Lesson 2: Valuation, Atomic Prices, Complete and Incomplete Markets

Economics of Finance

School of Economics, UNSW

The Law of One Price (LOP)

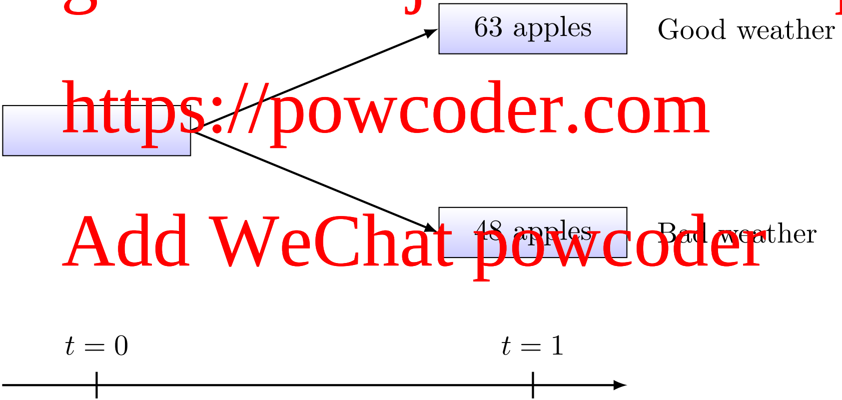
Definition: (LOP) *In an arbitrage-free economy with no transactions costs, any given time-state claim will sell for the same price, no matter how obtained. This holds for any 'package' of time-state claims.*

- In the 'real world' transactions costs are usually present;
- The lack of arbitrage opportunities only insures that prices for a given set of time-state claims will fall within a band narrow enough to preclude generating a positive profit *net of transactions costs* out of trading.

Valuation

Definition: *Valuation* is the process of determining the *present value* of a security or productive investment.

Example: How much is a tree worth today (at time 0)?



Present Value of a tree: $PV = 0.285 \cdot 63 + 0.665 \cdot 48 = 49.875$

Net Present Value

The *net present value* of a set of claims is based on future payments and any required payment in the present. If the Tree is purchased for 49.875 apples

$$NPV = \begin{pmatrix} 1.00 & 0.285 & 0.665 \end{pmatrix} \times \begin{pmatrix} -49.875 \\ 63 \\ 48 \end{pmatrix}$$

$$NPV = -49.875 + 0.285 \times 63 + 0.665 \times 48 = 0$$

The net present value of a fairly priced investment is zero.

Net Present Value (cont'd)

Assume you discover how to plant 60 apples in a way that will produce 100 apples if the weather is good and 50 apples if the weather is bad. Compute the net present value:

$$NPV = -60 + 0.285 \times 100 + 0.665 \times 50 = 1.75$$

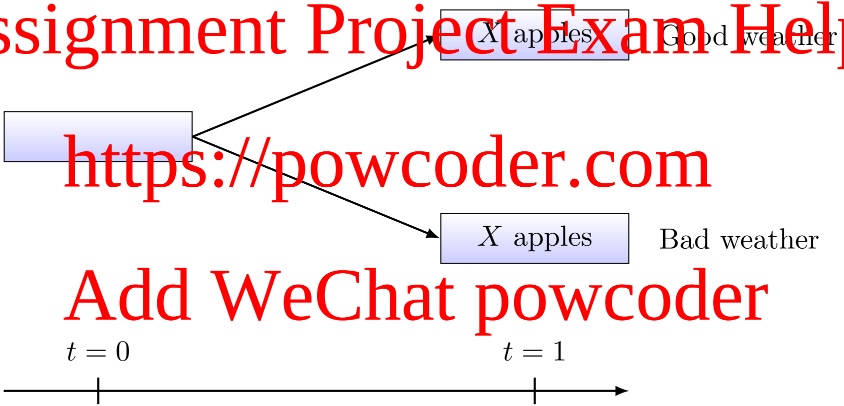
Should you do it? YES. Why?

Riskless Securities

Definition: A *riskless security* pays the same amount at a given time, no matter what state of the world occurs.

- A *riskless security* is equivalent to a bundle of equal amounts of atomic claims for a time period.
- In our example, a riskless security pays a fixed amount (say X apples) at time period 1, whether the weather has been good or bad.
- Equivalently, it is a bundle of X good weather apples (GA) and X bad weather apples (BA).

