Economics of Finance

Tutorial 7

1. Suppose an investor decides to construct a portfolio consisting of a risk-free asset that pays 6 percent and a stock index fund that has an expected rate of return of 12 percent and a standard deviation of 20 percent. Let x denote the proportion invested in the stock index fund. This implies that the proportion $(1-x)$ is invested in the risk-free asset. (In your answers use 6 for 6 percent etc).

(a) Find the equation for the efficient frontier. Graph the efficient frontier in (e_p, s_p) space where e_p is the expected return on the portfolio and s_p the standard deviation of the return on the portfolio.

Solution

Let e_{IF} and s_{IF} denote the expected rate of return and the standard deviation of the stock index fund. Let e_{RF} be the risk-free rate. Then, the expected return of the portfolio is

$$e_p = xe_{IF} + (1-x)e_{RF} = 12x + 6(1-x). \tag{1}$$

Notice, that the variance of a linear combination of two random variables $\widetilde{y}$ and $\widetilde{z}$ is given by

$$Var(a\widetilde{y} + b\widetilde{z}) = a^2 Var(\widetilde{y}) + b^2 Var(\widetilde{z}) + 2abCov(\widetilde{y}, \widetilde{z}).$$

The variance of the return on the risk-free asset is zero. The covariance of the return on the risk-free asset with any risky return is zero as well. Therefore, the variance of the portfolio is

$$v_p = x^2 \left(s_{IF}\right)^2$$

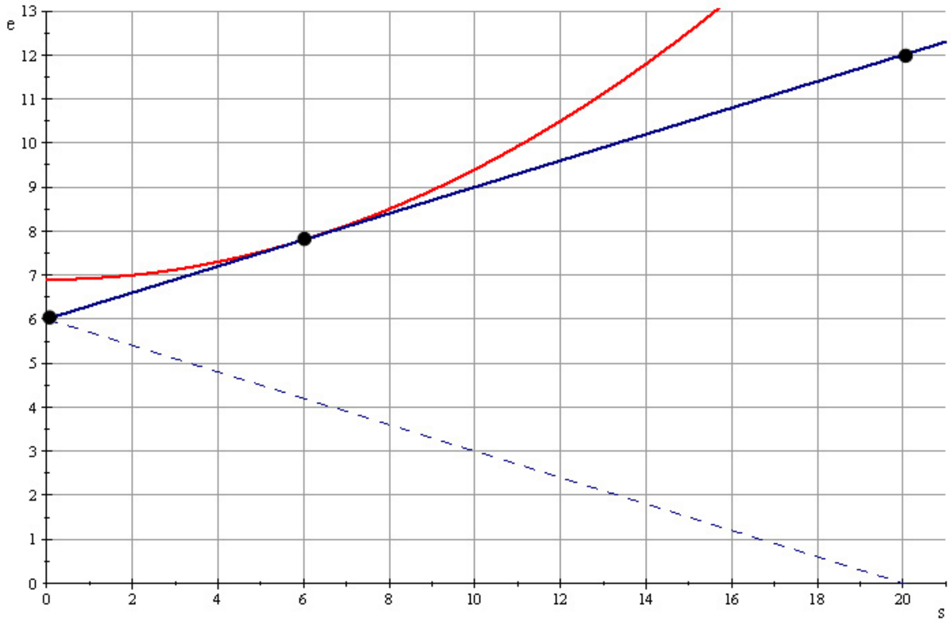
while its standard deviation is

$$s_p = |x| s_{IF} = 20|x|, \tag{2}$$

where $|x|$ denotes absolute value. Solving (2) for x and substituting it into equation (1) yields the following equation for the efficient frontier:

$$e_p = 6 + 0.3 s_p.$$

The graph of the frontier is the blue solid line in this figure.



(b) Suppose the expected utility of the investor is given by

$$Eu = e_p - 0.025 s_p^2. \tag{3}$$

What does expected utility imply in terms of preferences? What is the interpretation of the coefficient in front of s_p^2 in equation (3)?

Solution

Given this mean-variance expected utility, an investor prefers higher expected return, but dislikes risk (variance). The coefficient in front of s_p^2 indicates the rate at which the Investor is willing to trade expected value (or return) for variance. Thus, the risk tolerance t, the reciprocal of this value, is the rate at which the Investor is willing to trade variance for expected return.

