Economics of Finance

Tutorial 5

1. Two securities X and Y make the following payments (for each dollar invested) in the good and bad weather states:

	X	Y
B	1.10	0.80
G	1.00	1.50

Suppose the good outcome occurs with probability 0.6 and the bad outcome with probability 0.4.

(i) Compute the expected rate of return on securities X and Y.

Solution
The return is obtained as $r = (c - p)/p = c/p - 1$. Since p is normalized to 1, $r = c - 1$.

$$E(r_X) = \text{prob}(B) \times r_X(B) + \text{prob}(G) \times r_X(G) = 0.4 \times 0.10 + 0.6 \times 0 = 0.04$$

The expected return on X is 4 %.

$$E(r_Y) = \text{prob}(B) \times r_Y(B) + \text{prob}(G) \times r_Y(G) = 0.4 \times -0.2 + 0.6 \times 0.5 = 0.22$$

The expected return on Y is 22 %.

(ii) Compute the atomic security prices.

Solution
The atomic prices are given by

$$p_{atom} = (1 \;\; 1) \begin{pmatrix} 1.10 & 0.80 \\ 1.00 & 1.50 \end{pmatrix}^{-1} = (0.5882 \;\; 0.3529).$$

(iii) Compute the risk-free rate of return. Construct a portfolio of securities X and Y that pays the same amount in the good and bad states.

Solution
The discount factor is $df = p_B + p_G = 0.9412$. Risk-free rate of return is$1/df - 1 = 0.0625$. This is because if you buy one good and bad atomic security you are guaranteed one dollar in the next period. A portfolio that pays one dollar in each state can be found as follows:

$$n = Q^{-1}c = \begin{pmatrix} 1.10 & 0.80 \\ 1.00 & 1.50 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8235 \\ 0.1176 \end{pmatrix}.$$

That is, you should invest \$0.8235 in X and \$0.1176 in Y. To obtain \$K in each state you should invest K times these amounts in each security.

(iv) Compute the expected rate of return and risk premia of the *atomic* securities.

Solution
Returns of the atomics securities in each state of the world are

$$r_B(B) = \frac{1 - 0.5882}{0.5882} = 0.7,$$

$$r_B(G) = \frac{0 - 0.5882}{0.5882} = -1,$$

$$r_G(B) = \frac{0 - 0.3529}{0.3529} = -1,$$

$$r_G(G) = \frac{1 - 0.3529}{0.3529} = 1.8337.$$

Hence, the expected return of the bad atomic security is

$$E(r_B) = \text{prob}(B) \times r_B(B) + \text{prob}(G) \times r_B(G) = 0.4 \times 0.7 - 0.6 \times 1 = -0.32$$

The risk premia on the bad atomic security is $-0.32 - 0.0625 = -0.3825$. That is, the bad atomic security has a 38.25 percent risk discount. The expected return of the good atomic security is

$$E(r_G) = \text{prob}(B) \times r_G(B) + \text{prob}(G) \times r_G(G) = -0.4 \times 1 + 0.6 \times 1.8337 = 0.7$$

The risk premia on the good atomic security is $0.7 - 0.0625 = 0.6375$. That is, the good atomic security pays a 63.75 percent risk premium.

(v) Compute the forward prices of the atomic securities (risk neutral probabilities).

Solution
The forward prices of the atomic securities (risk neutral probabilities) are

$$f_B = p_B/df = 0.5882/0.9412 = 0.625$$
$$f_G = p_G/df = 0.3529/0.9412 = 0.375$$

(vi) Compare the forward price of each atomic security with the probability of that state being observed. Why are the forward prices and associated probabilities not equal?

Solution
Notice that

$$prob(B) = 0.4 \quad f_B = 0.625$$
$$prob(G) = 0.6 \quad f_G = 0.375$$

The forward price (risk neutral probability) of a dollar in the bad state is higher than the (physical) probability of the bad state being observed since investors are risk averse and dollars are scarcer (and hence more valuable) in the bad state than in the good state.

2. Consider the portfolio Z that makes the following payments in four different states (VB, B, G, VG). You are also given (physical) probabilities and forward prices (risk neutral probabilities) of each state

	c	prob	f
VB	60	0.3	0.5
B	10	0.2	0.3
G	10	0.1	0.1
VG	50	0.4	0.1

Suppose the risk-free rate of return is 5 percent.

(i) Compute the risk premium of portfolio Z.

Solution

$$\text{risk premium} = \left(\frac{ev}{fv} - 1\right)(1+i)$$

$ev = \text{prob} \cdot c = 0.3 \times 60 + 0.2 \times 10 + 0.1 \times 10 + 0.4 \times 50 = 41.$
$fv = f \cdot c = 0.5 \times 60 + 0.5 \times 10 + 0.1 \times 10 + 0.1 \times 50 = 39.$

$$\Rightarrow \text{ risk premium} = \left(\frac{41}{39} - 1\right)(1+0.05) = 0.05385$$

The risk premium of Z is 5.385 percent.

