Lesson 6: Arrow-Debreu Pricing: Multiple States, Stochastic Discount Factor, Heterogeneity, Pareto Optimality

Economics of Finance

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Pricing state-contingent claims

- Using the atomic state prices, often called, *pricing kernel*:

$$p = q \cdot c,$$

q - row vector of atomic state prices or pricing kernel
c - column vector of state contingent payments

- Using *risk-neutral measure*:

$$p = df \cdot \widetilde{E}(c),$$

c - random variable, realised value depends on a state,
$\widetilde{E}(\cdot)$ - expectation taken with respect to risk-neutral measure using risk-neutral probabilities $\widetilde{\pi}$

- Using *stochastic discount factor*:

$$p = E(m_1 c),$$

c - random variable, realised value depends on a state,
m_1 - stochastic discount factor,
$E(\cdot)$ - expectation taken with respect to physical probability measure using actual probabilities π

Heterogeneous agents: consumers' problem

- Each agent k maximises expected utility, U^k, given by

$$U^k = u^k\left(c_0^k\right) + \underbrace{\beta^k \sum_{s_1 \in S_1} \pi_{s_1} \cdot u^k\left(c_{s_1}^k\right)}_{\text{expected discounted future utility}}$$

- subject to period-0 constraint

$$c_0^k + \sum_{s_1 \in S_1} q_{s_1} \cdot a_{s_1}^k = e_0^k,$$

- and a series of period-1 constraints for every possible state:

$$c_{s_1}^k = a_{s_1}^k + e_{s_1}^k, \text{ for all } s_1 \in S_1$$

- Market clearing (now makes more sense)

$$\sum_{k=1}^{K} c_0^k = \sum_{k=1}^{K} e_0^k; \qquad \sum_{k=1}^{K} c_{s_1}^k = \sum_{k=1}^{K} e_{s_1}^k, \forall s_1$$

Characterisation of the Equilibrium

- From the first order conditions the prices of the atomic (Arrow-Debreu) securities

$$q_{s_1} = \beta^k \pi_{s_1} \frac{u^{k\,\prime}\left(c^k_{s_1}\right)}{u^{k\,\prime}\left(c^k_0\right)} =$$

$$= \beta^k \pi_{s_1} \frac{u^{k\,\prime}\left(e^k_{s_1} + a^k_{s_1}\right)}{u^{k\,\prime}\left(e^k_0 + a^k_0\right)}$$

  for all k and $s_1 \in S$.

- also impose **market clearing** which *implies* that

$$\sum_{k=1}^{K} a^k_{s_1} = 0, \, \forall s_1 \in S.$$

Pareto optimality

- Allocation of atomic (Arrow-Debreu) securities in which it is impossible to make any one consumer better off without making at least one consumer worse off.
- Under the first welfare theorem (we do not prove it here) *competitive* equilibrium (prices are taken as given) is equivalent to Pareto optimality.
- Some conditions: completeness - existence of atomic (Arrow-Debreu) securities for all states, no transaction costs, no externalities (utilities are independent).