Lecture 2: Valuation, Atomic Prices, Complete and Incomplete Markets

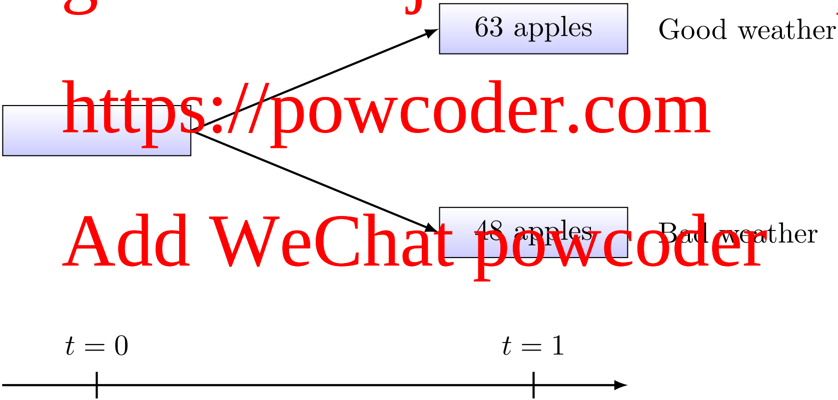
Economics of Finance

School of Economics, UNSW[1]

Valuation

Definition: *Valuation* is the process of determining the *present value* of a security or productive investment.

Example: How much is a tree worth today (at time 0)?



Present Value of a tree: $PV = 0.285 \cdot 63 + 0.665 \cdot 48 = 49.875$

In matrix notation

The vector of atomic prices: ($1 \times$states)

$$\mathbf{p} = \begin{array}{cc} \begin{pmatrix} 0.285 & 0.665 \end{pmatrix} \\ \text{Good Weather} \quad \text{Bad Weather} \end{array}$$

The vector of quantities: (states$\times 1$)

$$\mathbf{q} = \begin{pmatrix} 63 \\ 48 \end{pmatrix} \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array}$$

Present value

The Present Value:

$$PV = \mathbf{p} \cdot \mathbf{q} = \begin{pmatrix} 0.285 & 0.665 \end{pmatrix} \begin{pmatrix} 63 \\ 48 \end{pmatrix} = 49.875$$

Check the dimensions: $\underbrace{(1 \times states)}_{\mathbf{p}} \cdot \underbrace{(states \times 1)}_{\mathbf{q}} = \underbrace{(1 \times 1)}_{PV}$

MATLAB matrix operations

The following MATLAB commands will do the job:


```
>> p = [0.285 0.665]
p =
    0.2850 0.6650
>> q = [63 48]'
q =
    63
    48
>> PV = p*q
PV =
    49.8750
```

Net Present Value

The *net present value* of a set of claims is based on future payments and any required payment in the present. If the Tree is purchased for 49.875 apples

$$NPV = \begin{pmatrix} 1.00 & 0.285 & 0.665 \end{pmatrix} \times \begin{pmatrix} -49.875 \\ 63 \\ 48 \end{pmatrix}$$

$$NPV = -49.875 + 0.285 \times 63 + 0.665 \times 48 = 0$$

The net present value of a fairly priced investment is zero.

Net Present Value (cont'd)

Assume you discover how to plant 60 apples in a way that will produce 100 apples if the weather is good and 50 apples if the weather is bad. Compute the net present value:

