Lesson 3: Hedging with Minimum Cost, Multi-period Discounting, Bonds

Economics of Finance

School of Economics, UNSW

Hedging at Minimum Cost

Market is incomplete:

$$\underset{(3\times 1)}{\mathbf{c}} = \begin{array}{cl} \begin{pmatrix} 40 \\ 30 \\ 20 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{array} \end{array} \qquad \underset{(3\times 2)}{\mathbf{Q}} = \begin{array}{cl} \begin{pmatrix} 20 & 43 \\ 20 & 28 \\ 20 & 28 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

$$\underset{(1\times 2)}{\mathbf{ps}} = \begin{array}{c} \begin{pmatrix} 19 & 35 \end{pmatrix} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} \end{array}$$

Our objective is to construct the cheapest portfolio, $\mathbf{n}$, that will deliver as least as much as $\mathbf{c}$ in every state of nature.

Linear programming

- Our problem is to select a portfolio, $\mathbf{n}$, to minimize its cost, $\mathbf{p_S} \cdot \mathbf{n}$, subject to obtaining no less than the required state contingent payment, $\mathbf{c}$.
- A constrained optimization problem we have to solve is given by

$$\min_{\mathbf{n}} \mathbf{p_S} \cdot \mathbf{n} \text{ subject to } \mathbf{Q} \cdot \mathbf{n} \geq \mathbf{c}.$$

- Note: we use the sign $\geq$ to indicate that every element of vector $\mathbf{Q} \cdot \mathbf{n}$ is no less than the corresponding element of vector $\mathbf{c}$.
- We are facing a *linear programming* problem, or simply the problem of finding a vector that minimizes a linear function subject to linear constraints.

Hedging at Minimum Cost

Matlab functions provide us with several tools for solving linear programming problems. They solve this general problem (I use their notation):

$$\min_{\mathbf{x}} \mathbf{f}' \cdot \mathbf{x} \text{ subject to } \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}.$$

- `linprog(f, A, b)` in Matlab

Linear programming

We will ask this function to perform the simplest task:

- In our context, the role of $\mathbf{x}$ played by portfolio vector, $\mathbf{n}$. Note $\mathbf{f}$ is assumed as a column vector. The role of $\mathbf{f}'$ is performed by row vector $\mathbf{p_S}$.
- this function deals only with inequalities of the form: $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$. Let's multiply both sides of our constraint, $\mathbf{Q} \cdot \mathbf{n} \geq \mathbf{c}$, by -1 to obtain $-\mathbf{Q} \cdot \mathbf{n} \leq -\mathbf{c}$
- Hence, the role is $\mathbf{A}$ is played by $-\mathbf{Q}$, while the role of $\mathbf{b}$ is played by $-\mathbf{c}$.

Matlab: Hedging at Minimum Cost

Enter the data in MATLAB's command prompt:

```
>> Q = [20 43; 20 28; 20 28];
>> c = [40 30 20]';
>> ps = [19 35];
```

Use the linear programming function `linprog`

```
>> n = linprog(ps', -Q, -c)
```

The result of running these commands is:

```
n =
 0.5667
 0.6667
```

The price of the portfolio is:

```
>> p = ps*n
p = 34.1000
```

Wrapping up

Hedging involves fully covering contingent payments/liabilities and offsetting risks

- With complete market, this involves replicating desired payments/liabilities;
- With incomplete market, perfect hedging is can not be achieved;

- The ideal hedging then involves hedging with minimum cost;
- This can be done using Matlab "linprog" function.

- Completing the market will generally reduce deadweight loss associated with incomplete hedging.

Multi-period (Variable) Discount Factors

Definition: A nominal discount factor, $df(t)$, is the present value of one unit of currency to be paid with certainty at time t.

Notation:

- Discount factor $\{1 \times \text{periods}\}$:

$$\mathbf{df} = \begin{pmatrix} df(1) & df(2) & df(3) \end{pmatrix}$$

- Vector of cash flows known to be certain $\{\text{periods} \times 1\}$:

$$\mathbf{cf} = \begin{pmatrix} cf(1) \\ cf(2) \\ cf(3) \end{pmatrix}$$

PV

Discounted present value of the cash flows:

$$pv = \mathbf{df} \cdot \mathbf{cf}$$

Example: Coupon bonds with different maturities

- The Payment Matrix {periods×bonds}:

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{B1} & \text{B2} & \text{B3} \end{array} \\
\mathbf{Q} = & \begin{pmatrix} 103 & 4 & 3 \\ 0 & 104 & 3 \\ 0 & 0 & 103 \end{pmatrix}
\end{array}
\begin{array}{l} \text{Year 1} \\ \text{Year 2} \\ \text{Year 3} \end{array}
$$

