Lesson 4: Multiple Periods: Option Pricing and Put-call Parity

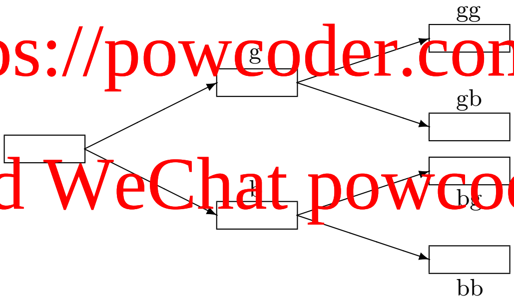
Economics of Finance

School of Economics, UNSW

Extending into multi-periods

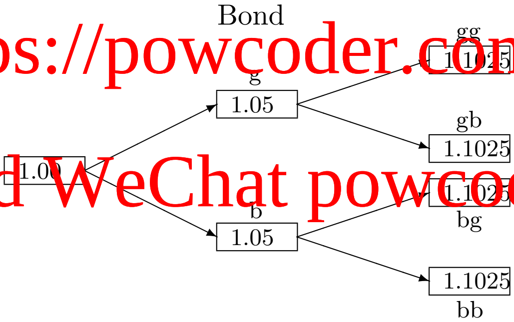
Time: Present (time 0); Future time periods (times 1 and 2)

State: Two possible realizations of uncertainty: good times and bad times



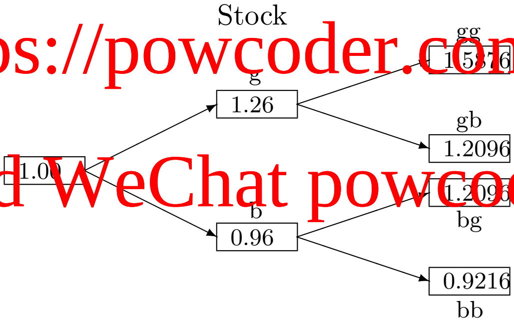
The Bond

- Two-period zero-coupon bond (no coupon payments)
- Its initial value is \$1.00.
- Its price increases 5% of its prior value in every period.



The Stock

- Its initial value is \$1.00. It pays no dividends.
- Its price increases 26% of its prior value in good times.
- Its price falls to 96% of its prior value in bad times.



Security revisited

The number of future states of the world equals six

- But seems we have only two ”securities”, bond and stock.
- What can we do?
- Recall our definition of ”security”, i.e. state contingent contract.

- Now that the time span has been extended into more than one periods, we need to extend the security space to accommodate them.

How?

Planned Acquisitions

Consider the following set of *planned acquisitions*

B0: Buy a Bond at period 0, sell it at the end of the next period;

S0: Buy a Stock at period 0, sell it at the end of the next period;

Bg: At period 1, if the state is **g**, buy a Bond, sell it at the end of the next period;

Sg: At period 1, if the state is **g**, buy a Stock, sell it at the end of the next period;

Bb: At period 1, if the state is **b**, buy a Bond, sell it at the end of the next period;

Sb: At period 1, if the state is **b**, buy a Stock, sell it at the end of the next period.

Matrix Notation

We write down the payment of these acquisitions in a matrix:

$$
\mathbf{Q} = \begin{array}{c c}
\begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} & \\
\begin{pmatrix}
1.05 & 1.26 & -1 & -1 & 0 & 0 \\
1.05 & 0.96 & 0 & 0 & -1 & -1 \\
0 & 0 & 1.05 & 1.26 & 0 & 0 \\
0 & 0 & 1.05 & 0.96 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.05 & 1.26 \\
0 & 0 & 0 & 0 & 1.05 & 0.96
\end{pmatrix} & \begin{array}{c} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array}
\end{array}
$$

Remarks

- A set of 6 column vectors;
- Each presenting a payment stream for a planned acquisition;

- Notice they are linearly independent, $det(\mathbf{Q}) \neq 0$
- Such linearly independent vector set is not unique, just like bond and stock is not the unique set of linearly independent securities in the one period world.

Price Vector

Price Vector entails:

$$\begin{array}{cccccccc} & & B0 & S0 & Bg & Sg & Bb & Sb \\ \mathbf{p}_S & = & (1.00 & 1.00 & 0.0 & 0.0 & 0.0 & 0.0) \end{array}$$

- Why are the strategies Bg, Sg, Bb, and Sb priced as 0?
- Any insights on the meaning of "price"?

