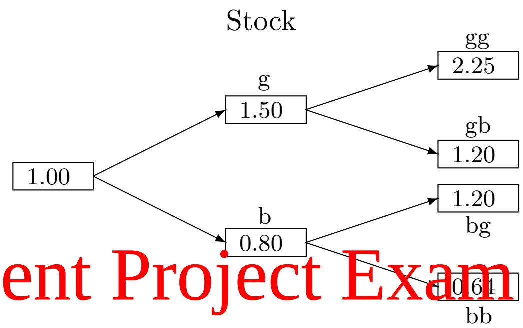
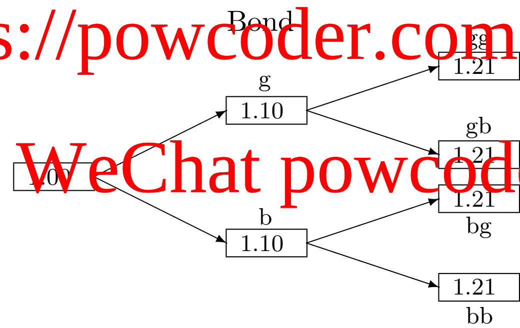
Economics of Finance

Tutorial 4 solution

1. Consider a three period binomial time-state model in which there are two securities, a bond and a stock. The payments made by these securities in each state are shown in the trees below:





(i) Write down the payment matrix, **Q**, and corresponding price vector, $\mathbf{p_S}$, derived from the following elemental payment combinations:

B0: Buy a Bond at period 0, sell it at the end of the next period;

S0: Buy a Stock at period 0, sell it at the end of the next period;

Bg: At period 1, if the state is **g**, buy a Bond, sell it at the end of the next period;

Sg: At period 1, if the state is **g**, buy a Stock, sell it at the end of the next period;

Bb: At period 1, if the state is **b**, buy a Bond, sell it at the end of the next period;

Sb: At period 1, if the state is **b**, buy a Stock, sell it at the end of the next period.

Solution

The $\mathbf{Q}$ matrix derived from the elemental payment combinations B0, S0, Bg, Sg, Bb, Sb is as follows:

$$\mathbf{Q} = \begin{pmatrix} 1.1 & 1.5 & -1 & -1 & 0 & 0 \\ 1.1 & 0.8 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1.1 & 1.5 & 0 & 0 \\ 0 & 0 & 1.1 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.1 & 1.5 \\ 0 & 0 & 0 & 0 & 1.1 & 0.8 \end{pmatrix}$$

The corresponding securities price vector is

$$\mathbf{p}_S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(ii) Compute the atomic security prices (i.e., the price of one dollar in each of the six future time-states: g, b, gg, gb, bg, bb). Write down the formula you used to derive the atomic security price vector.

Solution

The atomic security price vector can be derived from the $\mathbf{Q}$ matrix and its corresponding $\mathbf{p}_S$ vector:

$$\mathbf{p}_{atom} = \mathbf{p}_S \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 0.389\,61 & 0.519\,48 & 0.151\,80 & 0.202\,40 & 0.202\,40 & 0.269\,86 \end{pmatrix}$$

(iii) Write down the payment matrix, $\mathbf{Q}$, and corresponding price vector, $\mathbf{p_S}$, derived from the following elemental payment combinations:

B0: Buy a Bond at period 0, sell it at the end of the next period;

S0: Buy a Stock at period 0, sell it at the end of the next period;

Bb: At period 1, if the state is **b**, buy a Bond, sell it at the end of the next period;

Sb: At period 1, if the state is **b**, buy a Stock, sell it at the end of the next period.