- The Price Vector {1×bonds}:

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{B1} & \text{B2} & \text{B3} \end{array} \\
\mathbf{p} = & \begin{pmatrix} 100 & 101 & 98 \end{pmatrix}
\end{array}
$$

Inferring the discount function

- In matrix notation: $\mathbf{p} = \mathbf{df} \cdot \mathbf{Q}$
- If $\mathbf{Q}^{-1}$ exists, the discount function can be inferred as $\mathbf{df} = \mathbf{p} \cdot \mathbf{Q}^{-1}$;
- Since $\mathbf{Q}^{-1}$ exists, the discount function is

$$\underset{(1\times\text{Years})}{\mathbf{df}} = \begin{pmatrix} 100 & 101 & 98 \end{pmatrix} \begin{pmatrix} 103 & 4 & 3 \\ 0 & 104 & 3 \\ 0 & 0 & 103 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0.9708 & 0.9338 & 0.8959 \end{pmatrix}$$

- Any desired set of future certain payments over the next three years can be valued using this discount function.

Replicating bond portfolio

- To find a portfolio of the three bonds that will replicate a desired set of certain cash flows we can use

$$\underset{(\text{Years}\times\text{Bonds})}{\mathbf{Q}} \cdot \underset{(\text{Bonds}\times 1)}{\mathbf{n}} = \underset{(\text{Years}\times 1)}{\mathbf{c}}$$

- Let $\mathbf{c} = \begin{pmatrix} 300 & 200 & 100 \end{pmatrix}'$, then the replicating portfolio is

$$\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c} = \begin{pmatrix} 103 & 4 & 3 \\ 0 & 104 & 3 \\ 0 & 0 & 103 \end{pmatrix}^{-1} \begin{pmatrix} 300 \\ 200 \\ 100 \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} 2.8107 \\ 1.8951 \\ 0.97087 \end{pmatrix} & \begin{array}{c} \text{B1} \\ \text{B2} \\ \text{B3} \end{array} \end{array}$$

Multi-period Interest Rates

Investment grows from $V(0)$ to $V(t)$ in t periods,
$i(t)$ is the default-free interest rate for time t:

$$V(0)\,(1+i(t))^t = V(t)$$

Definition: The ratio of the ending value to the beginning value, $V(t)/V(0)$, is termed the (t-period) value relative.

- One dollar will grow to $1/df(t)$ dollars with certainty by time t, hence

$$(1+i(t))^t = \frac{V(t)}{V(0)} = \frac{1}{df(t)};\ i(t) = \left(\frac{1}{df(t)}\right)^{\frac{1}{t}} - 1.$$

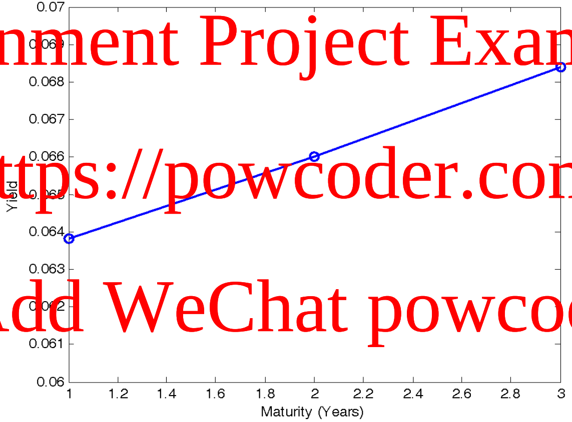
- Call $i(t)$ *multi-period interest rate*, or, *yield*.

Yield curve

The following Matlab code plot a term structure of interest rate given discount factors:

```
>>df = [0.94 0.88 0.82]
df =        0.9400 0.8800 0.8200
>> vr = 1./df
vr =        1.0638 1.1364 1.2195
>> i = vr.^(1./[1:3])-1
i =        0.0638 0.0660 0.0684
>>plot([1:3],i)
```

The plot


0.07
0.068
0.067
0.066
0.064
0.063
0.061
0.06
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
Maturity (Years)
Yield

Bond Yields (Yield to maturity)

Definition: Yield-to-maturity (YTM) is a *constant* interest rate that makes the present value of all the bond's payments equal its price.