(c) What is the investor's optimal portfolio, assuming that he/she is an expected utility maximiser? (In other words, find the investor's optimal choice for x). Show graphically the optimal portfolio choice in (e_p, s_p) space.

Solution

Consider the set of all portfolios that provide the level of the expected utility k. In (e_p, s_p) space, these portfolios will satisfy the following equation:

$$e_p = k + 0.025 s_p^2.$$

The slope of this indifference curve is given by

$$\frac{de_p}{ds_p} = 0.05 s_p.$$

Since the equation for the efficient frontier is $e_p = 6 + 0.3 s_p$, its slope is

$$\frac{de_p}{ds_p} = 0.3.$$

At the optimum the two slopes will be equated. Hence, it follows that $0.05 s_p = 0.3$ or $s_p = 6$. From the equation (2) it follows that $20x = 6$ or $x = 0.3$. From the equation of the frontier we obtain $e_p = 6 + 0.3 \cdot 6 = 7.8$. See the figure above for a graphical exposition.

(d) Consider the portfolio with $e_p = 8.4$ and $s_p = 8$. Show that this portfolio is efficient. Demonstrate that this portfolio is not optimal in the sense of maximizing the investor's expected utility. (Hint: Calculate the **certainty equivalent** of this portfolio and the one in part (c) and compare the two). Show graphically the certainty equivalent of this portfolio and the one in part (c) on the same graph in (e_p, s_p) space.

Solution

Since the equation for the efficient frontier is $e_p = 6 + 0.3 s_p$ it must be the case that an efficient portfolio with a standard deviation of 8 percent command the expected return of $6 + 0.3 \cdot 8 = 8.4$ percent. This is the case for the portfolio in question.

The utility level our investor will obtain from this portfolio is:

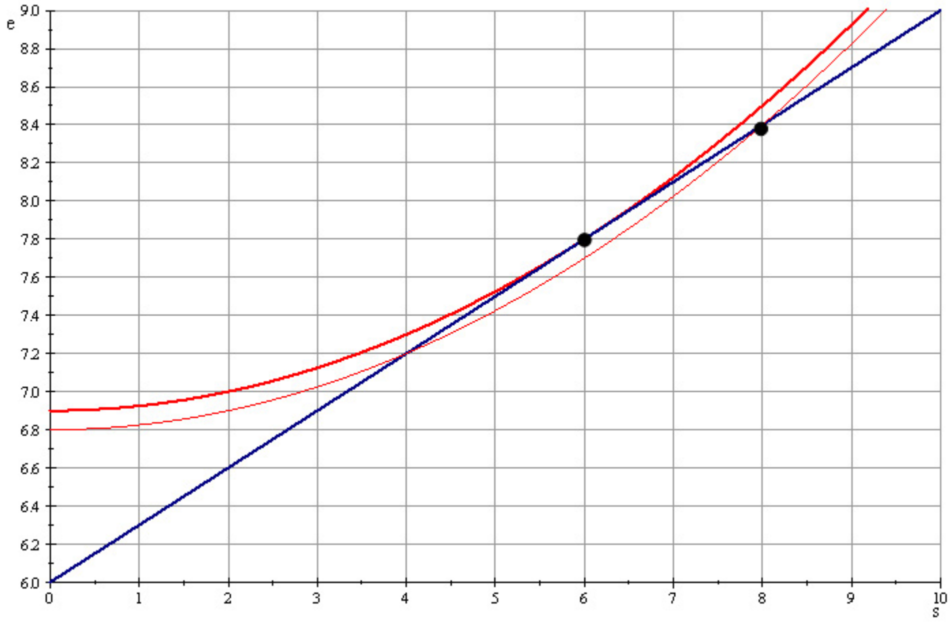
$$Eu_{d)} = e_p - 0.025 s_p^2 = 8.4 - 0.025 \cdot (8)^2 = 6.8$$

Notice from the utility function in (3) that a risk-free asset with a 6.8 percent rate of return will make our investor equally happy as this risky portfolio. Hence, the certainty equivalent of this portfolio is 6.8.

The utility level our investor will obtain from the portfolio in part (c) is:

$$Eu_{c)} = e_p - 0.025 s_p^2 = 7.8 - 0.025 \cdot (6)^2 = 6.9$$

Hence, the certainty equivalent of this portfolio is 6.9. Since, there is another efficient portfolio with a higher certainty equivalent for our investor, the portfolio with $e_p = 8.4$ and $s_p = 8$ is not optimal. See the figure for a graphical exposition.



