Lesson 5: Forward Prices, Physical and Risk-neutral Probabilities, Utility-based Models

Economics of Finance

School of Economics, UNSW

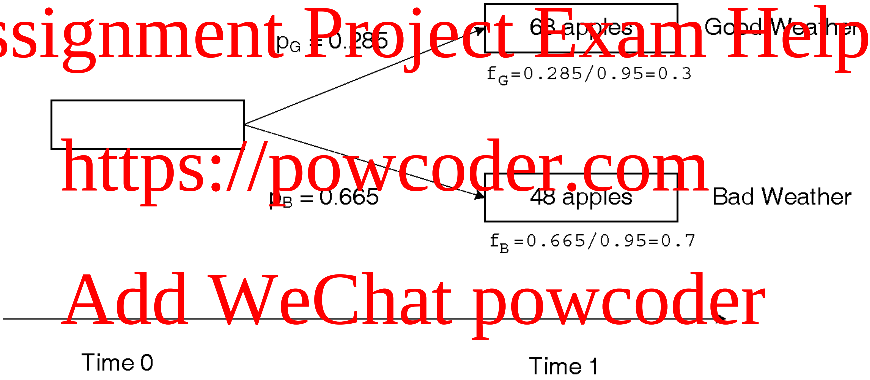
Forward Price

Definition: Forward price, $f(t)$, is the value of the payment at the time t.

Relation with present (spot) price:

$p = df(t)f(t) \Rightarrow f(t) = p/df(t) = p(1+i(t))^t$

Forward Atomic Prices



Note: Forward Atomic prices are positive and sum to 1. Why?

Forward Atomic Prices as Risk-neutral probabilities

If we assume that

- all investors agree on the same probabilities
- all investors are risk-neutral (value certain payoff as much as expected (average) payoff)

we can think about forward atomic prices as risk-neutral probabilities.

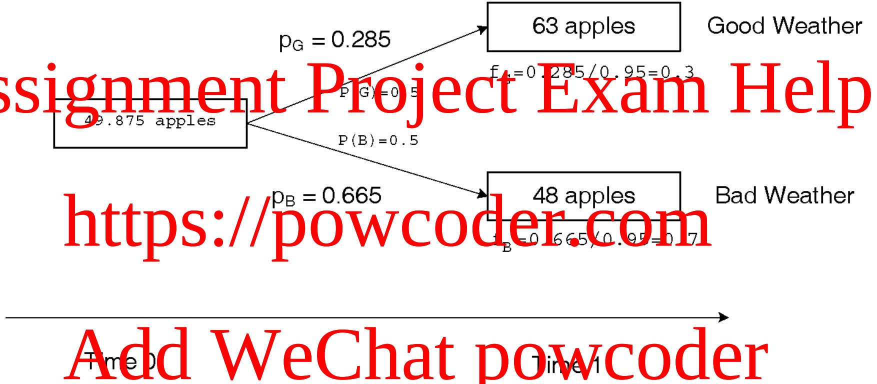
Expected value of discrete random variable X:

$$E(X) = \sum x_i P(X = x_i)$$

Forward value of the tree is expected payoff under risk-neural probabilities $f_{\text{tree}} = E_{\text{risk-neutral}}(c) = 63 \cdot 0.3 + 48 \cdot 0.7 = 52.5$

Note: investors are typically risk-averse and therefore there is a difference between *physical* and *risk-neutral* probabilities.

Physical probabilities



Expected payoff (wrt physical probability):
$E_{\text{physical}}(c_{\text{tree}}) = 63 \cdot 0.5 + 48 \cdot 0.5 = 55.5$

Expected return (wrt physical probability):
$E_{\text{physical}}(r_{\text{tree}}) = E(c_{\text{tree}})/p - 1 = 55.5/49.875 - 1 = 0.113$

