Economics of Finance

Tutorial 3 solution

1. Suppose there are three possible states of the world in the next period, denoted by good weather (GW), fair weather (FW) and bad weather (BW). Also, three securities are available on the market with payoffs in each state listed below.

	Bond	Stock
GW	20	50
FW	20	30
BW	20	0

The prices of the two securities are: $p_{Bond} = 19$, $p_{Stock} = 16.5$.

(i) Suppose an investor needs to hedge the following payments:

$$\underset{(states \times 1)}{\mathbf{c}} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix}.$$

Is it possible to perfectly replicate the portfolio? Why?

Solution

Trivial to show that perfect hedge is impossible, e.g., to replicate 0 in GW and 0 in FW, it requires 0 bond and 0 stock, which produces 0 in BW.

Minimum cost hedging portfolio can be constructed using Linear Programming in Matlab


```
>> Q =[20 50; 20 30; 20 0]
>> c =[0 0 20]'
>> ps = [19 16.5]
>> n = linprog (ps', -Q, -c);

n =

    1.0000
   -0.4000

>> h=Q*n

h =

     0
     8
    20

>> p =ps*n

p =

    12.4000
```

(ii) Suppose a dealer now offer a European put option on the stock which expires at period 1 . The strick price of the option is 20. The option is sold at 13, and the dealer does not allow shorting. Does the option help hedge the payments specified in part (i)?

Solution

Put option only get exercised when stock price goes below 20. The payment of European put option is given by:

$$\underset{(3\times1)}{\mathbf{c}_{option}} = \begin{pmatrix} \max\{(20-50),0\}=0 \\ \max\{(20-30),0\}=0 \\ \max\{(20-0),0\}=20 \end{pmatrix} \begin{matrix} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{matrix}$$

The option payment is identical to the target payment the investor wants to hedge. *Conditional on properly priced*, it could potentially help hedge the payment. However, notice that the payment of minimum hedging portfolio (h vector) dominates the option portfolio with an extra 8 apples in FW, while it costs $12.4 < 13$. The put option appears over priced and the dealer does not allow shorting. It does not help hedge the payments specified in part (i), as better (lower-costly) portfolio is already available without it.

(iii) In light of your answer to part (ii), the dealer hires you to provide a range of price for the option. What advice can you give?

Solution

Denote option price as p_{option}. a fairly priced security will generate all positive atomic security prices:

$$\mathbf{P}_{atom} = \mathbf{P}_S \cdot \mathbf{Q}^{-1}$$

$$\begin{pmatrix} p_G & p_F & p_B \end{pmatrix} = \begin{pmatrix} p_{bond} & p_{stock} & p_{option} \end{pmatrix} \cdot \begin{pmatrix} -0.075 & 0.125 & 0 \\ 0.05 & -0.05 & 0 \\ 0.075 & -0.125 & 0.05 \end{pmatrix}$$

Since $p_{bond} = 19$ and $p_{stock} = 16.5$, positive atomic prices require

$$\begin{aligned} p_G &= 19 \cdot -0.075 + 16.5 \cdot 0.05 + p_{option} \cdot 0.075 > 0, \\ p_F &= 19 \cdot 0.125 + 16.5 \cdot -0.05 + p_{option} \cdot -0.125 > 0, \\ p_B &= p_{option} \cdot 0.05 > 0, \end{aligned}$$

simplifying the inequalities, we have:

$$\begin{aligned} p_{option} &> \frac{19 \cdot 0.075 - 16.5 \cdot 0.05}{0.075} = 8, \\ p_{option} &< \frac{19 \cdot 0.125 + 16.5 \cdot -0.05}{0.125} = 12.4, \\ p_{option} &> 0, \end{aligned}$$

thus, $8 < p_{option} < 12.4$ is the range for fairly priced option.

(iv) Based on your analysis in part (i)-part (iii), comment on the role of financial engineering and financial market innovations.

Solution

Financial market innovations, particularly options and derivatives, provided properly engineered, will potentially help asset market allocation and reduce deadweight loss. In turn, they will provide positive impact on the financial market efficiency. However, poorly engineered financial market innovation will create noise and distort the pricing mechanism, which can be harmful.