$$NPV = -60 + 0.285 \times 100 + 0.665 \times 50 = 1.75$$

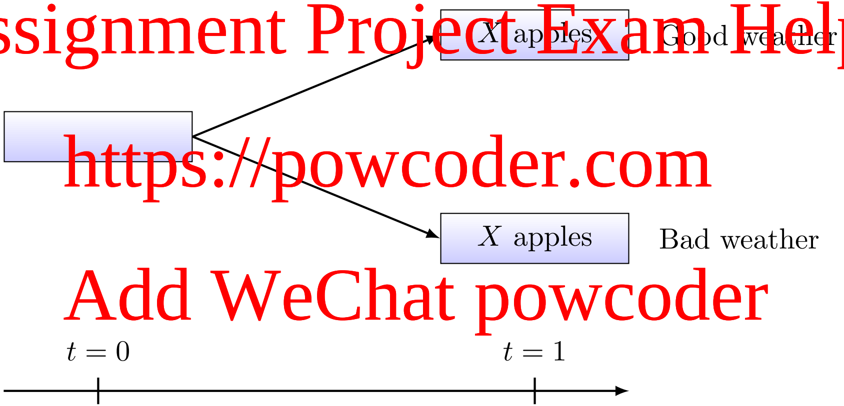
Should you do it? YES. Why?

Riskless Securities

Definition: A *riskless security* pays the same amount at a given time, no matter what state of the world occurs.

- A *riskless security* is equivalent to a bundle of equal amounts of atomic claims for a time period.
- In our example, a riskless security pays a fixed amount (say X apples) at time period 1, whether the weather has been good or bad.
- Equivalently, it is a bundle of X good weather apples (GA) and X bad weather apples (BA).

Riskless Securities



Riskless Securities

Question: What is the present value of a riskless security that pays 20 apples in time 1?

The vector of atomic prices: ($1 \times$states)

$$\mathbf{p} = \begin{array}{cc} \begin{pmatrix} 0.285 & 0.665 \end{pmatrix} \\ \text{Good Weather} \quad \text{Bad Weather} \end{array}$$

The vector of quantities: (states$\times 1$)

$$\mathbf{q} = \begin{pmatrix} 20 \\ 20 \end{pmatrix} \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array}$$

The Present Value:

$$PV = \mathbf{p} \cdot \mathbf{q} = \begin{pmatrix} 0.285 & 0.665 \end{pmatrix} \begin{pmatrix} 20 \\ 20 \end{pmatrix} = 0.285 \cdot 20 + 0.665 \cdot 20 = 19$$

The Discount Factor

Definition: *The discount factor* (for a certain date) represents the present value of a payment of one unit to be made with certainty at the specified future date.

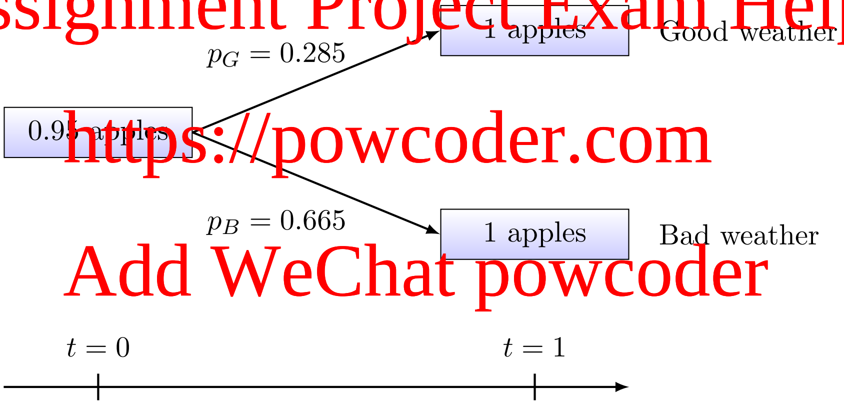
The tree example:

$$PV = \mathbf{p} \cdot \mathbf{q} = \underbrace{(0.285 + 0.665)}_{\text{sum of the atomic prices}} \cdot 20 = \underbrace{0.95}_{\text{df}(1)} \cdot 20 = 19$$

- The discount factor for a date in question equals to the sum of appropriate atomic prices (prices of basic atomic securities)

The Discount Factor

E.g. $df(1) = 0.95$



Financing Methods

Say you'd like to set up an apple firm which consists of an apple tree, i.e., you need 49.875 apples to purchase the tree.

