Solutions to Problem Set for Tutorial 1

1.(a) $\partial y/\partial x = \beta_{0,2}$, and so $\beta_{0,2}$ is the partial derivative of y with respect to x.

1.(b) Using the Chain rule: $\partial y/\partial x = x^{-1}\beta_{0,2}$. So $\beta_{0,2}$ gives the multiple of the proportional change in x by which y changes.

1.(c) Using the Chain rule: $\partial y/\partial x = y\beta_{0,2}$ and so $\beta_{0,2}$ is the proportionate change in y that is $100\beta_{0,2}$ is the percentage change in y; in this case $\beta_{0,2}$ is sometimes referred as the *semi-elasticity* of y with respect to x.

1.(d) Using the Chain rule: $\partial y/\partial x = (y/x)\beta_{0,2}$, and so $\beta_{0,2}$ is $(x/y)(\partial y/\partial x)$, the elasticity of y with respect to x.

1.(e) $\partial y/\partial x = \beta_{0,2} + 2\beta_{0,3}x$, and so $\beta_{0,2}$ is $\partial y/\partial x$ at $x = 0$.

2.(a) By definition $h(\theta) = \sum_{i=1}^{p} a_i\theta_i$ and so $\partial h(\theta)/\partial\theta_i = a_i$. Since $\partial h(\theta)/\partial\theta_i$ is the i^{th} element of $\partial h(\theta)/\partial\theta$ the result then follows immediately.

2.(b) By definition $g(\theta) = \sum_{i=1}^{p}\sum_{j=1}^{p} A_{i,j}\theta_i\theta_j$ where $A_{i,j}$ is the $i-j^{th}$ element of A. Therefore,

$$\begin{aligned}\partial g(\theta)/\partial\theta_s &= \sum_{i=1}^{p} A_{i,s}\theta_i + \sum_{j=1}^{p} A_{s,j}\theta_j \\ &= \theta' A_{\cdot s} + A_{s\cdot}\theta\end{aligned}$$

where $A_{s\cdot}$ and $A_{\cdot s}$ are the s^{th} row and column respectively of A. Now $A_{\cdot s} = B_{s\cdot}$ where $B = A'$, and so

$$\partial g(\theta)/\partial\theta_s = (A_{s\cdot} + B_{s\cdot})\,\theta \tag{1}$$

Since $\partial g(\theta)/\partial\theta_s$ is the s^{th} element of $\partial g(\theta)/\partial\theta$, it follows from (1) that $\partial g(\theta)/\partial\theta = (A + A')\theta$.

2.(c) From (1) it follows that

$$\partial^2 g(\theta)/\partial\theta_s\partial\theta_l = A_{s,l} + A_{l,s} \tag{2}$$

Since $\partial^2 g(\theta)/\partial\theta_s\partial\theta_l$ is the $s - l^{th}$ element of $\partial^2 g(\theta)/\partial\theta\partial\theta'$ the desired result follows directly from (2).

2.(d) If A is symmetric then $A = A'$ and so (b) becomes $\partial g(\theta)/\partial\theta = 2A\theta$, and (c) becomes $\partial^2 g(\theta)/\partial\theta\partial\theta' = 2A$.

3.(i) Following the hint, we have $c'X'Xc = b'b$ for $b = Xc$. By construction $b'b = \sum_{i=1}^{T} b_i^2$ where b_i is the i^{th} element of b, and so it follows immediately that $c'X'Xc \geq 0$. However, we must show this inequality is strict in order for $X'X$ to be pd. Note $b = Xc = \sum_{i=1}^{k} x_i c_i$ where x_i is the i^{th} column of X and c_i the i^{th} element of c, and so b is a linear combination of the columns of X. $rank(X) = k$ implies that the columns of X forms a linear independent set and so it must be that $b \neq 0$ and hence $b'b > 0$. This proves $c'X'Xc > 0$ for any non-zero vector c, and therefore that $X'X$ is pd.

3.(ii) If $rank(X) < k$ then for some choices of c we have $b = Xc \neq 0$ but for others $b = 0$. Therefore, $c'X'Xc \geq 0$ for all c, and so $X'X$ is positive semi-definite.

