Alastair Hall | **ECON61001: Econometric Methods** | Semester 1, 2020-21

Problem Set for Tutorial 5

To develop our large sample analysis, we assumed in class that $plim T^{-1}X'X = Q$*, a nonsingular matrix of finite constants. In this case you consider the behaviour of* $X'X$ *when a simple linear regression model with regressor equal to the time trend.*

1. Suppose that $x_t = [1, t]'$. Evaluate whether the following matrices converge to a finite, nonsingular matrix as $T \to \infty$: (a) $T^{-1}X'X$; (b) $T^{-2}X'X$; (c) $T^{-3}X'X$. *Hint:* $\sum_{t=1}^{T} t = T(T+1)/2$, $\sum_{t=1}^{T} t^2 = T(T+1)(2T+1)/6$.

The times series ε_t *is known as "white noise" if it has the following properties: (i)* $E[\varepsilon_t] = 0$ *for all* t*; (ii)* $Var[\varepsilon_t] = \sigma^2$*; (iii)* $Cov[\varepsilon_t, \varepsilon_s] = 0$ *for* $t \neq s$.

2. Let ε_t be a white noise process, and define the following three time series: $u_t = \varepsilon_t$, $v_t = (-1)^t\varepsilon_t$ and $w_t = \mathcal{I}(t = 10) + \varepsilon_t$ where $\mathcal{I}(\cdot)$ is an indicator function that equals one if the event ($t = 10$ here) in parentheses occurs and zero otherwise.

   (a) Is u_t weakly stationary? Explain.

   (b) Is v_t weakly stationary? Explain.

   (c) Is w_t weakly stationary? Explain.

   (d) Is v_t strongly stationary? Explain.

If the time series y_t *follows an Autoregressive Moving Average model of orders* (p, q)*, denoted* $ARMA(p, q)$ *then it is generated as follows:*

$$y_t = c + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \theta_p y_{t-p} + \varepsilon_t + \phi_1\varepsilon_{t-1} + \phi_2\varepsilon_{t-2} + \ldots + \phi_q\varepsilon_{t-q},$$

where ε_t *is a white noise process. Special case of this class of models are Autoregressive processes and Moving Average processes. If* y_t *follows an Autoregressive process of order* p *- denoted* $AR(p)$ *- then it is generated via:*

$$y_t = c + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \theta_p y_{t-p} + \varepsilon_t.$$

If y_t *follows a Moving Average process of order* q *- denoted* $MA(q)$ *- then it is generated is by the equation:*

$$y_t = c + \varepsilon_t + \phi_1\varepsilon_{t-1} + \phi_2\varepsilon_{t-2} + \ldots + \phi_q\varepsilon_{t-q}.$$

where ε_t *is a white noise process. In the next two questions, you consider the properties of two simple ARMA models, and your analysis provides an example of how the time series properties of MA and AR models are different.*

3. Consider the model the MA(1) model:

$$y_t \;=\; \varepsilon_t \;+\; \phi\varepsilon_{t-1}$$

where ε_t is white noise with variance σ^2.

(a) Show that $E[y_t] = 0$.

(b) Show that $Var[y_t] = \sigma^2(1 \,+\, \phi^2)$.

(c) Show that $Cov[y_t, y_{t-1}] = \phi\sigma^2$ and $Cov[y_t, y_{t-s}] = 0$ for all $s > 1$.

(d) Is y_t a weakly stationary process?

(e) What is the long run variance of y_t?

4. Consider the $AR(1)$ model:

$$y_t \;=\; \theta y_{t-1} \;+\; \varepsilon_t \tag{1}$$

where ε_t is white noise with variance σ^2. We assume $|\theta| < 1$ which is known in this context as the *stationarity condition.* Via back substitution in (1), we can show that:

$$\begin{aligned} y_t &= \theta y_{t-1} \;+\; \varepsilon_t \\ &= \theta\{\theta y_{t-2} \;+\; \varepsilon_{t-1}\} \;+\; \varepsilon_t \\ &= \theta^m y_{t-m} + \sum_{i=0}^{m-1} \theta^i \varepsilon_{t-i} \end{aligned} \tag{2}$$

If the stationarity condition holds and so $\lim_{m\to\infty} \theta^m = 0$ then it follows by letting $m \to \infty$ in (2) we can argue that y_t also has $MA(\infty)$ representation given by[1]

$$y_t \;=\; \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}.$$

Using the $MA(\infty)$ representation, answer the following questions.

(a) Show that $E[y_t] = 0$.

(b) Show that $Var[y_t] = \sigma^2/(1 - \theta^2)$.

(c) Show that $Cov[y_t, y_{t-s}] = \theta^s\sigma^2/(1 - \theta^2)$.

(d) Is y_t weakly stationary?

(e) What is the long run variance of y_t?

Hint: If $|h| < 1$ *then* $\sum_{i=0}^{\infty} h^i = (1 - h)^{-1}$.

[1] The details of the argument need not concern us here.