(ii) Compute the risk premia of the four atomic securities.

Solution

$$\text{risk premium}_{VB} = \left(\frac{\text{prob}_{VB}}{f_{VB}} - 1\right)(1+i) = \left(\frac{0.3}{0.5} - 1\right)(1+0.05) = -0.42$$

The VB atomic security has a risk discount of 42 percent.

$$\text{risk premium}_{B} = \left(\frac{\text{prob}_{B}}{f_{B}} - 1\right)(1+i) = \left(\frac{0.2}{0.3} - 1\right)(1+0.05) = -0.35$$

The B atomic security has a risk discount of 35 percent.

$$\text{risk premium}_{G} = \left(\frac{\text{prob}_{G}}{f_{G}} - 1\right)(1+i) = \left(\frac{0.1}{0.1} - 1\right)(1+0.05) = 0$$

The G atomic security has a zero risk premium.

$$\text{risk premium}_{VG} = \left(\frac{\text{prob}_{VG}}{f_{VG}} - 1\right)(1+i) = \left(\frac{0.4}{0.1} - 1\right)(1+0.05) = 3.15$$

The VG atomic security has a risk premium of 315 percent.

(iii) Will the *market* portfolio pay a risk premium in this case? Explain.

Solution
Assuming that for the market portfolio $c_{VG} > c_G > c_B > c_{VB}$, then it will pay a risk premium. This is because dollars in the VG state are worth less than dollars in the G state, which are worth less than dollars in the B state, which are worth less than dollars in the VB state, and the market portfolio pays out more in the states where dollars are worth less. The risk premium compensates investors for the fact that it pays out disproportionately in states where dollars are worth less.

3. (Expected utility) Suppose you are faced with the following gamble scenario:

- Consume 6000 with probability 0.4
- Consume 3000 with probability 0.6.

Suppose further that your utility function is:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where $\gamma = 1/2$. (Later will learn that γ is the coefficient of relative risk aversion. The higher γ, the more you dislike risk)

(a) What is your expected utility?

Solution
Notice that with $\gamma = 1/2$ the utility function becomes $U(c) = \frac{c^{1-1/2}}{1-1/2} = 2\sqrt{c}$. The expected utility of this gamble is therefore

$$\begin{aligned} EU &= \pi(c = 6000) \cdot U(6000) + \pi(c = 3000) \cdot U(3000) \\ &= 0.4 \cdot 2\sqrt{6000} + 0.6 \cdot 2\sqrt{3000} = 127.69 \end{aligned}$$

(b) What is your expected consumption?

Solution

$$\begin{aligned} E(c) &= \pi(c = 6000) \cdot 6000 + \pi(c = 3000) \cdot 3000 \\ &= 0.4 \cdot 6000 + 0.6 \cdot 3000 = 4200 \end{aligned}$$

(c) What is your attitude towards risk?

Solution
Compute utility associated with the expected consumption:

$$U(E(c)) = 2\sqrt{4200} = 129.61 > EU_{gamble},$$

i.e. with utility of consuming the amount of the expected consumption with certainty is higher than the expected utility of the risky gamble with the same expected consumption. This means you are *risk averse*.

(d) Certainty equivalent, CE, refers to the guaranteed amount of consumption that an individual would view as equally desirable as a risky gamble, that is $EU_{\text{gamble}} = u(CE)$. Compute the certainty equivalent of the gamble.

Solution
The certainty-equivalent c_{ce} is defined as

$$U(c_{ce}) = \pi(c = 6000) \cdot U(6000) + \pi(c = 3000) \cdot U(3000)$$

Hence

$$2\sqrt{c_{ce}} = 0.4 \cdot 2\sqrt{6000} + 0.6 \cdot 2\sqrt{3000},$$

which implies that $c_{ce} = 4076.4$.

Notice that the certainty equivalent is lower than the expected consumption. Therefore, the agent in question is risk-averse.

(e) Suppose now that the coefficient of relative risk aversion is $\gamma = 2$. Answer to the questions (a)–(d) above for with $\gamma = 2$. Are agents more or less tolerant to risk than before?

