Lecture 6: Arrow-Debreu Pricing: Multiple States, Stochastic Discount Factor, Heterogeneity, Pareto Optimality

Economics of Finance

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Back to our Arrow-Debreu Consumer's Problem:

- The problem:
  - Choose c_G, c_B, c_0, a_G, a_B
  - to maximize

$$u(c_0) + \beta \left[\pi_G \cdot u(c_G) + \pi_B \cdot u(c_B) \right]$$

  - subject to

$$c_0 + q_G \cdot a_G + q_B \cdot a_B = e_0,$$

$$c_G = a_G + e_G$$
$$c_B = a_B + e_B.$$

- Form the Lagrangian

$$\begin{aligned} L = {} & u(c_0) + \beta \left[\pi_G \cdot u(c_G) + \pi_B \cdot u(c_B) \right] \\ & - \lambda_0 \left[c_0 + q_G \cdot a_G + q_B \cdot a_B - e_0 \right] \\ & - \lambda_1 \left[c_G - a_G - e_G \right] \\ & - \lambda_2 \left[c_B - a_B - e_B \right] \end{aligned}$$

Solving the Consumer's Problem: (cont'd)

- Equate the partial derivatives of the Lagrangian to zero:
  - Partials w.r.t. c_G, c_B, c_0, a_G, a_B

$$
\begin{aligned}
\partial L/\partial c_0 &= u'(c_0) - \lambda_0 = 0 \\
\partial L/\partial c_G &= \beta\pi_G \cdot u'(c_G) - \lambda_1 = 0 \\
\partial L/\partial c_B &= \beta\pi_B \cdot u'(c_B) - \lambda_2 = 0 \\
\partial L/\partial a_G &= -\lambda_0 \cdot q_G + \lambda_1 = 0 \\
\partial L/\partial a_B &= -\lambda_0 \cdot q_B + \lambda_2 = 0
\end{aligned}
$$

  - Partial w.r.t. the multipliers $\lambda_0, \lambda_1, \lambda_2$ are just the constrains:

$$
\begin{aligned}
c_0 + q_G \cdot a_G + q_B \cdot a_B - e_0 &= 0 \\
c_G - a_G - e_G &= 0 \\
c_B - a_B - e_B &= 0
\end{aligned}
$$

Solving the Consumer's Problem: (cont'd)

- Expressing atomic prices as functions of consumption allocations:
  - The values of the multipliers:

$$\lambda_0 = u'(c_0)$$
$$\lambda_1 = \beta\pi_G \cdot u'(c_G)$$
$$\lambda_2 = \beta\pi_B \cdot u'(c_B)$$

  - The prices of the atomic (Arrow-Debreu) securities:

$$q_G = \frac{\lambda_1}{\lambda_0} = \beta\pi_G \frac{u'(c_G)}{u'(c_0)}$$
$$q_B = \frac{\lambda_2}{\lambda_0} = \beta\pi_B \frac{u'(c_B)}{u'(c_0)}$$

The Prices of Atomic (Arrow-Debreu) Securities

- Combine the solution to the consumer's problem with the market clearing conditions:
  - Atomic prices as functions of consumption allocations:

$$q_G = \frac{\lambda_1}{\lambda_0} = \beta \pi_G \frac{u'(c_G)}{u'(c_0)}$$

$$q_B = \frac{\lambda_2}{\lambda_0} = \beta \pi_B \frac{u'(c_B)}{u'(c_0)}$$

  - Market clearing conditions:

$$c_0 = e_0; c_G = e_G; c_B = e_B.$$

  - The prices of the atomic (Arrow-Debreu) securities:

$$q_G = \beta \pi_G \frac{u'(e_G)}{u'(e_0)}$$

$$q_B = \beta \pi_B \frac{u'(e_B)}{u'(e_0)}$$

Trade

Is there any trade of atomic (Arrow-Debreu) securities possible in this economy?

- Remember constraints

$$c_0 + q_G \cdot a_G + q_B \cdot a_B = e_0,$$
$$c_G = e_G + a_G,$$
$$c_B = e_B + a_B$$

- and market clearing

$$c_0 = e_0; c_G = e_G; c_B = e_B.$$

- gives us $a_G = a_B = 0$ in this equilibrium
- Since all agents are the same in this economy (represented by one representative agent) no trade is possible!

