Economics of Finance

Tutorial 8

1. Consider the following two stocks: stock 1 has expected return $e_1 = 3$ and standard deviation $s_1 = 3$; stock 2 has expected return $e_2 = 7$ and standard deviation $s_2 = 6$. Assume that the correlation between the returns on the two stocks is 0.25, i.e., the correlation coefficient $r_{12} = 0.25$.

(i) Write down an equation for the expected return (e_p) and the variance (v_p) of the return on the portfolio as a function of x only, where x is the proportion of invested wealth in *stock 2*.

Solution

In general, the expected return of the portfolio as a function of x is

$$e_p = (1-x)\,e_1 + xe_2 = e_1 + (e_2 - e_1)x,$$

while the variance of the returns is

$$v_p = (1-x)^2 v_1 + x^2 v_2 + 2(1-x)x s_1 s_2 r_{12}.$$

Substituting the given values yields

$$\begin{aligned} e_p &= 3 + 4x, \\ v_p &= 9(1-x)^2 + 36x^2 + 2(1-x)x(3 \times 6 \times 0.25) \\ &= 9\left(4x^2 - x + 1\right) \end{aligned}$$

(ii) Find the minimum variance portfolio. What is the expected return and variance of this portfolio?

Solution

Differentiating v_p with the respect to x and equating the result to zero yields

$$\frac{dv_p}{dx} = 9\,(8x-1) = 0,$$

from which it follows that $x = \frac{1}{8}$. Therefore $1 - x = \frac{7}{8}$,

$$v_p = 9\left(4x^2 - x + 1\right) = \frac{135}{16}$$

$$s_p = \frac{3}{4}\sqrt{15} = 2.9047$$

$$e_p = 3.5$$

(iii) Is Stock 1 on the efficient (portion of) frontier?

Solution
Stock 1 is not efficient because the minimum variance portfolio has a higher expected return and a smaller standard deviation (see Figure 1).

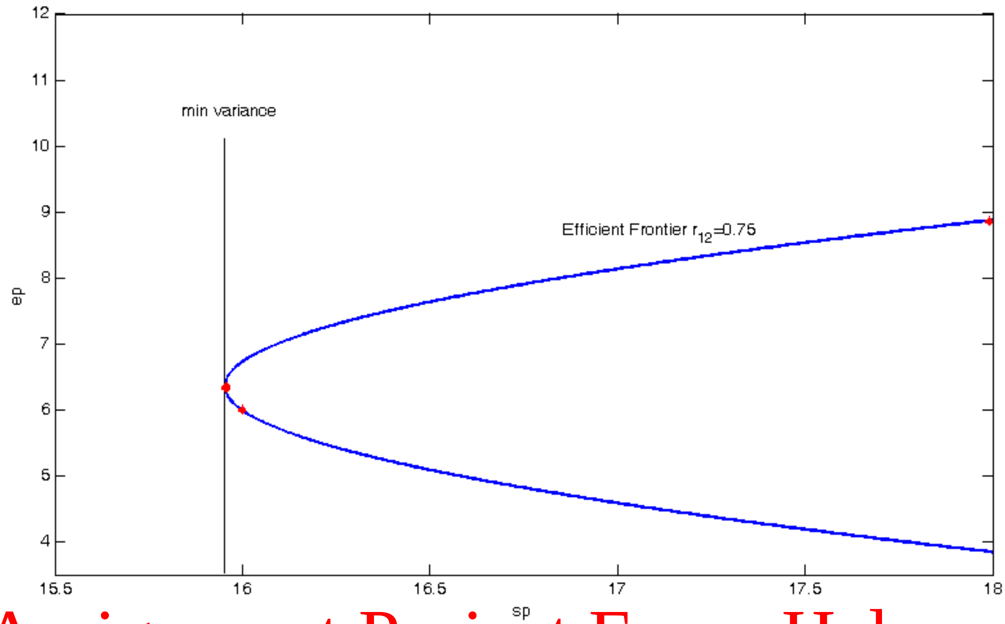

Figure 1: Minimum variance portfolio

(iv) Assume that the risk-free asset is now available and the risk-free rate is 2. What is the market portfolio? Compute its Sharpe ratio. Derive the capital market line?