There are two ways to finance this investment, issue *bonds* or issue *stocks*. Assume your firm issues a bond:

`The Apple Tree Firm promises to pay the holder 20 apples at the end of the year, no matter what the weather has been.`

This way the holder does not bear any face value risk (though other types of risk, e.g., default risk or interest rate risk, etc., remain).

Stock

If your firm issues a *stock*:


```
The Apple Tree Firm promises to pay the holder all
the apples left over after the bondholder has been
paid
```


This way the holder bear the risk of the apple production *net the issued bond payment*, BUT is entitled a voting right.

The bond represents the ownership of the money, i.e., *prior claim*; the stock represents the ownership of the firm, i.e., *residual claim*.

What is the bond worth? What is the stock worth? Payment vectors are:

$$\mathbf{q_{firm}} = \begin{pmatrix} 63 \\ 48 \end{pmatrix}; \ \ \mathbf{q_{bond}} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}; \ \ \mathbf{q_{stock}} = \begin{pmatrix} 43 \\ 28 \end{pmatrix}$$

The values are:

$$\begin{aligned} \mathbf{p} \times \mathbf{q_{firm}} &= \mathbf{49.875}; \\ \mathbf{p} \times \mathbf{q_{bond}} &= \mathbf{19.000}; \\ \mathbf{p} \times \mathbf{q_{stock}} &= \mathbf{30.875}; \end{aligned}$$

Note that $\mathbf{p} \times \mathbf{q_{firm}} = \mathbf{p} \times (\mathbf{q_{bond}} + \mathbf{q_{stock}})$. This is called *Principle of value additivity.*

Shareholding Structure

We know from principle of value additivity that

- either issuing bond or issuing stock or any proportion of each, it wouldn't change apple prices $\mathbf{p}$

- Neither does it change the apple tree production and firm's value $\mathbf{p} \times \mathbf{q_{firm}}$

How does shareholding structure matter?

Net worth

The key is shareholder's risk! Consider the instead of issuing 1 bond, you issue 2 bonds which takes a value of

$$19 \times 2 = 38,$$

which presents a liability of

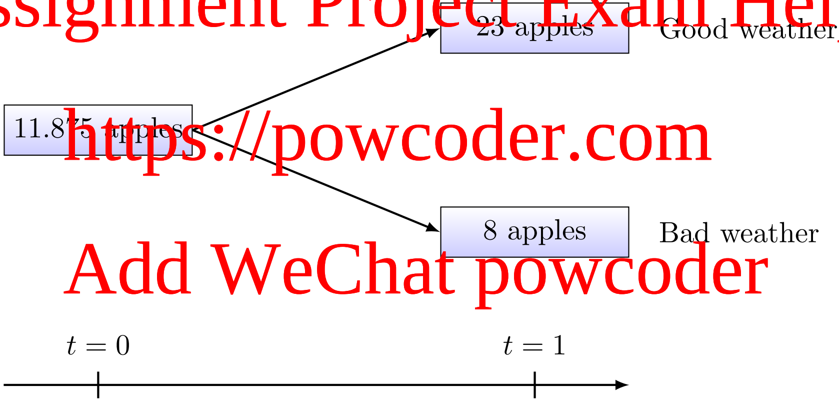
$$\mathbf{q_{bond}} \times \mathbf{2} = \begin{pmatrix} 40 \\ 40 \end{pmatrix}$$