Pricing a state

To price a unity of payment at each state, we can now use the formula we are familiar with: $\mathbf{p}_{atom} = \mathbf{p}_S \cdot \mathbf{Q}^{-1}$:

$$\begin{array}{cc} & \begin{array}{cccccc} g & b & gg & gb & bg & bb \end{array} \\ \mathbf{p}_{atom} = & \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix} \end{array}$$

Wrap up

- Extending time span necessarily extends the space of states;
- This, however, does not necessarily mean we need more than two securities;
- Instead, by manipulating with existing securities in various periods, we expand the *action space*;
- These actions creates linearly independent planned acquisitions. We call them “elementary strategies”;
- Each state can be price in a similar way to the atomic securities;
- Notice the set of elementary strategies may not be unique.

Definition

- Call option vs. Put option:
  - Call option: entitles the right to *buy* an underlying asset (say shares, foreign currency or commodity) at a specified *strike price, or, exercise price(X)*.
  - Put option: entitles the right to *sell* the underlying asset at a specified *strike price* X.
- European option vs. American Option
  - European put or call option: can be exercised only on expiry date.
  - American put or call option: can be exercised exercised on any date up to and including its expiration date.

Call option payment

Example: Consider a *call option* that entitles the right to buy the stock at $55. Strike price: $X = \$55$

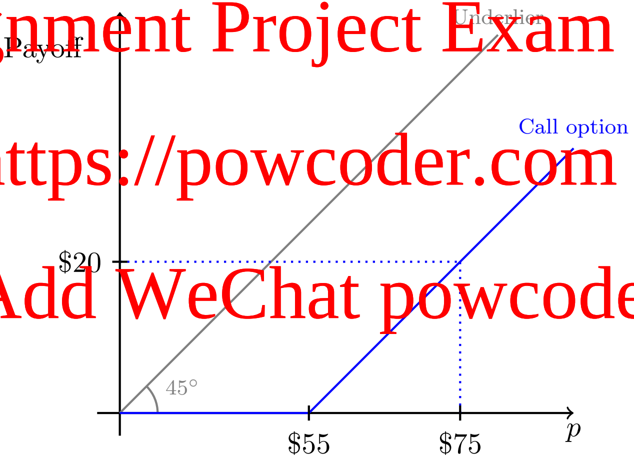
- Case 1: If the actual stock price is less than the strike price, $p < X$, then the option holder will *not* exercise the Call option. The payoff of exercising this Call option would be zero.

- Case 2: If the actual stock price in a year is more than the strike price, $p > X$, then it pays to exercise the Call option.

For example, if $p = \$75$, then the Call option's payoff if exercised is $75 - 55 = \$20$.
Note: No need to actually buy the stock to receive this payoff.

Long Call Payoff $= Max\{p - X, 0\}$

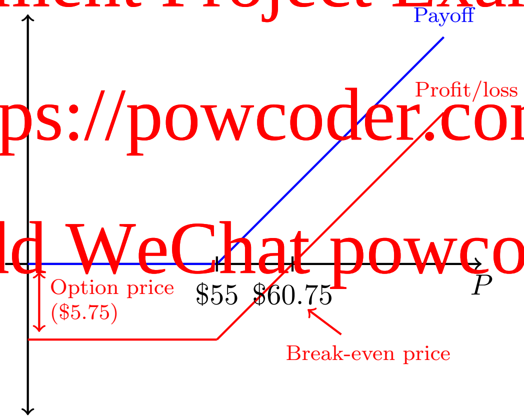
We plot the payoff of a call option with a given strike price as a function of price of the *underlying security ("underlier")*.



Overall profit

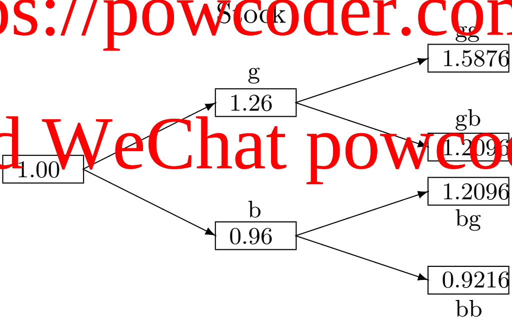
The overall profit/loss will also include the price of the option.

For example, if Call option price is $5.75.



