Lecture 9: Course overview

Economics of Finance

School of Economics, UNSW

A Road Map

Arbitrage-free Pricing

- financing company: stocks and bonds
- atomic or Arrow-Debreu securities
- bond valuation (duration)
- options: European, American
- CAPM and APT

Optimal decisions/allocation

- exploring arbitrage
- minimum cost hedging in incomplete markets
- optimal trade in Arrow-Debreu economy
- optimal portfolio allocations

The Law of One Price (LOP)

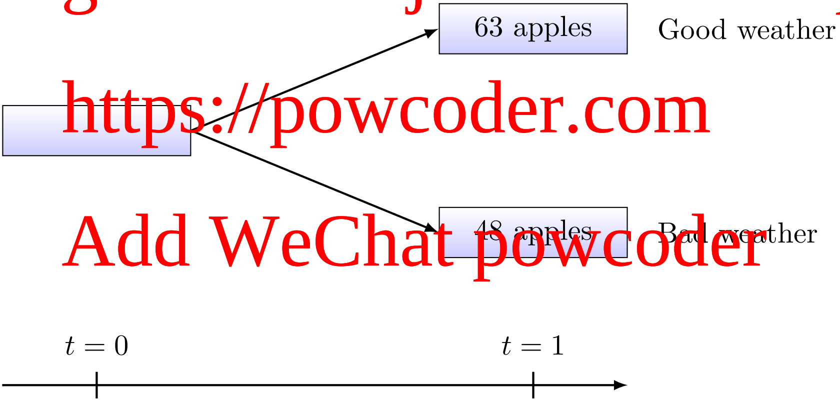
Definition: (LOP) *In an arbitrage-free economy with no transactions costs, any given time-state claim will sell for the same price, no matter how obtained. This holds for any 'package' of time-state claims.*

- In the 'real world' transactions costs are usually present;
- The lack of arbitrage opportunities only insures that prices for a given set of time-state claims will fall within a band narrow enough to preclude generating a positive profit *net of transactions costs* out of trading.

Valuation

Definition: *Valuation* is the process of determining the *present value* of a security or productive investment.

Example: How much is a tree worth today (at time 0)?



Present Value of a tree: $PV = 0.285 \cdot 63 + 0.665 \cdot 48 = 49.875$

Say you'd like to set up an apple firm which consists of an apple tree, i.e., you need 49.875 apples to purchase the tree.

There are two ways to finance this investment, issue *bonds* or issue *stocks*. Assume your firm issues a bond:

`The Apple Tree Firm promises to pay the holder 20 apples at the end of the year, no matter what the weather has been.`

This way the holder does not bear any face value risk (though other types of risk, e.g., default risk or interest rate risk, etc., remain).

Stock

If your firm issues a *stock*, instead:

`The Apple Tree Firm promises to pay the holder all the apples left over after the bondholder has been paid.`

This way the holder bear the risk of the apple production *net the issued bond payment*, BUT is entitled a voting right.

The bond represents the ownership of the money, i.e., *prior claim*, the stock represents the ownership of the firm, i.e., *residual claim*.

Principle of value additivity

$\mathbf{p_a} \times \mathbf{q_{firm}} = \mathbf{p_a} \times (\mathbf{q_{bond}} + \mathbf{q_{stock}})$.

Pricing future desired payment $\mathbf{c}$

With the payment matrix $\mathbf{Q}$ {states $\times$ securities}, having at least as many securities (with linearly *independent* payoffs) as states

and given $\mathbf{p}_S$ $\{1 \times \text{securities}\}$ security prices

we find **unique** price p for **any** desired payments $\mathbf{c}$ $\{\text{states} \times 1\}$

via replication portfolio $\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c}$ (from accounting $\mathbf{Qn} = \mathbf{c}$) and even simpler using atomic prices

$$p = \mathbf{p}_S \mathbf{n} = \mathbf{p}_S \mathbf{Q}^{-1}\mathbf{c} = \mathbf{p}_{atom}\mathbf{c} = df\,\mathbf{f}_{atom}\mathbf{c} = df\,E^*(c) = E(m_1 c),$$

where m_1 is the stochastic discount factor, and E and $\tilde{E}$ are expectations taken over physical and risk-neutral measures.

