Lecture 4: Multi-period Binomial Models, Options and Option Pricing

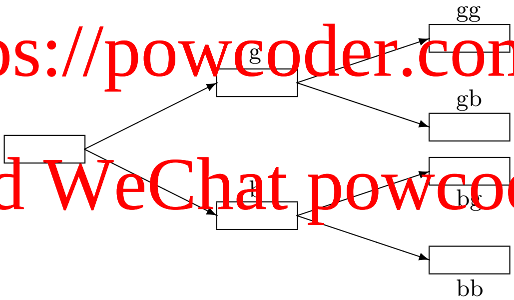
Economics of Finance

School of Economics, UNSW

Extending into multi-periods

Time: Present (time 0); Future time periods (times 1 and 2)

State: Two possible realizations of uncertainty: good times and bad times

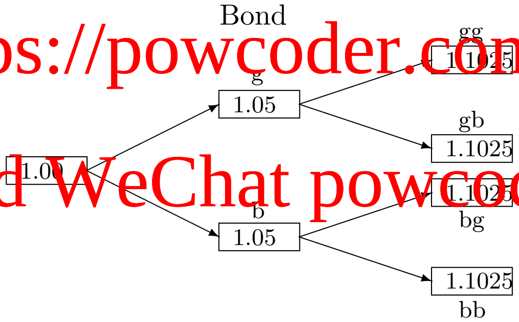


Multi-periods

- Notation:
  - Number of letters (g, gg) indicates time period;
  - Sequence of letters indicates the path taken to reach the node;
  - Notice the states of gb and bg can be identical.

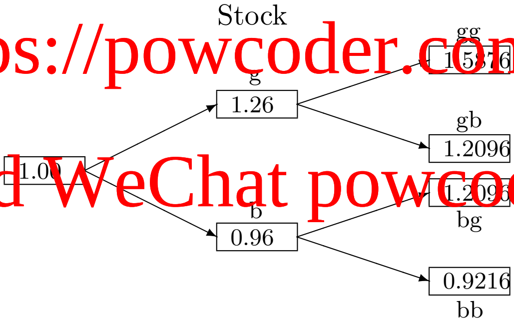
The Bond

- Two-period zero-coupon bond (no coupon payments)
- Its initial value is \$1.00.
- Its price increases 5% of its prior value in every period.



The Stock

- Its initial value is \$1.00. It pays no dividends.
- Its price increases 26% of its prior value in good times.
- Its price falls to 96% of its prior value in bad times.



Security revisited

The number of future states of the world equals six

- But seems we have only two "securities", bond and stock.
- What can we do?
- Recall our definition of "security", i.e. state contingent contract.
- Now that the time span has been extended into more than one periods, we need to extend the security space to accommodate them.

How?

Planned Acquisitions

Consider the following set of *planned acquisitions*

B0: Buy a Bond at period 0, sell it at the end of the next period;

S0: Buy a Stock at period 0, sell it at the end of the next period;

Bg: At period 1, if the state is **g**, buy a Bond, sell it at the end of the next period;

Sg: At period 1, if the state is **g**, buy a Stock, sell it at the end of the next period;

Bb: At period 1, if the state is **b**, buy a Bond, sell it at the end of the next period;

Sb: At period 1, if the state is **b**, buy a Stock, sell it at the end of the next period.

We write down the payment of these acquisitions in a matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \mathrm{B0} & \mathrm{S0} & \mathrm{Bg} & \mathrm{Sg} & \mathrm{Bb} & \mathrm{Sb} \end{array} \\
\mathbf{Q} = & \begin{pmatrix}
1.05 & 1.26 & -1 & -1 & 0 & 0 \\
1.05 & 0.96 & 0 & 0 & -1 & -1 \\
0 & 0 & 1.05 & 1.26 & 0 & 0 \\
0 & 0 & 1.05 & 0.96 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.05 & 1.26 \\
0 & 0 & 0 & 0 & 1.05 & 0.96
\end{pmatrix}
\begin{array}{c} \mathrm{g} \\ \mathrm{b} \\ \mathrm{gg} \\ \mathrm{gb} \\ \mathrm{bg} \\ \mathrm{bb} \end{array}
\end{array}
$$

Remarks

- A set of 6 column vectors;
- Each presenting a payment stream for a planned acquisition;

- Notice they are linearly independent, $det(\mathbf{Q}) \neq 0$
- Such linearly independent vector set is not unique, just like bond and stock is not the unique set of linearly independent securities in the one period world.

