Alastair Hall

ECON61001:
Econometric Methods

Semester 1 2020-21

Solutions to Problem Set for Tutorial 8

1.(a) We have:

$$u_i(\beta) \;=\; y_i \,-\, x_i'\beta \;=\; x_i'\beta_0 \,+\, u_i \,-\, x_i'\beta \;=\; u_i \,+\, x_i'(\beta_0 \,-\, \beta).$$

Therefore, it follows that

$$E[z_i u_i(\beta)] \;=\; E[z_i u_i(\beta_0)] \,+\, E[z_i x_i'](\beta_0 \,-\, \beta).$$

Since the population moment condition is valid and so $E[z_i u_i(\beta_0)] = 0$ it follows that

$$E[z_i u_i(\beta)] \;=\; E[z_i x_i'](\beta_0 \,-\, \beta). \tag{1}$$

We need conditions under which $E[z_i u_i(\beta)] \neq 0$ for all $\beta \neq \beta_0$. Notice that (1) represents a set of linear equations in $E[z_i u_i(\beta)]$ that is, they are of the form "$Av = b$" where $A = E[z_i x_i']$, $v = (\beta_0 - \beta)$ and $b = E[z_i u_i(\beta)]$. From linear algebra theory, the condition for $b \neq 0$ for $v \neq 0$ iff A is full column rank. Since $E[z_i x_i']$ is 2×2 here, the condition is that $rank\{E[z_i x_i']\} = 2$.

1.(b) Using the hint, $E[z_i x_i']$ is full rank if it is nonsingular or equivalently if $det\{E[z_i x_i']\} \neq 0$. Since

$$E[z_i x_i']) \;=\; E\begin{bmatrix} 1 & x_{2,i} \\ z_{2,i} & z_{2,i}x_{2,i} \end{bmatrix},$$

it follows that

$$det\{E[z_t x_t']\} = E[z_{2,i}x_{2,i}] - E[x_{2,i}]\,E[z_{2,i}] \;=\; Cov(z_{2,i}, x_{2,i}).$$

Therefore, the relevance condition is that $Corr(z_{2,i}, x_{2,i}) \neq 0$.

1.(c) The relevance condition is satisfied if the instrument $z_{2,i}$ is linearly related to the variable for which it is used as an instrument $x_{2,i}$.

2.(a) Multiplying out and using , we have:

$$Q_{IV}(\beta) \;=\; y'Z(Z'Z)^{-1}Z'y - y'Z(Z'Z)^{-1}Z'X\beta - \beta'X'Z(Z'Z)^{-1}Z'y + \beta'X'Z(Z'Z)^{-1}Z'X\beta.$$

Since $\beta'X'Z(Z'Z)^{-1}Z'y$ is a scalar and so equal to its transpose, it follows that:

$$\beta'X'Z(Z'Z)^{-1}Z'y \;=\; (\beta'X'Z(Z'Z)^{-1}Z'y)' \;=\; y'Z(Z'Z)^{-1}Z'X\beta,$$

(using $(AB)' = B'A'$). Therefore, we can write:

$$Q_{IV}(\beta) \;=\; y'Z(Z'Z)^{-1}Z'y \,-\, 2y'Z(Z'Z)^{-1}Z'X\beta \,+\, \beta'X'Z(Z'Z)^{-1}Z'X\beta.$$

Using the the matrix differentiation results in Tutorial 1 Question 3, we have:

$$\frac{\partial(y'Z(Z'Z)^{-1}Z'X\beta)}{\partial\beta} = X'Z(Z'Z)^{-1}Z'y,$$
$$\frac{\partial(\beta'X'Z(Z'Z)^{-1}Z'X\beta)}{\partial\beta} = 2X'Z(Z'Z)^{-1}Z'X\beta,$$

where the last result uses the symmetry of $X'(Z'Z)^{-1}X$. Therefore, we have:

$$\frac{\partial Q_{IV}(\beta)}{\partial\beta} = -2X'Z(Z'Z)^{-1}Z'y + 2X'Z(Z'Z)^{-1}Z'X\beta.$$