B02: Buy a Bond at period 0, sell it at the end of period 2;

S02: Buy a Stock at period 0, sell it at the end of period 2;

Solution

The $\mathbf{Q}$ matrix derived from the elemental payment combinations B0, S0, Bb, Sb, B02, S02 is as follows:

$$\mathbf{Q} = \begin{pmatrix} 1.1 & 1.5 & 0 & 0 & 0 & 0 \\ 1.1 & 0.8 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.21 & 2.25 \\ 0 & 0 & 0 & 0 & 1.21 & 1.20 \\ 0 & 0 & 1.1 & 1.5 & 1.21 & 1.20 \\ 0 & 0 & 1.1 & 0.8 & 1.21 & 0.64 \end{pmatrix}$$

The corresponding securities price vector is

$$\mathbf{p}_S = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(iv) Verify the atomic security prices computed using the payment matrix $\mathbf{Q}$ in part (iii) is the same as the one found using the payment matrix $\mathbf{Q}$ in part (i).

Solution
The atomic security price vector derived from the $\mathbf{Q}$ matrix from part (iii) is the same as the vector derived from the $\mathbf{Q}$ matrix in part (i). They are different representations of the same market.

(v) Suppose an investor wants to obtain the following time-state payments:

$$\mathbf{c} = \begin{pmatrix} 0 & 10 & 20 & 20 & 30 & 40 \end{pmatrix}'.$$

The vector of payment combination holdings, $\mathbf{n}$, is calculated as follows: $\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c}$. Calculate $\mathbf{n}$ for both the $\mathbf{Q}$ matrices considered in (i) and (iii) above.

Solution
Using the elemental payment combinations from part (i) we get:

$$\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c} = \begin{pmatrix} 1.1 & 1.5 & -1 & -1 & 0 & 0 \\ 1.1 & 0.8 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1.1 & 1.5 & 0 & 0 \\ 0 & 0 & 1.1 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.1 & 1.5 \\ 0 & 0 & 0 & 0 & 1.1 & 0.8 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 10 \\ 20 \\ 20 \\ 30 \\ 40 \end{pmatrix} = \begin{pmatrix} 63.839 \\ -34.694 \\ 18.182 \\ 0 \\ 46.753 \\ -14.286 \end{pmatrix}$$

Using the elemental payment combinations from part (ii) we get:

$$\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c} = \begin{pmatrix} 1.1 & 1.5 & 0 & 0 & 0 & 0 \\ 1.1 & 0.8 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.21 & 2.25 \\ 0 & 0 & 0 & 0 & 1.21 & 1.20 \\ 0 & 0 & 1.1 & 1.5 & 1.21 & 1.20 \\ 0 & 0 & 1.1 & 0.8 & 1.21 & 0.64 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 10 \\ 20 \\ 20 \\ 30 \\ 40 \end{pmatrix} = \begin{pmatrix} 47.310 \\ -34.694 \\ 28.571 \\ -14.286 \\ 16.529 \\ 0 \end{pmatrix}$$

(vi) Take each vector $\mathbf{n}$ from part (v) and calculate how much of the bond and stock the investor must buy or sell in aggregate in each state in period 1 to implement this dynamic strategy? Show your workings, and verify that both $\mathbf{n}$ vectors are describing the same overall strategy.

Solution
Taking the elemental payment combinations from part (i), the dynamic strategy is implemented as follows:

Period 0:	Buy 63.839 of Bond (row 1 of vector **n**) Sell 34.694 of Stock (row 2 of vector **n**)
Period 1 (good)	▶ Sell previously purchased Bonds for $63.839 \times 1.1 = 70.223$ ▶ Buy previously sold Stocks for $34.694 \times 1.5 = 52.041$ ▶ Buy 18.182 worth of Bond (row 3 of vector **n**) ▶ Sell/Buy 0 of Stock (row 4 of vector **n**) *Overall change in holdings after state* g: ▶ Sell 52.041 worth of Bonds ($-70.223 + 18.182 = -52.041$) ▶ Buy 52.041 worth of Stocks *Cash flow in state* g: $-52.041 + 52.041 = 0$ (row 1 of vector **c**)
Period 1 (bad)	▶ Sell previously purchased Bonds for $63.839 \times 1.1 = 70.223$ ▶ Buy previously sold Stocks for $34.694 \times 0.8 = 27.755$ ▶ Buy 46.753 worth of Bond (row 5 of vector **n**) ▶ Sell 14.286 worth of Stock (row 6 of vector **n**) *Overall change in holdings after state* b: ▶ Sell 23.47 worth of Bonds ($-70.223 + 46.753 = -23.47$) ▶ Buy 13.469 worth of Stocks ($27.755 - 14.286 = 13.469$) *Cash flow in state* b: $23.47 - 13.469 = 10.00$ (row 2 of vector **c**)

