Economics of Finance

Tutorial 6

1. (A-D Economy with Representative Agent) Consider a world in which there are only two dates: 0 and 1. At date 1 there are two possible states of nature: a bad weather state (B) and a good weather state (G). The B state occurs with probability $\pi_B = 0.3$, while the G state occurs with probability $\pi_G = 0.7$.

There is one non-storable consumption good – say apples. The endowment of apples at time 0, e_0, is 1. At time 1 the endowment of apples is state dependent. In the G state, the endowment, e_G, is 2. In the B state, the endowment, e_B, equals 0.7.

There is one representative consumer in this economy. The consumer's preferences over apples are given by

$$u(c_0) + \beta \sum_{s_1 \in \{B,G\}} \pi_{s_1} u(c_{s_1}),$$

where the instantaneous utility function is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma = 2$. The consumer's time discount factor, β, is 0.98.

In this economy the only traded securities are atomic (Arrow-Debreu) securities. One unit of 'G security' sells at time 0 at a price q_G and pays one unit of consumption at time 1 if state 'G' occurs and nothing otherwise. One unit of 'B security' sells at time 0 at a price q_B and pays one unit of consumption in state 'B' only.

(a) Write down the consumer's set of budget constraints.

Solution

Let a_G denote the quantity of Arrow securities obtained at time 0 what pay 1 unit of consumption at time 1 in GW state and nothing otherwise. The budget constraints can be written as follows:

Time 0:

$$c_0 + q_G \cdot a_G + q_B \cdot a_B = e_0, \tag{1}$$

Time 1:

$$c_G = 1 \cdot a_G + e_G, \tag{2}$$

$$c_B = 1 \cdot a_B + e_B. \tag{3}$$

In more general notation (1) reads as

$$c_0 + \sum_{s_1 \in \{G,B\}} q_{s_1} \cdot a_{s_1} = e_0, \tag{4}$$

while constraints (2)-(3) can be generalised to

$$c_{s_1} = a_{s_1} + e_{s_1}, \text{ for all } s_1 \in \{G, B\}. \tag{5}$$

(b) Define a *Market Equilibrium* in this economy. Is there any trade of atomic (Arrow-Debreu) securities possible in this economy?

Solution

A *Market Equilibrium* in this economy is defined as an allocation c_0, c_G, c_B, a_G, a_B and prices q_G, q_B, such that: (1) given the prices, the allocation solves

$$\max u(c_0) + \beta \sum_{s_1 \in S_1} \pi_{s_1} u(c_{s_1}),$$

subject to

$$c_0 + \sum_{s_1 \in \{G,B\}} q_{s_1} \cdot a_{s_1} = e_0,$$

$$c_{s_1} = a_{s_1} + e_{s_1}, \text{ for all } s_1 \in \{G, B\};$$

(2) Markets clear in every period and state:

$$c_0 = e_0,$$
$$c_{s_1} = e_{s_1}, \text{ for all } s_1 \in \{G, B\}.$$

Note: since all agents are the same in this economy (represented by one representative agent) no trade is possible. This is also obvious when you compare budget constraints with market clearing condition.

(c) Write down the Lagrangian for the consumer's optimisation problem and find the first order necessary conditions.

Solution
The Lagrangian for the consumer's problem can be written as

$$\begin{aligned}\mathcal{L} &= u(c_0) + \beta \sum_{s_1 \in \{G,B\}} \pi_{s_1} u(c_{s_1}) \\ &\quad - \lambda_0 \left(c_0 + \sum_{s_1 \in \{G,B\}} q_{s_1} \cdot a_{s_1} - e_0 \right) \\ &\quad - \sum_{s_1 \in \{G,B\}} \lambda_{s_1} \left(c_{s_1} - a_{s_1} - e_{s_1} \right).\end{aligned}$$

