Lecture 5: Put-call parity, Forward Price, Expected utility, Constraint optimization

Economics of Finance

School of Economics, UNSW

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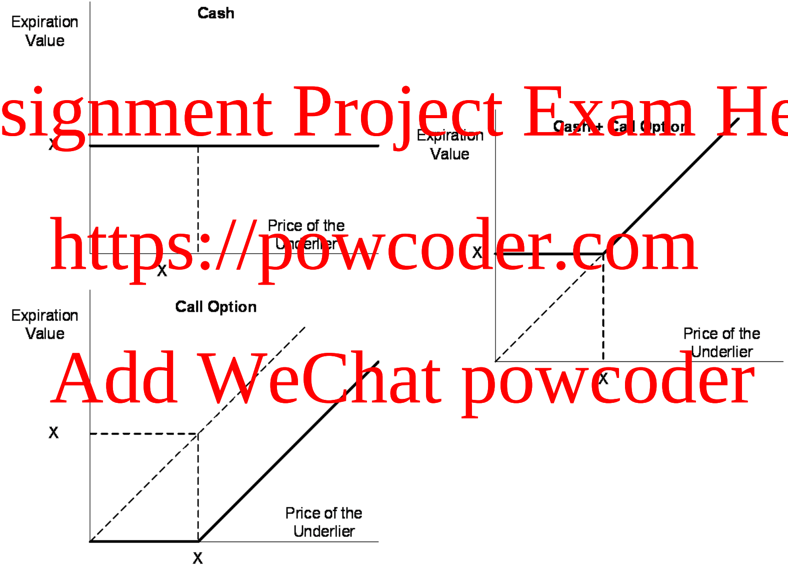
- the same strike price;
- the same expiration date.

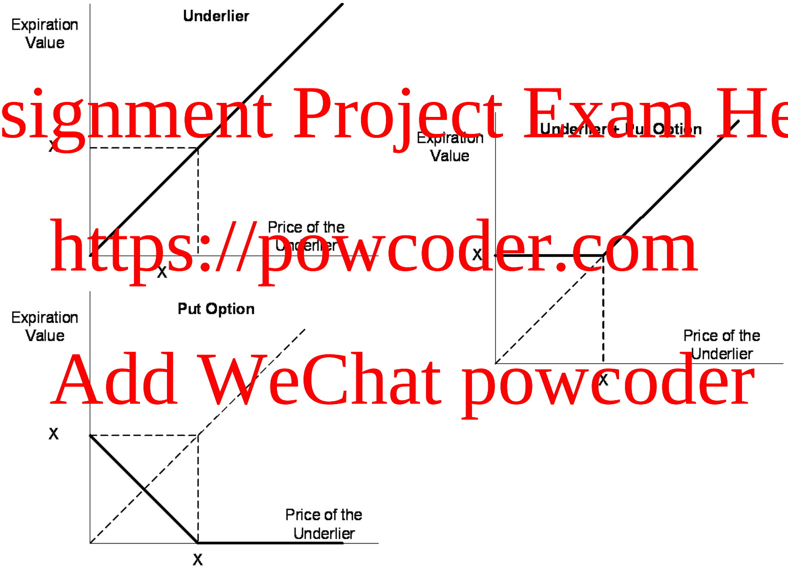
The relationship is derived using arbitrage arguments. Consider two portfolios consisting of:

- The Call option and an amount of cash equal to the present value of the strike price.
- The Put option and the underlying stock.

Put-Call Parity: Cash and Call



Put-Call Parity: Underlier and Put



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- Irrespective of the value of the underlier at expiration, each portfolio will have the same value as the other.
- If the two portfolios are going to have the same value at expiration, then they must have the same value today. Otherwise, an investor could make an arbitrage profit.