Price Vector

Price Vector entails:

$$\begin{array}{cccccccc} & & \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \\ \mathbf{p}_G & = & (1.00 & 1.00 & 0.0 & 0.0 & 0.0 & 0.0) \end{array}$$

- Why are the strategies Bg, Sg, Bb, and Sb priced as 0?
- Any insights on the meaning of "price"?

Pricing a state

To price a unit of payment at each state, we can now use the formula we are familiar with: $\mathbf{p}_{atom} = \mathbf{p}_S \cdot \mathbf{Q}^{-1}$:

$$\begin{array}{cccccccc} & & g & b & gg & gb & bg & bb \\ \mathbf{p}_{atom} & = & \Big(0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \Big) \end{array}$$

Wrap up

- Extending time span necessarily extends the space of states;
- This, however, does not necessarily mean we need more than two securities;
- Instead, by manipulating with existing securities in various periods, we expand the *action space*;
- These actions creates linearly independent planned acquisitions. We call them "elementary strategies";
- Each state can be price in a similar way to the atomic securities;
- Notice the set of elementary strategies may not be unique.

Definition

- Call option vs. Put option:
  - Call option: entitles the right to *buy* an underlying asset (say shares, foreign currency or commodity) at a specified *strike price, or, exercise price(X).*
  - Put option: entitles the right to *sell* the underlying asset at a specified *strike price* X.

- European option vs. American Option
  - European put or call option: can be exercised only on expiry date.
  - American put or call option: can be exercised on any date up to and including its expiration date.

Terminology

In-the-money, at the money and out-of the money.
Denote p as the market price:

	Call Option	Put Option
In-the-money	p $>$ X	p $<$ X
At-the-money	p $=$ X	p $=$ X
Out-of-the-money	p $<$ X	p $>$ X

Example: General Electric options (March 2013)

Name	Last	Change
(GE) GENERAL ELECTRIC CO.	$23.24	-0.13(-0.56%)

Options Chain Add to Portfolio

Chain Type: Calls and Puts · Chain Type: All · Expiration: Mar 2013 · View Chain

Calls								Puts						
Contract Name	Last Trade	Change	Bid	Ask	Volume	Interest	Strike Price	Contract Name	Last Trade	Change	Bid	Ask	Volume	Interest
GE\13C28\17.0	0.00	0.00	5.30	6.55	0.00	0.00	17.00	GE\13O28\17.0	0.00	0.00	0.00	0.04	0.00	0.00
GE\13C28\18.0	0.00	0.00	5.00	5.30	0.00	0.00	18.00	GE\13O28\18.0	0.00	0.00	0.00	0.04	0.00	0.00
GE\13C28\19.0	0.00	0.00	4.00	4.55	0.00	0.00	19.00	GE\13O28\19.0	0.00	0.00	0.00	0.04	0.00	0.00
GE\13C28\19.5	0.00	0.00	3.50	3.85	0.00	0.00	19.50	GE\13O28\19.5	0.00	0.00	0.00	0.04	0.00	0.00
GE\13C28\20.0	0.00	0.00	3.00	3.50	0.00	0.00	20.00	GE\13O28\20.0	0.00	0.00	0.00	0.04	0.00	0.00
GE\13C28\20.5	[illegible]	-0.04	[illegible]	[illegible]	[illegible]	[illegible]	20.50	GE\13O28\20.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
GE\13C28\21.0	[illegible]	0.00	1.75	[illegible]	[illegible]	[illegible]	21.00	GE\13O28\21.0	0.00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
GE\13C28\21.5	1.70	-0.14	1.74	1.77	149	276	21.50	GE\13O28\21.5	0.01	0.00	0.01	0.01	200	200
GE\13C28\22.0	1.17	-0.21	1.24	1.27	701	20	22.00	GE\13O28\22.0	0.02	-0.01	0.01	0.02	308	436
GE\13C28\22.5	0.73	-0.11	0.75	0.78	1,174	154	22.50	GE\13O28\22.5	0.02	-0.01	0.01	0.02	370	828
GE\13C28\23.0	0.34	-0.04	0.30	0.32	1,921	969	23.00	GE\13O28\23.0	0.08	0.00	0.06	0.07	3,407	2,570
GE\13C28\23.5	0.04	-0.08	0.04	0.05	11,436	1,918	23.50	GE\13O28\23.5	0.29	0.02	0.29	0.31	1,119	340
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	221	24.00	[illegible]	[illegible]	0.08	[illegible]	[illegible]	229	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	2	24.50	[illegible]	0.00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
GE\13C28\25.0	0.00	0.00	0.00	0.04	0.00	0.00	25.00	[illegible]	1.75	0.02	1.74	1.77	31	0.00
GE\13C28\25.5	0.00	0.00	0.00	0.04	0.00	0.00	25.50	GE\13O28\25.5	0.00	0.00	1.97	2.68	0.00	0.00
GE\13C28\26.0	0.00	0.00	0.00	0.04	0.00	0.00	26.00	GE\13O28\26.0	0.00	0.00	2.47	3.05	0.00	0.00
GE\13C28\26.5	0.00	0.00	0.00	0.04	0.00	0.00	26.50	GE\13O28\26.5	0.00	0.00	2.97	3.50	0.00	0.00
GE\13C28\27.0	0.00	0.00	0.00	0.04	0.00	0.00	27.00	GE\13O28\27.0	0.00	0.00	3.45	4.00	0.00	0.00
GE\13C28\27.5	0.00	0.00	0.00	0.04	0.00	0.00	27.50	GE\13O28\27.5	0.00	0.00	3.95	4.50	0.00	0.00
GE\13C28\28.0	0.00	0.00	0.00	0.04	0.00	0.00	28.00	GE\13O28\28.0	0.00	0.00	4.45	5.00	0.00	0.00
GE\13C28\29.0	0.00	0.00	0.00	0.04	0.00	0.00	29.00	GE\13O28\29.0	0.00	0.00	5.45	7.20	0.00	0.00
GE\13C28\30.0	0.00	0.00	0.00	0.04	0.00	0.00	30.00	GE\13O28\30.0	0.00	0.00	5.25	8.20	0.00	0.00