To obtain the first order conditions we must differentiate the Lagrangian with respect to c_0, $\{c_{s_1}\}_{s_1 \in \{G,B\}}$, and $\{a_{s_1}\}_{s_1 \in \{G,B\}}$ and equate the derivatives to zero. The corresponding optimality conditions are

$$u'(c_0) - \lambda_0 = 0, \tag{6}$$

$$\beta \pi_{s_1} u'(c_{s_1}) - \lambda_{s_1} = 0, \text{ for } s_1 \in \{G, B\}, \tag{7}$$

$$-\lambda_0 q_{s_1} + \lambda_{s_1} = 0, \text{ for } s_1 \in \{G, B\}. \tag{8}$$

From (13) we get

$$q_{s_1} = \frac{\lambda_{s_1}}{\lambda_0} \tag{9}$$

Substituting the expressions for the multipliers λ_0 and λ_{s_1} from (11) and (12) into (14) we obtain the pricing equation for the Arrow securities:

$$q_{s_1} = \beta \pi_{s_1} \frac{u'(c_{s_1})}{u'(c_0)}. \tag{10}$$

(d) Find the prices of the atomic (Arrow-Debreu) securities. What is the one-period discount factor and risk-free interest rate in this economy?

Solution
Since market clearing requires that $c_0 = e_0$, and $c_{s_1} = e_{s_1}$, for $s_1 = G, B$, the prices of the Arrow securities must satisfy

$$q_{s_1} = \beta \pi_{s_1} \frac{u'(e_{s_1})}{u'(e_0)}, \text{ for all } s_1 \in \{G, B\}.$$

Since we have only two possible states at time 1, there are only two Arrow securities to price:

$$q_G = \beta \pi_G \frac{u'(e_G)}{u'(e_0)}$$

$$q_B = \beta \pi_B \frac{u'(e_B)}{u'(e_0)}$$

Since the instantaneous utility is $u(c) = -1/c$, the marginal utility of consumption is $u'(c) = 1/c^2$. Therefore,

$$q_G = \beta \pi_G \left(\frac{e_0}{e_G}\right)^2 = 0.98 \cdot 0.7 \cdot \left(\frac{1}{2}\right)^2 = 0.1715$$
$$q_B = \beta \pi_B \left(\frac{e_0}{e_B}\right)^2 = 0.98 \cdot 0.3 \cdot \left(\frac{1}{0.7}\right)^2 = 0.6$$

The one period discount factor, df is the sum of the corresponding Arrow securities

$$\begin{aligned} df &= q_G + q_B \\ &= 0.1715 + 0.6 = 0.7715 \end{aligned}$$

One period riskless interest rate, i, is therefore

$$i = \frac{1}{df} - 1 = \frac{1}{0.7715} - 1 = 0.2962,$$

or 29.62%. Hopefully not what the RBA has in mind :)

(e) How is the financial market discount factor, df different from consumer's time preferences discount factor, β?

Solution

df represents the ratio the agent converts future apple to present apple with certainty. β on the other hand, represents how the agent values future utility generated by future consumption. From equation (10), df contains β, while also incorporating physical probability and the ratio between marginal utilities as well.

2. (General Equilibrium A-D Economy) Consider a world in which there are only two dates: 0 and 1. At date 1 there are two possible states of nature: a bad weather state (B) and a good weather state (G). That is: $s_1 \in S_1 = \{G, B\}$, while the state at date zero is known: call it s_0 if you will. The B state occurs with probability $\pi_B = 1/3$, while the G state occurs with probability $\pi_G = 2/3$.

There is one non-storable consumption good - say apples. There are two consumers in this economy. Their preferences over apples are exactly the same and are given by the following expected utility function

$$\frac{1}{2} c_0^k + \beta \sum_{s_1 \in S_1} \pi_{s_1} \ln\left(c_{s_1}^k\right),$$

where subscript $k = 1, 2$ denotes consumers. The consumer's time discount factor, β, is 0.9.

