Lecture 7: Modern Portfolio Theory

Economics of Finance

School of Economics, UNSW

Recap

We have studied competitive market theories

- An *utility function* has been specified on the space of consumptions
- Agents make consumption choice to maximize expected utility

We learnt

- Asset price is the ratio between marginal utilities (values of securities/value of money)
- Trading improves Pareto efficiency
- Financial market plays important social roles:
- *consumption smoothing*
- *risk sharing*

Roadmap

- Until now a general framework which captures *risk* and relevant trade-offs seems absent
- Consider small number of discrete outcomes (Good, Bad) and small number of securities
- In reality there are many securities and their returns are better approximated by continuous variable
- We will discuss situation which involves many possible financial instruments with a more general (continuous) measure of risks.

Measure of risk

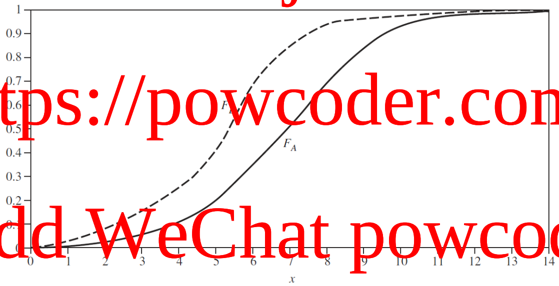
Standard deviation is a measure of *risk*

If investors are risk averse, they would prefer:

- higher expected returns for a given level of standard deviation
- lower standard deviations for a given level of expected return
- Portfolios that provide the maximum expected return for a given standard deviation and the minimum standard deviation for a given expected return are called *efficient portfolios*.
- All others are inefficient.

First-order Stochastic Dominance

If $F_A(x) \leq F_B(x)$, $\forall x$ and $F_A(x) < F_B(x)$ for at least for some x, random variable A first-order stochastically dominates B.



Expected utility: $EU(A) \geq EU(B)$ for any preferences.

First-order Stochastic Dominance: examples

- Y has an expected return of 10 and a standard dev. of 15
- X has an expected return of 14 and a standard dev. of 15
- X *first-order stochastically dominates* Y

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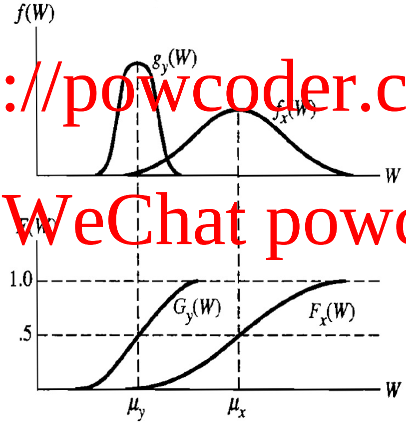
Variance does not have to be the same though.



Second-order Stochastic Dominance

If $\int_{-\infty}^{x}[F_A(t) - F_B(t)]dt \geq 0$, $\forall x$ and strict inequality at least for some x, B second-order stochastically dominates A.

- When they have the same mean, but A has a higher var than B (A is a mean-preserving spread of B).
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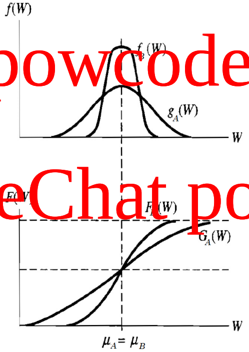

Expected utility: $EU(B) \geq EU(A)$ for *risk-averse* preferences.

Expected utility

The *mean-variance* expected utility takes the form:

$$Eu = e - \frac{v}{t} = e - \frac{s^2}{t},$$

- e is the expected return
- s is the standard deviation of the expected return,
- $t = 2/c$ is the investor's *risk tolerance* and c is risk aversion.
- t or c can be time-varying and wealth-dependent, but for simplicity we assume they are constant

Mean-variance expected utility can be exactly derived from several basic utilities, e.g. from negative exponential (constant absolute risk aversion, CARA) utility $u = 1 - e^{-cr}$ and assuming normality of $r \sim N(e, s^2)$.

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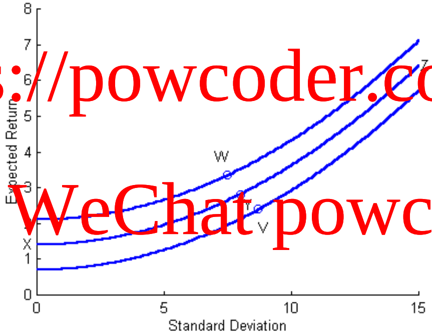
A sketch of proof (optional): from the properties of log-normal distribution $E\left(e^{-cr}\right) = e^{-ce+cs^2/2}$. Using monotonic transformation, $-\ln(1 - Eu)/c$, yields the result.