4.(i) The normal equations can be written as: $X'(y - X\hat{\beta}_T) = X'e = 0$. Substituting for X we have:

$$X'e = \begin{bmatrix} \iota_T' \\ X_2' \end{bmatrix} e = \begin{bmatrix} \iota_T' e \\ X_2' e \end{bmatrix} = 0.$$

These are a system of $k \times 1$ equations, the first of which is $\iota_T' e = 0$. Since $\iota_T' e = \sum_{t=1}^{T} e_t$, it follows immediately that $\bar{e} = 0$.

4.(ii) From part (i), it follows that $\iota_T'(y - X\hat{\beta}_T) = 0$ which in turn implies $T^{-1}\iota_T'(y - X\hat{\beta}_T) = 0$ (as $T > 0$) and so

$$T^{-1}\iota_T' y = T^{-1}\iota_T' X\hat{\beta}_T \tag{3}$$

Since the left hand side of (3) is $\bar{y}$ and the right-hand side of (3) is $\bar{\hat{y}}$, it follows that $\bar{y} = \bar{\hat{y}}$.

5. Prove $A^{-1} = B$ by showing B satisfies $AB = BA = I$. Now if $AB = I$ then

$$A_{1,1}B_{1,1} + A_{1,2}B_{2,1} = I \tag{4}$$

$$A_{1,1}B_{1,2} + A_{1,2}B_{2,2} = 0 \tag{5}$$

$$A_{2,1}B_{1,1} + A_{2,2}B_{2,1} = 0 \tag{6}$$

$$A_{2,1}B_{1,2} + A_{2,2}B_{2,2} = I \tag{7}$$

and so we now verify that the stated formulae for $B_{i,j}$ satisfy these equation. Substituting $B_{1,2} = -A_{1,1}^{-1}A_{1,2}B_{2,2}$ into the left-hand side of (5) gives

$$\begin{aligned} A_{1,1}(-A_{1,1}A_{1,2}B_{2,2}) + A_{1,2}B_{2,2} &= -A_{1,2}B_{2,2} + A_{1,2}B_{2,2} \\ &= 0 \end{aligned}$$

and so $B_{1,2} = -A_{1,1}^{-1}A_{1,2}B_{2,2}$ satisfies (5). Substituting $B_{2,1} = -A_{2,2}^{-1}A_{2,1}B_{1,1}$ into the left-hand side of (6) gives

$$\begin{aligned} A_{2,1}B_{1,1} + A_{2,2}(-A_{2,2}A_{2,1}B_{1,1}) &= A_{2,1}B_{1,1} - A_{2,1}B_{1,1} \\ &= 0 \end{aligned}$$

and so $B_{2,1} = -A_{2,2}^{-1}A_{2,1}B_{1,1}$ satisfies (6). Substituting $B_{1,2} = -A_{1,1}^{-1}A_{1,2}B_{2,2}$ into the left-hand side of (7) gives

$$A_{2,1}(-A_{1,1}^{-1}A_{1,2}B_{2,2}) + A_{2,2}B_{2,2} = I$$

which implies

$$(A_{2,2} \; - \; A_{2,1}A_{1,1}^{-1}A_{1,2})B_{2,2} \; = \; I$$

and so $B_{2,2} = (A_{2,2} \; - \; A_{2,1}A_{1,1}^{-1}A_{1,2})^{-1}$ satisfies (7). Finally substituting $B_{2,1} = -A_{2,2}^{-1}A_{2,1}B_{21}$ into the left-hand side of (4) gives

$$A_{1,1}B_{1,1} \; + \; A_{1,2}(-A_{2,2}A_{2,1}B_{1,1}) \; + \; A_{2,2}B_{2,2} \; = \; I$$

which implies

$$(A_{1,1} \; - \; A_{1,2}A_{2,2}^{-1}A_{2,1})B_{1,1} \; = \; I$$

and so $B_{1,1} = (A_{1,1} \; - \; A_{1,2}A_{2,2}^{-1}A_{2,1})^{-1}$ satisfies (4).

Since A and B are nonsingular, $AB = I \;\Rightarrow\; BAB = B \;\Rightarrow\; BA = BB^{-1} = I$. Therefore $B = A^{-1}$.

Notice that if we begin with $BA = I$ then it is possible to derive alternative – but equivalent – representations for $B_{i,j}$ in terms of $A_{i,j}$ for $i \neq j$. These are

$$\begin{aligned} B_{1,2} &= -B_{1,1}A_{1,2}A_{2,2}^{-1} && (8) \\ B_{2,1} &= -B_{2,2}A_{2,1}A_{1,1}^{-1} && (9) \\ & && (10) \end{aligned}$$