in period 1. Net worth of the share is now

$$49.875 - 38 = 11.875$$

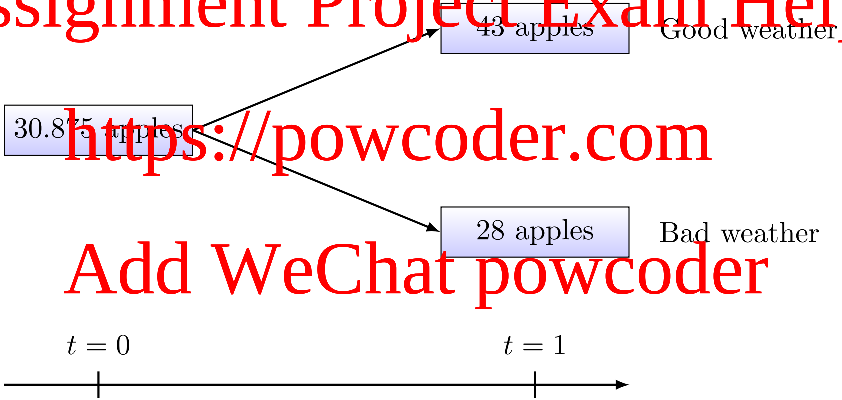
Shareholder

The shareholder's payment structure is now:



Shareholder II

Instead of:



Take-outs

While avoiding diluting voting right, issuing more bond makes shareholder's net worth more risky.

- In practice, the way to finance an investment depends on the director board's attitude towards risks.
- An aggressive, dictating director board is more likely to issue more unites of bonds, and bears more risk;

- An modest, cooperative director board however, is more likely to issue less bonds, issue more stocks and share risks with other shareholders;

- Usually it also involve more complex factors, e.g., the bond market capacity, possibility of merger and acquisition, or even political factors.

Inferring Atomic Security Prices

- Until now we have assumed that dealers stand ready to buy and sell basic atomic securities;
- Even though there are financial instruments that resemble atomic securities (e.g. insurance policy) this assumption is not very realistic.

- We will relax this assumption and consider the world in which only two securities are traded on a regular basis:
  - The common stock of the Apple Tree Firm.
  - The riskless bond of the Apple Tree Firm.

Inferring Atomic Security Prices

Let $\mathbf{Q}$ {states $\times$ securities} be the payment matrix of the two securities:

$$\underset{(2\times 2)}{\mathbf{Q}} = \begin{array}{cl} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

Let $\mathbf{p}_S$ {1$\times$securities} be a vector of security prices:

$$\underset{(1\times 2)}{\mathbf{p_S}} = \begin{array}{c} \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} \end{array}$$

Let $\mathbf{n}$ {securities$\times$1} be a vector of *portfolio holdings*:

$$\underset{(2\times 1)}{\mathbf{n}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{array}{l} \text{number of Bonds} \\ \text{number of Stocks} \end{array}$$

Inferring Atomic Security Prices

Let $\mathbf{c}$ {states$\times$1} be the vector of payments in each state, then it must hold that

$$\underset{(\text{states}\times\text{securities})}{\mathbf{Q}} \cdot \underset{(\text{securities}\times 1)}{\mathbf{n}} = \underset{(\text{states}\times 1)}{\mathbf{c}}$$

In our example the above identity reads as

$$\underset{(2\times 2)}{\mathbf{Q}} \cdot \underset{(2\times 1)}{\mathbf{n}} = \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 106 \\ 76 \end{pmatrix}$$

where the vector of state-contingent payments is

$$\underset{(2\times 1)}{\mathbf{c}} = \begin{pmatrix} 106 \\ 76 \end{pmatrix} \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array}$$

Obtaining a desired portfolio

Question: What portfolio $\mathbf{n}$ will provide a desired set of state state-contingent payments $\mathbf{c}$?

If the payoff matrix $\mathbf{Q}$ is invertible, then the answer is simple.

$$\underset{(\text{securities}\times 1)}{\mathbf{n}} = \underset{(\text{securities}\times\text{states})}{\mathbf{Q}^{-1}} \cdot \underset{(\text{states}\times 1)}{\mathbf{c}}$$

Note: If a matrix $\mathbf{Q}$ satisfies the following conditions:

(i) $\mathbf{Q}$ is a square matrix i.e. its number of rows equals to its number of columns;

(ii) $\mathbf{Q}$ is non-singular i.e. its rows/columns are linearly independent;

then Q^{-1} exists.

Obtaining a desired portfolio

Example: Suppose we wish to have 845 apples if the weather is good and 620 if the weather is bad.