Indifference curves

Now fix a given expected utility level $Eu \equiv U_\ell$

$$e = U_\ell + \frac{s^2}{t}$$

This is an upward-sloping, convex curve in terms of s.



To maintain a fixed utility level, a higher return is required for a higher risk.

Certainty Equivalent

$$e = U_\ell + \frac{s^2}{t}$$

- when $s = 0$, $e = U_\ell$, i.e., a fixed amount of return which is equally satisfying
- this is *certainty equivalent* - certain return which gives the same utility level as the expected utility of risky return

Risk tolerance and premium

Re-arrange,

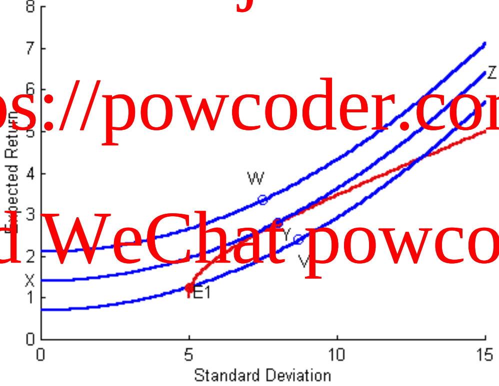
$$e - U_\ell = \frac{s^2}{t}$$

is the *risk premium* required by the individual investor

- given a fixed s, a greater t means a less risk premium;
- the more tolerant agent is, the less risk premium she/he would require for taking the risk.

Optimal Portfolio Choice

- Investor will maximize the utility (blue indifference curves)
- Given the $e - s$ opportunities (red) available on the market
  – efficient frontier of a portfolio



Why is efficient frontier concave?

Some history

A 15-pages PhD thesis by Harry Max Markowitz in 1954

- titled "Portfolio selection"
- a ground-breaking, insightful contribution

This inspired a sequence of research

- William Sharpe, among others, formulated theories as the CAPM today
- Sharpe (1964): 'Capital asset prices – a theory of market equilibrium under conditions of risk'
- Markowitz, Sharpe and Miller shared **1990 Nobel prize**

Market opportunities

Set of opportunities is presented by portfolio of assets.
Example with two assets:

- Let R_1 and R_2 are random returns of two assets
- Expected values $E(R_1) = e_1$ and $E(R_2) = e_2$;
- Variances $\mathrm{Var}(R_1) = v_1$ and $\mathrm{Var}(R_2) = v_2$
- Correlation $\mathrm{Corr}(R_1, R_2) = \rho_{12}$

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Let x denotes the proportion of asset 1 in the portfolio, then:

- $E(xR_1 + (1-x)R_2) = xe_1 + (1-x)e_2$
- $\mathrm{Var}(xR_1 + (1-x)R_2) = x^2 v_1 + (1-x)^2 v_2 + 2x(1-x)\sqrt{v_1 v_2}\rho_{12}$
- $\mathrm{StDev}(xR_1 + (1-x)R_2) = \sqrt{\mathrm{Var}(xR_1 + (1-x)R_2)}$

Solve for x in Var or StDev equation and substitute to get to $e - s$ opportunities space.

Deriving Frontier

$$e = xe_1 + (1-x)e_2$$
$$v = x^2 v_1 + (1-x)^2 v_2 + 2x(1-x)\sqrt{v_1 v_2}\rho_{12}$$

- Solve out x:

$$x = \frac{e - e_2}{e_1 - e_2} = e \cdot \frac{1}{e_1 - e_2} - \frac{e_2}{e_1 - e_2}$$

- on the other hand solve out v as a function of x:

$$\begin{aligned} v(x) &= v_1 x^2 + \left(1 + x^2 - 2x\right) v_2 + \left(2x - 2x^2\right)\sqrt{v_1 v_2}\rho_{12} \\ &= \left(v_1 + v_2 - 2\sqrt{v_1 v_2}\rho_{12}\right) x^2 + 2x\left(\sqrt{v_1 v_2}\rho_{12} - v_2\right) + v_2; \end{aligned}$$

