Lecture 3: Hedging with Minimum Cost Multi-period Discounting, Bonds

Economics of Finance

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Hedging

Consider following. Markets are *incomplete*: number of states is higher then the number of linearly independent securities.

$$\underset{(3\times 2)}{\mathbf{Q}} = \begin{array}{cl} \begin{pmatrix} 20 & 43 \\ 20 & 28 \\ 20 & 28 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array} \qquad \underset{(1\times 2)}{\mathbf{ps}} = \begin{array}{c} \begin{pmatrix} 19 & 35 \end{pmatrix} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} \end{array}$$

Suppose that an investor asks an investment firm to create a product with the following payment:

$$\underset{(3\times 1)}{\mathbf{c}} = \begin{pmatrix} 40 \\ 30 \\ 20 \end{pmatrix} \begin{array}{l} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{array}$$

Questions: How to do it? What should the firm charge?

Hedging in Incomplete Market

- The Problem: No matter how many bonds and stocks are chosen, the payments in the "Fair Weather" state and the "Bad Weather" state will be the same.
- Suppose the firm will select 40 in 'GW' and 30 in 'FW' or 'BW' to cover all outflows. We can do it by:

$$\underset{(2\times2)}{\mathbf{Q}} = \begin{array}{cc} \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix} & \begin{array}{l} \text{Good W.} \\ \text{Fair or Bad W.} \end{array} \\ \text{Bond} \quad \text{Stock} & \end{array} \qquad \underset{(2\times1)}{\mathbf{c}} = \begin{pmatrix} 40 \\ 30 \end{pmatrix} \begin{array}{l} \text{Good W.} \\ \text{Fair or Bad W.} \end{array}$$

$$\mathbf{n} = \mathbf{Q}^{-1} \cdot \mathbf{c} = \begin{pmatrix} 20 & 43 \\ 20 & 28 \end{pmatrix}^{-1} \begin{pmatrix} 40 \\ 30 \end{pmatrix} = \begin{pmatrix} 0.56667 \\ 0.66667 \end{pmatrix} \begin{array}{l} \text{Bond} \\ \text{Stock} \end{array}$$

$$\mathbf{p} = \mathbf{p_S} \cdot \mathbf{n} = \begin{pmatrix} 19 & 35 \end{pmatrix} \begin{pmatrix} 0.56667 \\ 0.66667 \end{pmatrix} = 34.10$$

Different Scenarios

- It costs 34.10 PA to purchase a portfolio, **n**, that would cover all future outflows:
  - The firm will receive 10 in 'BW' state;
  - Investor will be assured to receive all the promised payments.
- Since there is extra 10 BA, the firm will be happy to sell the product for 34.10PA

New product on the market

What if the new product is offered for 32PA?

$$\underset{(3\times3)}{\mathbf{Q}} = \begin{array}{c} \begin{pmatrix} 20 & 43 & 40 \\ 20 & 28 & 30 \\ 20 & 28 & 20 \end{pmatrix} \\ \text{Bond} \quad \text{Stock} \quad \text{Product} \end{array} \begin{array}{l} \text{Good W.} \\ \text{Fair W.} \\ \text{Bad W.} \end{array} \qquad \underset{(1\times3)}{\mathbf{ps}} = \begin{array}{c} \begin{pmatrix} 19 & 35 & 32 \end{pmatrix} \\ \text{Bond} \quad \text{Stock} \quad \text{Product} \end{array}$$

The atomic security prices can be found:

$$\mathbf{p}_{atomic} = \mathbf{ps} \cdot \mathbf{Q}^{-1} = \begin{pmatrix} 19 & 35 & 32 \end{pmatrix} \begin{pmatrix} 20 & 43 & 40 \\ 20 & 28 & 30 \\ 20 & 28 & 20 \end{pmatrix}^{-1}$$

$$= \begin{array}{c} \begin{pmatrix} 0.56 & 0.18 & 0.21 \end{pmatrix} \\ \text{Good W.} \quad \text{Fair W} \quad \text{. Bad W.} \end{array}$$

Completing the market

The new product, *linearly independent* of the existing ones, completes the market.