If any security (derivative, option) deviates from this pricing, we have **arbitrage**, can construct profitable strategy using replicating portfolio.

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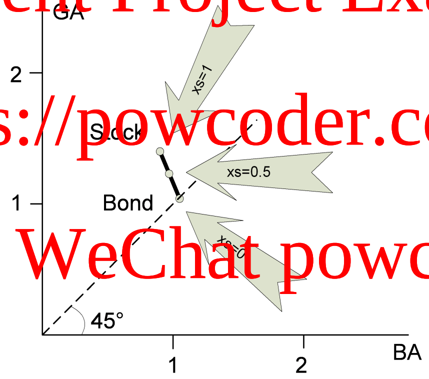
Reality check 1 How do we get $\mathbf{Q}$?
Reality check 2 Infinite number of states (always incomplete)?

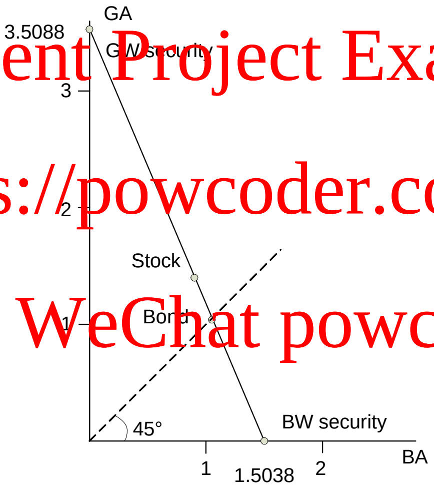
Value relative

How much payment can you get with a dollar?
Consider Bond and Stock:



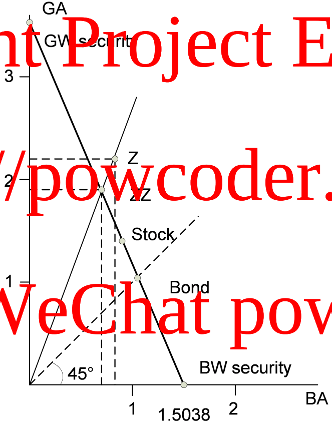
Opportunity Set

Make linear combination of value relatives for Bond and Stock:



Arbitrage Opportunity

When value relative is above/below the opportunity set, there are arbitrage opportunities:



Z is preferred to **ZZ**. Example of realising arbitrage: short-sell replicated **ZZ** to buy **Z** with proceeds. Note **ZZ** period 1 payments are fully covered by **Z** plus there is risk-free profit.

Hedging with Minimum Cost

Hedging is a technology to construct a portfolio and offset risks in all future states.

- Essentially same technology of payment replication as arbitrage;
- Arbitrage: replicate an asset that is over/under valued in the market and generate profit;
- Hedging: given an asset position, offset all risks and liability by replicating this position.

In an incomplete market, hedging entails a constrained optimization problem:

$$\min_{\mathbf{n}} \mathbf{p_S} \cdot \mathbf{n} \text{ subject to } \mathbf{Q} \cdot \mathbf{n} \geq \mathbf{c}.$$

Bonds

- The multiperiod discount factor **df**
- The multiperiod certain cash flow **cf**
- Bond's value $PV = \mathbf{df} \cdot \mathbf{cf}$, many bonds: $\mathbf{p} = \mathbf{df} \cdot \mathbf{Q}$
- Multi-period interest rate, yield curve $i(t) = \left(\frac{1}{df(t)}\right)^{\frac{1}{t}} - 1$
- Yield-to-maturity is a *constant* interest rate, such that the present value of all bond's payments equals its price.
- Duration
  - The average waiting period for a bond to be paid back.