(e) Suppose the expected utility of the investor is given by

$$Eu = e_p - 0.01 s_p^2. \tag{4}$$

What is the investor's optimal portfolio in this case? Compare your answer to the optimal portfolio you found in part (c). Comment.

Solution

Consider the set of all portfolios that provide the level of the expected utility k. In (e_p, s_p) space, these portfolios will satisfy the following equation:

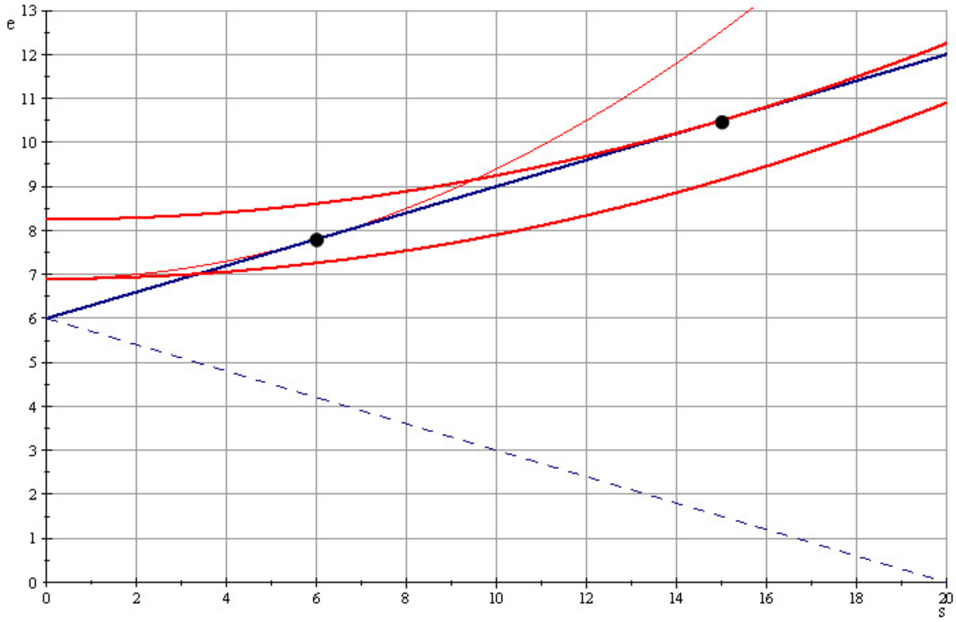
$$e_p = k + 0.01 s_p^2.$$

The slope of this indifference curve is given by

$$\frac{de_p}{ds_p} = 0.02 s_p.$$

As before, the slope of the efficient frontier is 0.3 At the optimum the two slopes must be equated: $0.02 s_p = 0.3$. Therefore $s_p = 15$. From the equation (2) it follows that $20x = 15$ or $x = 0.75$. From the equation of the frontier we obtain $e_p = 6 + 0.3 \cdot 15 = 10.5$.

The new investor is less risk averse. His risk tolerance $t = 100$, while the tolerance of the investor in part b) is $t = 40$. Notice, that the new investor is 2.5 time more risk tolerant, he/she invested 2.5 times more wealth into risky asset, and his/her optimal portfolio has 2.5 higher standard deviation than that of the original investor. See the figure for a graphical exposition.



(f) Is there a relationship between the standard deviation of the optimal portfolio you found in part (c) and (e) and the investors degree of risk tolerance. If so, can that relationship be described precisely in mathematical terms?

Solution

If the efficient portfolio is linear in mean/standard deviation space, there is a one-to-one mapping between risk undertaken and risk tolerance, assuming efficient investment strategies are utilised. Let the efficient frontier be:

$$e = a + bs.$$

Then

$$s = \frac{e - a}{b},$$

$$v = s^2 = \frac{(e - a)^2}{b^2}.$$

The rate at which v can be substituted for e along the frontier is thus:

$$\frac{dv}{de} = \frac{2\,(e - a)}{b^2} = \left(\frac{2}{b}\right)\frac{(e - a)}{b} = \left(\frac{2}{b}\right)s.$$

For an investment strategy to be optimal, this must equal the investor's risk tolerance t:

$$t = \left(\frac{2}{b}\right)s$$

It follows that, compared with an "average investor": an Investor with twice the risk tolerance should take twice the risk; an Investor with half the risk tolerance should take half the risk.