Taking the elemental payment combinations from part (ii), the dynamic strategy is implemented as follows:

Period 0:	▶ Buy 47.310 of Bond (row 1 of vector $\mathbf{n}$) ▶ Sell 34.694 of Stock (row 2 of vector $\mathbf{n}$) ▶ Buy 16.529 of Bond (row 5 of vector $\mathbf{n}$) ▶ Buy/Sell 0 Stock (row 6 of vector $\mathbf{n}$) *Overall change in holdings after Period 0* ▶ Buy 63.839 of Bond ▶ Sell 34.694 of Stock

Period 1 (good)	▶ Sell previously purchased Bonds for $47.310 \times 1.1 = 52.041$ ▶ Buy previously sold Stocks for $34.694 \times 1.5 = 52.041$ *Overall change in holdings after state* g: ▶ Sell 52.041 worth of Bonds ▶ Buy 52.041 worth of Stocks *Cash flow in state* g: $52.041 - 52.041 = 0$ (row 1 of vector $\mathbf{c}$)

Period 1 (bad)	▶ Sell previously purchased Bonds for $47.310 \times 1.1 = 52.041$ ▶ Buy previously sold Stocks for $34.694 \times 0.8 = 27.755$ ▶ Buy 28.571 worth of Bond (row 3 of vector $\mathbf{n}$) ▶ Sell 14.286 worth of Stock (row 4 of vector $\mathbf{n}$) *Overall change in holdings after state* b: ▶ Sell 23.47 worth of Bonds ($-52.041 + 28.571 = -23.47$) ▶ Buy 13.469 worth of Stocks ($27.755 - 14.286 = 13.469$) *Cash flow in state* b: $23.47 - 13.469 = 10.00$ (row 2 of vector $\mathbf{c}$)

In each state, the aggregate purchases and sales derived from both portfolio vectors, $\mathbf{n}$, are equivalent.

(vii) Compute the arbitrage-free price of the time-state payment vector $\mathbf{c}$. Explain how you arrived at your answer.

Solution

The time-state payment vector, $\mathbf{c}$, can be priced via the atomic price vector, which tells us the present value of dollars in each of the future time-states:

$$\mathbf{p} = \mathbf{p}_{atom} \cdot \mathbf{c} = 29.1449$$

2. Consider a three period binomial time-state model in which there are two securities, a bond and a stock. The payments made by the stock in each state are shown in the tree below. The bond pays 10 percent interest each period.

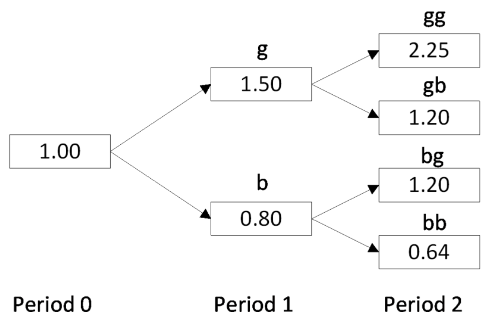


Figure 1: Lattice for the Stock

(i) Calculate the price in Period 0 of an American Call (buy) option that expires at the end of Period 2 with an exercise price of 1.60. Explain how you arrived at your answer.