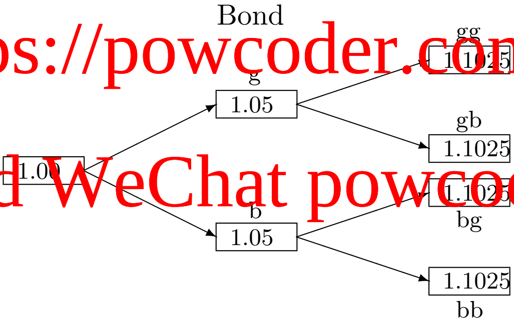
$$D = \sum_t \frac{t\, cf_t}{(1+y)^t\, p_b}$$

  - Frequently used is the *modified duration* – negative relative change in bond's value to per unit of interest rate change

$$md = \frac{D}{(1+y)}$$

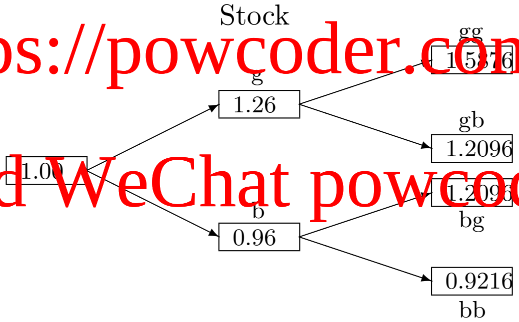
Multi-periods: Bond

- Two-period zero-coupon bond (no coupon payments)
- Its initial value is \$1.00.
- Its price increases 5% of its prior value in every period.



Multi-periods: Stock

- Its initial value is \$1.00. It pays no dividends.
- Its price increases 26% of its prior value in good times.
- Its price falls to 96% of its prior value in bad times.



Planned Acquisitions

We write down the payment of these acquisitions in a matrix:

$$
\mathbf{Q} = \begin{array}{c} \begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} \\ \begin{pmatrix} 1.05 & 1.26 & -1 & -1 & 0 & 0 \\ 1.05 & 0.96 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1.05 & 1.26 & 0 & 0 \\ 0 & 0 & 1.05 & 0.96 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.05 & 1.26 \\ 0 & 0 & 0 & 0 & 1.05 & 0.96 \end{pmatrix} \end{array} \begin{array}{c} \\ \text{g} \\ \text{b} \\ \text{gg} \\ \text{gb} \\ \text{bg} \\ \text{bb} \end{array}
$$

Price Vector:

$$
\mathbf{p}_S = \begin{array}{c} \begin{array}{cccccc} \text{B0} & \text{S0} & \text{Bg} & \text{Sg} & \text{Bb} & \text{Sb} \end{array} \\ \begin{pmatrix} 1.00 & 1.00 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix} \end{array}
$$

To price a unit of payment at each state use $\mathbf{p}_{atom} = \mathbf{p}_S \cdot \mathbf{Q}^{-1}$

$$
\mathbf{p}_{atom} = \begin{array}{c} \begin{array}{cccccc} \text{g} & \text{b} & \text{gg} & \text{gb} & \text{bg} & \text{bb} \end{array} \\ \begin{pmatrix} 0.2857 & 0.6666 & 0.0816 & 0.1904 & 0.1904 & 0.4444 \end{pmatrix} \end{array}
$$