Put-Call Parity

(Cont'd)

Accordingly, we have the price equality:

$$p_{call} + PV(X) = p_{put} + p_{underlier} \tag{1}$$

where:

- p_{call} is the current market value of the call;
- $PV(X)$ is the present value of the strike price, X;
- p_{put} is the current market value of the put;
- $p_{underlier}$ is the current market value of the underlying stock.

Note: "Current" refers to Period 0 since you are evaluating today prices

Put-Call Parity: An example

- We have priced a European Call option that gives the holder a right to Buy the Stock at Period 2 at the Exercise Price, $X = 1.10$. We found its price to be $\mathbf{p}_{\text{Call}} = 0.0816$.

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- Consider a European Put option that gives the holder a right to sell the Stock at Period 2 at the Exercise Price, $X = 1.10$.

The cash flow associated with the Put option:

$$\mathbf{c} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0.1784 \end{pmatrix}'$$

The atomic prices are still the same:

$$\begin{array}{cccccccc} & & \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \\ \mathbf{p}_{atom} & = & \big(0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444\big) \end{array}$$

The value of the Put option is:

$$\mathbf{p}_{\text{Put}} = \mathbf{p}_{atom} \cdot \mathbf{c} = 0.4444 \cdot 0.1784 = 0.0793$$

Put-Call Parity: An example (cont'd)

According to the Put-Call parity we have

$$p_{call} + PV(X) = p_{put} + p_{underlier}$$

Notice that $PV(X) = df(2) \cdot X$, where $df(2)$ is the discount factor for Period 2. $df(2)$ is the present value of one certain dollar received at Period 2. It must equal to the sum of atomic security prices for states: gg, gb, bg and bb.

$$df(2) = 0.0816 + 0.1904 + 0.1904 + 0.4444 = 0.9070$$

$$PV(X) = df(2) \cdot X = 0.9070 \cdot 1.10 = 0.9977$$

Therefore

$$\begin{aligned} p_{call} &= p_{put} + p_{underlier} - PV(X) \\ &= 0.0793 + 1 - 0.9977 = 0.0816 \end{aligned}$$

This is the same value as the one we found before.

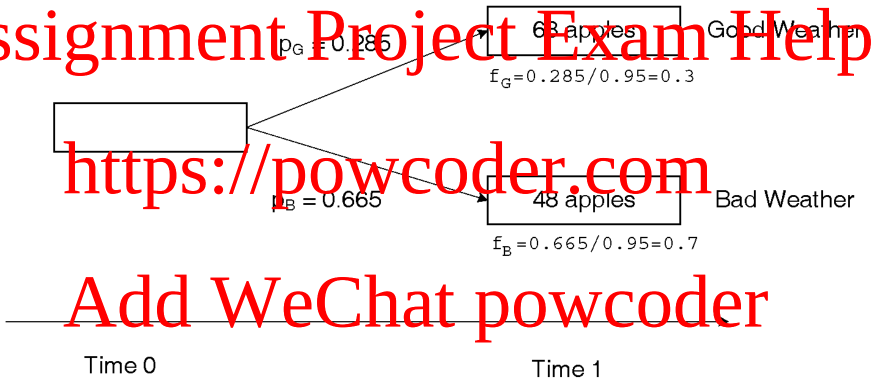
Forward Price

Definition: Forward price, $f(t)$, is the value of the payment at the time t.

Relation with present (spot) price:

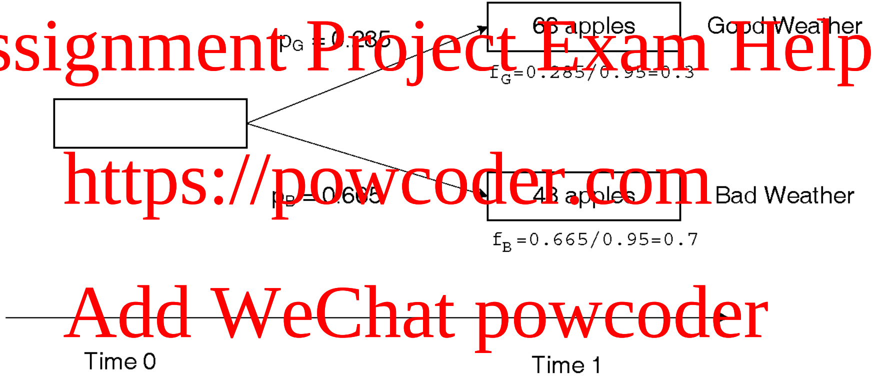
$p = df(t)f(t) \Rightarrow f(t) = p/df(t) = p(1 + i(t))^t$

