Economics of Finance

Tutorial 2

1. An apple tree firm offers for sale a bond and stock. An apple tree produces 70GA and 45BA. The bond pays 20GA and 20BA. The stock pays 50GA and 25BA. The price of the bond is 18PA, and the price of the stock is 30PA.

(i) Construct and graph the opportunity set for future apples per present apple. Plot the bond and stock on the opportunity set. Calculate the arbitrage-free atomic prices and also plot them on the opportunity set.

Solution
The opportunity set describes what combinations of GA and BA can be obtained for 1PA. Let $\mathbf{Q}$ {states*securities} be the payment matrix of the two securities:


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Q:                  Bond  Stock
     Good Weather   20     50
     Bad Weather    20     25
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Let $\mathbf{p}_S$ {1*securities} be a vector of security prices:

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ps:      Bond  Stock
          18     30
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The matrix of the value relatives for the two securities will indicate their coordinates in the opportunity set.

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VR:                   Bond   Stock
     Good Weather   20/18    50/30
     Bad Weather    20/18    25/30
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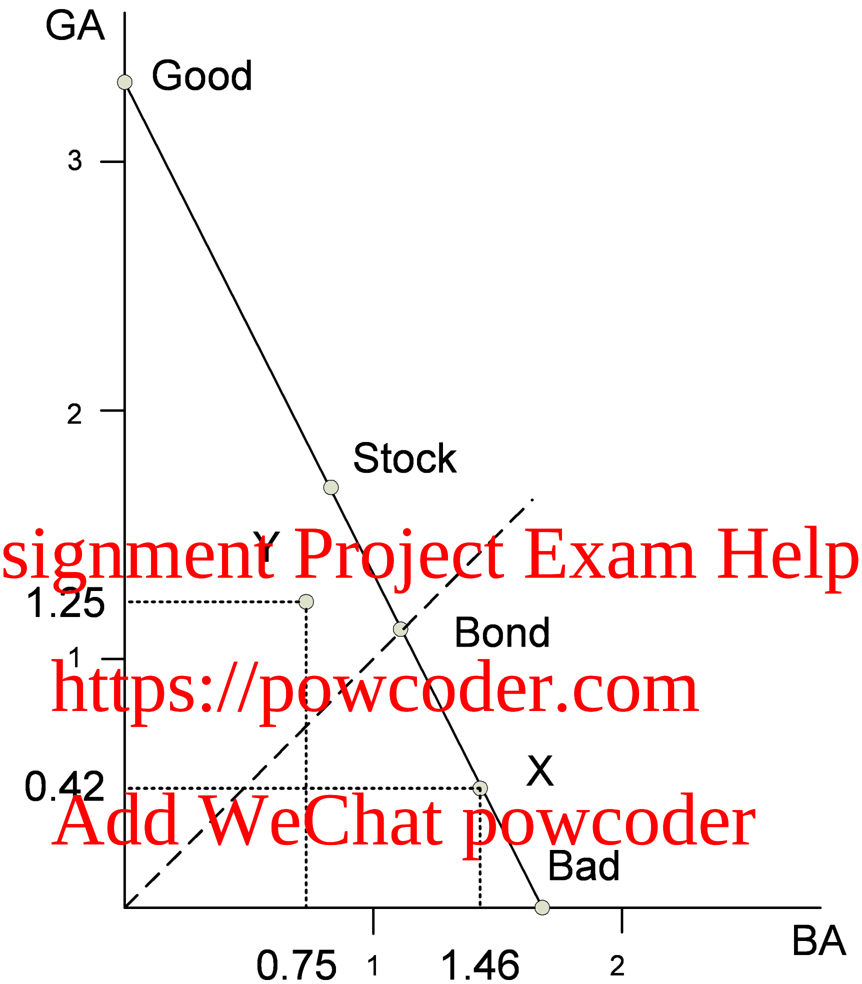
In plain words, 1PA can buy 1/18th of a bond which yields 20/18 GA and 20/18BA. The numbers 1.11 (GA) and 1.11 (BA) are the coordinates for the bond B in the opportunity set. 1PA buys 1/30th of a stock which yields 50/30GA and 25/30BA. The numbers 1.67GA and 0.83BA are the coordinates for the stock S in the opportunity set.