The consumers differ in their endowments which are given in the table below:

Consumers	Endowments		
	$t = 0$	t=1	t=1
		B	G
Consumer 1	10	1	2
Consumer 2	5	4	6

In this economy the only traded securities are basic Arrow securities. One unit of 'G security' sells at time 0 at a price q_G and pays one unit of consumption at time 1 if state 'G' occurs and nothing otherwise. One unit of 'B security' sells at time 0 at a price q_B and pays one unit of consumption in state 'B' only.

i) Write down the consumer's budget constraint including amounts of Arrow securities purchased a_{s_1}, where $s_1 \in S_1$ for all times and states.

Solution
Let a_G^k denote the quantity of Arrow securities obtained at time 0 by consumer k which provides 1 unit of consumption at time 1 in G state and nothing otherwise. The budget constraints can be written as follows:

Time 0:

$$c_0^1 + q_G \cdot a_G^1 + q_B \cdot a_B^1 = e_0^1,$$
$$c_0^2 + q_G \cdot a_G^2 + q_B \cdot a_B^2 = e_0^2.$$

Time 1:

$$c_B^1 = 1 \cdot a_B^1 + e_B^1,$$
$$c_G^1 = 1 \cdot a_G^1 + e_G^1,$$
$$c_B^2 = 1 \cdot a_B^2 + e_B^2,$$
$$c_G^2 = 1 \cdot a_G^2 + e_G^2.$$

ii) Define a *Market Equilibrium* in this economy. Is there any trade of Arrow securities possible in this economy?

Solution
A *Market Equilibrium* in this economy is defined as an allocation $c_0^k, \{c_{s_1}^k\}_{s_1 \in S_1}, \{a_{s_1}^k\}_{s_1 \in S_1}$ and prices $\{q_{s_1}\}_{s_1 \in S_1}$ such that: (1) given the prices $\{q_{s_1}\}_{s_1 \in S_1}$ the allocation solves

$$\max \frac{1}{2} c_0^k + \beta \sum_{s_1 \in s_1} \pi_{s_1} \ln\left(c_{s_1}^k\right),$$

subject to

$$c_0^k + \sum_{s_1 \in S_1} q_{s_1} \cdot a_{s_1}^k = e_0^k,$$

$$c_{s_1}^k = a_{s_1}^k + e_{s_1}^k, \text{ for all } s_1 \in S_1 \text{ and for all consumers } i;$$

(2) Markets clear in every period and state:

$$\sum_k c_0^k = \sum_k e_0^k,$$
$$\sum_k c_{s_1}^k = \sum_k e_{s_1}^k, \text{ for all } s_1 \in S_1.$$

iii) Write down the Lagrangian for the consumer's optimisation problem and find the first order necessary conditions.

Solution
The Lagrangian for the consumer's problem can be written as

$$\begin{aligned}
\mathcal{L} = &\frac{1}{2} c_0^k + \beta \sum_{s_1 \in s_1} \pi_{s_1} \ln\left(c_{s_1}^k\right) \\
&- \lambda_0^k \left(c_0^k + \sum_{s_1 \in S_1} q_{s_1} \cdot a_{s_1}^k - e_0^k \right) \\
&- \sum_{s_1 \in S_1} \lambda_{s_1}^k \left(c_{s_1}^k - a_{s_1}^k - e_{s_1}^k \right).
\end{aligned}$$

To obtain the first order conditions we must differentiate the Lagrangian with respect to c_0^k, $\left\{c_{s_1}^k\right\}_{s_1 \in S_1}$, and $\left\{a_{s_1}^k\right\}_{s_1 \in S_1}$ and equate the derivatives to zero. The corresponding optimality conditions are

$$\frac{1}{2} - \lambda_0^k = 0, \tag{11}$$

$$\beta \pi_{s_1} \frac{1}{c^k} - \lambda_{s_1}^k = 0, \text{ for } s_1 \in S_1, \tag{12}$$

$$-\lambda_0^k q_{s_1} + \lambda_{s_1}^k = 0, \text{ for } s_1 \in S_1. \tag{13}$$

