Lesson 0: Refresher on Matrices

Economics of Finance

School of Economic, UNSW

Review of matrix algebra

Matrices: A combination of vectors with the same dimension.

Features:

- square/non-square
- invertible/non-invertible

Operations:

- transpose
- addition/subtraction
- multiplication
- inverse

Matrix Operations

Some tips:

- Addition/subtraction needs to be between matrices with identical shape:

$$A_{m\times n} \pm B_{m\times n}$$

- Dimension of the line vector in front of $\times$ needs to match with the dimension of the column vector after $\times$:

$$A_{l\times m} \times B_{m\times n}$$

- Most times we need to transpose a matrix because of the above two rules.

Matrix inversion

Matrix inversion is closely related to linear independency.

Definition

A set of n-dimensional column vectors, $\{A_1, A_2, ...A_m\}$ are linearly independent iff. the unique solution of the linear equation set:

$$\lambda_1 A_1 + \lambda_2 A_2 + ... + \lambda_m A_m = \mathbf{0}$$

is $\lambda = \mathbf{0}$

A square matrix, $A_{n\times n}$, is invertible iff its column vectors, $\{A_1, A_2, ...A_n\}$ are linearly independent.

Why does it matter?

A square matrix is invertible iff an arbitrary linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has the unique solution.

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Existence and Uniqueness of Solution

Suppose there exists a $\lambda \neq \mathbf{0}$ s.t.

$$\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_n a_n = \mathbf{0},$$

then we have two scenarios:

- No solution;
- There exists $\mathbf{x}$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{b}$, then for any $\alpha \neq 0$, $\mathbf{x}_\alpha = \mathbf{x} + \alpha\lambda$ also satisfies $\mathbf{A}\mathbf{x}_\alpha = \mathbf{b}$.

The function either have no solution or infinite solutions.

Number of linearly independent λ is called *degree of freedom*.

Interchangeable Concepts

For a *square* matrix $\mathbf{A}_{n \times n}$, the following statements are equivalent:

- invertible
- non-singular
- full rank
- linearly independent in columns
- determinant is non-zero
- an arbitrary function $\mathbf{Ax} = \mathbf{b}$ has unique solution.

Example

$$M = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{pmatrix}$$

1. Square? Yes.
2. Linearly independent? No, third column is a linear combination of the first and second columns $\rightarrow C_3 = C_1 + 2C_2$.
3. Check determinant is zero

Thus, not invertible