Solution

The **Q** matrix derived from the elemental payment combinations B0, S0, Bg, Sg, Bb, Sb is as follows:

$$
\mathbf{Q} = \begin{array}{c} \begin{matrix} B0 & S0 & Bg & Sg & Bb & Sb \end{matrix} \\ \begin{pmatrix} 1.1 & 1.5 & -1 & -1 & 0 & 0 \\ 1.1 & 0.8 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1.1 & 1.5 & 0 & 0 \\ 0 & 0 & 1.1 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.1 & 1.5 \\ 0 & 0 & 0 & 0 & 1.1 & 0.8 \end{pmatrix} \end{array} \begin{array}{c} \\ g \\ b \\ gg \\ gb \\ bg \\ bb \end{array}
$$

The corresponding securities price vector is

$$\mathbf{p}_S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The atomic security price vector can be derived from either **Q** matrix and its corresponding $\mathbf{p}_S$ vector:

$$\mathbf{p}_{atom} = \mathbf{p}_S \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 0.3896 & 0.5195 & 0.1518 & 0.2024 & 0.2024 & 0.2699 \end{pmatrix}$$

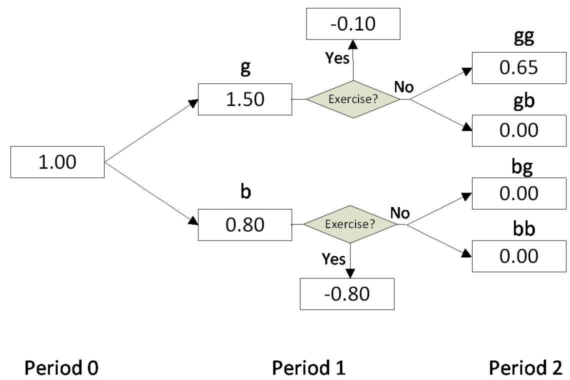
If this Call option is held until Period 2, it should be left to expire worthless in all but State gg. In this case, the option will be worth \$0.65 since it can be used to buy a stock worth \$2.25 for only \$1.60. Figure below shows the situation diagrammatically.

The values in the boxes for time Period 2 indicate the cash flows if the option is held until time Period 2 and then exercised optimally. In this case, the optimal strategy is simple. Exercising the option in Period 1 will only lead to a loss. Hence, whatever state occurs in Period 1 the option should be kept till Period 2. The vector of state contingent cash flows corresponding this Call option is

$$\mathbf{c}_{Call} = \begin{pmatrix} 0 & 0 & 0.65 & 0 & 0 & 0 \end{pmatrix}';$$

and its arbitrage-free price is

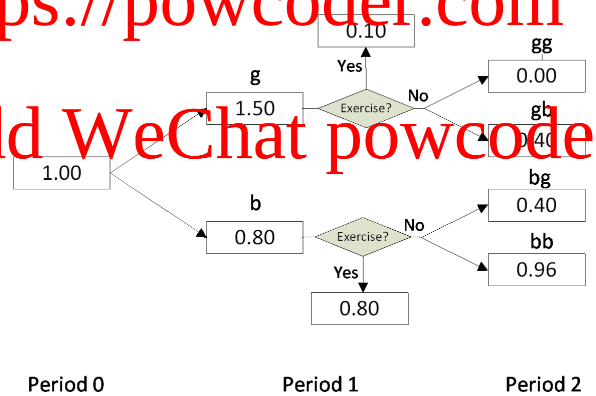
$$\mathbf{p}_{Call} = \mathbf{p}_{atom} \cdot \mathbf{c}_{Call} = 0.0987.$$



(ii) Calculate the price in Period 0 of an American Put (sell) option that expires at the end of Period 2 with an exercise price of 1.60. Explain how you arrived at your answer.

Solution

If this Put option is held until Period 2 it should be exercises in all but State gg. For instance, the option will be worth \$0.40 in State gb or bg since it can be used to sell a stock worth only \$1.20 for \$1.60. Figure below shows the situation diagrammatically. The values in the boxes for Period 2 indicate the cash flows if the option is held until Period 2 and then exercised optimally.



