R,T}$. Recall that

$$\hat{\beta}_{R,T} = \hat{\beta}_T + (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(r - R\hat{\beta}_T) \tag{6}$$

$$\hat{\sigma}^2_{R,T} = e_R' e_R/(T - k + n_r) \tag{7}$$

where $\hat{\beta}_T$ is the OLS estimator of β_0 based on (5) and $e_R = y - X\hat{\beta}_{R,T}$.

(a) Define $Q = I - X(X'X)^{-1}X' + X(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R(X'X)^{-1}X'$. Show that Q is an orthogonal projection matrix.

(b) Show that $e_R' e_R = u' Q u$.

(c) Show that $E[\hat{\sigma}^2_{R,T}] = \sigma_0^2$.

(d) Show that $(T - k + n_r)\hat{\sigma}^2_{R,T}/\sigma_0^2 \sim \chi^2_{T-k+n_r}$.

(e) Show that $\hat{\sigma}^2_{R,T}$ is a more efficient estimator of σ_0^2 than the OLS estimator, $\hat{\sigma}^2_T$.
Hint: If $v \sim \chi^2_n$ then $E[v] = n$ and $Var[v] = 2n$.

8.(a) For conformable matrices, A, B and C show that: $tr(ABC) = tr(BCA) = tr(CAB)$ where $tr(\cdot)$ denotes the trace operator.

8.(b) Consider $f = x'Az$ where x is a $m \times 1$ vector, A is a $m \times n$ matrix and z is a $n \times 1$ vector. Derive and state the dimensions of: (i)$\partial f/\partial x$; (ii)$\partial f/\partial z$.

9. Consider the linear regression model

$$y_i = x_i'\beta_0 + u_i$$

where $x_i = (1, x_{2,i}')'$, $\{(u_i, x_{2,i}')', \ i = 1, 2, \dots N\}$ forms an independent and identically distributed (i.i.d.) sequence, $E[x_i x_i'] = Q$, a finite, positive definite matrix of constants, $E[u_i|x_i] = 0$, $Var[u_i|x_i] = \sigma_0^2$, a positive, finite constant. Consider the statistic

$$S_N = \sum_{i=1}^{N} u_i x_i' M_N \sum_{i=1}^{N} x_i u_i$$

where M_N is a $k \times k$ random matrix.

(a) Propose a choice of M_N such that $S_N \xrightarrow{d} \chi^2_k$ being sure to justify your answer carefully. *Hint: you may quote the generic form of the Weak Law of Large Numbers, $N^{-1}\sum_{i=1}^N z_i \xrightarrow{p} \mu_z$, but must verify the specific forms of the limits relevant to the quantities analyzed in your answer; you may quote the generic forms of the Central Limit Theorem, $N^{-1/2}\sum_{i=1}^N (z_i - \mu_z) \xrightarrow{d} N(0, V_z)$ but must verify the specific forms of μ_z and V_z limits relevant to the quantities analyzed in your answer.*

(b) Suppose now that $Var[u_i|x_i] = h(x_i)$. Is it still true that $S_N \xrightarrow{d} \chi^2_k$ for your choice of M_N in part (a)? Justify your answer briefly but formal derivations are not required.