The Setup: Three Period Binomial Model

- Two-period zero-coupon bond with initial value of \$1.00. Its price increases 5% of its prior value in every period.
- The Stock pays no dividends. Its initial value is \$1.00.
- Its price increases 26% of its prior value in good times.
- Its price falls to 96% of its prior value in bad times.



Computing atomic (state) prices

- The Payment Matrix:

$$
\mathbf{Q} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccccc}
\text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb}
\end{array}
$$

$$
\mathbf{Q} = \begin{pmatrix}
1.05 & 1.26 & -1 & -1 & 0 & 0 \\
1.05 & 0.96 & 0 & 0 & -1 & -1 \\
0 & 0 & 1.05 & 1.26 & 0 & 0 \\
0 & 0 & 1.05 & 0.96 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.05 & 1.26 \\
0 & 0 & 0 & 0 & 1.05 & 0.96
\end{pmatrix}
\begin{array}{c} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array}
$$

- The Price Vector:

$$
\begin{array}{cccccc}
\text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb}
\end{array}
$$

$$
\mathbf{p}_S = \begin{pmatrix} 1.00 & 1.00 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}
$$

- The atomic prices $\mathbf{p}_{atom} = \mathbf{p_S} \cdot \mathbf{Q}^{-1}$:

$$
\begin{array}{cccccc}
\text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb}
\end{array}
$$

$$
\mathbf{p}_{atom} = \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix}
$$

Alternative and better way to compute atomic (state) prices

- The Payment Matrix:

$$\mathbf{Q} = \begin{array}{c} \begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} \\ \begin{pmatrix} 1.05 & 1.26 & -1.05 & -1.26 & 0 & 0 \\ 1.05 & 0.96 & 0 & 0 & -1.05 & -0.96 \\ 0 & 0 & 1.1025 & 1.5876 & 0 & 0 \\ 0 & 0 & 1.1025 & 1.2096 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.1025 & 1.2096 \\ 0 & 0 & 0 & 0 & 1.1025 & 0.9216 \end{pmatrix} \end{array} \begin{array}{c} \\ \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array}$$

- The Price Vector:

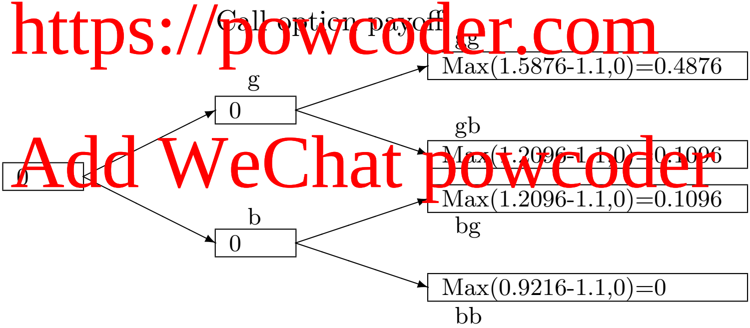
$$\mathbf{p}_S = \begin{array}{c} \begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} \\ \begin{pmatrix} 1.00 & 1.00 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix} \end{array}$$

- The atomic prices $\mathbf{p}_{atom} = \mathbf{p_S} \cdot \mathbf{Q}^{-1}$:

$$\mathbf{p}_{atom} = \begin{array}{c} \begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\ \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix} \end{array}$$

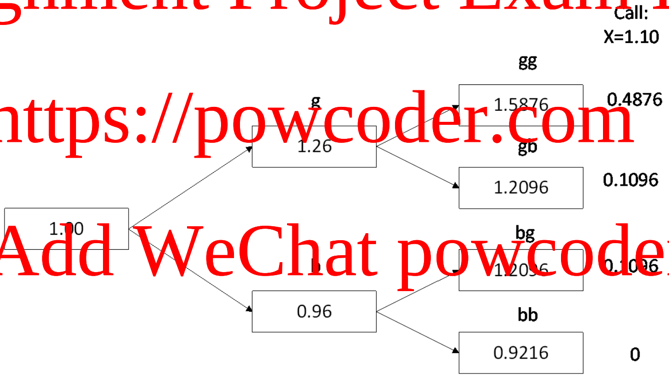
European Call Option

The matrix c can be derived from the payoff of call options at the end of each state by using Max (S-X,0) where X is given by $1.1 in the example. Since the European call option will be likely exercised at T=2 (i.e:expiry date), payoff at T=1 (ie: **g** and **b** states) will be **zero.**



Example: European Call Option

Consider a European Call option that gives the holder a right to buy the Stock at Period 2 at the Exercise Price, $X = 1.10$.