- The vector of state-contingent payments is:

$$\underset{(2\times 1)}{\mathbf{f}} = \begin{array}{cc} \begin{pmatrix} 845 \\ 620 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array} \end{array}$$

- The payoff matrix is $\mathbf{Q}$ is invertible since its determinant is different from zero:

$$\det(\mathbf{Q}) = \det \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} = 20 \cdot 28 - 20 \cdot 43 = -300 \neq 0$$

The portfolio vector that delivers the desired state-contingent payoffs is given by $\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c}$

$$\underset{(\text{securities}\times 1)}{\mathbf{n}} = \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix}^{-1} \begin{pmatrix} 845 \\ 620 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 10 \\ 15 \end{pmatrix} \end{array} \begin{array}{l} \text{Bonds} \\ \text{Stocks} \end{array}$$

Atomic Security Prices

To obtain payment $\mathbf{c}$, we can buy a portfolio $\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c}$. This portfolio will cost us

$$\mathbf{p} = \mathbf{p_S} \cdot \mathbf{n} = \left[\mathbf{p_S} \cdot \mathbf{Q^{-1}}\right] \mathbf{c}$$

Therefore, we can infer atomic security prices from the prices and payments of the traded securities:

$$\underset{(1\times\text{states})}{\mathbf{p}_{atom}} \equiv \underset{(1\times\text{securities})}{\mathbf{p_S}} \cdot \underset{(\text{securities}\times\text{states})}{\mathbf{Q}^{-1}}$$

$\mathbf{Q^{-1}}$ revisited

Recall that $\mathbf{Qn} = \mathbf{c}$ and that $\mathbf{n} = \mathbf{Q^{-1}c}$. Say you'd wish to find a portfolio such that $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This portfolio is given by the first column of $\mathbf{Q^{-1}}$. In our example

$$\mathbf{n} = \mathbf{Q}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.0933 & 0.1433 \\ 0.0667 & -0.0667 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.0933 \\ 0.0667 \end{pmatrix}$$

What is the present value of $\mathbf{n}$?

$$\mathbf{p_s n} = \begin{pmatrix} 19.0 & 30.875 \end{pmatrix}\begin{pmatrix} -0.0933 \\ 0.0667 \end{pmatrix}$$

$$19 \times (-0.0933) + 30.875 \times 0.0667 = -1.7727 + 2.0593 = 0.285$$

Atomic Security Prices

Example: How much would it cost to get 845 GA and 620 BA?

$$\mathbf{p} = \mathbf{p_S} \cdot \mathbf{n} = \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \begin{pmatrix} 10 \\ 15 \end{pmatrix} = 653.125$$

The prices of the atomic securities can be inferred from

$$\mathbf{p}_{atom} = \mathbf{p_S} \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix}^{-1} = \begin{array}{cc} \begin{pmatrix} 0.285 & 0.665 \end{pmatrix} \\ \text{Good W.} \quad \text{Bad W.} \end{array}$$

Using inferred prices of the atomic securities we can price $\mathbf{c}$ as follows

$$\mathbf{p} = \mathbf{p}_{atom} \cdot \mathbf{c} = \begin{pmatrix} 0.285 & 0.665 \end{pmatrix} \begin{pmatrix} 845 \\ 620 \end{pmatrix} = 653.125$$

Example in MATLAB

```
> > Q = [20 43; 20 28];
>> ps = [19 30.875];
>> c = [845; 620];
>> n = inv(Q)*c
n =
     10.0000
     15.0000
>> p = ps*n
p =
     653.1250
>> p_atom = ps*inv(Q)
p_atom =
     0.2850 0.6650
```

Another look

In our example $\mathbf{c} = \begin{pmatrix} 845 \\ 620 \end{pmatrix}$, so given $\mathbf{Q}$, our problem is

$$\begin{aligned} 20n_1 + 43n_2 &= 845 \\ 20n_1 + 28n_2 &= 620 \end{aligned}$$

We could use the first equation

$$n_1 = \left(\frac{845}{20} - \frac{43}{20}n_2\right)$$

and use the result in the second

$$20\left(\frac{845}{20} - \frac{43}{20}n_2\right) + 28n_2 = 845 - 15n_2 = 620; n_2 = 15.$$

The Opportunity Set

Suppose you have a dollar.
What opportunity can you get from the market?