Forward Atomic Prices



Note: Forward Atomic prices are positive and sum to 1. Why?

Using Forward Atomic Prices



Forward value of the tree is $f_{\text{tree}} = 63 \cdot 0.3 + 48 \cdot 0.7 = 52.5$
Present value of the tree is $p_{\text{tree}} = 52.5 \cdot 0.95 = 49.875$

Forward Atomic Prices as Risk-neutral probabilities

If we assume that

- all investors agree on the same probabilities
- all investors are risk-neutral (value certain payoff as much as expected (average) payoff)

we can think about forward atomic prices as risk-neutral probabilities.

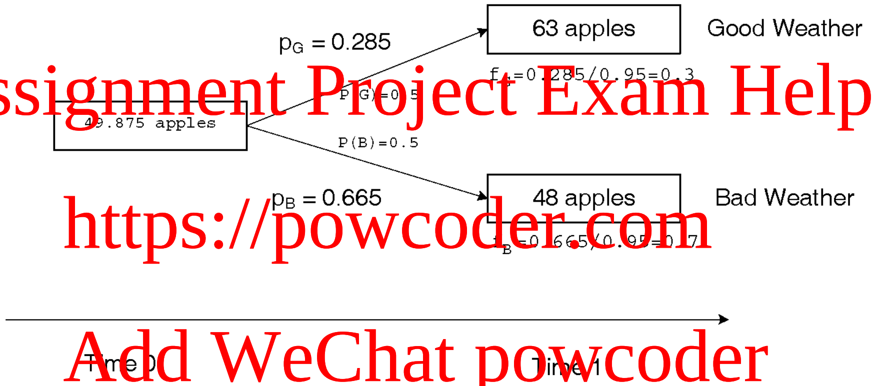
Expected value of discrete random variable X:

$$E(X) = \sum x_i P(X = x_i)$$

Forward value of the tree is expected payoff under risk-neural probabilities $f_{\text{tree}} = E_{\text{risk-neutral}}(c) = 63 \cdot 0.3 + 48 \cdot 0.7 = 52.5$

Note: investors are typically risk-averse and therefore there is a difference between *physical* and *risk-neutral* probabilities.

Physical probabilities



Expected payoff (wrt physical probability):
$E_{\text{physical}}(c_{\text{tree}}) = 63 \cdot 0.5 + 48 \cdot 0.5 = 55.5$

Expected return (wrt physical probability):
$E_{\text{physical}}(r_{\text{tree}}) = E(c_{\text{tree}})/p - 1 = 55.5/49.875 - 1 = 0.113$

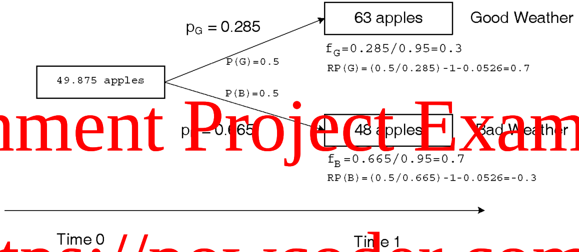
Risk premium

Expected return of the risky tree:
$E_{\text{physical}}(r_{\text{tree}}) = E(c)/p - 1 = 55.5/49.875 - 1 = 0.113$

Return of the riskless asset:
$r_{\text{riskless}} = 1/df - 1 = i = 1/0.95 - 1 = 0.053$

Risk premium: difference between expected risky return and riskless return $E_{\text{physical}}(r_{\text{tree}}) - r_{\text{riskless}} = 0.113 - 0.053 = 0.06$

Atomic risk premia



- Risk premium of the GW atomic security is positive, 0.7, because the forward price of 1 GW apple is lower than the physical probability of GW state. We value GAs not that much because they are more abundant.