Arrow-Debreu Consumer's Problem: Multiple States

- The setting:
  - two periods 0 and 1;
  - multiple states in period 1 indexed by s_1
  - set of all possible states in period 1 is S_1, so that $s_1 \in S_1$, e.g., $S_1 = \{\text{Good}, \text{Fair}, \text{Bad}\}$
- The problem:
  - Choose c_0, c_{s_1}, a_{s_1}, for all $s_1 \in S_1$
  - to maximize

$$u(c_0) + \beta \sum_{s_1 \in S_1} \pi_{s_1} \cdot u(c_{s_1})$$

  - subject to

$$c_0 + \sum_{s_1 \in S_1} q_{s_1} \cdot a_{s_1} = e_0,$$

$$c_{s_1} = a_{s_1} + e_{s_1}, \text{ for all } s_1 \in S_1$$

Solving the Consumer's Problem

- Form the Lagrangian

$$L = u(c_0) + \beta \sum_{s_1 \in S_1} \pi_{s_1} \cdot u(c_{s_1}) -$$

$$- \lambda_0 \Big(c_0 + \sum_{s_1 \in S_1} q_{s_1} \cdot a_{s_1} - e_0 \Big) - \sum_{s_1 \in S_1} \lambda_{s_1} (c_{s_1} - a_{s_1} - e_{s_1})$$

- Equate the partial derivatives of the Lagrangian to zero:
  - Partials w.r.t. c_0, c_{s_1}, a_{s_1}, for all $s_1 \in S_1$

$$\partial L / \partial c_0 = u'(c_0) - \lambda_0 = 0$$

$$\partial L / \partial c_{s_1} = \beta \pi_{s_1} \cdot u'(c_{s_1}) - \lambda_{s_1} = 0, \text{ for all } s_1 \in S_1$$

$$\partial L / \partial a_{s_1} = -\lambda_0 \cdot q_{s_1} + \lambda_{s_1} = 0, \text{ for all } s_1 \in S_1$$

  - Partial w.r.t. the multipliers λ_0, λ_{s_1} are just the constrains:

$$c_0 + \sum_{s_1 \in S_1} q_{s_1} \cdot a_{s_1} - e_0 = 0$$

$$c_{s_1} - a_{s_1} - e_{s_1} = 0, \text{ for all } s_1 \in S_1.$$

The Prices of Atomic (Arrow-Debreu) Securities

- Expressing atomic prices as functions of consumption allocations:

$$q_{s_1} = \frac{\lambda_{s_1}}{\lambda_0} = \beta \pi_{s_1} \frac{u'(c_{s_1})}{u'(c_0)}, \text{ for all } s_1 \in S_1.$$

- Combine with the market clearing conditions:

$$c_0 = e_0;\ c_{s_1} = e_{s_1}, \text{ for all } s_1 \in S_1.$$

- The prices of the atomic (Arrow-Debreu) securities:

$$q_{s_1} = \beta \pi_{s_1} \frac{u'(e_{s_1})}{u'(e_0)}, \text{ for all } s_1 \in S_1.$$

Stochastic Discount Factor

- The prices of the atomic (Arrow-Debreu) securities:

$$q_{s_1} = \beta \pi_{s_1} \frac{u'(e_{s_1})}{u'(e_0)}, \text{ for all } s_1 \in S_1.$$

- The discount factor of a specific period is the sum of all atomic security prices in this period

$$df(1) = \sum_{s_1 \in S_1} q_{s_1} = \sum_{s_1 \in S_1} \beta \pi_{s_1} \frac{u'(e_{s_1})}{u'(e_0)}$$

- The stochastic discount factor, m_1, is a random variable
  - its value is unknown at $t = 0$;
  - its value at time 1 is $m_{s_1} = \beta \frac{u'(e_{s_1})}{u'(e_0)}$, if state s_1 is realized;

- Then the discount factor is

$$df(1) = \sum_{s_1 \in S_1} \beta \pi_{s_1} \frac{u'(e_{s_1})}{u'(e_0)} = \sum_{s_1 \in S_1} \pi_{s_1} m_{s_1} = E[m_1]$$

where $E[\cdot]$ is the expectation operator.

Forward atomic prices and risk neutral probabilities

- The (spot) prices of the atomic (Arrow-Debreu) securities:

$$q_{s_1} = \beta \pi_{s_1} \frac{u'(e_{s_1})}{u'(e_0)}, \text{ for all } s_1 \in S_1.$$

- The forward prices of the atomic (Arrow-Debreu) securities:

$$f_{s_1} = \frac{q_{s_1}}{df(1)} = \frac{q_{s_1}}{\sum_{s_1 \in S_1} q_{s_1}} = \pi_{s_1} \frac{u'(e_{s_1})}{u'(e_0)} \Big/ \sum_{s_1 \in S_1} \pi_{s_1} \frac{u'(e_{s_1})}{u'(e_0)}, \forall s_1 \in S_1.$$

- The forward prices are often called risk neutral probabilities

$$\widetilde{\pi}_{s_1} = f_{s_1} \forall s_1 \in S_1.$$

If agents are risk neutral, their utility is linear $u' = \text{const}$ and f_{s_1} simplifies to

$$\widetilde{\pi}_{s_1} = \pi_{s_1} \Big/ \sum_{s_1 \in S_1} \pi_{s_1} = \pi_{s_1}, \forall s_1 \in S_1.$$