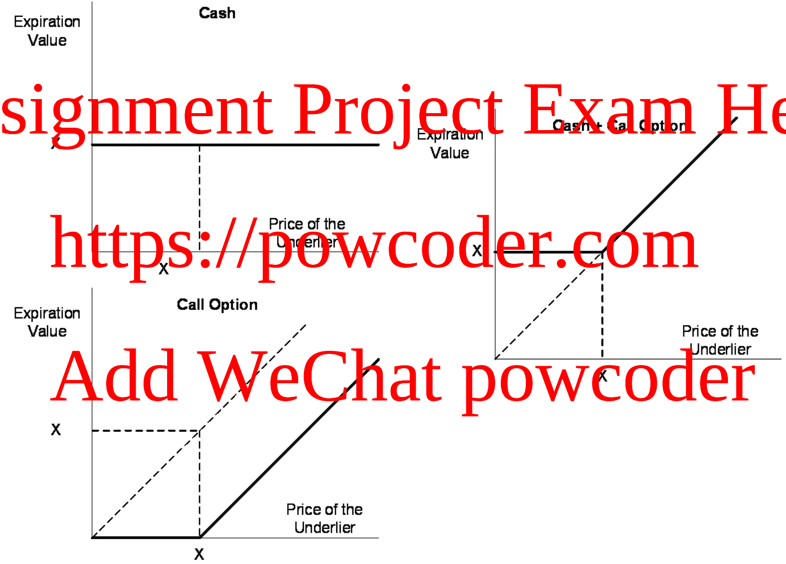
Options

- Call option vs. Put option:
  - Call option: entitles the right to *buy* an underlying asset (say shares, foreign currency or commodity) at a specified *strike price, or, exercise price*(X).
  - Put option: entitles the right to *sell* the underlying asset at a specified *strike price* X.
- European option vs. American Option
  - European put or call option: can be exercised only on expiration date.
  - American put or call option: can be exercised exercised on any date up to and including its expiration date.

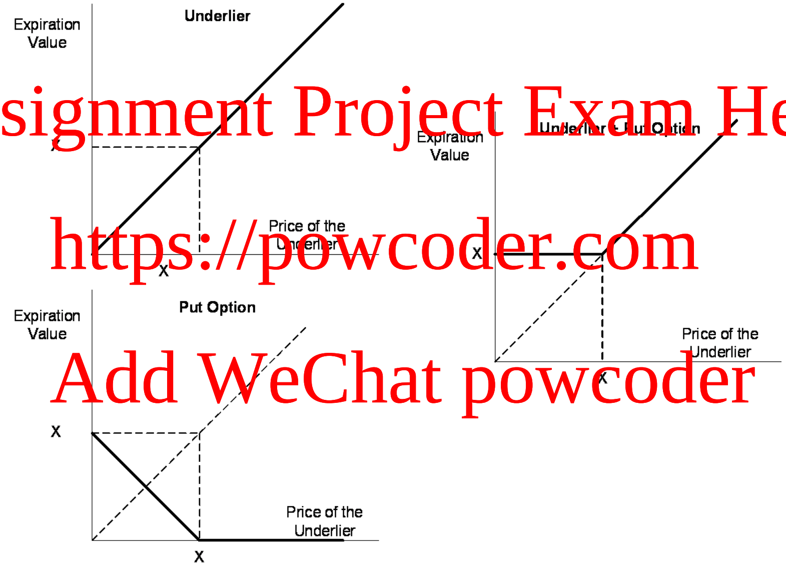
Pricing:

- European: use atomic security prices to price all net payoffs *at the end of its life*
- American: must consider a possibility of an earlier exercise

European-style option: Cash and Call



European-style option: Underlier and Put



Put-Call Parity

$$p_{Call} + PV(X) = p_{Put} + p_{underlier}$$

- Irrespective of the value of the underlier at expiration, both portfolios will have the same payoffs;
- Law of One Price – same payoffs should have the same price
- Only applies to European options.

Arrow-Debreu equilibrium:

- Individual expected utility maximisation problem

$$\max_{c_0, c_{s_1}, a_{s_1} \forall s_1 \in S_1} \quad U = u(c_0) + \beta \sum_{s_1 \in S_1} \pi_{s_1} \cdot u(c_{s_1})$$

- Given budget constraints *under all states*;

$$c_0 + \sum_{s_1 \in S_1} q_{s_1} \cdot a_{s_1} = e_0,$$

$$c_{s_1} = a_{s_1} + e_{s_1}, \forall s_1 \in S_1$$

- Prices taken as given;

- Set up Lagrangian and take first order conditions;
- Derive prices: $q_{s_1} = \frac{\lambda_{s_1}}{\lambda_0} = \beta \pi_{s_1} \frac{u'(c_{s_1})}{u'(c_0)}, \forall s_1 \in S_1$
- Combine with market clearing: $\sum_k c_0 = \sum_k e_0$; $\sum_k c_{s_1} = \sum_k e_{s_1}, \forall s_1 \in S_1$
- Characterise equilibrium c, a, qs.

Summary of Gains from Trade and Pareto improvement

- Heterogeneous consumers benefit from trade relatively to *autarky*
  - Consumption smoothing
  - Risk sharing due to difference in endowments
  - Risk sharing due to difference in risk aversions
- *Competitive* equilibrium (prices are taken as given) leads to Pareto improvement – making at least one consumer better off without making anyone worse off .