Call option payment

Example: Consider a *call option* that entitles the right to buy the stock at \$55. Strike price, $X = \$55$

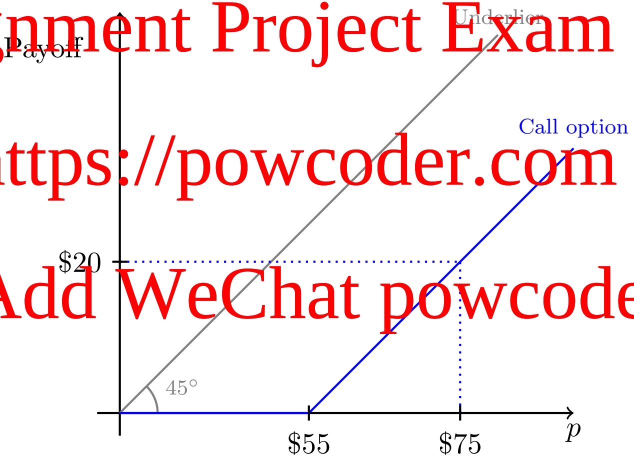
- Case 1: If the actual stock price is less than the strike price, $p < X$, then the option holder will *not* exercise the Call option. The payoff of exercising this Call option would be zero.

- Case 2: If the actual stock price in a year is more than the strike price, $p > X$, then it pays to exercise the Call option.

For example, if $p = \$75$, then the Call option's payoff if exercised is $75 - 55 = \$20$.
Note: No need to actually buy the stock to receive this payoff.

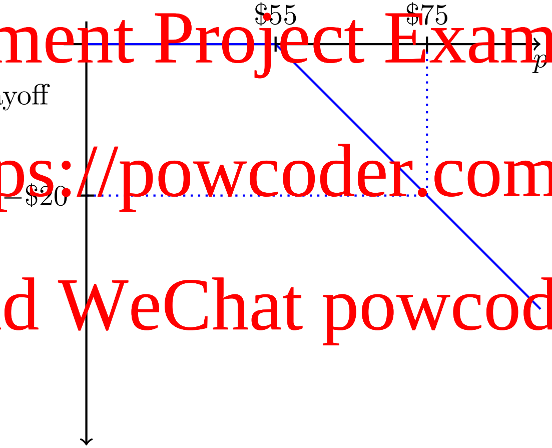
Long Call Payoff $= Max\{p - X, 0\}$

We plot the payoff of a call option with a given strike price as a function of price of the *underlying security ("underlier")*.