Should the option be exercised at the end of Period 1? Suppose, the state is g. Exercising the Put would yield the following vector of cash flows:

$$\mathbf{c}^g_{Exer.} = \begin{pmatrix} 0.10 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}';$$

Keeping the option till Period 2 would yield the following vector of cash flows:

$$\mathbf{c}^g_{Keep} = \begin{pmatrix} 0 & 0 & 0 & 0.40 & 0 & 0 \end{pmatrix}';$$

Pricing the two alternatives gives the answer to the question whether the option should be exercised in State g:

$$\mathbf{p}_{atom} \cdot \mathbf{c}^g_{Exer.} = 0.0390 < \mathbf{p}_{atom} \cdot \mathbf{c}^g_{Keep} = 0.0810.$$

The answer is NO! If the state is Good in Period 1, the option should be kept till Period 2.

Now suppose, the state is b. Exercising the Put would yield the following vector of cash flows:

$$\mathbf{c}^{b}_{Exer.} = \begin{pmatrix}0 & 0.80 & 0 & 0 & 0 & 0\end{pmatrix}';$$

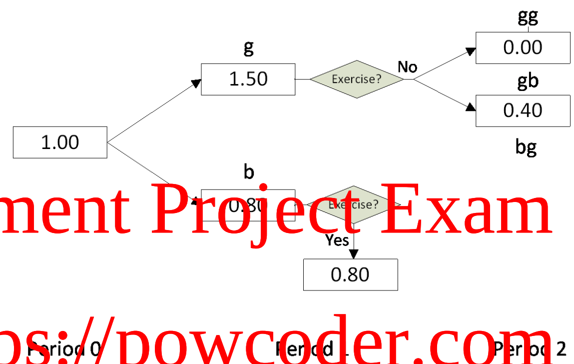
Keeping the option till Period 2 would yield the following vector of cash flows:

$$\mathbf{c}^{b}_{Keep} = \begin{pmatrix}0 & 0 & 0 & 0 & 0.40 & 0.96\end{pmatrix}';$$

Once again, pricing the two alternatives tell us whether the option should be exercised in State b:

$$\mathbf{p}_{atom} \cdot \mathbf{c}^{b}_{Exer.} = 0.4156 > \mathbf{p}_{atom} \cdot \mathbf{c}^{b}_{Keep} = 0.3400.$$

Hence, if the state is Bad in Period 1, the option should be exercised. The figure below shows the option tree after it has been modified to include only the optimal paths:



The vector of state contingent cash flows corresponding this Put option is

$$\mathbf{c}_{Put} = \begin{pmatrix}0 & 0.80 & 0 & 0.40 & 0 & 0\end{pmatrix}';$$

and its arbitrage-free price is

$$\mathbf{p}_{Put} = \mathbf{p}_{atom} \cdot \mathbf{c}_{Put} = 0.4965.$$

3. An investor owns 2,000 shares of Walmart Inc, now selling at \$110. The investor has a one-year investment horizon and is concerned of a downside risk. She considers buying a call or put option. A one-year call with a strike price of \$110 is now selling at \$1.50 whilst the put option with similar strike price and expiry date is selling at \$0.41.

i) Given the above data, devise the protective strategy.

Solution
A protective put strategy involves the combination of a long stock and long put options. Since the investor holds 2000 shares, he needs to hold 2000 put options to offset the risk of stock price going down.

ii) Explain why in this situation a call option is more expensive than a put option.

Solution
Given the market is arbitrage free, recall put-call parity:

$$C + df\,X = S_0 + P,$$

where C is the call option price, P is the put option price, X is the strike price, df is the discount factor, and S_0 is the stock price. Rearrange the equation yields:

$$C - P = S_0 - df X,$$

The price difference between call option and put option depends on the difference between the current stock price and discounted value of the strike price. Realise $S_0 = X = \$110$,we have:

$$C - P = X(1 - df),$$

The price difference between call option and put option in this context reflects the interest on a discounted bond with a face value of the strike price. In general $df < 1$, so a positive price difference between call option and put option can be expected.

iii) Under which conditions a call option price will be exactly equal to a put option price.

Solution
Following our analysis of part ii), in this context $C = P$ if $df = 1$, i.e., call option price equal to put option price only if discount factor is 1, i.e., interest rate is 0.