Perfect hedging can be achieved when market is complete:

$$\mathbf{n} = \mathbf{Q}^{-1} \cdot \mathbf{c} = \begin{pmatrix} 20 & 43 & 40 \\ 20 & 28 & 30 \\ 20 & 28 & 20 \end{pmatrix}^{-1} \begin{pmatrix} 40 \\ 30 \\ 20 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \begin{array}{l} \text{Bond} \\ \text{Stock} \\ \text{Product} \end{array}$$

Hedging at Minimum Cost

Recall our original example. Market is incomplete:

$$\underset{(3\times 1)}{\mathbf{c}} = \begin{array}{cc} \begin{pmatrix} 40 \\ 30 \\ 20 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{array} \end{array} \qquad \underset{(3\times 2)}{\mathbf{Q}} = \begin{array}{cc} \begin{pmatrix} 20 & 43 \\ 20 & 28 \\ 20 & 28 \end{pmatrix} & \begin{array}{l} \text{Good Weather} \\ \text{Fair Weather} \\ \text{Bad Weather} \end{array} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} & \end{array}$$

$$\underset{(1\times 2)}{\mathbf{ps}} = \begin{array}{c} \begin{pmatrix} 19 & 35 \end{pmatrix} \\ \begin{array}{cc} \text{Bond} & \text{Stock} \end{array} \end{array}$$

Our objective is to construct the cheapest portfolio, $\mathbf{n}$, that will deliver as least as much as $\mathbf{c}$ in every state of nature.

Linear programming

- Our problem is to select a portfolio, $\mathbf{n}$, to minimize its cost, $\mathbf{p_S} \cdot \mathbf{n}$, subject to obtaining no less than the required state contingent payment, $\mathbf{c}$.

- A constrained optimization problem we have to solve is given by

$$\min_{\mathbf{n}} \mathbf{p_S} \cdot \mathbf{n} \text{ subject to } \mathbf{Q} \cdot \mathbf{n} \geq \mathbf{c}.$$

- Note: we use the sign $\geq$ to indicate that every element of vector $\mathbf{Q} \cdot \mathbf{n}$ is no less than the corresponding element of vector $\mathbf{c}$.

- We are facing a *linear programming* problem, or simply the problem of finding a vector that minimizes a linear function subject to linear constraints.

Matlab functions provide us with several tools for solving linear programming problems. They solve this general problem (I use their notation):

$$\min_{\mathbf{x}} \mathbf{f}' \cdot \mathbf{x} \text{ subject to } \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}.$$

- `linprog(f, A, b)` in Matlab

We will ask this function to perform the simplest task:

- In our context, the role of $\mathbf{x}$ played by portfolio vector, $\mathbf{n}$. Note $\mathbf{f}$ is assumed as a column vector. The role of $\mathbf{f}'$ is performed by row vector $\mathbf{p_S}$.
- this function deals only with inequalities of the form: $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$. Let's multiply both sides of our constraint, $\mathbf{Q} \cdot \mathbf{n} \geq \mathbf{c}$, by -1 to obtain $-\mathbf{Q} \cdot \mathbf{n} \leq -\mathbf{c}$
- Hence, the role is $\mathbf{A}$ is played by $-\mathbf{Q}$, while the role of $\mathbf{b}$ is played by $-\mathbf{c}$.

Enter the data in MATLAB's command prompt:

```
>> Q = [20 43; 20 28; 20 28];
>> c = [40 30 20]';
>> ps = [19 35];
```

Use the linear programming function `linprog`

```
>> n = linprog(ps', -Q, -c)
```

The result of running these commands is:

```
n =
 0.5667
 0.6667
```

The price of the portfolio is:

```
>> p = ps*n
p = 34.1000
```

Wrapping up

Hedging involves fully covering contingent payments/liabilities and offsetting risks

- With complete market, this involves replicating desired payments/liabilities;
- With incomplete market, perfect hedging is can not be achieved;

- The ideal hedging then involves hedging with minimum cost;
- This can be done using Matlab "linprog" function;

- Completing the market will generally reduce deadweight loss associated with incomplete hedging.

The Discount Factor

Definition: *The discount factor* (for a certain date) represents the present value of a payment of one unit to be made with **certainty** at the specified future date.

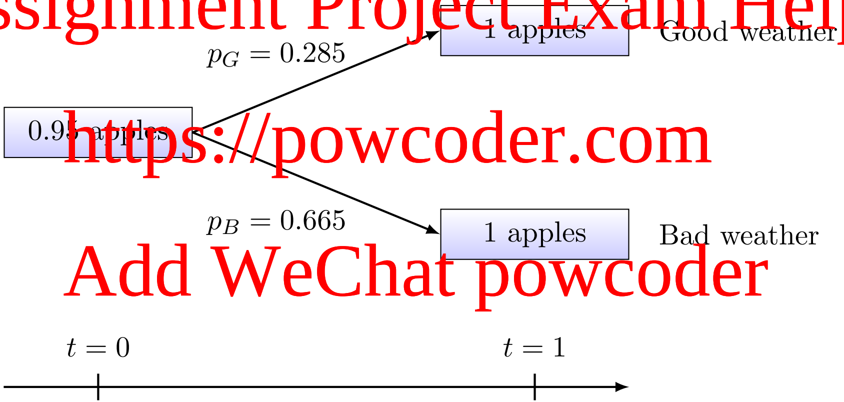
The tree example:

$$PV = \mathbf{p} \cdot \mathbf{q} = \underbrace{(0.285 + 0.665)}_{\text{sum of the atomic prices}} \cdot 20 = \underbrace{0.95}_{\text{df}(1)} \cdot 20 = 19$$