Risk neutral probabilities and Stochastic discount Factor

Forward price and stochastic discount factor

- Forward prices – risk-neutral probabilties

$$f_{s_1} = \frac{q_{s_1}}{df(1)} = \frac{q_{s_1}}{\sum_{s_1 \in S_1} q_{s_1}} = \pi_{s_1} \frac{u'(c_{s_1})}{u'(c_0)} \Big/ \sum_{s_1 \in S_1} \pi_{s_1} \frac{u'(c_{s_1})}{u'(c_0)}, \forall s_1 \in S_1$$

- Stochastic df

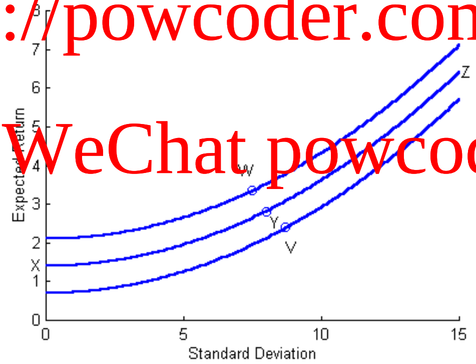
$$m_{s_1} = \beta \frac{u'(c_{s_1})}{u'(c_0)}$$

such that $df(1) = E(m_1)$

Expected mean-variance utility

$$Eu = e - (s^2/t),$$

- increasing w.r.t. e (monotonicity);
- decreasing w.r.t. s (risk aversion);
- t *risk tolerance*;

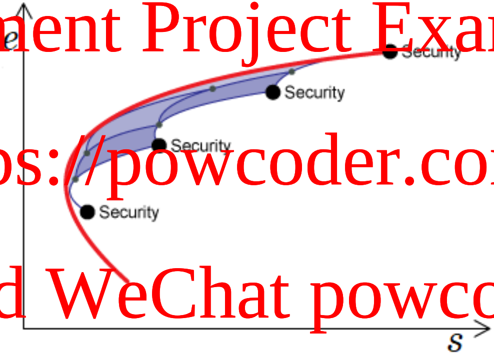


Market opportunities

Market opportunities are presented by portfolios of available securities

- the stochastically dominant section is called *efficient frontier*
- a single stock market: two symmetric line sectors from the risk free rate
- two general stocks: a rightward parabola in $e - v$ space, or a hyperbola in $e - s$ space

Efficient frontier

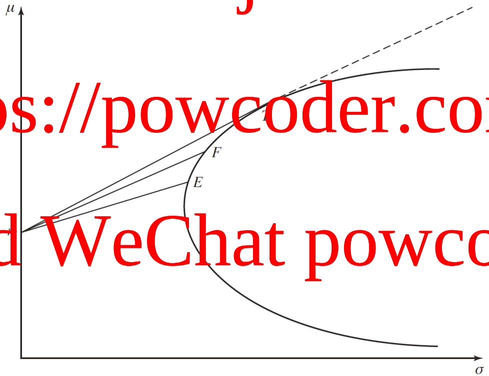


Sharpe ratio

Sharpe ratio:

$$S = \frac{e - r_f}{s},$$

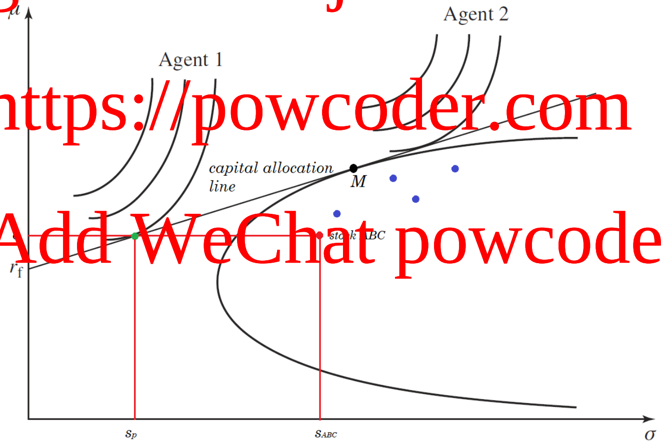
measures marginal return for risk.