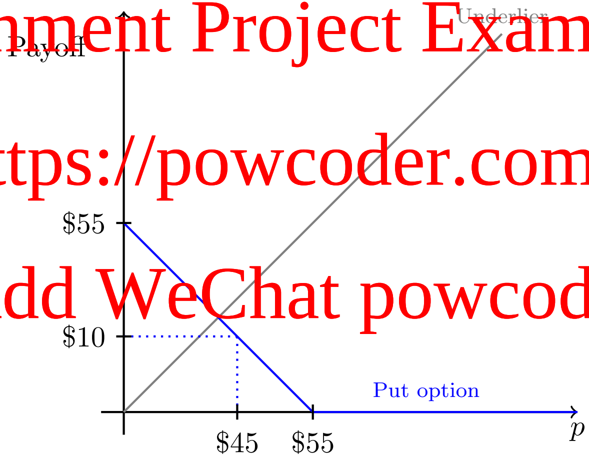
Short Call Payoff $= -Max\{p - X, 0\}$

Payoffs of *selling* call option.



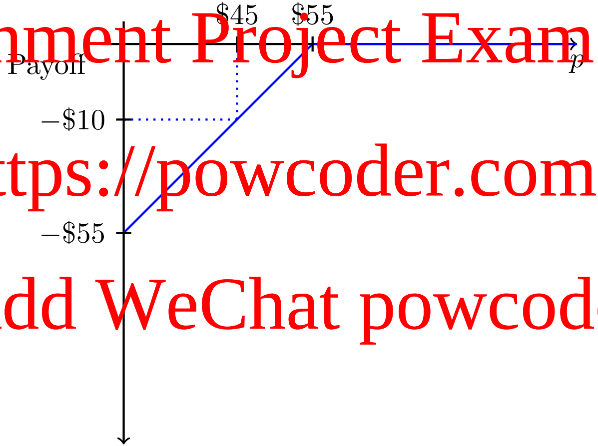
$$\text{Long Put Payoff} = Max\{X - p, 0\}$$

Consider a *put* option, where $X = \$55$ and $P = \$45$. For the option buyer:



$$\text{Short Put Payoff} = -Max(X - S, 0)$$

Similarly, payoffs of a seller of *put* option



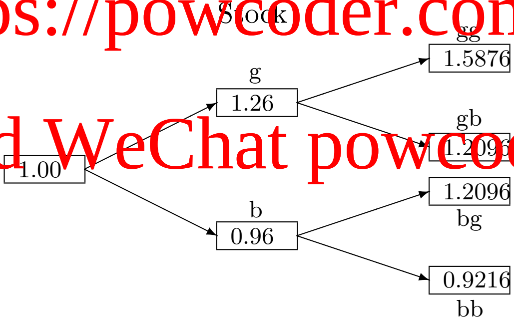
Overall profit

The overall profit/loss will also include the price of the option.

For example, if Call option price is \$5.75.

The Setup: Three Period Binomial Model

- Two-period zero-coupon bond with initial value of \$1.00. Its price increases 5% of its prior value in every period.
- The Stock pays no dividends. Its initial value is \$1.00.
- Its price increases 26% of its prior value in good times.
- Its price falls to 96% of its prior value in bad times.



Computing atomic (state) prices

- The Payment Matrix:

$$
\mathbf{Q} = \begin{array}{c} \\ \\ \end{array}
\begin{array}{cc}
\begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} & \\
\begin{pmatrix}
1.05 & 1.26 & -1 & -1 & 0 & 0 \\
1.05 & 0.96 & 0 & 0 & -1 & -1 \\
0 & 0 & 1.05 & 1.26 & 0 & 0 \\
0 & 0 & 1.05 & 0.96 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.05 & 1.26 \\
0 & 0 & 0 & 0 & 1.05 & 0.96
\end{pmatrix} &
\begin{array}{c} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array}
\end{array}
$$

- The Price Vector:

$$
\begin{array}{c}
\begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} \\
\mathbf{p}_S = \begin{pmatrix} 1.00 & 1.00 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}
\end{array}
$$