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- substitute x in $v(x)$, we have a functional form of $v(e)$
- notice that

$$\left(v_1 + v_2 - 2\sqrt{v_1 v_2}\rho_{12}\right) = Var(R_1 - R_2) > 0;$$

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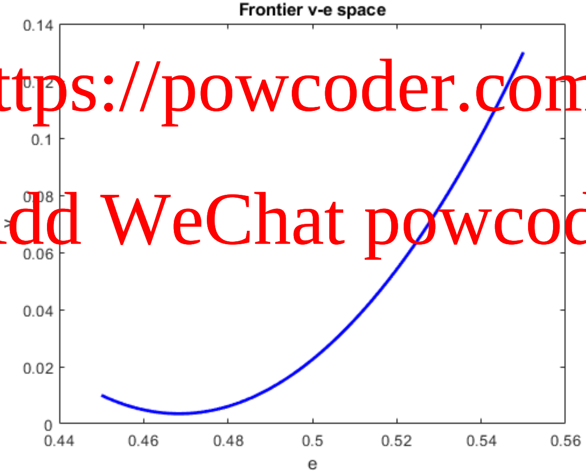
- v is a parabola with an upward opening in $v - e$ space

Deriving Frontier (cont)

$$v\left(e\right) = ae^2 + be + c,$$

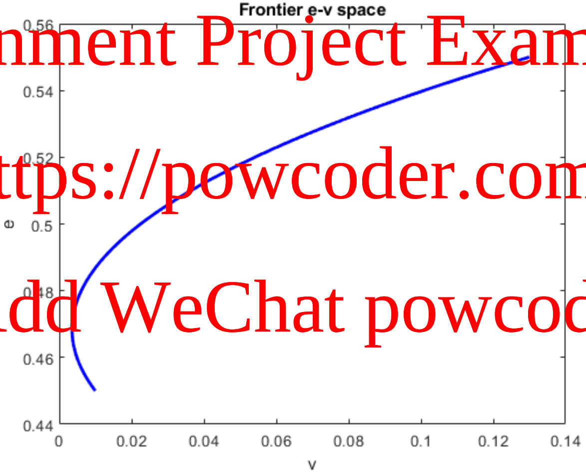
where $a > 0, b, c$ are constants depending on

$e_1 = 0.45, e_2 = 0.55, v_1 = 0.01, v_2 = 0.0225, \rho_{12} = -0.5.$



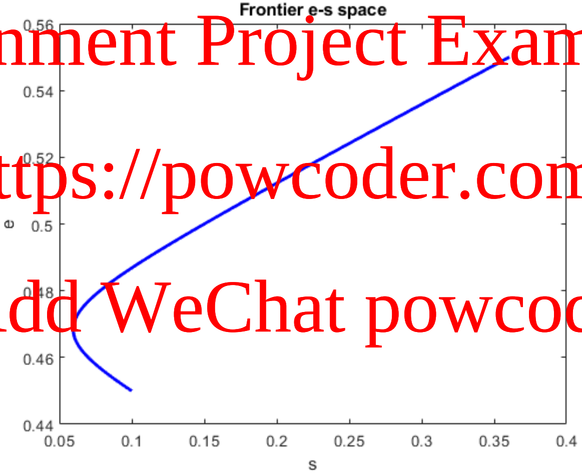
Mirror it to get to $e - v$ space

$e(v)$ is the inverse of quadratic function



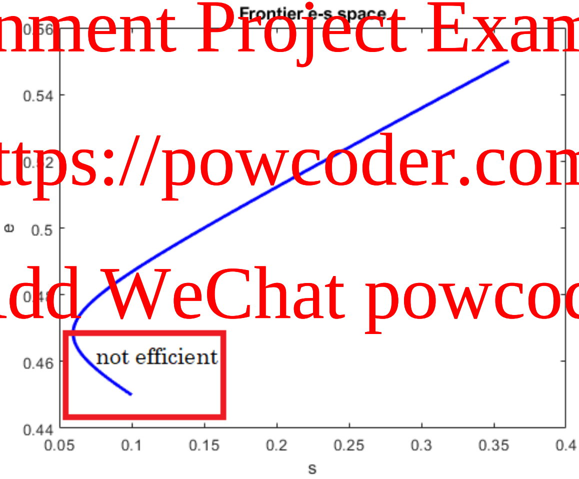
“Squeeze” it further to get to $e - s$ space

$e(s) = e(\sqrt{v})$ nice concave function



Efficient frontier

A higher risk should be rewarded with a higher expected return, if it not the case that is not an efficient investment.