- The discount factor for a date in question equals to the sum of appropriate atomic prices (prices of basic atomic securities)

The Discount Factor

E.g. $df(1) = 0.95$



Constant Interest Rate

Assume interest rate, i, is constant for now. df(1) is the present value (at time 0) of 1 for certain in one period. Hence $df(1) \cdot (1+i) = 1$.

$$df(1) = \frac{1}{1+i}.$$

df(2) is the present value (at time 0) of 1 for certain in *two* periods. If interest rate is constant, $df(2) \cdot (1+i)^2 = 1$, or

$$df(2) = \frac{1}{(1+i)^2}.$$

To generalise,

$$df(n) = \frac{1}{(1+i)^n}.$$

PV

The present value of an asset that produces a cash flow, C_1, a year from now is

$$PV = df(1) \times C_1 = \frac{1}{1+i} \times C_1$$

$i = 0.0526$ or 5.26 percent and $C_1 = \$100$. Then

$$PV = \frac{1}{1+i} \times C_1 = \frac{100}{1.0526} = \$95.00$$

The PV of a cash flow two years from now (assuming i constant) is: $PV = df(2) \times C_2 = \frac{1}{(1+i)^2} \times C_2$. Say $C_2 = \$100$: $PV = \frac{100}{(1.0526)^2} = 90.2554$.

Multi-period (Variable) Discount Factors

Definition: A nominal discount factor, $df(t)$, is the present value of one unit of currency to be paid with certainty at time t.

Notation:

- Discount factor $\{1\times\text{periods}\}$:

$$\mathbf{df} = \begin{pmatrix} df(1) & df(2) & df(3) \end{pmatrix}$$

- Vector of cash flows known to be certain $\{\text{periods}\times 1\}$:

$$\mathbf{cf} = \begin{pmatrix} cf(1) \\ cf(2) \\ cf(3) \end{pmatrix}$$

PV

Discounted present value of the cash flows:

$$pv = \mathbf{df} \cdot \mathbf{cf}$$

Example: Coupon bonds with different maturities

- The Payment Matrix {periods×bonds}:

$$\mathbf{Q} = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{cc} \begin{array}{ccc} \text{B1} & \text{B2} & \text{B3} \end{array} & \\ \begin{pmatrix} 103 & 4 & 3 \\ 0 & 104 & 3 \\ 0 & 0 & 103 \end{pmatrix} & \begin{array}{l} \text{Year 1} \\ \text{Year 2} \\ \text{Year 3} \end{array} \end{array}$$

- The Price Vector {1×bonds}:

$$\mathbf{p} = \begin{array}{c} \begin{array}{ccc} \text{B1} & \text{B2} & \text{B3} \end{array} \\ \begin{pmatrix} 100 & 101 & 98 \end{pmatrix} \end{array}$$

Discount factors

The Question: What are the discount factors for Year 1, Year 2, Year 3?

- The price of each bond should equal its discounted present value:

$$100 = df\,(1) \cdot 103$$

$$101 = df\,(1) \cdot 4 + df\,(2) \cdot 104$$

$$98 = df\,(1) \cdot 3 + df\,(2) \cdot 3 + df\,(3) \cdot 103$$

Inferring the discount function

- In matrix notation: $\mathbf{p} = \mathbf{df} \cdot \mathbf{Q}$
- If $\mathbf{Q}^{-1}$ exists, the discount function can be inferred as $\mathbf{df} = \mathbf{p} \cdot \mathbf{Q}^{-1}$;
- Since $\mathbf{Q}^{-1}$ exists, the discount function is

$$\underset{(1\times\text{Years})}{\mathbf{df}} = \begin{pmatrix} 100 & 101 & 98 \end{pmatrix} \begin{pmatrix} 103 & 4 & 3 \\ 0 & 104 & 3 \\ 0 & 0 & 103 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 0.9708 & 0.9338 & 0.8959 \end{pmatrix}$$

- Any desired set of future certain payments over the next three years can be valued using this discount function.