- The atomic prices $\mathbf{p}_{atom} = \mathbf{p_S} \cdot \mathbf{Q}^{-1}$:

$$
\begin{array}{c}
\begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\
\mathbf{p}_{atom} = \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix}
\end{array}
$$

Alternative way to compute atomic (state) prices

- The Payment Matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} \\
\mathbf{Q} = & \begin{pmatrix}
1.05 & 1.26 & -1.05 & -1.26 & 0 & 0 \\
1.05 & 0.96 & 0 & 0 & -1.05 & -0.96 \\
0 & 0 & 1.1025 & 1.5876 & 0 & 0 \\
0 & 0 & 1.1025 & 1.2096 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.1025 & 1.2096 \\
0 & 0 & 0 & 0 & 1.1025 & 0.9216
\end{pmatrix}
\begin{array}{c} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array}
\end{array}
$$

- The Price Vector:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} \\
\mathbf{p}_S = & \begin{pmatrix} 1.00 & 1.00 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}
\end{array}
$$

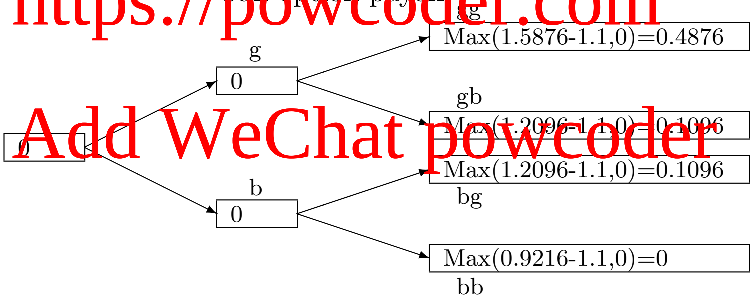
- The atomic prices $\mathbf{p}_{atom} = \mathbf{p_S} \cdot \mathbf{Q}^{-1}$:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\
\mathbf{p}_{atom} = & \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix}
\end{array}
$$

European Call Option

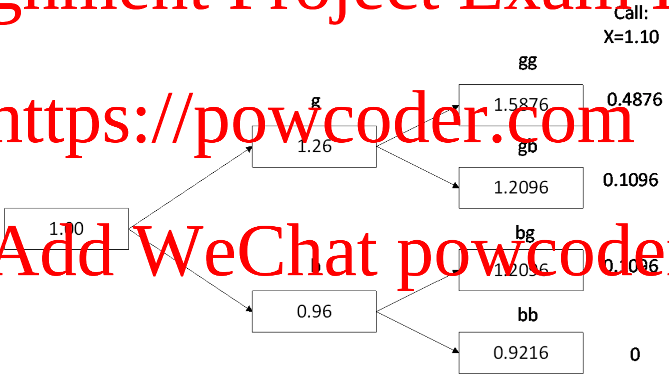
The matrix c can be derived from the payoff of call options at the end of each state by using Max (S-X,0) where X is given by \$1.1 in the example. Since the European call option will be likely exercised at T=2 (i.e.expiry date), payoff at T=1 (ie: **g** and **b** states) will be **zero.**

Call option payoff



Example: European Call Option

Consider a European Call option that gives the holder a right to buy the Stock at Period 2 at the Exercise Price, $X = 1.10$.



Pricing a European Call Option

The cash flow associated with the Call option:

$$\mathbf{c} = \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 0.4876 \\ 0.1096 \\ 0.1096 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array}$$

The atomic prices are still the same:

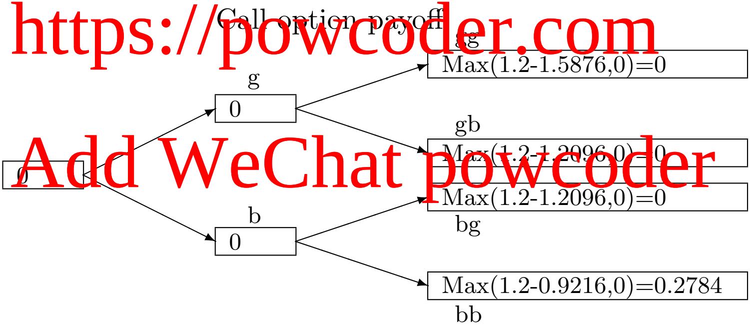
$$\mathbf{p}_{atom} = \begin{array}{c} \begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\ \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix} \end{array}$$