The FOC are

$$\left.\frac{\partial Q_{IV}(\beta)}{\partial\beta}\right|_{\beta=\hat{\beta}_{IV}} = 0,$$

and so imply:

$$-X'Z(Z'Z)^{-1}Z'y + X'Z(Z'Z)^{-1}Z'X\hat{\beta}_{IV} = 0$$

Given that $X'Z$ and $Z'Z$ are full rank, $X'Z(Z'Z)^{-1}Z'X$ is nonsingular, it follows that:

$$\hat{\beta}_{IV} = \left(X'Z(Z'Z)^{-1}Z'X\right)^{-1} X'Z(Z'Z)^{-1}Z'y.$$

(b) If $q = k$, then $X'Z$ is square matrix of full rank and so nonsingular. Using the result that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, we have:

$$\left(X'Z(Z'Z)^{-1}Z'X\right)^{-1} = (Z'X)^{-1}\{(Z'Z)^{-1}\}^{-1}(X'Z)^{-1} = (Z'X)^{-1}Z'Z(X'Z)^{-1},$$

and substituting this result into the formula for $\hat{\beta}_{IV}$, we obtain:

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'Z(X'Z)^{-1}X'Z(Z'Z)^{-1}Z'y.$$

Using in turn that $(X'Z)^{-1}X'Z = I_k$ and $Z'Z(Z'Z)^{-1} = I_k$, the formula simplifies to:

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'y.$$

3. A suitable test statistic is

$$W_N^{(IV)} = N(R\hat{\beta}_{IV} - r)'[R\hat{V}_{IV}R']^{-1}(R\hat{\beta}_{IV} - r)$$

where $\hat{V}_{IV} = \hat{M}^{-1}\hat{Q}_{xz}\hat{Q}_{zz}^{-1}\hat{\Omega}_h\hat{Q}_{zz}^{-1}\hat{Q}_{xz}'\hat{M}^{-1}$, $\hat{M} = \hat{Q}_{xz}\hat{Q}_{zz}^{-1}\hat{Q}_{xz}'$, $\hat{Q}_{xz} = N^{-1}\sum_{i=1}^N z_i x_i'$, $\hat{\Omega}_h = N^{-1}\sum_{i=1}^N \hat{u}_i^2 z_i z_i'$ and $\hat{u}_i = y_i - x_i'\hat{\beta}_{IV}$.

Aside: Under the null hypothesis, we have:

$$N^{1/2}(R\hat{\beta}_{IV} - r) \xrightarrow{d} N(0, RV_{IV}R')$$

where $V_{IV} = M^{-1}Q_{xz}Q_{zz}^{-1}\Omega_h Q_{zz}^{-1}Q'_{xz}M'$*,* $M = Q_{xz}Q_{zz}^{-1}Q'_{xz}$ *and* $\Omega_h = E[h(z_i)z_i z_i']$*. Using the WLLN, we have* $\hat{Q}_{xz} \xrightarrow{p} Q_{xz}$ *and* $\hat{Q}_{zz} \xrightarrow{p} Q_{zz}$*. Using similar arguments to White's heteroscedasticity covariance matrix estimator, we have* $\hat{\Omega}_h \xrightarrow{p} \Omega_h$*. Therefore, using Slutsky's Theorem, we can deduce that* $R\hat{V}_{IV}R' \xrightarrow{p} RV_{IV}R'$*. So using Lemma 3.6 in the Lecture notes,* $W_{IV} \xrightarrow{d} \chi^2_{n_r}$*.*

A suitable decision rule is to reject H_0 at the (approximate) $100\alpha\%$ significance level if $W_N^{(IV)} > c_{n_r}(\alpha)$ where $c_{n_r}(\alpha)$ is the $100(1-\alpha)^{th}$ of the $\chi^2_{n_r}$ distribution.

