Alastair Hall | **ECON61001: Econometric Methods** | Autumn 2020

Problem Set for Tutorial 1

In the first lecture, we saw how the linear regression model subsumes models in which the dependent variable and regressors may be functions of certain underlying variables. In this question we explore how the functional form of the variables affects the interpretation of the parameters.

1. Let x be a continuous random variable and assume u is independent of x. Derive $\partial y/\partial x$ in the following models.

   (a) $y = \beta_{0,1} + \beta_{0,2}x + u$.

   (b) $y = \beta_{0,1} + \beta_{0,2}ln(x) + u$.

   (c) $ln(y) = \beta_{0,1} + \beta_{0,2}x + u$.

   (d) $ln(y) = \beta_{0,1} + \beta_{0,2}ln(x) + u$.

   (e) $y = \beta_{0,1} + \beta_{0,2}x + \beta_{0,3}x^2 + u$.

   For models (a)-(e), deduce the interpretation of $\beta_{0,2}$.

The next two questions consider certain linear algebra results that are used in our derivation of the OLS estimator.

2. Let $h(\theta) = a'\theta$ and $g(\theta) = \theta' A\theta$ where θ and a are $p \times 1$ vectors and A is a $p \times p$ matrix. Show that:

   (a) $\partial h(\theta)/\partial\theta = a$.

   (b) $\partial g(\theta)/\partial\theta = (A + A')\theta$.

   (c) $\partial^2 g(\theta)/\partial\theta\partial\theta' = A + A'$.

   (d) If A is symmetric, then show how the results in parts (b) and (c) can be simplified.

   Hint: by definition, $\partial h(\theta)/\partial\theta$ is $p\times 1$ vector whose i^{th} element is $\partial h(\theta)/\partial\theta_i$ where θ_i is the i^{th} element of θ. So the results can be shown by deriving the form of $\partial h(\theta)/\partial\theta_i$ and then using this to deduce the appropriate result for $\partial h(\theta)/\partial\theta$. Similarly, $\partial^2 g(\theta)/\partial\theta_i\partial\theta_j$ is the $i-j^{th}$ element of $\partial^2 g(\theta)/\partial\theta\partial\theta'$.

3.(i) Recall that a $k \times k$ matrix M is positive definite if $c'Mc > 0$ for any non-null $k \times 1$ vector c. Define X to be a $T \times k$ matrix with $rank(X) = k$. Show that $X'X$ is a positive definite matrix. *Hint: Show $c'X'Xc = b'b$ for a certain choice of b and then deduce the result by considering the properties of b.*

3.(ii) Suppose now that $rank(X) < k$ what can be said about the sign of $c'X'Xc$?

This question considers a property of OLS estimators for the linear model,

$$y \; = \; X\beta_0 \, + \, u$$

where all definitions and dimensions are the same as our discussion in the lectures. It is assumed that the model includes an intercept and so $X = [\iota_T, X_2]$ *where* ι_T *is a* $T \times 1$ *vector of ones.*

4. Let $\hat{y} = X\hat{\beta}_T$ with t^{th} element $\hat{y}_t$, $e = y - \hat{y}$ with t^{th} element e_t, $\bar{y} = T^{-1}\sum_{t=1}^{T} y_t$, $\bar{\hat{y}} = T^{-1}\sum_{t=1}^{T} \hat{y}_t$, and $\bar{e} = T^{-1}\sum_{t=1}^{T} e_t$. By considering the first order conditions of OLS estimation, show that: (i) $\bar{e} = 0$; (ii) $\bar{y} = \bar{\hat{y}}$. *Hint: part (ii) follows from part (i).*

In this question, you establish the partitioned matrix inversion result to which we appealed in our discussionof the Frisch-Waugh-Lovell Theorem.

5. Consider the partitioned matrix

$$A \; = \; \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

where $|A_{ii}| \neq 0$. Prove that $A^{-1} = B$ where B is a similarly partitioned matrix with

$$\begin{aligned} B_{1,1} &= (A_{1,1} \, - \, A_{1,2}A_{2,2}^{-1}A_{2,1})^{-1}, \\ B_{2,2} &= (A_{2,2} \, - \, A_{2,1}A_{1,1}^{-1}A_{1,2})^{-1}, \\ B_{1,2} &= -A_{1,1}^{-1}A_{1,2}B_{2,2}, \\ B_{2,1} &= -A_{2,2}^{-1}A_{2,1}B_{1,1}. \end{aligned}$$