Riskless Securities



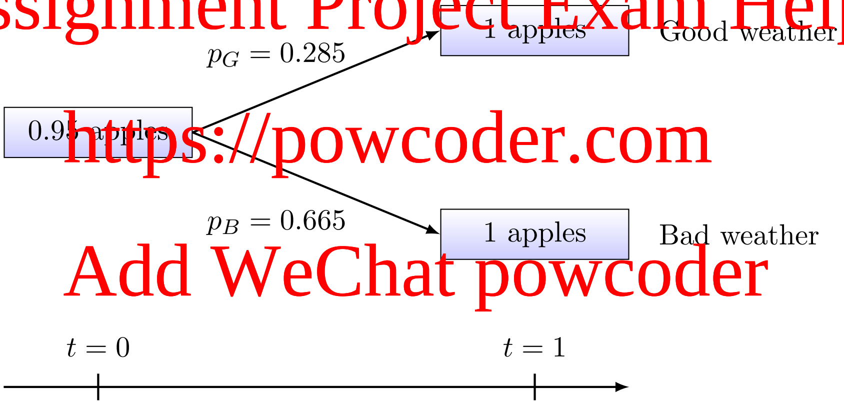
The Discount Factor

Definition: *The discount factor* (for a certain date) represents the present value of a payment of one unit to be made with certainty at the specified future date.

- The discount factor for a date in question equals to the sum of appropriate atomic prices (prices of basic atomic securities).

The Discount Factor

E.g. $df(1) = 0.95$



Inferring Atomic Security Prices

Let $\mathbf{Q}$ {states $\times$ securities} be the payment matrix of the two securities:

$$\underset{(2\times 2)}{\mathbf{Q}} = \begin{array}{cc} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

Let $\mathbf{p}_S$ {1$\times$securities} be a vector of security prices:

$$\underset{(1\times 2)}{\mathbf{p_S}} = \begin{array}{c} \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} \end{array}$$

Let $\mathbf{n}$ {securities$\times$1} be a vector of *portfolio holdings*:

$$\underset{(2\times 1)}{\mathbf{n}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{array}{l} \text{number of Bonds} \\ \text{number of Stocks} \end{array}$$

Inferring Atomic Security Prices

Let $\mathbf{c}$ {states×1} be the vector of payments in each state, then it must hold that

$$\underset{(\text{states}\times\text{securities})}{\mathbf{Q}} \cdot \underset{(\text{securities}\times 1)}{\mathbf{n}} = \underset{(\text{states}\times 1)}{\mathbf{c}}$$

In our example the above identity reads as

$$\underset{(2\times 2)}{\mathbf{Q}} \cdot \underset{(2\times 1)}{\mathbf{n}} = \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 106 \\ 76 \end{pmatrix}$$

where the vector of state-contingent payments is

$$\underset{(2\times 1)}{\mathbf{c}} = \begin{pmatrix} 106 \\ 76 \end{pmatrix} \begin{array}{c} \text{Good Weather} \\ \text{Bad Weather} \end{array}$$

Obtaining a desired portfolio

Question: What portfolio $\mathbf{n}$ will provide a desired set of state state-contingent payments $\mathbf{c}$?

If the payoff matrix $\mathbf{Q}$ is invertible, then the answer is simple.

$$\underset{(\text{securities}\times 1)}{\mathbf{n}} = \underset{(\text{securities}\times\text{states})}{\mathbf{Q}^{-1}} \cdot \underset{(\text{states}\times 1)}{\mathbf{c}}$$

Note: If a matrix $\mathbf{Q}$ satisfies the following conditions:

(i) $\mathbf{Q}$ is a square matrix i.e. its number of rows equals to its number of columns;

(ii) $\mathbf{Q}$ is non-singular i.e. its rows/columns are linearly independent;

then Q^{-1} exists.

Atomic Security Prices

To obtain payment $\mathbf{c}$, we can buy a portfolio $\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c}$. This portfolio will cost us

$$\mathbf{p} = \mathbf{p_S} \cdot \mathbf{n} = \left[\mathbf{p_S} \cdot \mathbf{Q^{-1}}\right] \mathbf{c}$$

Therefore, we can infer atomic security prices from the prices and payments of the traded securities:

$$\underset{(1\times\text{states})}{\mathbf{p}_{atom}} \equiv \underset{(1\times\text{securities})}{\mathbf{p_S}} \cdot \underset{(\text{securities}\times\text{states})}{\mathbf{Q}^{-1}}$$

$\mathbf{Q^{-1}}$ revisited

Recall that $\mathbf{Qn} = \mathbf{c}$ and that $\mathbf{n} = \mathbf{Q^{-1}c}$. Say you'd wish to find a portfolio such that $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This portfolio is given by the first column of $\mathbf{Q^{-1}}$. In our example

$$\mathbf{n} = \mathbf{Q^{-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.0933 & 0.1433 \\ 0.0667 & -0.0667 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.0933 \\ 0.0667 \end{pmatrix}$$

What is the present value of $\mathbf{n}$?

$$\mathbf{p_s n} = \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \begin{pmatrix} -0.0933 \\ 0.0667 \end{pmatrix}$$

$$19 \times (-0.0933) + 30.875 \times 0.0667 = -1.7727 + 2.0593 = 0.285$$

Atomic Security Prices

Example: How much would it cost to get 845 GA and 620 BA?