Example: A bond is selling for \$97.84 and provides a certain vector of cash flows:

$$\mathbf{cf} = \begin{pmatrix} 6 \\ 6 \\ 106 \end{pmatrix} \begin{array}{l} \text{Year 1} \\ \text{Year 2} \\ \text{Year 3} \end{array}$$

Yield-to-maturity, y, must satisfy

$$\frac{6}{1+y} + \frac{6}{(1+y)^2} + \frac{106}{(1+y)^3} = 97.84$$

Computing Bond Yields

- **Problem:** We need to solve a non-linear equation!
- **Solution:** Use numerical techniques and tools, e.g. Octave (Matlab) `fsolve` function

- `fsolve` function looks for a solution to equation $f(y) = 0$

$$f(y) = \frac{6}{1+y} + \frac{6}{(1+y)^2} + \frac{106}{(1+y)^3} - 97.84$$

Matlab code


```
y0=0.06;  % Coupon rate is an initial guess
y=fsolve(@(y)(6/(1+y)+6/(1+y)^2+106/(1+y)^3-97.84),y0);
y =
    0.0682
Optimization terminated:  first-order optimality is
less than options.TolFun.
```

Duration

- We have all types of measurements about bond's return (yield, YTM, df, etc.) till now;

- In some applications we are interested in how sensitive is the bond value towards market interest rate;
- Consider a vector of certain cash flows associated with a bond is:

$$\mathbf{cf} = \begin{pmatrix} 6 \\ 6 \\ 106 \end{pmatrix} \begin{array}{l} \text{Year 1} \\ \text{Year 2} \\ \text{Year 3} \end{array}$$

- The market discount function is:

$$\underset{(1\times \text{Years})}{\mathbf{df}} = \begin{pmatrix} 0.94 & 0.88 & 0.82 \end{pmatrix}$$

Periodical Values and Weights

The present value of each year's cash flow:

$$\underset{(1\times\text{Years})}{\mathbf{v}} = \begin{pmatrix} df(1)\cdot cf(1) & df(2)\cdot cf(2) & df(3)\cdot cf(3) \end{pmatrix}$$
$$= \begin{pmatrix} 5.64 & 5.28 & 86.92 \end{pmatrix}$$

$w(t)$ is the fraction of the bond's present value paid in year t:

$$\underset{(1\times\text{Years})}{\mathbf{w}} = \begin{pmatrix} \frac{df(1)\cdot cf(1)}{\mathbf{df}\cdot\mathbf{cf}} & \frac{df(2)\cdot cf(2)}{\mathbf{df}\cdot\mathbf{cf}} & \frac{df(3)\cdot cf(3)}{\mathbf{df}\cdot\mathbf{cf}} \end{pmatrix}$$
$$= \begin{pmatrix} 5.64/97.84 & 5.28/97.84 & 86.92/97.84 \end{pmatrix}$$
$$= \begin{pmatrix} 0.0576 & 0.0540 & 0.8884 \end{pmatrix}$$

Duration

The bond's duration is the average waiting time for (coupon) payments:

$$D = \sum_{t=1}^{3} t \cdot w(t) = 1 \cdot w(1) + 2 \cdot w(2) + 3 \cdot w(3)$$
$$= 1 \cdot 0.0576 + 2 \cdot 0.0540 + 3 \cdot 0.8884 = 2.8308$$

Duration using Bond yield

Duration of a Bond is often calculated using yield-to-maturity.

$$D = \sum_{t=1}^{3} t \cdot \tilde{w}(t) = \sum_{t=1}^{3} t \cdot \frac{cf(t)/\left(1+y\right)^t}{\sum_{t=1}^{3} cf(t)/\left(1+y\right)^t}$$

$$= \sum_{t=1}^{3} t \cdot \frac{cf(t)/\left(1+y\right)^t}{P_{bond}}$$

In our example: $y = 0.0682$, $P_{bond} = 97.84$ therefore

$$D = \left(1 \cdot \frac{6}{1.0682} + 2 \cdot \frac{6}{\left(1.0682\right)^2} + 3 \cdot \frac{106}{\left(1.0682\right)^3}\right) / 97.84$$

$$= 2.8315$$

Modified Duration

Let $v(t) = cf(t)/(1+y)^t$ and note

$$\frac{dv(t)}{dy} = -t \cdot cf(t) \cdot (1+y)^{-t-1} \Rightarrow$$

$$dv(t) = -t \cdot v(t) \cdot \frac{dy}{(1+y)} \Rightarrow$$

$$\sum_{t=1}^{N} dv(t) = -\sum_{t=1}^{N} t \cdot v(t) \cdot \frac{dy}{(1+y)} \Rightarrow$$

$$\sum_{t=1}^{N} \frac{dv(t)}{v} = -\left(\sum_{t=1}^{N} t \cdot \frac{v(t)}{v}\right) \cdot \frac{dy}{(1+y)} \Rightarrow$$

$$\frac{dv}{v} = -md \cdot dy$$

Modified Duration

- Modified Duration, **md**: $md = \frac{D}{(1+y)}$;
- It measures the (negative) relative change in the value of the bond per marginal change in its own yield-to-maturity.
- Or, in short, the *interest rate risk* of the bond.