The value of the Call option is:

$$\mathbf{p}_{\text{Call}} = \mathbf{p}_{atom} \cdot \mathbf{c} = 0.0816$$

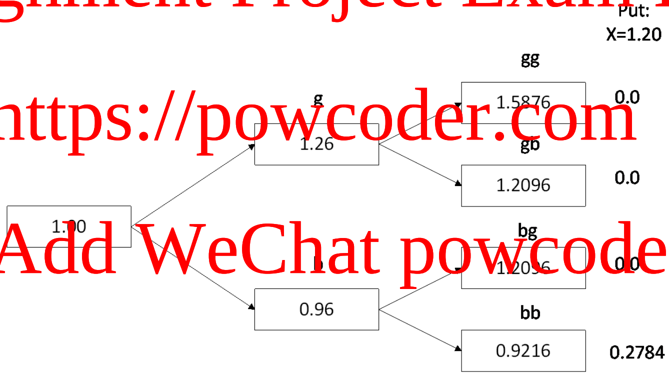
Example: European Put Option

The matrix c can be derived from the payoff of put options at the end of each state by using Max (X-S,0) where X is given by \$1.2 in the example. Since the European put option will be likely exercised at T=2 (i.e.expiry date), payoff at T=1 (i.e. **g** and **b** states) will be **zero.**



Example: European Put Option

Consider a European Put option that gives the holder a right to sell the Stock in Period 2 at the Exercise Price, $X = 1.20$.



Pricing a European Put Option

The cash flow associated with the Put option:

$$\mathbf{c} = \begin{array}{cc} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.2784 \end{pmatrix} & \begin{array}{c} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array} \end{array}$$

The atomic prices are still the same:

$$\begin{array}{rc} & \begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\ \mathbf{p}_{atom} = & \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix} \end{array}$$

The value of the Put option is:

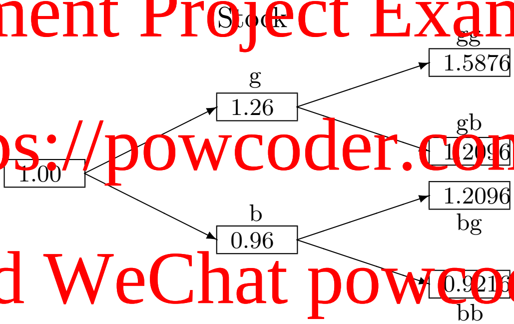
$$\mathbf{p}_{\text{Put}} = \mathbf{p}_{atom} \cdot \mathbf{c} = 0.4444 \cdot 0.2784 = 0.1237$$

American Put Option

Consider an American Put option that allows the holder to sell the Stock at a price of $X = 1.20$ at either Period 1 or Period 2.

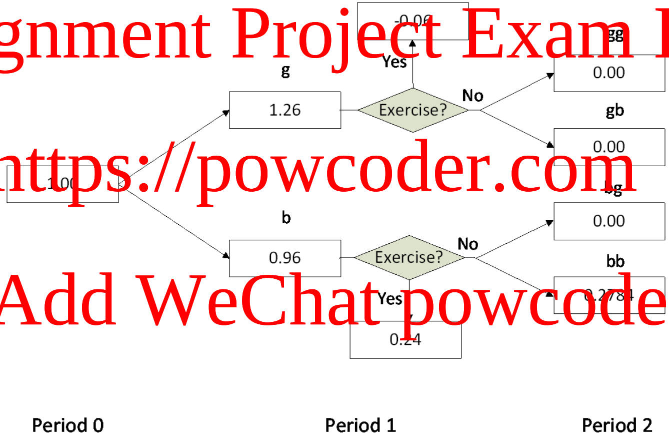
- Two-period zero-coupon bond with initial value of \$1.00. Its price increases 5% of its prior value in every period.
- The Stock pays no dividends. Its initial value is \$1.00.
- Its price increases 26% of its prior value in good times.
- Its price falls to 96% of its prior value in bad times.

Stock payment



American Put Option

Put payoff is $\max\{X - S, 0\}$, where $X = 1.2$



American Put Option

Do we exercise the option in Period 1 when the stock price has risen?

- The answer: keep it till Period 2! Exercising in state **g** will result in a negative cash flow.

Do we exercise the option in Period 1 when the stock price has fallen?