Pricing a European Call Option

The cash flow associated with the Call option:

$$\mathbf{c} = \begin{array}{cc} \begin{pmatrix} 0 \\ 0 \\ 0.4876 \\ 0.1096 \\ 0.1096 \\ 0 \end{pmatrix} & \begin{array}{c} \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array} \end{array}$$

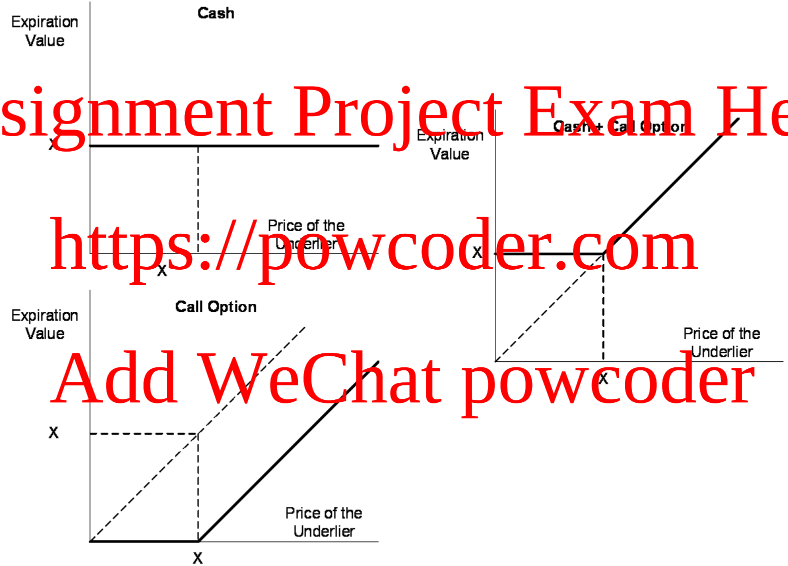
The atomic prices are still the same:

$$\mathbf{p}_{atom} = \begin{array}{c} \begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\ \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix} \end{array}$$

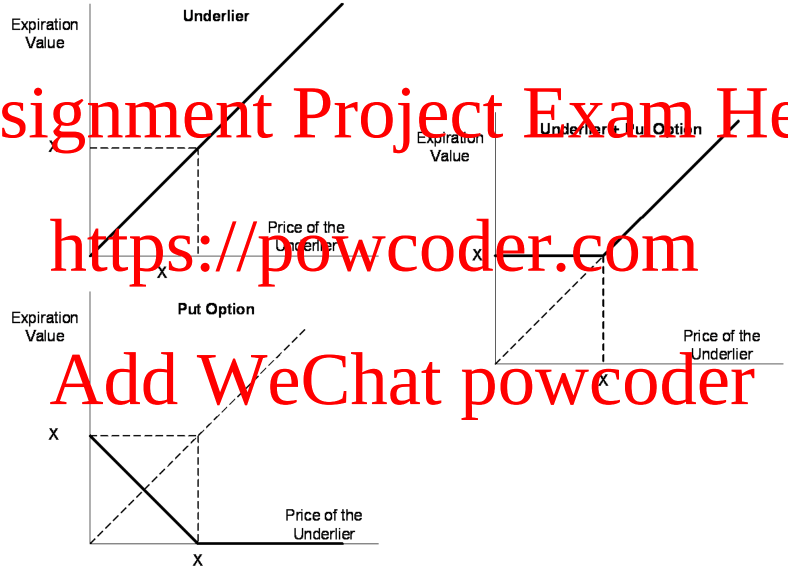
The value of the Call option is:

$$\mathbf{p}_{\text{Call}} = \mathbf{p}_{atom} \cdot \mathbf{c} = 0.0816$$

Put-Call Parity: Cash and Call



Put-Call Parity: Underlier and Put



Put-Call Parity

- The two portfolios (call + cash and put + underlier) have identical expiration values.
- If the two portfolios are going to have the same value at expiration, then they must have the same value today. Otherwise, an investor could make an arbitrage profit
- Therefore, we have the price equality

$$p_{call} + PV(X) = p_{put} + p_{underlier} \tag{1}$$

where:

  - p_{call} is the current market value of the call;
  - $PV(X)$ is the present value of the strike price, X;
  - p_{put} is the current market value of the put;
  - $p_{underlier}$ is the current market value of the underlying stock.