The vector of the atomic prices $\mathbf{p}_{atom}$ can be found as

$$\mathbf{p}_{atom} = \mathbf{p}_S \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 18 & 30 \end{pmatrix} \begin{pmatrix} 20 & 50 \\ 20 & 25 \end{pmatrix}^{-1} = \begin{pmatrix} 0.3 & 0.6 \end{pmatrix}.$$

To summarize, 1GA costs 0.3PA, and 1BA costs 0.6PA. In other words, 3.33GA cost 1PA and 1.67BA cost 1PA. These numbers indicate the coordinates for the points where the opportunity set cuts the GA and BA axes.

Figure 1: Opportunity set



(ii) Holding the payments made by both the bond and stock in each state fixed, and the price of the bond fixed, calculate the range of prices of the stock that generate positive prices for both the atomic securities. (Hint: Observe the relative positions of the bond and stock on the graph, in terms of security payoffs and prices).

Solution

Let c_B^G and c_B^B denote the payments made by the bond in the good and bad states. Similarly, let c_S^G and c_S^B denote the payments made by the stock in the good and bad states. Finally, let p_B and p_S denote the prices of the bond and stock.

The trick to this question is to note that the slope of the opportunity set is downward sloping and the coordinates on the opportunity set represent different ratios of bundles of payments to price.

In addition, recall as illustrated in the previous question, the coordinates for the atomic security points cut the GA and BA axes in the positive range.

Assuming that the stock pays more in the good state than in the bad state, then we will observe positive prices for both the atomic securities if the following two inequalities are both satisfied:

$$\frac{c_B^G}{p_B} < \frac{c_S^G}{p_S},$$
$$\frac{c_B^B}{p_B} > \frac{c_S^B}{p_S}.$$

The first inequality implies that the stock pays a higher return per PA invested if the good state is observed. The second inequality states that the bond pays a higher return per PA invested if the bad state is observed. If the stock pays a higher return in both states, then you can earn arbitrage profits by going long on the stock and short on the bond. If the bond pays a higher return in both states, then you should go long on the bond and short on the stock. If both inequality signs are reversed, then everything is fine again, except that the stock now pays more in the bad state than in the good state. This means effectively that the good and bad states should be relabelled.

In the question you are asked to construct bounds on p_S. Rearranging the inequality constraints with p_S on the left hand side, we obtain the following:

$$p_S < p_B \frac{c_S^G}{c_B^G},$$
$$p_S > p_B \frac{c_S^B}{c_B^B}.$$

Substituting the values $c_B^G = 20$, $c_B^B = 20$, $c_S^G = 50$, $c_S^B = 25$ and $p_B = 18$, we obtain that $p_S < 45$ and $p_S > 22.5$. The actual price $p_S = 30$ clearly lies within this range.

Alternatively,

$$\mathbf{P}_{atom} = \mathbf{P}_S \cdot \mathbf{Q}^{-1}$$

$$\begin{pmatrix} p_G \\ p_B \end{pmatrix} = \begin{pmatrix} 18 & p_S \end{pmatrix} \cdot \begin{pmatrix} -0.05 & 0.10 \\ 0.04 & -0.04 \end{pmatrix}$$

since $p_B = 18$.

Therefore positive atomic prices require

$$p_G = 18 \cdot -0.05 + p_S \cdot 0.04 > 0$$
$$p_B = 18 \cdot 0.10 + p_S \cdot -0.04 > 0,$$

thus $p_S > 22.5$ and $p_S < 45$.