- Risk premium of the BW atomic security is negative (risk discount), -0.3, because the forward price of 1 BW apple is higher than the physical probability of BW state. This is like buying an insurance to cover your consumption in BW state.
- Remember that the whole tree still carried risk premium.

Two different perspectives on asset pricing

- *Relative* Pricing - covered up until now
  - assuming arbitrage-free environment and a competitive market which eliminates any arbitrage;
  - pricing using the Law-of-One-Price and replicating portfolios;
  - relying on existing securities for market completeness;
  - atomic (state) prices used to price any future state-contingent payoffs $p_{atom} = p_S \times Q^{-1}$

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- Pricing from *microfoundations* - from now on
  - expected utility optimisation
  - assumptions on preferences, i.e., functional form of the utility function;
  - market is completed by introducing securities;
  - market clearing: matching aggregate demand/supply;
  - explains how we arrive at the equilibrium.

Preference

Economics studies individual choice:

- *Preference* relation describes ordering of choices, e.g., $a \succsim b$, where a and b are not necessarily numbers, $a$pple is preferred or indifferent to $b$anana.
- We use a utility function, $u(\cdot)$ to represent the preference relation,

$$u\left(a\right) \geqslant u\left(b\right) \Leftrightarrow a \succsim b,$$

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- Utility function $u(\cdot)$ gives us relative numbers such that above holds (can be strict, i.e., $>$ and $a \succ b$)
- Monotonic transformation of u does not change ordering, e.g., $\tilde{u} = \gamma u + c$, for $\gamma \geq 0$ represents the same preference relation as u. Why?

Uncertainty

- Two periods: today (time 0) and future (time 1)
- Today's state of nature s_0 is known.
- Set of possible future events - good weather (G) and bad weather (B): $S = \{G, B\}$.
- G occurs with probability $\pi(s_1 = G)$; B with probability $\pi(s_1 = B)$.

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- G occurs with probability $\pi(s_1 = G)$; B with probability $\pi(s_1 = B)$.
- Aim – design optimal state-contingent consumption plan:
  - $c(s_0)$– consumption at time 0.
  - $c(s_1 = G)$−consumption at time 1 if state is G;
  - $c(s_1 = B)$−consumption at time 1 if state is B;

Expected Utility: an introduction

- A consumer has a time and state separable utility function over consumption $c(s_0)$ and $c(s_1)$ – each period-state *instantaneous* utility function $u(c)$ does not depend on other period-state consumption directly.
- Consumers discount future expected utility with *time discount factor* $\beta \in (0, 1)$ which represent time preferences. The lower the β, the more impatient are the consumers.

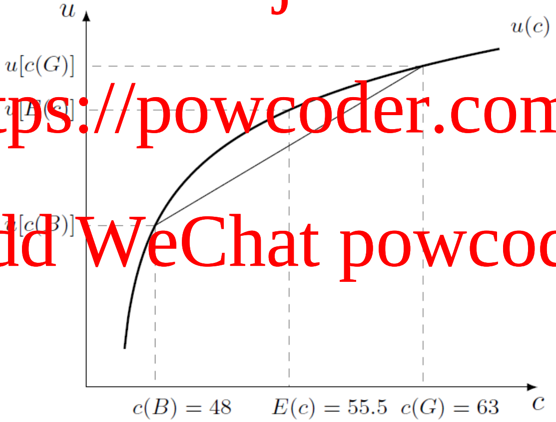
- The period utility function $u(c)$ is assumed to be strictly increasing and concave, i.e., $u' > 0$ and $u'' \leq 0$;
- The consumer maximises expected utility, U, given by

$$U = u\left(c(s_0)\right) + \underbrace{\beta\left[\pi(G) \cdot u\left(c(G)\right) + \pi(B) \cdot u\left(c(B)\right)\right]}_{\text{expected discounted future utility}}$$