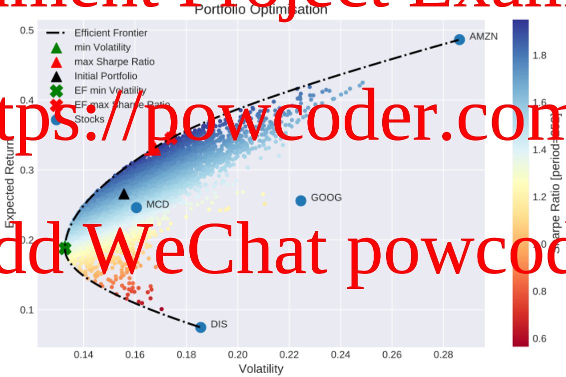


Efficient Frontier: many securities

Create "portfolios of portfolios" and select the most *efficient* combinations, higher expected return for the same variance (first-order stochastic dominance), lower variance for the same expected return (second-order stochastic dominance).

Example: efficient frontier

Based on historical daily data (two years 2015–2017) of four stocks: Macdonald's, Disney, Amazon, Microsoft.
Black triangle – example of portfolio with equal shares



Software: module FinQuant for Python.

Risky asset and risk-free asset

When R_2 is risk free, $e - s$ frontier is a *straight line*

- $E(R_1) = e_1$ and $E(R_2) = e_f < e_1$
- $\text{Var}(R_1) = v_1$, $\text{Var}(R_2) = 0$ and $\rho_{12} = 0$
- $e = xe_1 + (1-x)e_f = e_f + x(e_1 - e_f)$
- $s = |x|\sqrt{v_1}$
- $e - s$ curve: $e = e_f \pm \frac{(e_1 - e_f)s}{\sqrt{v_1}}$
- $e = \begin{cases} e_f + \frac{e_1 - e_f}{\sqrt{v_1}} s, & \text{efficient} \\ e_f - \frac{e_1 - e_f}{\sqrt{v_1}} s, & \text{not efficient} \end{cases}$
- efficient section: a straight line with slope $\frac{e_1 - e_f}{\sqrt{v_1}}$

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- $\mathrm{Var}(R_1) = v_1$, $\mathrm{Var}(R_2) = 0$ and $\rho_{12} = 0$
- $e = xe_1 + (1-x)e_f = e_f + x(e_1 - e_f)$
- $s = |x|\sqrt{v_1}$
- $e - s$ curve: $e = e_f + \frac{(e_1 - e_f)s}{\sqrt{v_1}}$
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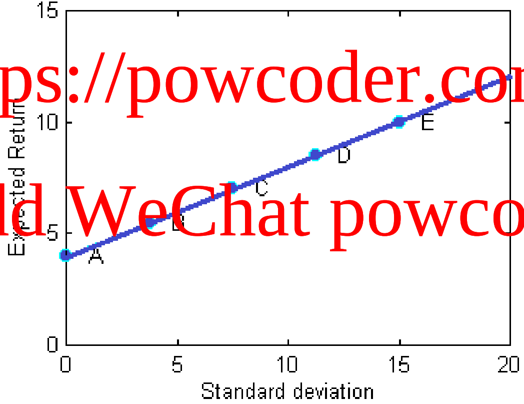
- efficient section: a straight line with slope $\frac{e_1 - e_f}{\sqrt{v_1}}$
  Interpretation:
  $e_1 - e_f$: excess return, or, *risk premium*
  $s_1 = \sqrt{v_1}$: standard deviation, or, *risk*
  $S = \frac{e_1 - e_f}{s_1}$: Sharpe ratio, *marginal return on unit of risk*

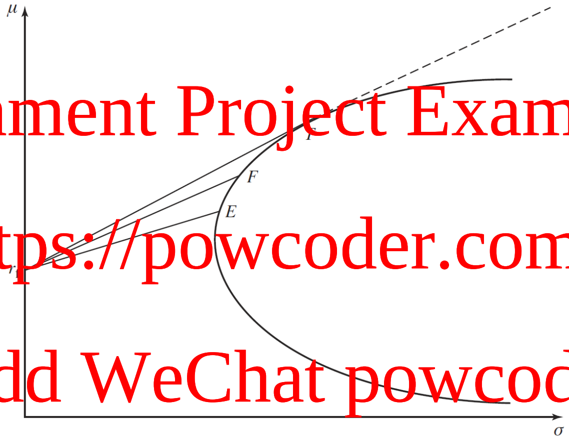
Example: e-s space

$e_1 = 10, s_1 = 15$ and $e_f = 4$

$$\begin{aligned} e &= 10x + 4(1-x) \\ s &= 15x \end{aligned}$$



Combine portfolios of risky assets with a risk-free asset



The point of tangency T is called "Market portfolio", the best portfolio of risky assets on the market you can use to combine with the risk-free asset.
What about combinations with F and E portfolios?