Value Relative

Definition: Value relative associated with a given state of nature is the future payment per unit invested that will be received if that state occurs.

In our example the matrix for the value relatives is:

$$\underset{(2\times 2)}{\mathbf{vr}} = \begin{array}{cc} \begin{pmatrix} 20/19 & 43/30.875 \\ 20/19 & 28/30.875 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

$$= \begin{array}{cc} \begin{pmatrix} 1.0526 & 1.3927 \\ 1.0526 & 0.9069 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

Octave(Matlab)

The following script will do the job in the command prompt:

```
>> Q = [20 43; 20 28];
>> ps = [19 30.875];
>> vr = Q./[ps; ps];
```

Value relative and return

Value of relative is the percentage value of an ending value in terms of initial value.

E.g., if the weather is good, the value relative of a GA is `1/0.285 = 3.5088`. If the weather is bad, the value relative of a GA is `0/0.285 = 0`.

Return is value relative, net 100%

$$\texttt{return} = \texttt{vr} - 1$$

An atomic security returns `-100%` in all states except the one it corresponds to.

The Opportunity Set

Definition: The opportunity set is the set of feasible future payoffs available with a wealth of one unit of present value.

Example: The opportunities for future apples for a present apple invested:

$\mathbf{Q}$ {states*securities} is the payment matrix of the two securities:

$$\underset{(2\times 2)}{\mathbf{Q}} = \begin{array}{cc} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} & \begin{array}{c} \text{Good Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

$\mathbf{p}_S$ {1*securities} is a vector of security prices:

$$\underset{(1\times 2)}{\mathbf{p_S}} = \begin{array}{c} \begin{pmatrix} 19.0 & 30.875 \end{pmatrix} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} \end{array}$$

Derivative securities

By combining existing securities (the bond and the stock), one can synthesize a security that does not exist (e.g. a state-contingent claim).

The result is often termed a *derivative* security.

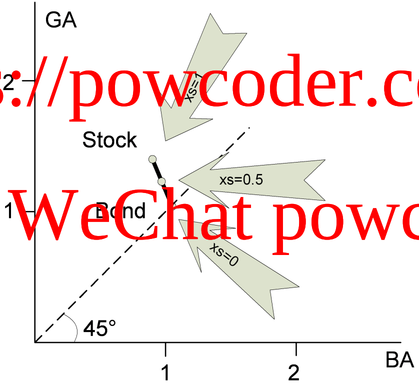
Opportunity set

- Let $\mathbf{vrb} = \begin{pmatrix} 1.0526 \\ 1.0526 \end{pmatrix}$ be the value relative for the bond;
- Let $\mathbf{vrs} = \begin{pmatrix} 1.3927 \\ 0.9069 \end{pmatrix}$ be the value relative for the stock;
- Let xs denote a proportion of wealth invested in the stock, then the value relative for the portfolio is given by

$$\begin{aligned} \mathbf{vrp} &= xs \cdot \mathbf{vrs} + (1 - xs) \cdot \mathbf{vrb} \\ &= xs \cdot \begin{pmatrix} 1.3927 \\ 0.9069 \end{pmatrix} + (1 - xs) \cdot \begin{pmatrix} 1.0526 \\ 1.0526 \end{pmatrix} \end{aligned}$$

How much can we get from one apple?

By choosing a portfolio that includes positive (long) positions in the Bond and in the Stock with a total present value of 1 apple, an investor can obtain any position on the line segment connecting the two securities in the Figure



Shorting securities

What about negative (short) positions in either security?