Risk Aversion

$u(c)$ is assumed to be strictly increasing and concave, e.g., $u(c) = ln(c)$.

$$u(c)' > 0, \;\; u(c)'' \leq 0$$



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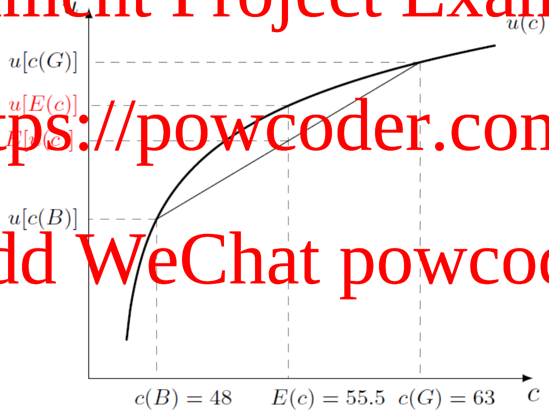
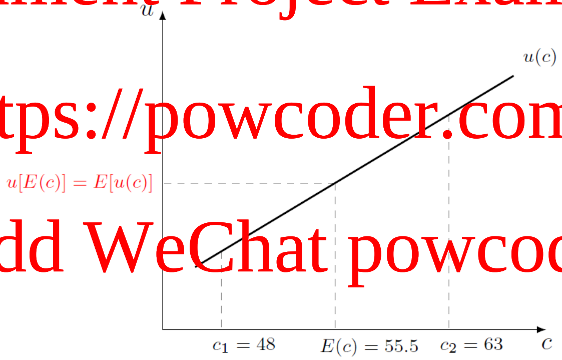


Risk aversion $\Leftrightarrow u[E(c)] > E[u(c)]$

Risk Neutrality

$u(c)$ is assumed linear, i.e., $u(c) = a + b\,c$.

$$u(c) = b, \text{a constant; } u(c)'' = 0$$



Risk neutrality $\Leftrightarrow u[E(c)] = E[u(c)]$

Background

Since Adam Smith (1776), “The Wealth of Nations”, economists strive to prove the existence of “invisible hands”

- Advocates free trade;
- Individual’s selfish decision drives the aggregate economy;

Edgeworth (1881), “Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences”.

- Two people, two goods;
- Free trade is allowed;
- He suspects the outcome is always social desirable;
- Pareto (1906) confirmed this social desirability, now known as “Pareto Efficiency”;

Arrow-Debreu Approach

Arrow, K. J., Debreu, G. (1954). "Existence of an equilibrium for a competitive economy" filled this important gap.

- Proved Adam Smith's and Edgeworth's congestion in a more general context.
- Two separate Nobel prizes were awarded for this ground breaking contribution;
  - Arrow (1972); Debreu (1983)

A Sketch

Assumptions:

- Everyone is self-interested and optimises own utility;

$$U = u\left(c(s_0)\right) + \underbrace{\beta\left[\pi(G) \cdot u\left(c(G)\right) + \pi(B) \cdot u\left(c(B)\right)\right]}_{\text{expected discounted future utility}}$$

- subject to budget constraints;
- free trade is allowed;
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- market clears (demand=supply);

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- subject to budget constraints;
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- market clears (demand=supply);

Results:

- Not only such outcome is Pareto efficient (First Fundamental Theory of Welfare);
- *Any* Pareto efficient outcome can be produced by such economic environment (Second Fundamental Theory of Welfare).

Endowments:

- There is an (exogenously given) supply or endowment of a **non-storable** consumption good at each time and state;
- At $t = 0$, the consumer does not know which state will realise in the future.