2. Consider the following three bonds that make the coupon payments listed below:

$$\begin{array}{l} \\ \text{Year 1} \\ \text{Year 2} \\ \text{Year 3} \end{array} \begin{array}{c} \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{pmatrix} 100 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 105 & 100 \end{pmatrix} \end{array}$$

The prices of these bonds are as follows: $p_{B1} = 95$, $p_{B2} = 88$, $p_{B3} = 75$.
(i) Compute the discount factors for Years 1, 2 and 3.

Solution
The discount factors can be computed as follows:

$$\mathbf{df} = \mathbf{p}_B \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 95 & 88 & 75 \end{pmatrix} \begin{pmatrix} 100 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 105 & 100 \end{pmatrix}^{-1} = \begin{pmatrix} 0.95 & 0.9 & 0.75 \end{pmatrix}.$$

That is, $df(1) = 0.95, df(2) = 0.9, df(3) = 0.75$.

(ii) Suppose an investor wants to receive the following payment vector:

$$c = \begin{pmatrix} 50 \\ 10 \\ 20 \end{pmatrix}$$

Construct a portfolio of the three bonds that generates this payment vector. What is the arbitrage-free price of this portfolio?

Solution

The replicating portfolio can be found as follows:

$$\mathbf{n} = \mathbf{Q}^{-1} \cdot \mathbf{c} = \begin{pmatrix} 100 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 105 & 100 \end{pmatrix}^{-1} \begin{pmatrix} 50 \\ 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 2.0 \\ -1.9 \end{pmatrix}$$

In other words, the investor should buy 0.4 units of bond 1, 2 units of bond 2, and sell 1.9 units of bond 3. The arbitrage-free price of this portfolio is

$$p_c = \mathbf{p}_B \cdot \mathbf{n} = df \cdot c,$$

$$p_c = \mathbf{p}_B \cdot \mathbf{n} = \begin{pmatrix} 95 & 88 & 75 \end{pmatrix} \begin{pmatrix} 0.4 \\ 2.0 \\ -1.9 \end{pmatrix} = 71.5,$$

$$p_c = \mathbf{df} \cdot \mathbf{c} = \begin{pmatrix} 0.95 & 0.9 & 0.75 \end{pmatrix} \begin{pmatrix} 50 \\ 10 \\ 20 \end{pmatrix} = 71.5.$$

(iii) Compute the interest rates $i(1)$, $i(2)$ and $i(3)$. Explain in words the interpretation on $i(3)$.

Solution
The interest rates can be computed using the formula:

$$i(t) = \left(\frac{1}{df(t)} \right)^{\frac{1}{t}} - 1.$$

Hence,

$$i(1) = \left(\frac{1}{0.95}\right) - 1 = 0.0526,$$
$$i(2) = \left(\frac{1}{0.9}\right)^{\frac{1}{2}} - 1 = 0.0541,$$
$$i(3) = \left(\frac{1}{0.75}\right)^{\frac{1}{3}} - 1 = 0.1006.$$

The interpretation of $i(3)$ is the following: measures the average annual rate of return an investor would receive if she invested an amount for three years.

(iv) Compute the duration and the modified duration of the three bonds. How do you interpret these numbers?

Solution
Duration measures the interest rate risks associated with the bond. To compute duration of the bonds one should start by calculating the present value of every cash flow. The following MATLAB code will do the job:


```
>> pv = Q.*[df' df' df']

pv =

   95.00000    4.75000    0.00000
    0.00000    4.50000    0.00000
    0.00000   78.75000   75.00000
```


Calculate fractions of each Bond's present value (i.e its price) paid in each period:

```
>> w = pv./ [p; p; p]

w =

   1.00000   0.05398   0.00000
   0.00000   0.05114   0.00000
   0.00000   0.89489   1.00000
```

Finally, compute the weighted averages of the payment times with the weight given above:

```
>> D = [1:3]*w

D =

   1.0000   2.8409   3.0000
```

The modified duration is given by

$$mD = \frac{D}{1+y}.$$

Use the ysolve function in octave to solve YTM for bond 2:

```
>> y=fsolve(@(y)(5/(1+y)+5/(1+y)^2+105/(1+y)^3-88),0)

y =  0.098092
```

The modified durations are:

```
>>D =[1.0000   2.8409   3.0000]
y=[0.0526,0.098092,0.1006]
mD=D./(1+y)

mD =

0.95003   2.58712   2.72579
```