- Consider the cash flows that correspond to the two decisions:

$$
\mathbf{c}_{exercise} = \begin{array}{cl} \left(\begin{array}{c} 0 \\ 0.24 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right) & \begin{array}{l} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array} \end{array}
\qquad
\mathbf{c}_{keep} = \begin{array}{cl} \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.2784 \end{array}\right) & \begin{array}{l} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array} \end{array}
$$

American Put Option

The atomic prices are as before:

$$\begin{array}{ccccccc} & \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \\ \mathbf{p}_{atom} = & \big(0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444\big) \end{array}$$

The present values of the two cash flows are:

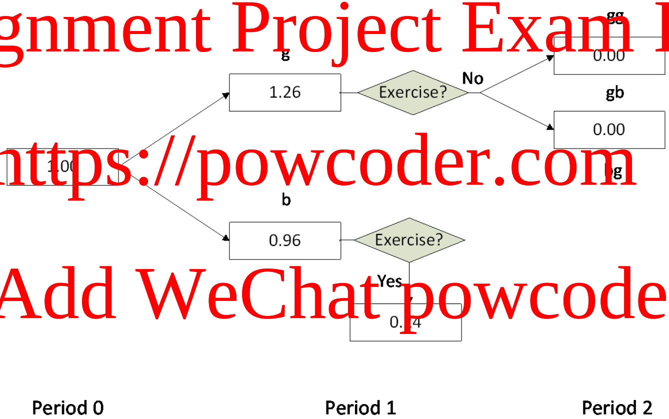
$$\mathbf{p}_{atom} \cdot \mathbf{c}_{exercise} = 0.6666 \cdot 0.24 = 0.16$$

$$\mathbf{p}_{atom} \cdot \mathbf{c}_{keep} = 0.4444 \cdot 0.2784 = 0.1237$$

The decision: Exercise the option in Period 1 if the stock price has fallen.

American Put Option

The simplified decision tree include only optimal paths is:



American put option pricing using atomic security

The cash flow associated with the Put option:

$$\mathbf{c} = \begin{array}{cc} \begin{pmatrix} 0 \\ 0.24 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array} \end{array}$$

The atomic prices are still the same:

$$\mathbf{p}_{atom} = \begin{array}{c} \begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\ \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix} \end{array}$$

Option price

The value of the Put option is:

$$\mathbf{p}_{put} = \mathbf{p}_{atom} \cdot \mathbf{c} = 0.6666 \cdot 0.24 = 0.16$$

When does American option early exercise?

We know that American options provide more "options" of early exercising.

- In general, this implies that an American option is more worthy than European option with identical underlier and strike price;
- How general is this intuition?
- The fact is quite disappointing.
- In most cases, the American option value is exactly the same as the corresponding European option.
- Early exercise is rare

Put-Call Parity

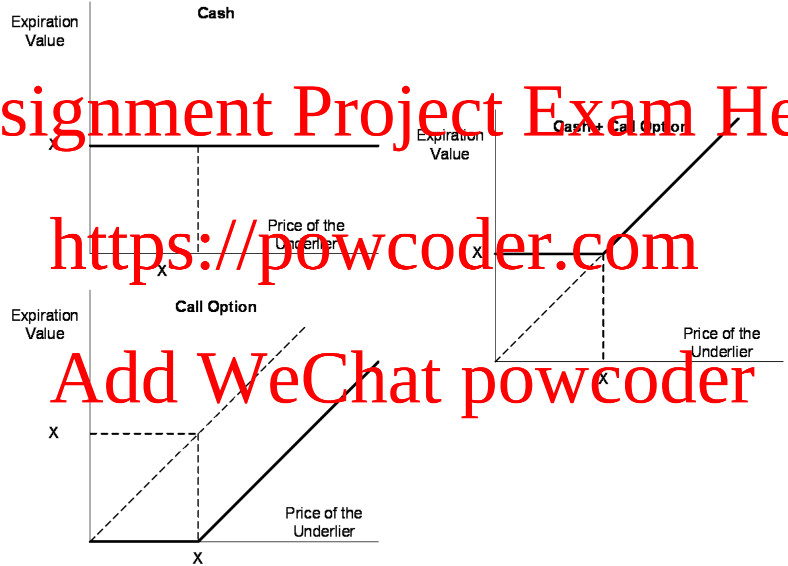
Put-Call Parity is a relationship, first identified by Stoll (1969), that must exist between the prices of European Put and Call options that both have:

- the same underlying stock;
- the same strike price;
- the same expiration date.