$$\mathbf{p} = \mathbf{p_S} \cdot \mathbf{n} = \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \begin{pmatrix} 10 \\ 15 \end{pmatrix} = 653.125$$

The prices of the atomic securities can be inferred from

$$\mathbf{p}_{atom} = \mathbf{p_S} \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix}^{-1} = \begin{array}{cc} \begin{pmatrix} 0.285 & 0.665 \end{pmatrix} \\ \text{Good W.} \quad \text{Bad W.} \end{array}$$

Using inferred prices of the atomic securities we can price $\mathbf{c}$ as follows

$$\mathbf{p} = \mathbf{p}_{atom} \cdot \mathbf{c} = \begin{pmatrix} 0.285 & 0.665 \end{pmatrix} \begin{pmatrix} 845 \\ 620 \end{pmatrix} = 653.125$$

The Opportunity Set

Suppose you have a dollar.
What opportunity can you get from the market?

Value Relative

Definition: Value relative associated with a given state of nature is the future payment per unit invested that will be received if that state occurs.

In our example the matrix for the value relatives is:

$$\underset{(2\times 2)}{\mathbf{vr}} = \begin{array}{cc} \begin{pmatrix} 20/19 & 43/30.875 \\ 20/19 & 28/30.875 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

$$= \begin{array}{cc} \begin{pmatrix} 1.0526 & 1.3927 \\ 1.0526 & 0.9069 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

Value relative and return

Value of relative is the percentage value of an ending value in terms of initial value.

E.g., if the weather is good, the value relative of a GA is `1/0.285 = 3.5088`. If the weather is bad, the value relative of a GA is `0/0.285 = 0`.

Return is value relative, net 100%

$$\texttt{return} = \texttt{vr} - 1$$

An atomic security returns `-100%` in all states except the one it corresponds to.

The Opportunity Set

Definition: The opportunity set is the set of feasible future payoffs available with a wealth of one unit of present value.

Example: The opportunities for future apples for a present apple invested:
Q {states*securities} is the payment matrix of the two securities:

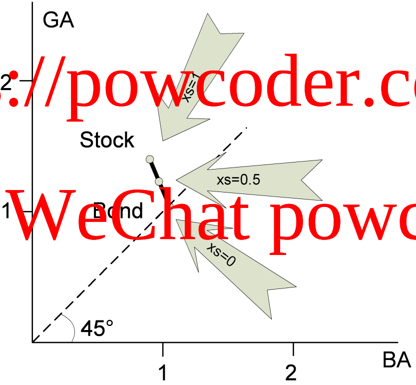
$$\underset{(2\times 2)}{\mathbf{Q}} = \begin{array}{cc} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} & \begin{array}{c} \text{Good Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

$\mathbf{p}_S$ {1*securities} is a vector of security prices:

$$\underset{(1\times 2)}{\mathbf{p_S}} = \begin{array}{c} \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} \end{array}$$

How much can we get from one apple?

By choosing a portfolio that includes positive (long) positions in the Bond and in the Stock with a total present value of 1 apple, an investor can obtain any position on the line segment connecting the two securities in the Figure



Shorting securities

What about negative (short) positions in either security?

Opportunity Set Frontier

Suppose, one can take negative positions in a security as long as investor's overall portfolio does not lead to negative net payments in any state of the world

- Then, an investor can obtain any point on the line through **Bond** and **Stock** extended all the way to the axes (see the next figure)

- Value relative of the atomic securities can be found as follows:

$$\mathbf{p}_a = \mathbf{p}_S \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 19 & 30.875 \end{pmatrix} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix}^{-1} = \begin{pmatrix} 0.285 & 0.665 \end{pmatrix}$$

$$\underset{(2\times 2)}{\mathbf{vr}_{atom}} = \begin{array}{cc} \begin{pmatrix} 1/0.285 & 0/0.665 \\ 0/0.285 & 1/0.665 \end{pmatrix} \\ \text{GW claim} \quad \text{BW claim} \end{array} = \begin{array}{cc} \begin{pmatrix} 3.5088 & 0 \\ 0 & 1.5038 \end{pmatrix} \\ \text{GW claim} \quad \text{BW claim} \end{array} \begin{array}{l} \text{Good W.} \\ \text{Bad W.} \\ \end{array}$$

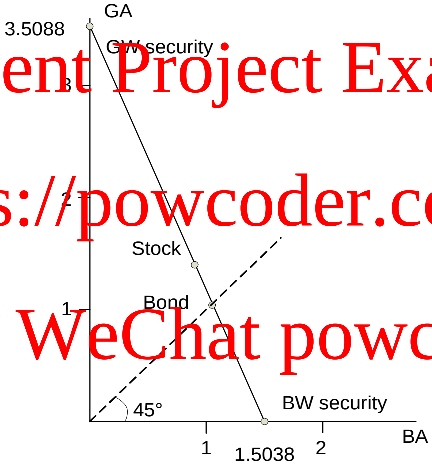
Plotting Opportunity Set

If we add to vr the value relatives of the atomic securities we obtain:

	Good	Bond	Stock	Bad
Good Weather	3.5088	1.0526	1.3927	0
Bad Weather	0	1.0526	0.9069	1.5038

These are points we can plot in the space of GA and BA.