From (13) we get

$$q_{s_1} = \frac{\lambda_{s_1}^k}{\lambda_0^k} \tag{14}$$

Substituting the expressions for the multipliers λ_0^k and $\lambda_{s_1}^k$ from (11) and (12) into (14) we obtain the system of consumers' equilibrium pricing equations for the Arrow securities:

$$q_{s_1} = \beta \pi_{s_1} \frac{2}{c_{s_1}^1},$$

$$q_{s_1} = \beta \pi_{s_1} \frac{2}{c_{s_1}^2}.$$

iv) Find the prices of the atomic (Arrow-Debreu) securities and quantities traded.

Solution

In order to find the Arrow securities prices we need to combine the above consumers' equilibrium pricing equations with the market clearing conditions, i.e.,

$$c_0^1 + c_0^2 = 16,$$
$$c_B^1 + c_B^2 = 5,$$
$$c_G^1 + c_G^2 = 8.$$

We need to solve the following system of equations:

$$q_B = 0.9 \frac{1}{3} \frac{2}{c_B^1},$$
$$q_B = 0.9 \frac{1}{3} \frac{2}{c_B^2},$$
$$q_G = 0.9 \frac{2}{3} \frac{2}{c_G^1},$$
$$q_G = 0.9 \frac{2}{3} \frac{2}{c_G^2},$$
$$c_B^1 + c_B^2 = 5,$$
$$c_G^1 + c_G^2 = 8,$$

which transforms into the following system after diving the corresponding $q(\cdot)$ equations:

$$c_B^1 = c_B^2,$$
$$c_G^1 = c_G^2,$$
$$c_B^1 + c_B^2 = 5,$$
$$c_G^1 + c_G^2 = 8.$$

Hence, $c_B^1 = c_B^2 = 2.5$ and $c_G^1 = c_G^2 = 4$

Therefore,

$$q_B = 0.9\frac{1}{3}\frac{2}{2.5} = 0.24,$$
$$q_G = 0.9\frac{2}{3}\frac{2}{4} = 0.3$$

Quantities of the Arrow securities traded can be found from the budget constrains

Time 1:

$$c_B^1 = 1 \cdot a_B^1 + 1,$$
$$c_B^2 = 1 \cdot a_B^2 + 4,$$
$$c_G^1 = 1 \cdot a_G^1 + 2,$$
$$c_G^2 = 1 \cdot a_G^2 + 6.$$

Hence,

$$a_B^1 = 1.5,$$
$$a_B^2 = -1.5,$$
$$a_G^1 = 2,$$
$$a_G^2 = -2.$$

Time 0 consumption can also be found from the budget constrains

Time 0:

$$c_0^1 + 0.3 \cdot 2 + 0.24 \cdot 1.5 = 10,$$
$$c_0^2 + 0.3 \cdot (-2) + 0.24 \cdot (-1.5) = 5.$$

Hence, $c_0^1 = 9.04$ and $c_0^2 = 5.96$.

v) Find the risk neutral probabilities, and the stochastic discount factors of this economy. What does your results suggest?

Solution

Denote discount factor as df. Forward prices are given by:

$$f_{s_1} = \frac{q_{s_1}}{df} = \left(\frac{24}{54}, \frac{30}{54}\right) = \left(\frac{4}{9}, \frac{5}{9}\right) = (0.4444, 0.5556)$$

while stochastic discount factors are:

$$m_{s_1} = \frac{q_{s_1}}{\pi_{s_1}} = \left(\frac{0.24}{0.3333}, \frac{0.30}{0.6667}\right) = (0.72, 0.45)\,;$$

realise $f_G < \pi_G$ and $f_B > \pi_B$ while $m_G < df$ and $m_B > df$. This suggests that agents in the economy are risk averse. Notice that $(\ln(c))'' = -\frac{1}{c^2}$, this indeed match with our intuition.

vi) Suppose that trade is not allowed in the economy. Compare the utilities of consumer without trade and with trade. Is there a Pareto improvement due to trade?