2. Consider the following two stocks: stock 1 has expected return $e_1 = 6$ and standard deviation $s_1 = 16$; stock 2 has expected return $e_2 = 10$ and standard deviation $s_2 = 20$. Assume that the correlation between the returns on the two stocks is 0.75, that is, the correlation coefficient $r_{12} = 0.75$.

(a) Write down an equation for the expected return (e_p) and the variance (v_p) of the return on the portfolio as a function of x_2 only, where x_2 is the proportion of invested wealth in stock 2.

Solution

In general, the expected return of the portfolio as a function of x_2 is

$$e_p = (1 - x_2)e_1 + x_2 e_2 = e_1 + (e_2 - e_1)x_2,$$

while the variance of the returns is

$$v_p = (1 - x_2)^2 v_1 + x_2^2 v_2 + 2(1 - x_2)x_2 s_1 s_2 r_{12}.$$

Substituting the given values yields

$$\begin{aligned} e_p &= 6 + 4x_2, \\ v_p &= (1 - x_2)^2 \cdot 16^2 + x_2^2 \cdot 20^2 + 2(1 - x_2)x_2 .16 \cdot 20 \cdot 0.75 \\ &= 16\left(11x_2^2 - 2x_2 + 16.0\right) \end{aligned}$$

(b) Find the minimum variance portfolio. What is the expected return and variance of this portfolio?

Solution

Differentiating v_p with the respect to x_2 and equating the result to zero yields

$$\frac{dv_p}{dx_2} = 16\,(22x_2 - 2) = 0,$$

from which it follows that $x_2 = \frac{1}{11}$. Therefore $x_1 = \frac{10}{11}$,

$$v_p = 16\left(11\left(\frac{1}{11}\right)^2 - 2\left(\frac{1}{11}\right) + 16.0\right) = 16 \cdot 15\frac{10}{11} = 254.55$$

$$s_p = \sqrt{254.55} = 15.955$$

$$e_p = 6 + 4\frac{1}{11} = 6.3636$$

(c) Is Stock 1 on the efficient frontier? Draw a graph in expected return-standard deviation space to illustrate your answer.

Solution

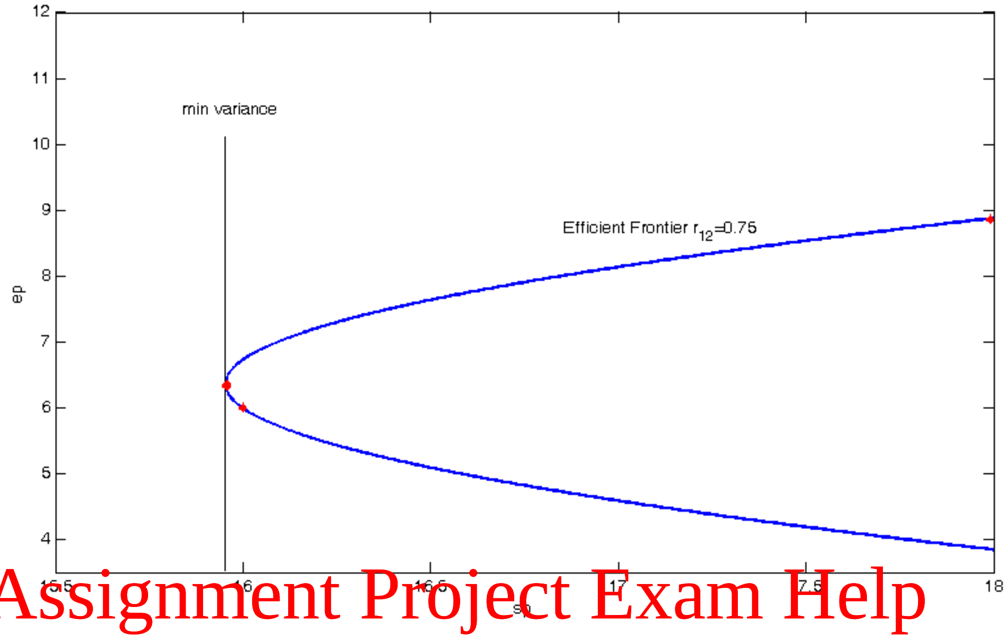


Figure 1: Minimum variance portfolio

Stock 1 is not efficient because the minimum variance portfolio has a higher expected return and a smaller standard deviation (see Figure 1).