- Notation (Endowments):
  - $e(s_0)$ - the initial endowment of consumption good;
  - $e(s_1 = G)$ - the quantity of the consumption good consumer receives (say apples from a tree) at time 1 if the realized state is Good Weather;
  - $e(s_1 = B)$ - the endowment available at time 1 in the Bad Weather state;

Market structure:

- The consumer can freely borrow or lend in a complete set of atomic (Arrow-Debreu) securities.
- We assume the existence of two securities: Bad Weather security and Good Weather security.
  - One unit of 'G security' sells at time 0 at a price $q(s_0, s_1 = G)$ and pays one unit of consumption at time 1 if state 'G' occurs and nothing otherwise.
  - One unit of 'B security' sells at time 0 at a price $q(s_0, s_1 = B)$ and pays one unit of consumption in state 'B' only.
- In this notation: s_0 refers to the state when securities are traded; $s_1 = G$ refers to a particular realization of the state s_1 when the security pays off.

Flow budget constraints: Time 0

- In the first period the consumer has initial endowment $e(s_0)$. They can consume or buy Arrow-Debreu securities:

$$
\begin{aligned}
c(s_0) + q(s_0, s_1 = G) \cdot a(s_0, s_1 = G) \\
+ q(s_0, s_1 = B) \cdot a(s_0, s_1 = B) = e\,(s_0)
\end{aligned}
$$

  - $a(s_0, s_1 = G)$ – quantity G securities acquired in state s_0;
  - $a(s_0, s_1 = B)$ – quantity B securities acquired in state s_0;

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$$+ q(s_0, s_1 = B) \cdot a(s_0, s_1 = B) = e(s_0)$$

  - $a(s_0, s_1 = G)$ – quantity G securities acquired in state s_0;
  - $a(s_0, s_1 = B)$ – quantity B securities acquired in state s_0;
- In our two-period model all trades occur in state s_0. The only uncertainty is about the realization of the state s_1. Therefore, we can use simplified notation:

  - for atomic security prices: q_G, q_B
  - for quantities of the atomic security purchased (sold): a_G, a_B

$$c_0 + q_G \cdot a_G + q_B \cdot a_B = e_0$$

Flow budget constraints: Time 1

- If the realized state at time 1 is Good Weather:
  - Each of a_G G atomic securities pays off 1 unit of consumption;
  - Bad Weather atomic securities do not pay off at all;
  - Consumer receives an endowment corresponding to G state: e_G and consumes every unit of consumption they have got:

$$c_G = 1 \cdot a_G + 0 \cdot a_B + e_G,$$

Flow budget constraints: Time 1

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- If the realized state at time 1 is Bad Weather:
  - Each of a_B B atomic securities pays off 1 unit of consumption;
  - G atomic securities do not pay off at all;
  - Consumer receives an endowment corresponding to B state: e_B and consumes every unit of consumption they have got:

$$c_B = 0 \cdot a_G + 1 \cdot a_B + e_B.$$

Market Equilibrium:

- A *Market Equilibrium* in this economy is defined as an allocation c_0, c_G, c_B, a_G, a_B and prices q_G, q_B such that:
  - Given the prices, the allocation solves the consumer's problem of maximizing expected utility

$$u(c_0) + \beta [\pi_G \cdot u(c_G) + \pi_B \cdot u(c_B)]$$

subject to a sequence of budget constraints

$$c_0 + q_G \cdot a_G + q_B \cdot a_B = e_0,$$

$$c_G = a_G + e_G,$$
$$c_B = a_B + e_B.$$

  - Prices are such that markets clear in every period and state:

$$c_0 = e_0; c_G = e_G; c_B = e_B,$$

Constrained Optimization: A refresher

- To deal with the consumer's problem we have to maximize a function subject to several equality constraints.
- Consider a problem of choosing x, y, z to maximize the objective function f subject to equality constraints g_1 and g_2 :

$$\max_{x,y,z} f(x, y, z)$$

$$\text{s.t. } g_1(x, y, z) = b_1,$$

$$g_2(x, y, z) = b_2,$$

where b_1 and b_2 are constants.