Sharpe ratio

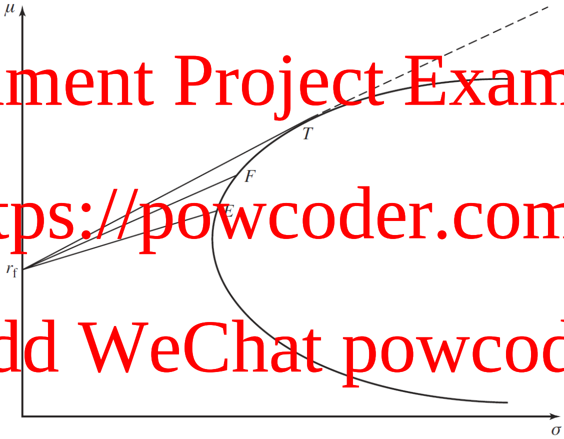
William Sharpe proposed the following *Sharpe ratio*

$$S = \frac{e - e_f}{s}$$

where $e - e_f$ is excess return and s is standard deviation (risk) of a risky asset/portfolio.

- reflect the tradeoff between excess return and risk
- simple way to compare different stocks/portfolios
- usually $S > 1$ is acceptable, $S > 2$ is very good

Sharpe ratio of the Market portfolio



$S_M = \frac{e_M - e_f}{s_M}$ is the slope to the tangent line and therefore the *best Sharpe ratio* available on the market

Investor's problem

Investor's problem: optimal wealth Y_0 allocation between the market portfolio and risk-free asset (under CARA). Y_0 is often normalised to 1.

$$\max_x Eu = xe_M + (Y_0 - x)e_f - \frac{1}{t}x^2 v_M, \tag{1}$$

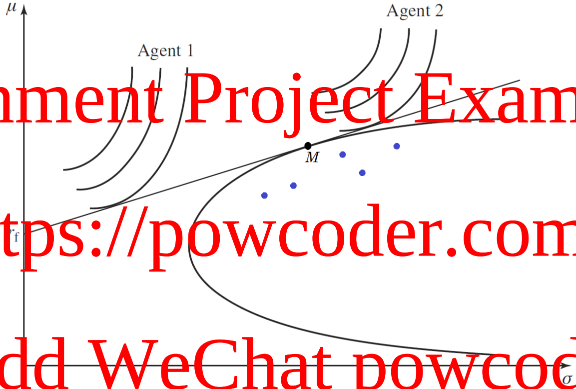
where x is amount of wealth invested in risky asset.

Take FOC, we get:

$$x^* = \frac{t}{2}\frac{e_M - e_f}{v_M} = \frac{t}{2}\frac{S_M}{\sqrt{v_M}}$$

More tolerant investors choose more risky asset, but all investors get the same best Sharpe ratio

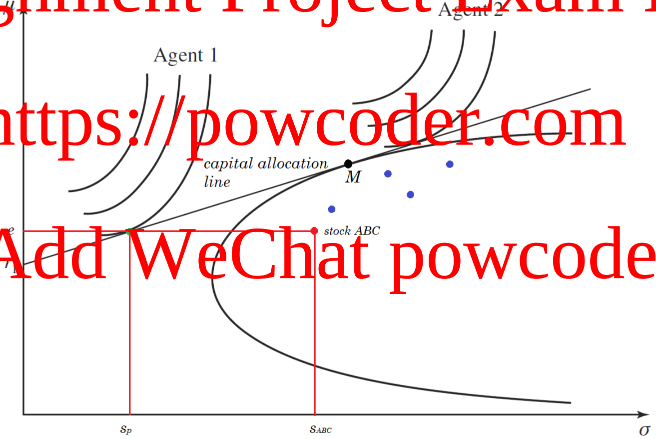
Optimal investment



All investors invest in the combination of the risky-free asset and market portfolio. The share of the market portfolio and risk-free asset is determined by their risk tolerance (risk-aversion).

Capital allocation line

A maximum Sharpe ratio is obtained for any portfolio on the straight line from r_f tangent with the efficient frontier at M. This line is called *capital allocation line* (CAL).



Separation theorem

Separation (or two-fund) theorem: an optimal investor's risky portfolio is identified *separately* from their risk preferences; investors hold only a combination of two assets (funds): the market portfolio and the risk-free asset.