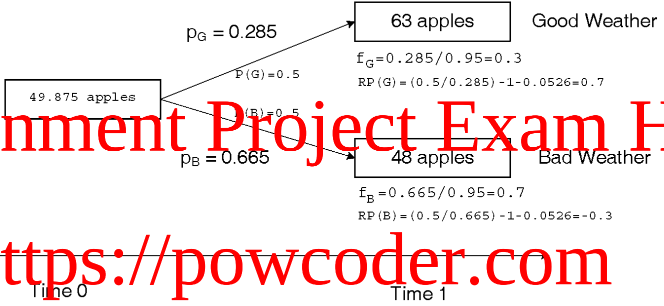
Risk premium

Expected return of the risky tree:
$E_{\text{physical}}(r_{\text{tree}}) = E(c)/p - 1 = 55.5/49.875 - 1 = 0.113$

Return of the riskless asset:
$r_{\text{riskless}} = 1/df - 1 = i = 1/0.95 - 1 = 0.053$

Risk premium: difference between expected risky return and riskless return $E_{\text{physical}}(r_{\text{tree}}) - r_{\text{riskless}} = 0.113 - 0.053 = 0.06$

Atomic risk premia



- Risk premium of the GW atomic security is positive, 0.7, because the forward price of 1 GW apple is lower than the physical probability of GW state. We value GAs not that much because they are more abundant.
- Risk premium of the BW atomic security is negative (risk discount), -0.3, because the forward price of 1 BW apple is higher than the physical probability of BW state. This is like buying an insurance to cover your consumption in BW.
- Remember that the whole tree still carried risk premium

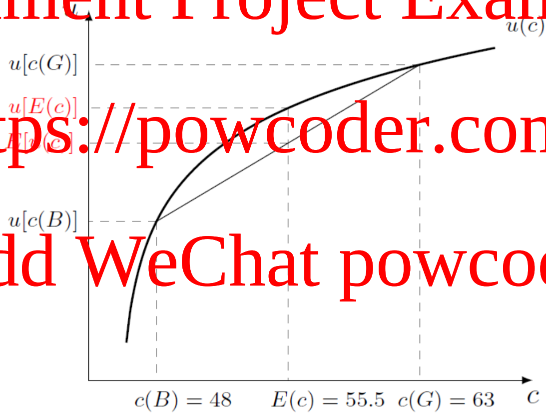
Two different perspectives on asset pricing

- *Relative* Pricing - covered up until now
  - assuming arbitrage-free environment and a competitive market which eliminates any arbitrage;
  - pricing using the Law-of-One-Price and replicating portfolios;
  - relying on existing securities for market completeness;
  - atomic (state) prices used to price any future state-contingent payoffs $p_{atom} = p_{\mathrm{S}} \times Q^{-1}$
- Pricing from *microfoundations* - from now on
  - expected utility optimisation
  - assumptions on preferences, i.e., functional form of the utility function;
  - market is completed by introducing securities;
  - market clearing: matching aggregate demand/supply;
  - explains how we arrive at the equilibrium.

Risk Aversion

$u(c)$ is assumed to be strictly increasing and concave, e.g., $u(c) = ln(c)$.

$$u(c)' > 0, \; u(c)'' \leq 0$$

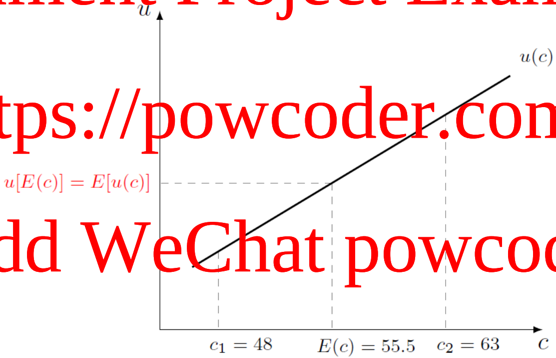


Risk aversion $\Leftrightarrow u[E(c)] > E[u(c)]$

Risk Neutrality

$u(c)$ is assumed linear, i.e., $u(c) = a + b\,c$.

$$u(c) = b, \text{a constant; } u(c)'' = 0$$



Risk neutrality $\Leftrightarrow u[E(c)] = E[u(c)]$

A Sketch

Assumptions:

- Everyone is self-interested and optimises own utility;

$$U = u\left(c(s_0)\right) + \underbrace{\beta\left[\pi(G)\cdot u\left(c(G)\right) + \pi(B)\cdot u\left(c(B)\right)\right]}_{\text{expected discounted future utility}}$$

- subject to budget constraints;

- free trade is allowed;
- everyone take price as given;
- market clears (demand=supply);

Results:

- Not only such outcome is Pareto efficient (First Fundamental Theory of Welfare);
- *Any* Pareto efficient outcome can be produced by such economic environment (Second Fundamental Theory of Welfare).