Step 1: Set up a Lagrangian

We want to "translate" the constrained maximization problem into a unconditional maximisation question.

The Lagrangian, $\mathcal{L}(x, y, z, \lambda_1, \lambda_2)$, contains:

- the objective function f
- *minus* Lagrange multiplier for constraint 1, λ_1, times the difference between LHS and RHS of the constraint 1;

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The Lagrangian, $\mathcal{L}(x, y, z, \lambda_1, \lambda_2)$, contains:

- the objective function f
- *minus* Lagrange multiplier for constraint 1, λ_1, times the difference between LHS and RHS of the constraint 1;
- *minus* Lagrange multiplier for constraint 2, λ_2, times the difference between LHS and RHS of the constraint 2;

$$\begin{aligned}\mathcal{L}\left(x, y, z, \lambda_1, \lambda_2\right) = f\left(x, y, z\right) - \lambda_1\left[g_1\left(x, y, z\right) - b_1\right] \\ - \lambda_2\left[g_2\left(x, y, z\right) - b_2\right]\end{aligned}$$

Step 2: First order conditions

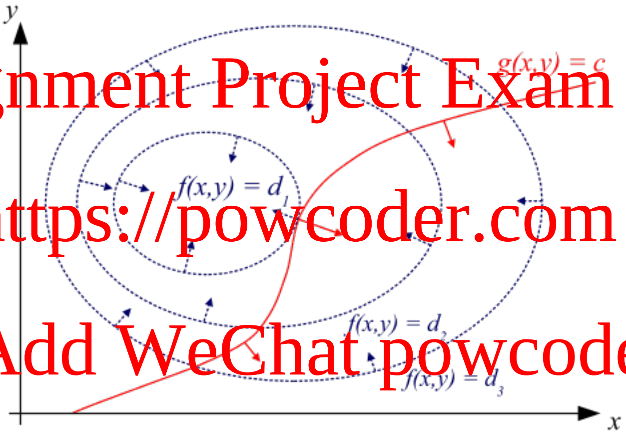
Then we take partial derivatives of the Lagrangian with respect to its every argument and equate them to zero;

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda_1 \frac{\partial g_1}{\partial x} - \lambda_2 \frac{\partial g_2}{\partial x} \equiv 0; \\
\frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda_1 \frac{\partial g_1}{\partial y} - \lambda_2 \frac{\partial g_2}{\partial y} \equiv 0; \\
\frac{\partial \mathcal{L}}{\partial z} &= \frac{\partial f}{\partial z} - \lambda_1 \frac{\partial g_1}{\partial z} - \lambda_2 \frac{\partial g_2}{\partial z} \equiv 0; \\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= -g_1(x, y, z) + b_1 \equiv 0; \\
\frac{\partial \mathcal{L}}{\partial \lambda_2} &= -g_2(x, y, z) + b_2 \equiv 0.
\end{aligned}
$$

Now we have five equations with five unknowns: $x, y, z, \lambda_1, \lambda_2$ and, hence, can find the solution.

An illustration: two-dimensional optimisation problem

An illustration: contour lines



At the optimum, the tangency point of $f(x, y)$ and $g(x, y)$,
$\nabla f = \lambda \nabla g$,

What does λ mean?

When we set up Lagrangian, we are looking for x, y, z that satisfies:

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2;$$

where $\nabla(\cdot) \equiv \left(\frac{\partial(\cdot)}{\partial x}, \frac{\partial(\cdot)}{\partial y}, \frac{\partial(\cdot)}{\partial z}\right)$ is the gradient vector.

λs measure the importance of each constraint, i.e., how much the maximum value will change if we marginally change the value of that constraint.