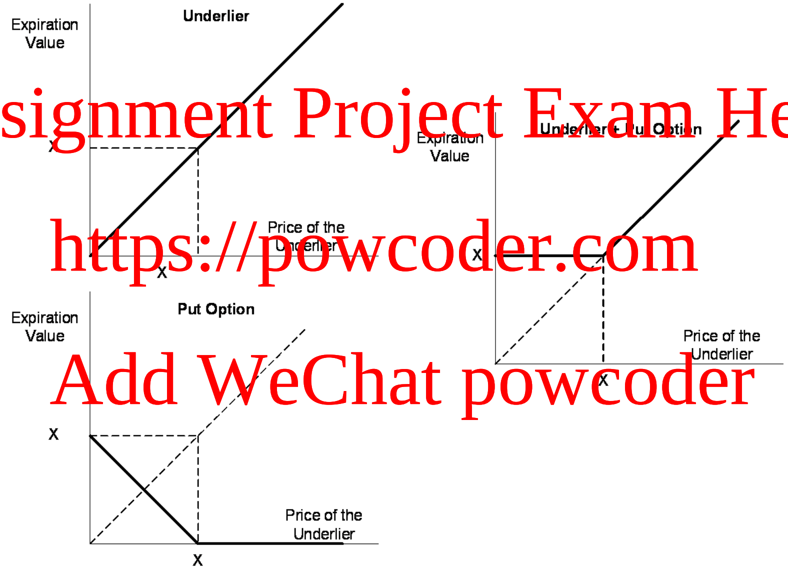
The relationship is derived using arbitrage arguments. Consider two portfolios consisting of:

- The Call option and an amount of cash equal to the present value of the strike price.
- The Put option and the underlying stock.

Put-Call Parity: Cash and Call



Put-Call Parity: Underlier and Put



Put-Call Parity

- The two portfolios (call + cash and put + underlier) have identical expiration values.

- Irrespective of the value of the underlier at expiration, each portfolio will have the same value as the other.
- If the two portfolios are going to have the same value at expiration, then they must have the same value today. Otherwise, an investor could make an arbitrage profit

Put-Call Parity

(Cont'd)

Accordingly, we have the price equality:

$$p_{call} + PV(X) = p_{put} + p_{underlier} \quad (1)$$

where:

- p_{call} is the current market value of the call;
- $PV(X)$ is the present value of the strike price, X;
- p_{put} is the current market value of the put;
- $p_{underlier}$ is the current market value of the underlying stock.

Note: "Current" refers to Period 0 since you are evaluating today prices

Put-Call Parity: An example

- We have priced a European Call option that gives the holder a right to Buy the Stock at Period 2 at the Exercise Price, $X = 1.10$. We found its price to be $\mathbf{p}_{\text{Call}} = 0.0816$.
- Consider a European Put option that gives the holder a right to sell the Stock at Period 2 at the Exercise Price, $X = 1.10$.

The cash flow associated with the Put option:

$$\mathbf{c} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0.1784 \end{pmatrix}'$$

The atomic prices are still the same:

$$\begin{array}{cc} & \begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\ \mathbf{p}_{atom} = & \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix} \end{array}$$

The value of the Put option is:

$$\mathbf{p}_{\text{Put}} = \mathbf{p}_{atom} \cdot \mathbf{c} = 0.4444 \cdot 0.1784 = 0.0793$$

Put-Call Parity: An example (cont'd)

According to the Put-Call parity we have

$$p_{call} + PV(X) = p_{put} + p_{underlier}$$

Notice that $PV(X) = df(2) \cdot X$, where $df(2)$ is the discount factor for Period 2. $df(2)$ is the present value of one certain dollar received at Period 2. It must equal to the sum of atomic security prices for states: gg, gb, bg and bb.

$$df(2) = 0.0816 + 0.1904 + 0.1904 + 0.4444 = 0.9070$$

$$PV(X) = df(2) \cdot X = 0.9070 \cdot 1.10 = 0.9977$$

Therefore

$$\begin{aligned} p_{call} &= p_{put} + p_{underlier} - PV(X) \\ &= 0.0793 + 1 - 0.9977 = 0.0816 \end{aligned}$$

This is the same value as the one we found before.