4. First consider the orthogonality condition $E[z_t u_t] = 0$. Here we have:

$$E[z_t u_t] \;=\; \begin{bmatrix} E[u_t] \\ E[y_{t-2}u_t] \end{bmatrix}.$$

and so we need to show that $E[u_t] = 0$ and $E[y_{t-2}u_t] = 0$. From Tutorial 5 Question 3 part (a), $E[u_t] = 0$. Now consider $y_{t-2}u_t$. Using the hint, it follows that $E[y_{t-2}u_t]$ is a linear function $E[u_t]$ and $E[u_t u_{t-j}]$ for $j = 2, 3, \ldots$. Using Tutorial 5 Question 3 parts (a) & (c), it follows that $E[u_t] = 0$ and $E[u_t u_{t-j}] = 0$ for $j = 2, 3, \ldots$. Therefore, $E[y_{t-2}u_t] = 0$.

Now consider the relevance condition. Here the relevance condition is that $rank\{E[z_t x_t']\} = 2$. Using Question 1 above, this condition is equivalent to $Corr(y_{t-1}, y_{t-2})$. Since we are given that the process is weakly stationary we can use the hint to deduce that

$$Corr(y_{t-1}, y_{t-2}) \;=\; \frac{(\phi + \beta_{0,2})(1 + \phi\beta_{0,2})}{1 + 2\phi\beta_{0,2} + \phi^2}.$$

It can be recognized that the relevance condition is satisfied because we are given that $\beta_{0,2}$ and ϕ are both less than one in absolute value and $\phi \neq -\beta_{0,2}$.

5.(a) The relevance condition is that $rank\{E[z_i x_i]\} = 1$ and as $E[z_i x_i]$ is a scalar this is equivalent to $E[z_i x_i] \neq 0$. However, we are given that $E[z_i x_i] = 0$, and so the relevance condition fails.

5.(b) Following the same steps as led to (1) above, we have

$$E[z_i u_i(\beta)] \;=\; E[z_i x_i](\beta_0 \;-\; \beta).$$

and so as $E[z_i x_i] = 0$, it follows that $E[z_i u_i(\beta)] = 0$ for all β.

5.(c) If $E[z_i u_i(\beta)] = 0$ for β then being told that $E[z_i u_i(\beta_0)] =$ tells us nothing unique about β_0. In other words, the IV estimation is based on a moment condition that is uninformative about β_0. Therefore intuition suggests that $\hat{\beta}_{IV}$ is not consistent. (In fact, the IV estimator does not converge to a constant in this case but to a random variable. To see this, we write:

$$\hat{\beta}_{IV} \;=\; \beta_0 \;+\; \frac{\sum_{i=1}^N z_i u_i}{\sum_{i=1}^N z_i x_i}. \qquad (2)$$

Recall that when the instruments satisfy the orthogonality and relevance conditions then we establish consistency by dividing the numerator and denominator of the ratio of the right-hand side of (2) by N and then applying the WLLN to $N^{-1}\sum_{i=1}^{N} z_i u_i$ and $N^{-1}\sum_{i=1}^{N} z_i x_i$. However, this approach runs in to problems here because terms are converging to zero. Instead to develop an analysis of the large sample behaviour of the estimator, we divide the numerator and denominator of the fraction by $N^{1/2}$. Then applying the CLT, we have:

$$\left[\begin{array}{c} N^{-1/2}\sum_{i=1}^{N} z_i u_i \\ N^{-1/2}\sum_{i=1}^{N} z_i x_i \end{array}\right] \xrightarrow{p} \left[\begin{array}{c} \xi_1 \\ \xi_2 \end{array}\right] \sim N(0,\, \Psi),$$

for some pd matrix Ψ. It then follows that $\hat{\beta}_{IV} \xrightarrow{p} \beta_0 + v$ where $v = \xi_1/\xi_2$.)