Solution

Market portfolio is the portfolio that maximizes the Sharpe ratio. To construct the market portfolio, consider the following problem:

$$\max_{\langle x \rangle} \left\{ \frac{e_p - R_f}{\sqrt{v_p}} \right\}$$

where $v_p = 9(4x^2 - x + 1)$ and $e_p = 3 + 4x$.
FOC yields:

$$\frac{4}{\sqrt{4x^2 - x + 1}} = \frac{1}{2} \frac{(4x+1)(8x-1)}{(4x^2 - x + 1)^{\frac{3}{2}}}$$

Which solves:

$$4\left(4x^2 - x + 1\right) = \frac{1}{2}(4x+1)(8x-1);$$
$$x = \frac{3}{4};$$

Sharpe ratio equals to excess return standardized (divided) by standard deviation of risky asset (or risky portfolio). Sharpe ratio is the slope of the asset allocation line. In here, it yields:

$$S = \frac{8}{3\sqrt{10}} \approx 0.84;$$

Capital market line is then

$$e_p = 2 + 0.84 s_p$$

(v) Draw the security market line. What is β for the stock 2? What is stock 2's idiosyncratic risk in terms of standard deviation?

Solution

Denote M as market portfolio. The market portfolio entails:

$$e_M = 6; s_M = \frac{3\sqrt{10}}{2};$$

SML is:

$$e = r_f + \beta\left(e_M - r_f\right) = 2 + 4\beta$$

β can be found by:

$$\begin{aligned}
\beta_2 &= \frac{cov(M, R_2)}{var(M)} \\
&= \frac{1}{var(M)} cov\left(\frac{1}{4}R_1 + \frac{3}{4}R_2, R_2\right) \\
&= \frac{1}{var(M)}\left[\frac{1}{4}cov\left(R_1, R_2\right) + \frac{3}{4}cov\left(R_2, R_2\right)\right] \\
&= 1.25;
\end{aligned}$$

Alternatively, using SML, we have:

$$\beta_2 = \frac{e_2 - r_f}{e_M - r_f} = 1.25,$$

which mirrors our result.
Idiosyncratic risk is given by:

$$s_2^i = s_2 - \beta_2 s_M = 6 - \frac{15\sqrt{10}}{8} = 0.07$$

2. Consider the multifactor model (arbitrage pricing theory, APT) with only two factors. The risk premiums on the factor 1 and factor 2 portfolios are 5% and 6%, respectively. Stock A has a beta of 1.2 on factor 1, and a beta of 0.7 on factor 2. The expected return on stock A is 17%.

Statistical arbitrage is a strategy which exploits mispricing of expected returns, but it is different from the *pure arbitrage* strategy which covers all risks. Statistical arbitrage involves risks as there is no guaranty that each particular realized returns are similar to expected returns. Most of trading strategies are some variants of statistical arbitrage, but as such they are not risk-free.

(i) If no statistical arbitrage opportunities exist, what is the risk-free rate of return?

Solution

Arbitrage pricing theory suggest fair expected return of an asset is determined by a factor model. Here we are given a simple two-factor model:

$$r_A = r_f + \beta_{A,1} f_1 + \beta_{A,2} f_2 + \varepsilon_A$$

Taking the expectation $e_A = E(r_A)$ we obtain

$$e_A = r_f + \beta_{A,1} f_1 + \beta_{A,2} f_2$$

Therefore, $r_f = e_j - \beta_1 f_1 + \beta_2 f_2 = 17 - 1.2 \cdot 5 - 0.7 \cdot 6 = 6.8\%$

(ii) What an investor would do to take advantage of a (statistical) arbitrage opportunity?

Solution

If it happens that the current return of stock A is higher or lower than the expected return implied by APT, there is an opportunity for statistical arbitrage. If the actual return is higher than the expected return, the stock is undervalued, that is, its current price is too low. To realise the statistical arbitrage, you need to buy the stock and short-sell the replicating portfolio using the portflolio of the risk factor portfolios with *beta*-weights. This way, you have zero net position and opportunity to profit if mispricing is corrected. Statistically, it will be corrected (if our model is right), yet is each specific realization, statistical arbitrage may lead to a loss.

(iii) Recall strategies involving trading dual-listed shares (the last question of Assignment 1). Do these provide statistical or pure arbitrage opportunities?

Solution

Dual-listed shares represent claims on the same company and hence in theory should be valued similarly (up to tax differences). Yet, as we remember from the research article by de Jong, Rosenthal and van Dijk (2009), the timing of correction is not certain and hence this is the situation closer to the statistical arbitrage: the model is correct on average, but there is some noise preventing us from realising the pure arbitrage.