Solution
Notice that with $\gamma = 2$ the utility function becomes $U(c) = \frac{c^{1-2}}{1-2} = -\frac{1}{c}$. The expected utility of this gamble is therefore

$$\begin{aligned} EU &= \pi(c = 6000) \cdot U(6000) + \pi(c = 3000) \cdot U(3000) \\ &= 0.4 \cdot \frac{-1}{6000} + 0.6 \cdot \frac{-1}{3000} = -2.6667 \times 10^{-4} \end{aligned}$$

To see what happened to the agents attitude to risk, let us compute his/her certainty equivalent of the gamble. The certainty-equivalent c_{ce} is defined as

$$U(c_{ce}) = \pi(c = 6000) \cdot U(6000) + \pi(c = 3000) \cdot U(3000)$$

Hence

$$\frac{-1}{c_{ce}} = 0.4 \cdot \frac{-1}{6000} + 0.6 \cdot \frac{-1}{3000} = -2.6667 \times 10^{-4}$$

which implies that $c_{ce} = \frac{-1}{-2.6667 \times 10^{-4}} = 3750.0$. The agent is now willing to accept a smaller amount for sure instead of a risky gamble. The agent is more risk averse now ($\gamma = 2$) than in the first part of the exercise ($\gamma = 1/2$). This result is general. The agents become less tolerant to risk as their RRA coefficient γ is increasing.

4. (The Role of Finance) Consider an economy in which a representative agent lives for two periods, year 0 and year 1. The representative agent derives utility from consumption and their time discount rate is β. Suppose there is *no uncertainty.* The agent life-time utility is given by:

$$U(c_0, c_1) = \ln(c_0) + \beta \ln(c_1),$$

The agent receives an initial endowment, e, at time zero and recieves income (say from labor) in period zero and one, y_0 and y_1, respectively. The agent can save, s, or borrow (negative s) money at interest rate i.

(a) Write down the maximization problem in detail.

Solution
The maximization problem entails:

$$\max_{\langle c_0, c_1 \rangle} \left\{ u(c_0) + \beta u(c_1) \right\},$$

$$s.t. \; a_0 + y_0 + \frac{y_1}{1+i} = c_0 + \frac{c_1}{1+i};$$

(b) Write down the Lagrangian that represents the maximization problem.

Solution
Construct the Lagrangian:

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \lambda \left\{ a_0 + y_0 + \frac{y_1}{1+i} - c_0 - \frac{c_1}{1+i} \right\};$$

(c) Derive the first order conditions.

Solution
Take FOCs of the Lagrangian:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_0} &= 0 : u'(c_0) = \lambda; \\
\frac{\partial \mathcal{L}}{\partial c_1} &= 0 : \beta u'(c_1) = \frac{\lambda}{1+i}; \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 0 : a_0 + y_0 + \frac{y_1}{1+i} = c_0 + \frac{c_1}{1+i}.
\end{aligned}$$

From the FOCs, we obtain the so-called Euler's equation:

$$u'(c_0) = \beta (1+i) \, u'(c_1);$$

(d) Interpret the trade-offs you find.

Solution
The Euler equation states that the maximization condition of the consumer's utility. Explicitly, it states that marginal utility derived from current consumption is equivalent to the marginal utility derived from future consumption weighted by the hybrid discount factor, where the hybrid discount factor contains the time preference discount factor, β, and the gross interest rate, $(1+i)$.

(e) Solve for equilibrium consumption, and saving/borrowing.

Solution
Using ln utility function, $u'(c) = \frac{1}{c}$, now Euler Equation entails:

$$\frac{c_1}{c_0} = \beta (1 + i);$$

Combine with budget constraint, we can solve:

$$\begin{aligned}
c_0^* &= \frac{1}{1+\beta}\left(a_0 + y_0 + \frac{y_1}{1+i}\right); \\
c_1^* &= \frac{\beta(1+i)}{1+\beta}\left(a_0 + y_0 + \frac{y_1}{1+i}\right);
\end{aligned}$$

Saving entails:

$$S = a_0 + y_0 - c_0;$$

in such equilibrium, saving yields:

$$\begin{aligned}
S^* &= a_0 + y_0 - c_0^*; \\
&= \frac{1}{1+\beta}\left[\beta (a_0 + y_0) - \frac{y_1}{1+i}\right];
\end{aligned}$$

(f) Suppose $y_0 = 0.4$, $e = 0.6$ and $y_1 = 3$, $\beta = 0.98$ while $i = 0.05$. Compare the welfare (utility) of equilibrium with financing options (saving/borrowing available) and without. Comment on your result.

Solution
With borrow and lending,

$$\begin{aligned}
U &= \ln c_0^* + \beta \ln c_1^* \\
&= \ln\left(\frac{1 + 3/1.05}{1.98}\right) + 0.98 \cdot \ln\left[\frac{0.98 \cdot 1.05 \cdot (1 + 3/1.05)}{1.98}\right] \\
&= 1.32;
\end{aligned}$$

without borrow and lending:

$$\begin{aligned}
U &= \ln c_0^* + \beta \ln c_1^* \\
&= \ln(1) + 0.98 \cdot \ln(3) \\
&= 1.10;
\end{aligned}$$

This shows allowing borrow and lending generally expands welfare. Intuitively, borrow and lending are facilitated by financial system, our result shows an efficient and effective financial system helps improving social welfare.