(iii) A new security X appears on the market that pays 10GA and 35BA and has a price of 24PA. Plot this security on the opportunity set. Does it provide any arbitrage opportunities? If so, design a profitable arbitrage strategy.

Solution

1PA buys 1/24th X which yields 10/24GA and 35/24BA. These numbers 0.42 and 1.46 provide the coordinates for X in the opportunity set. Note that security X lies on the opportunity set frontier.

This means that there are no arbitrage opportunities. To see that, notice that the security X can be derived from the existing Bond and Stock. The cash flow vector associated with the security X is given by

$$\mathbf{c}_X = \begin{pmatrix} 10 \\ 35 \end{pmatrix}.$$

Therefore, the replicating portfolio will be

$$\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c}_X = \begin{pmatrix} 20 & 50 \\ 20 & 25 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 35 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

The same time-state payoffs as provided by X can be obtained by buying 3 bonds and selling 1 stock. The total price of this portfolio is $3 \times 18 - 1 \times 30 = 24PA$. This means that X is trading at its arbitrage free price.

(iv) Another security Y appears on the market that pays 50GA and 30BA and has a price of 40PA. Plot this security on the opportunity set. Does it provide any arbitrage opportunities? If so, design a profitable arbitrage strategy.

Solution

1PA buys 1/40th Y which yields 50/40GA and 30/40BA. These numbers 1.25 and 0.75 provide the coordinates for Y in the opportunity set.

The security Y creates arbitrage opportunities if it can be shorted since it lies below the opportunity set frontier. The same time-state payoffs can be obtained from the bond and stock as follows:

$$\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c}_X = \begin{pmatrix} 20 & 50 \\ 20 & 25 \end{pmatrix}^{-1} \begin{pmatrix} 50 \\ 30 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.8 \end{pmatrix}$$

The price of the replicating portfolio is $18 \times 0.5 + 30 \times 0.8 = 33$PA. Hence, the arbitrage-free price of any security providing the same time-state payments as Y must be 33PA. Since Y is traded for 40PA, you will make a profit of 7PA by buying 0.5 bonds, 0.8 stocks and selling 1Y. Furthermore, you will remain perfectly hedged.

2. Suppose there are three possible states of the world in the next period, denoted by good weather (GW), fair weather (FW) and bad weather (BW). Also, three securities are available on the market with payoffs in each state listed below.

	Bond	Stock	Security X
GW	20	43	26
FW	20	35	10
BW	20	30	0

(i) Suppose an investor desires the following portfolio:

$$\underset{(states \times 1)}{\mathbf{c}} = \begin{pmatrix} 10 \\ 15 \\ 20 \end{pmatrix}.$$

Is it possible? Why?

Solution

The payment matrix below

$$\mathbf{Q} = \begin{pmatrix} 20 & 43 & 26 \\ 20 & 35 & 10 \\ 20 & 30 & 0 \end{pmatrix}$$

is singular. This means that the payment vectors associated with the three securities are not linearly independent. Indeed the time-state payments of X can be replicated by buying 2 units of the stock and selling 3 units of the bond.

In general, this means that it will be impossible to construct a portfolio that generates the time-state payoffs in **c**. In special cases, however, it might be possible. To check this, construct a portfolio that delivers the desired time-state claims for two of the three states. Then, calculate the payoff of the constructed portfolio in the remaining state. Check is the payoff equals to the desired one. For instance, to obtain 10GA and15FA one needs to obtain the following portfolio:

$$\begin{pmatrix} 20 & 43 \\ 20 & 35 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 15 \end{pmatrix} = \begin{pmatrix} 1.8438 \\ -0.625 \end{pmatrix} \begin{array}{l} \text{bond} \\ \text{stock} \end{array}$$

Buying today 1.8438 bonds and shorting -0.625 stocks will generate the following payoff in the bad weather state:

$$\begin{pmatrix} 20 & 30 \end{pmatrix} \begin{pmatrix} 1.8438 \\ -0.625 \end{pmatrix} = 18.125$$

The constructed portfolio does not generate the required payoff in the bad state (i.e., 20). Hence, it is impossible to generate the state-contingent payoffs in **c** by trading the bond, the stock and the security X.