Solution

In case of no trade the only consumption possible is derived from the corresponding endowments of the consumers. By inputing the endowments into the expected utility equation we find that $EU^1 = 5.4159$, $EU^2 = 3.9909$. While if we allow for trade the expected utilities become $EU^1 = 5.6267$, $EU^2 = 4.0867$. The utilities of both consumers have increased after trade, which means that trade leads to the Pareto improvement.

To see this, we compare allocations before trade and after trade. Before trading, recall endowment table:

Consumers	Endowments		
	$t = 0$	t=1	t=1
		B	G
Consumer 1	10	1	2
Consumer 2	5	4	6

Recall expected utility formula:

$$EU = \frac{1}{2}c_0^k + \beta \sum_{s_1 \in S_1} \pi_{s_1} \ln\left(c_{s_1}^k\right),$$

Substitute allocations in endowment table:

$$EU^1 = \frac{1}{2}(10) + 0.9[\frac{1}{3}\ln(1) + \frac{2}{3}\ln(2)] = 5.4159,$$

$$EU^2 = \frac{1}{2}(5) + 0.9[\frac{1}{3}\ln(4) + \frac{2}{3}\ln(6)] = 3.9909,$$

Now consider after trading:

Consumers	Endowments		
	$t = 0$	t=1	t=1
		B	G
Consumer 1	9.04	2.5	4
Consumer 2	5.96	2.5	4

In equilibrium the consumer 1 buys 1.5 B securities and buys 2 G securities from the consumer 2. Cost of the transaction:

$$1.5 \times q_B + 2 \times q_G = 1.5 \times 0.24 + 2 \times 0.3 = 0.96.$$

Initial endowment for consumer 1 will be reduced by 0.96. After trade, new $c_0^1 = 10 - 0.96 = 9.04$. Then consumer 2 will be better by 0.96. As a result, $c_0^2 = 5 + 0.96 = 5.96$

$$EU^1 = \frac{1}{2}(9.04) + 0.9[\frac{1}{3}\ln(2.5) + \frac{2}{3}\ln(4)] = 5.6267,$$

$$EU^2 = \frac{1}{2}(5.96) + 0.9[\frac{1}{3}\ln(2.5) + \frac{2}{3}\ln(4)] = 4.0867,$$

Note: In this example, trading gains are realised through two channels: inter-temporal reallocation and risk sharing. The first channel allows agents to transfer resources from relatively abundant period to relative scarce period, while the second one allows agents to transfer risk from relatively risk aversion people to more risk liking ones.

vii) Suppose that instead of the atomic securities there are two securities, e.g. a stock and a bond, available for trade in this economy. Their payments are given by the following matrix.

$$\begin{array}{cc} & \begin{array}{cc} B & S \end{array} \\ \begin{array}{c} G \\ B \end{array} & \begin{pmatrix} 1 & 1.5 \\ 1 & 0.8 \end{pmatrix} \end{array}.$$

Find their arbitrage-free prices at time 0.

Solution

Denote the payment matrix of bond and stock as Q, realise Q is full rank, i.e., market is complete, and we can replicate any state-contingent claim, including atomic securities, with bond and stock. The replication portfolio is the inverse of matrix Q:

$$Q^{-1} = \begin{pmatrix} -1.14 & 2.14 \\ 1.43 & -1.43 \end{pmatrix},$$

i.e., atomic securities can be replicated by combinations of bonds and stocks. It takes shorting 1.14 of bond and longing 1.43 of stock to replicate good weather security, and longing 2.14 of bond and shorting 1.43 of stock gives a bad weather security.

Now that atomic securities are replicated, our previous analysis using atomic securities follows. The arbitrage-free prices of bond and stock then entail:

$$p_s = p_{atom} \times Q = (0.30\ 0.24) \times \begin{pmatrix} 1 & 1.5 \\ 1 & 0.8 \end{pmatrix} = (0.54\ 0.64).$$