Opportunity Set Frontier

Suppose, one can take negative positions in a security as long as investor's overall portfolio does not lead to negative net payments in any state of the world

- Then, an investor can obtain any point on the line through **Bond** and **Stock** extended all the way to the axes (see the next figure)

- Value relative of the atomic securities can be found as follows:

$$\mathbf{p}_a = \mathbf{p}_S \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 19 & 30.875 \end{pmatrix} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix}^{-1} = \begin{pmatrix} 0.285 & 0.665 \end{pmatrix}$$

$$\underset{(2\times 2)}{\mathbf{vr}_{atom}} = \begin{array}{c} \begin{pmatrix} 1/0.285 & 0/0.665 \\ 0/0.285 & 1/0.665 \end{pmatrix} \\ \text{GW claim} \quad \text{BW claim} \end{array} = \begin{array}{cc} \begin{array}{c} \begin{pmatrix} 3.5088 & 0 \\ 0 & 1.5038 \end{pmatrix} \\ \text{GW claim} \quad \text{BW claim} \end{array} & \begin{array}{l} \text{Good W.} \\ \text{Bad W.} \\ \end{array} \end{array}$$

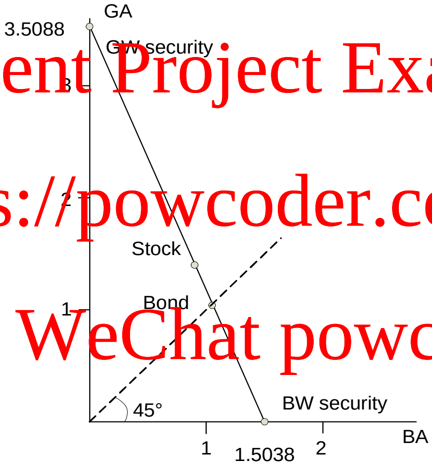
Plotting Opportunity Set

If we add to `vr` the value relatives of the atomic securities we obtain:

	Good	Bond	Stock	Bad
Good Weather	3.5088	1.0526	1.3927	0
Bad Weather	0	1.0526	0.9069	1.5038

These are points we can plot in the space of GA and BA.

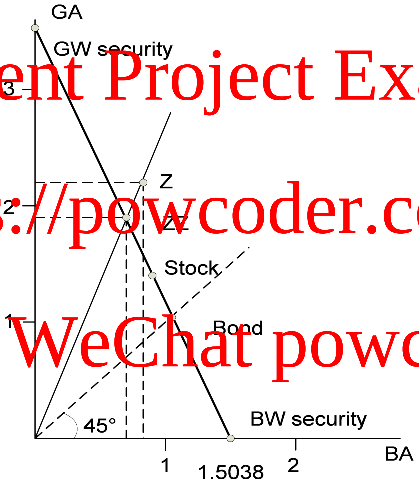
The Opportunity Set



The Opportunity Set: Remarks

- Taking a negative position in the Stock amounts to signing a document of the form: “I promise to pay the holder whatever the firm (tree) pays its stockholders”
- By combining (in the right proportions):
  - a long position in the Bond with a short position in the Stock one can construct a pure “Bad Weather Claim”
  - a short position in the Bond with a long position in the Stock one can construct a pure “Good Weather Claim”
- By combining existing securities (the Bond and the Stock) one can synthesize a security that does not exist (e.g. a Good Weather claim). The result is termed a *derivative security*, since it is derived from the existing securities.

Arbitrage Opportunities



Arbitrage Opportunities (cont.)

- Any security not priced in accordance with the atomic prices implied by the traded securities will present an opportunity for arbitrage
- For example, imagine a security Z appears outside the opportunity set frontier
  - Draw a line through Z to the origin; Denote ZZ the point where the line intersects the opportunity set frontier;
  - Payments ZZ can be obtained by a portfolio of the Bond and the Stock worth 1 PA;
  - Sell ZZ short, and use the proceeds (1 PA) to buy Z;
  - Z pays more that ZZ (per apple invested) in every state of the world, hence we obtain **arbitrage opportunity**