(ii) The prices of the three securities are as follows: $p_{Bond} = 10$, $p_{Stock} = 20$, $p_X = 15$. Are there arbitrage opportunities? If so, design a profitable arbitrage strategy; if not, explain why.

Solution

Yes, there are arbitrage opportunities. Buying two stocks and selling short three bonds generates the same time-state payment vector as holding one unit of X. Buying the replicating portfolio (i.e., buying two stocks and shorting three bonds) and selling short X, yields a perfectly hedged portfolio and a \$5 profit.

That you make a profit of \$5 can be deduced as follows: $-2 \times 20 + 10 \times 3 + 1 \times 15 = 5$. That you are fully hedged can be demonstrated as follows:

	2S	-3B	-X	
GW	86	-60	-26	=0
FW	70	-60	-10	=0
BW	60	-60	0	=0

.

(iii) Suppose security X disappears from the market.

(a) Construct a portfolio that pays 10 in GW and 15 in FW. What is the arbitrage-free price of this portfolio? What is its payoff in the bad weather state?

Solution
This case has already been worked out above. The required portfolio consists of 1.84375 bonds and -0.625 stocks. The arbitrage-free price of this portfolio is $10 \times 1.84375 - 20 \times 0.625 = 5.9375$. Its payoff in the bad weather state is $20 \times 1.84375 - 30 \times 0.625 = 18.125$.

(b) Construct a portfolio that pays 10 in GW and 20 in BW. What is the arbitrage-free price of this portfolio? What is its payoff in the fair weather state?

Solution
The required portfolio is derived as follows:

$$\begin{pmatrix} 20 & 43 \\ 20 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 2.1538 \\ -0.769 \end{pmatrix}$$

The arbitrage-free price of this portfolio is $10 \times 2.1538 - 20 \times 0.769 = 6.158$.

Its payoff in the fair weather state is $20 \times 2.153846 - 35 \times 0.76923 = 16.15385$.

(c) Construct a portfolio that pays 15 in FW and 20 in BW. What is the arbitrage-free price of this portfolio? What is its payoff in the good weather state?

Solution
The required portfolio is derived as follows:

$$\begin{pmatrix} 20 & 35 \\ 20 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 15 \\ 20 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -1 \end{pmatrix}$$

The arbitrage-free price of this portfolio is $10 \times 2.5 - 20 \times 1 = 5$.

Its payoff in the good weather state is $20 \times 2.5 - 43 \times 1 = 7$.

(d) Is it possible for an investment firm to provide the time-state claims in (i) to a client without incurring any risk? Explain.

Solution
No it is not possible for an investment firm to provide the time-state claims in (i) without

incurring risk. In other words, the investment firm cannot fully hedge this position using just the bond and stock. For example, the investment firm could buy 1.103448×1.84375 bonds and sell 1.103448×0.625 stocks, and then provide **c** at a price of 6.551724. The investment firm would then cover its costs and receive an additional 0.103448×10 in good weather, 0.103448×15 in fair weather and 0 in bad weather. The fact that its total payoff differs across states means that it is not fully hedged.

(e) What is the minimum cost to for the investor? (Hint: use optimisation function.)

Solution

Linear Programming in Matlab

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>> Q =[20 43; 20 35; 20 30]
>> c =[10 15 20]'
>> ps = [10 20]
>> n = linprog (ps', -Q, -c);

n =

    2.1538
   -0.7692

>> p =ps*n

p =

    6.1538
```


(f) What is the minimum cost if X becomes available? Comment on your results.

Solution

With X appears in the economy it is impossible to find minimum, because as we verified, the market present arbitrage opportunity. The investor's optimum strategy consists of exploiting the arbitrage opportunity as much as possible and generate profit. No cost incurs and solution is unbounded.