Pricing state-contingent claims

- Using the atomic state prices, often called, *pricing kernel*:

$$p = q \cdot c,$$

q - row vector of atomic state prices or pricing kernel
c - column vector of state-contingent payments

- Using *risk-neutral measure*:

$$p = df \cdot \widetilde{E}(c),$$

c - random variable, realised value depends on a state,
$\widetilde{E}(\cdot)$ - expectation taken with respect to risk-neutral measure using risk-neutral probabilities $\widetilde{\pi}$

- Using *stochastic discount factor*:

$$p = E(m_1 c),$$

c - random variable, realised value depends on a state,
m_1 - stochastic discount factor,
$E(\cdot)$ - expectation taken with respect to physical probability measure using actual probabilities π

Heterogeneity

Homogeneity (representative agent) is clearly a simplified assumption.

- Heterogeneity either in *endowments* or in *preferences* (incl. β), or *both* is necessary for trade

- Consider K agents, each indexed by k;
- with utilities u^k
- each agent k chooses optimal c_0^k and $c_{s_1}^k$ given endowments e_0^k and $e_{s_1}^k$ for all s_1.

Consumers' Problem

- Each agent k maximises expected utility, U^k, given by

$$U^k = u^k\left(c_0^k\right) + \underbrace{\beta^k \sum_{s_1 \in S_1} \pi_{s_1} \cdot u^k\left(c_{s_1}^k\right)}_{\text{expected discounted future utility}}$$

- subject to period-0 constraint

$$c_0^k + \sum_{s_1 \in S_1} q_{s_1} \cdot a_{s_1}^k = e_0^k,$$

- and a series of period-1 constraints for every possible state:

$$c_{s_1}^k = a_{s_1}^k + e_{s_1}^k, \text{ for all } s_1 \in S_1$$

- Market clearing (now makes more sense)

$$\sum_{k=1}^{K} c_0^k = \sum_{k=1}^{K} e_0^k; \qquad \sum_{k=1}^{K} c_{s_1}^k = \sum_{k=1}^{K} e_{s_1}^k, \ \forall s_1$$

Market clearing conditions

Homogeneous consumers (representative agent):

$$c_0(s_0) = e_0(s_0);\ c_1(s_1) = e_1(s_1), \forall s_1 \in S_1.$$

- Goods from endowment are non-storable and there is no trade, so consume all you can.

Heterogeneous consumers:

$$\sum_{k=1}^{K} c_0^k(s_0) = \sum_{k=1}^{K} e_0^k(s_0); \qquad \sum_{k=1}^{K} c_{s_1}^k = \sum_{k=1}^{K} e_{s_1}^k, \forall s_1$$

- Trade is possible
- The total number of goods from all endowments in each time-state must equal the total number of goods consumed.
- Agents may use atomic (Arrow-Debreu) securities to shift consumption between time-states, but all endowment must be consumed jointly in the respective time-state.

Characterisation of the Equilibrium

- From the first order conditions the prices of the atomic (Arrow-Debreu) securities

$$q_{s_1} = \beta^k \pi_{s_1} \frac{u^{k\prime}\left(c^k_{s_1}\right)}{u^{k\prime}\left(c^k_0\right)} =$$

$$= \beta^k \pi_{s_1} \frac{u^{k\prime}\left(e^k_{s_1} + a^k_{s_1}\right)}{u^{k\prime}\left(e^k_0 - \sum_{s_1 \in S_1} q_{s_1} \cdot a^k_{s_1}\right)}$$

for all k and $s_1 \in S$.

- also impose **market clearing** which *implies* that

$$\sum_{k=1}^{K} a^k_{s_1} = 0, \forall s_1 \in S.$$

Example: Heterogeneous Consumers

- Consider a world in which there are two periods: 0 and 1.
- In period 1 there are two possible states of nature: a good weather state (G) and a bad weather state (B). That is $s_1 \in S_1 = \{G, B\}$. They are equally probable, i.e., $\pi_G = \pi_B = 1/2$
- There are two consumers in this economy.
- Their preferences over apples are exactly the same and are given by the following expected utility function:

$$\frac{1}{2} c_0^k + \beta \sum_{s_1 \in S_1} \pi_{s_1} \ln\left(c_{s_1}^k\right).$$

  where subscript $k = 1, 2$ denotes consumers.
- The consumer's time discount factor $\beta = 0.9$.

Example: Heterogeneous Endowment

The consumers are identical in every way (e.g. utility function and discount factor) except in their endowments which are given in the table below:

Consumers	Endowments		
	$t = 0$	$t = 1$	
		G	B
Consumer 1	4	4	2
Consumer 2	4	2	1

There is some *inequality*: consumer 1 has better endowments in both states.

