L4 Optional Reading: Why are American Options RARELY early exercised?

Economics of Finance

School of Economics, UNSW

Binomial Model

Binomial states: $s \in \{u, d\}$, where u : stock price goes up, d : stock price goes down, e.g., $u = 1.3$, $d = 0.6$.

Atomic security prices: $\{p\ q\}$, where $p \to u$, $q \to d$.

Interest rate: r

Arbitrage free implies:

$$\begin{aligned} p + q &= \frac{1}{1+r}; \\ up + dq &= 1 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} p &= \frac{1+r-d}{(1+r)(u-d)}; \\ q &= \frac{u-1-r}{(1+r)(u-d)} \end{aligned}$$

Options

Option: strike price K, stock price S;

Intrinsic value (IV): the value of exercising immediately;

Time value (TV): the value of holding the option until next period.

American option holder solves $\max\{IV, TV\}$, i.e.,

$$\begin{cases} IV > TV: & \text{early excercise;} \\ IV \leq TV: & \text{hold} \end{cases}$$

Call Option: $IV = S - X$

Case I: $dS \geqslant X$:

$$\begin{aligned} TV &= p(uS - X) + q(dS - X) \\ &= (up + dq)S - (p + q)X \\ &= S - \frac{X}{1+r} > IV; \end{aligned}$$

Case II: $dS < X$: $IV = p(dS - X)$.

$$\begin{aligned} TV - IV &= p(uS - X) - (S - X) \\ &= (up - 1)S + (1 - p)X \\ &= (1 - p)X - dqS \\ &> (1 - p - q)X > 0; \end{aligned}$$

Sum up, $TV > IV$ for either Case I or Case II. Early exercise is never preferable for call option.

Put option: $IV = X - S$

Case 1: $uS \leqslant X$, or $S \leqslant \frac{X}{u}$:

$$\begin{aligned} TV &= p(X - uS) + q(X - dS) \\ &= (up + dq)S - (p + q)X \\ &= \frac{X}{1+r} - S < IV; \end{aligned}$$

Early exercise is preferable;

Put option (cont')

Case II: $uS > X$, or $S > \frac{X}{u}$: $TV = q\left(X - dS\right)$.

$$\begin{aligned} TV - IV &= q\left(X - dS\right) - \left(X - S\right) \\ &= \left(1 - dq\right) S - \left(1 - q\right) X \\ &= upS - \left(1 - q\right) X; \end{aligned}$$

Let $\tilde{S} = \frac{(1-q)X}{up} = \left(1 + \frac{r(a-d)}{1+r-d}\right) \frac{X}{u} > \frac{X}{u}$, then

$$TV \begin{cases} > IV, & \text{if } S \geqslant \tilde{S} \\ < IV, & \text{if } \frac{X}{u} < S < \tilde{S} \end{cases}$$

Sum up, early exercise is preferable only if $S < \tilde{S}$, i.e., when the put option is deeply in the money.