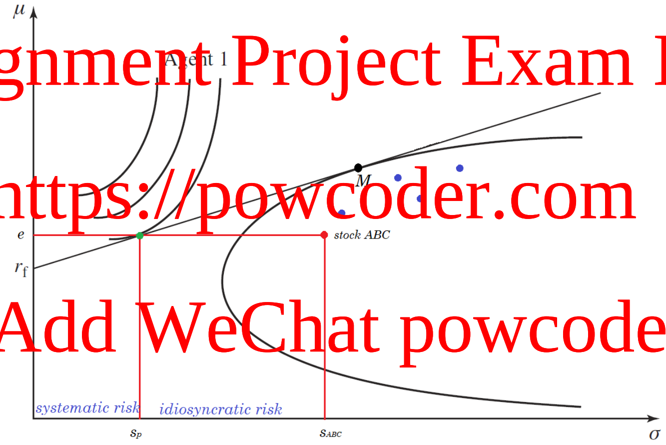
$S_M = \frac{e_M - r_f}{s_M}$ is the slope to the tangent line and therefore the *best Sharpe* ratio available on the market

Capital allocation line and separation theorem

All investors invest in the combination of the risky-free asset and the *same* market portfolio. The share of the market portfolio and risk-free asset is determined by their risk tolerance (risk-aversion)



Systematic vs Idiosyncratic risk



Capital Asset Pricing Model

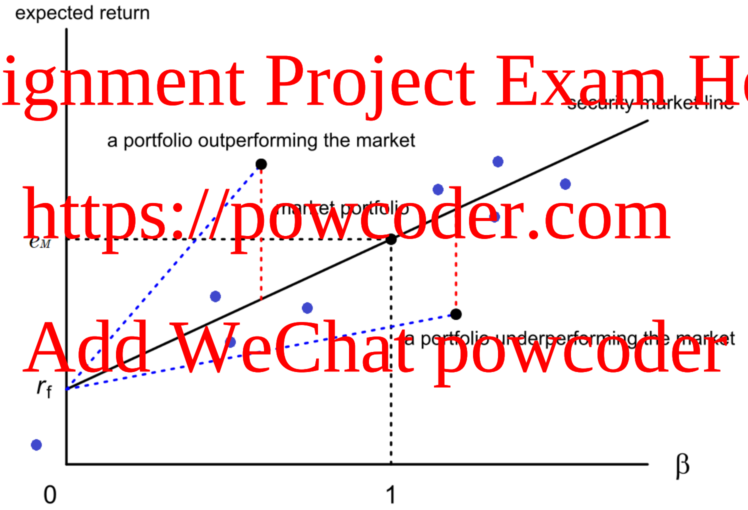
Capital asset pricing model (CAPM) is a model used to determine an appropriate expected return of any asset.

- only systematic risk is valued
- replicate any desired expected asset return e_j using the market portfolio (fraction β_j) and the risk-free asset (fraction $1–\beta_j$)

$$e_j = r_f + \beta_j(e_M - r_f)$$

- β_j: sensitivity of asset j to market movements

Security market line



Factor models

CAPM provides good benchmark, but reality is more complicated: market risk is just one factor, but there are others

$$R_j = r_f + \beta_{j,1} f_1 +, \ldots, + \beta_{j,K} f_K + \varepsilon_j,$$

- R_j is the expected return of the asset (or portfolio) j
- ε_j idiosyncratic, unexplained part of return
  $E(\varepsilon_j) = 0, \; E(R_j) = e_j$
- r_f is the risk-free rate
- f_k is the factor risk premium
- $\beta_{j,k}$ is the sensitivity of portfolio j to factor k
- K is the number of factors.

This is not *pure* arbitrage, but *statistical* arbitrage

Final exam

- Monday, 15 August, 13:00-17:00 time frame
- Two hours + 10 minute reading time
- Comprehensive – covers all course material
- Open book, but *individual* exam
- Multiple-choice, compute questions
- Exam counts for 45% of the overall mark

my Feedback matters

- The best way to thank me is to fill in myExperience survey
- We take this *very seriously* so please make your comments clear and constructive