The Opportunity Set



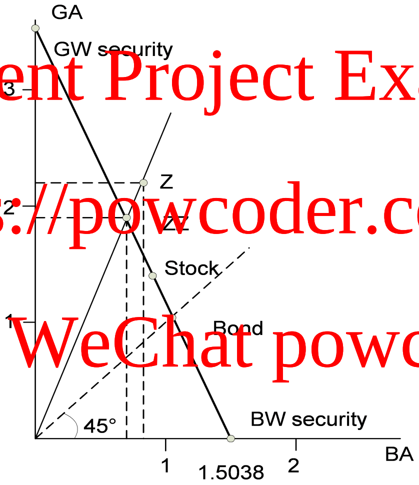
The Opportunity Set: Remarks

- Taking a negative position in the Stock amounts to signing a document of the form: “I promise to pay the holder whatever the firm (tree) pays its stockholders”

- By combining (in the right proportions):
  - a long position in the Bond with a short position in the Stock one can construct a pure “Bad Weather Claim”
  - a short position in the Bond with a long position in the Stock one can construct a pure “Good Weather Claim”

- By combining existing securities (the Bond and the Stock) one can synthesize a security that does not exist (e.g. a Good Weather claim). The result is termed a *derivative security*, since it is derived from the existing securities.

Arbitrage Opportunities



Arbitrage Opportunities (cont.)

- Any security not priced in accordance with the atomic prices implied by the traded securities will present an opportunity for arbitrage
- For example, imagine a security Z appears outside the opportunity set frontier
  - Draw a line through Z to the origin; Denote ZZ the point where the line intersects the opportunity set frontier;
  - Payments ZZ can be obtained by a portfolio of the Bond and the Stock worth 1 PA;
  - Sell ZZ short, and use the proceeds (1 PA) to buy Z;
  - Z pays more that ZZ (per apple invested) in every state of the world, hence we obtain **arbitrage opportunity**

Hedging

Consider following. Markets are *incomplete*: number of states is higher then the number of linearly independent securities.

$$\underset{(3\times 2)}{\mathbf{Q}} = \begin{array}{cc} \begin{pmatrix} 20 & 43 \\ 20 & 28 \\ 20 & 28 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array} \qquad \underset{(1\times 2)}{\mathbf{ps}} = \begin{array}{c} \begin{pmatrix} 19 & 35 \end{pmatrix} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} \end{array}$$

Suppose that an investor asks an investment firm to create a product with the following payment:

$$\underset{(3\times 1)}{\mathbf{c}} = \begin{pmatrix} 40 \\ 30 \\ 20 \end{pmatrix} \begin{array}{l} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{array}$$

Questions: How to do it? What should the firm charge?

Hedging in Incomplete Market

- The Problem: No matter how many bonds and stocks are chosen, the payments in the "Fair Weather" state and the "Bad Weather" state will be the same!

- Suppose the firm will select 40 in 'GW' and 30 in 'FW' or 'BW' to cover all outflows. We can do it by:

$$\underset{(2\times 2)}{\mathbf{Q}} = \begin{array}{cc} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} & \begin{array}{l} \text{Good W.} \\ \text{Fair or Bad W.} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array} \qquad \underset{(2\times 1)}{\mathbf{c}} = \begin{pmatrix} 40 \\ 30 \end{pmatrix} \begin{array}{l} \text{Good W.} \\ \text{Fair or Bad W.} \end{array}$$

$$\mathbf{n} = \mathbf{Q}^{-1} \cdot \mathbf{c} = \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix}^{-1} \begin{pmatrix} 40 \\ 30 \end{pmatrix} = \begin{pmatrix} 0.56667 \\ 0.66667 \end{pmatrix} \begin{array}{l} \text{Bond} \\ \text{Stock} \end{array}$$

$$\mathbf{p} = \mathbf{p_S} \cdot \mathbf{n} = \begin{pmatrix} 19 & 35 \end{pmatrix} \begin{pmatrix} 0.56667 \\ 0.66667 \end{pmatrix} = 34.10$$

Different Scenarios

- It costs 34.10 PA to purchase a portfolio, **n**, that would cover all future outflows:
  - The firm will receive 10 in 'BW' state;
  - Investor will be assured to receive all the promised payments.

- Since there is extra 10 BA, the firm will be happy to sell the product for 34.10PA