Questions

- What is the equilibrium condition?
- Equilibrium price and trading volume?
- Welfare gain from free trade?

Solving for Equilibrium

Equilibrium prices (same and taken for both consumers)

$$q_{s_1} = \beta^k \pi_{s_1} \frac{u^{k\prime}(c_{s_1})}{u^{k\prime}(c_0)}$$

$$q_G = \beta \pi_G \frac{1/c_G^1}{1/2},\ q_G = \beta \pi_G \frac{1/c_G^2}{1/2} \Rightarrow c_G^1 = c_G^2$$

$$q_B = \beta \pi_B \frac{1/c_B^1}{1/2},\ q_B = \beta \pi_B \frac{1/c_B^2}{1/2} \Rightarrow c_B^1 = c_B^2$$

Clearing conditions: $c_G^1 + c_G^2 = 6$; $c_B^1 + c_B^2 = 3$

Equilibrium consumption: $c_G^1 = c_G^2 = 3$; $c_B^1 = c_B^2 = 1.5$

Equilibrium trades: $a_G^1 = -a_G^2 = -1$; $a_B^1 = -a_B^2 = -0.5$

Gains from trade

$$U^1_{\text{trade}} = 2.9768 > U^1_{\text{autarky}} = 2.9357$$

$$U^2_{\text{trade}} = 2.3768 > U^2_{\text{autarky}} = 2.3110$$

Consumers mutually benefit from trade. Why?

Consumption smoothing is an important driving force.

How does discount factor, β^k, affect the allocation in this case?

Example: two symmetric agents

- two agents (1,2); two periods, two states (A,B) in period 1 $\pi_A = \pi_B = 1/2$
- same preferences (u and β)
- same endowments in period 0 $e_0^1 = e_0^2 = e_0$
- different (but "symmetrical") endowments in period 1: $e_A^1 < e_B^1$, $e_B^2 < e_A^2$, $e_A^1 = e_B^2 = e$, $e_B^1 = e_A^2 = E$.

Consumers	Endowments		
	$t = 0$	$t = 1$	
		A	B
Consumer 1	z	e	E
Consumer 2	z	E	e

Example: two symmetric agents

- two agents (1,2); two periods, two states (A,B) in period 1 $\pi_A = \pi_B = 1/2$
- same preferences (u and β)

- same endowments in period 0 $e_0^1 = e_0^2 = e_0$
- different (but "symmetrical") endowments in period 1: $e_A^1 < e_B^1$, $e_B^2 < e_A^2$, $e_A^1 = e_B^2 = e$, $e_B^1 = e_A^2 = E$.

- from *market clearing* $a_A^1 = -a_A^2$, $a_B^1 = -a_B^2$
- in symmetric equilibrium, $a_A^1 = -a_B^1 = -a_A^2 = a_B^2 = a$
- The prices of the Arrow-Debreu securities: $q_A = q_B$

$$q_A = \frac{1}{2}\beta \frac{u'\left(e_A^1 + a\right)}{u'\left(e_0\right)} = \frac{1}{2}\beta \frac{u'\left(e_A^2 - a\right)}{u'\left(e_0\right)},$$

where a is such that
$u'\left(e_A^1 + a\right) = u'\left(e_A^2 - a\right) \Rightarrow a = \frac{1}{2}\left(e_A^2 - e_A^1\right) = \frac{1}{2}\left(E - e\right).$

Gains from trade

- Because of differences in endowments trade is mutually beneficial
- Compare utilities U^1 and U^2 when $a = 0$ (autarky) vs. utilities with optimal a found from optimisation.
- Which one is higher (at least not smaller) and how do we know?
- Consumer are reducing the risk of consuming smaller amounts if their own "bad" state realises by giving up some of the consumption in the "good" state.
- In other words, consumers are *Risk Sharing*.

Difference in risk aversion

- More risk-averse consumers would hedge their consumption against the B state.
- More risky consumers would sell B Arrow-Debreu securities and boost their consumption today or in G in hope that B is not going to realise.
- This is also a form of risk sharing.

Summary of Gains of Trade

- Consumption smoothing
- Risk sharing due to difference in endowments
- Risk sharing due to difference in risk aversions

Pareto optimality

- Allocation of atomic (Arrow-Debreu) securities in which it is impossible to make any one consumer better off without making at least one consumer worse off.

- Under the first welfare theorem (we do not prove it here) *competitive* equilibrium (prices are taken as given) is equivalent to Pareto optimality.

- Some conditions: completeness - existence of atomic (Arrow-Debreu) securities for all states, no transaction costs, no externalities (utilities are independent).