Endowments:

- There is an (exogenously given) supply or endowment of a **non-storable** consumption good at each time and state;
- At $t = 0$, the consumer does not know which state will realise in the future.
- Notation (Endowments):
  - $e(s_0)$ - the initial endowment of consumption good;
  - $e(s_1 = G)$ - the quantity of the consumption good consumer receives (say apples from a tree) at time 1 if the realized state is Good Weather;
  - $e(s_1 = B)$ - the endowment available at time 1 in the Bad Weather state;

Market structure:

- The consumer can freely borrow or lend in a complete set of atomic (Arrow-Debreu) securities.
- We assume the existence of two securities: Bad Weather security and Good Weather security.
  - One unit of 'G security' sells at time 0 at a price $q(s_0, s_1 = G)$ and pays one unit of consumption at time 1 if state 'G' occurs and nothing otherwise.
  - One unit of 'B security' sells at time 0 at a price $q(s_0, s_1 = B)$ and pays one unit of consumption in state 'B' only.
- In this notation: s_0 refers to the state when securities are traded; $s_1 = G$ refers to a particular realization of the state s_1 when the security pays off.

Flow budget constraints: Time 0

- In the first period the consumer has initial endowment $e(s_0)$. They can consume or buy Arrow-Debreu securities:

$$c(s_0) + q(s_0, s_1 = G) \cdot a(s_0, s_1 = G) + q(s_0, s_1 = B) \cdot a(s_0, s_1 = B) = e\,(s_0)$$

  - $a(s_0, s_1 = G)$ – quantity G securities acquired in state s_0;
  - $a(s_0, s_1 = B)$ – quantity B securities acquired in state s_0;
- In our two-period model all trades occur in state s_0. The only uncertainty is about the realization of the state s_1. Therefore, we can use simplified notation:

  - for atomic security prices: q_G, q_B
  - for quantities of the atomic security purchased (sold): a_G, a_B

$$c_0 + q_G \cdot a_G + q_B \cdot a_B = e_0$$

Flow budget constraints: Time 1

- If the realized state at time 1 is Good Weather:
  - Each of a_G G atomic securities pays off 1 unit of consumption;
  - Bad Weather atomic securities do not pay off at all;
  - Consumer receives an endowment corresponding to G state: e_G and consumes every unit of consumption they have got:

$$c_G = 1 \cdot a_G + 0 \cdot a_B + e_G.$$

- If the realized state at time 1 is Bad Weather:
  - Each of a_B B atomic securities pays off 1 unit of consumption;
  - G atomic securities do not pay off at all;
  - Consumer receives an endowment corresponding to B state: e_B and consumes every unit of consumption they have got:

$$c_B = 0 \cdot a_G + 1 \cdot a_B + e_B.$$

Market Equilibrium:

- A *Market Equilibrium* in this economy is defined as an allocation c_0, c_G, c_B, a_G, a_B and prices q_G, q_B such that:
  - Given the prices, the allocation solves the consumer's problem of maximizing expected utility

$$u(c_0) + \beta \left[\pi_G \cdot u(c_G) + \pi_B \cdot u(c_B) \right]$$

subject to a sequence of budget constraints

$$c_0 + q_G \cdot a_G + q_B \cdot a_B = e_0,$$

$$c_G = a_G + e_G,$$
$$c_B = a_B + e_B.$$

  - Prices are such that markets clear in every period and state:

$$c_0 = e_0; c_G = e_G; c_B = e_B,$$