Replicating bond portfolio

- To find a portfolio of the three bonds that will replicate a desired set of certain cash flows we can use

$$\underset{(\text{Years}\times\text{Bonds})}{\mathbf{Q}} \cdot \underset{(\text{Bonds}\times 1)}{\mathbf{n}} = \underset{(\text{Years}\times 1)}{\mathbf{c}}$$

- Let $\mathbf{c} = \begin{pmatrix} 300 & 200 & 100 \end{pmatrix}'$, then the replicating portfolio is

$$\mathbf{n} = \mathbf{Q}^{-1}\mathbf{c} = \begin{pmatrix} 103 & 4 & 3 \\ 0 & 104 & 3 \\ 0 & 0 & 103 \end{pmatrix}^{-1} \begin{pmatrix} 300 \\ 200 \\ 100 \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} 2.8107 \\ 1.8951 \\ 0.97087 \end{pmatrix} & \begin{array}{c} \text{B1} \\ \text{B2} \\ \text{B3} \end{array} \end{array}$$

Multi-period Interest Rates

Investment grows from $V(0)$ to $V(t)$ in t periods,
$i(t)$ is the default-free interest rate for time t:

$$V(0)(1+i(t))^t = V(t)$$

Definition: The ratio of the ending value to the beginning value, $V(t)/V(0)$, is termed the (t-period) value relative.

- One dollar will grow to $1/df(t)$ dollars with certainty by time t, hence

$$(1+i(t))^t = \frac{V(t)}{V(0)} = \frac{1}{df(t)};\ i(t) = \left(\frac{1}{df(t)}\right)^{\frac{1}{t}} - 1.$$

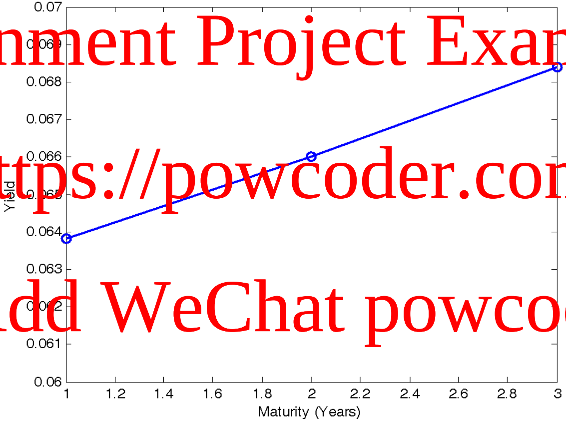
- Call $i(t)$ *multi-period interest rate*, or, *yield*.

Yield curve

The following Matlab code plot a term structure of interest rate given discount factors:

```
>>df = [0.94 0.88 0.82]
df =        0.9400 0.8800 0.8200
>> vr = 1./df
vr =        1.0638 1.1364 1.2195
>> i = vr.^(1./[1:3])-1
i =        0.0638 0.0660 0.0684
>>plot([1:3],i)
```

The plot


0.07
0.068
0.067
0.066
0.064
0.063
0.061
0.06
Yield
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
Maturity (Years)

Bond Yields (Yield to maturity)

Definition: Yield-to-maturity (YTM) is a *constant* interest rate that makes the present value of all the bond's payments equal its price.

Example: A bond is selling for \$97.84 and provides a certain vector of cash flows:

$$\mathbf{cf} = \begin{pmatrix} 6 \\ 6 \\ 106 \end{pmatrix} \begin{array}{l} \text{Year 1} \\ \text{Year 2} \\ \text{Year 3} \end{array}$$

Yield-to-maturity, y, must satisfy

$$\frac{6}{1+y} + \frac{6}{(1+y)^2} + \frac{106}{(1+y)^3} = 97.84$$

Computing Bond Yields

- **Problem:** We need to solve a non-linear equation!
- **Solution:** Use numerical techniques and tools, e.g. Octave (Matlab) `fsolve` function

- `fsolve` function looks for a solution to equation $f(y) = 0$

$$f(y) = \frac{6}{1+y} + \frac{6}{(1+y)^2} + \frac{106}{(1+y)^3} - 97.84$$

Matlab code


```
y0=0.06;  % Coupon rate is an initial guess
y=fsolve(@(y)(6/(1+y)+6/(1+y)^2+106/(1+y)^3-97.84),y0);
y =
    0.0682
Optimization terminated:  first-order optimality is
less than options.TolFun.
```

Duration

- We have all types of measurements about bond's return (yield, YTM, df, etc.) till now;
- In some applications we are interested in how sensitive is the bond value towards market interest rate;
- Consider a vector of certain cash flows associated with a bond is:

$$\mathbf{cf} = \begin{pmatrix} 6 \\ 6 \\ 106 \end{pmatrix} \begin{array}{l} \text{Year 1} \\ \text{Year 2} \\ \text{Year 3} \end{array}$$

- The market discount function is:

$$\underset{(1\times\text{Years})}{\mathbf{df}} = \begin{pmatrix} 0.94 & 0.88 & 0.82 \end{pmatrix}$$

Periodical Values and Weights

The present value of each year's cash flow:

$$\underset{(1\times \text{Years})}{\mathbf{v}} = \begin{pmatrix} df(1)\cdot cf(1) & df(2)\cdot cf(2) & df(3)\cdot cf(3) \end{pmatrix}$$
$$= \begin{pmatrix} 5.64 & 5.28 & 86.92 \end{pmatrix}$$

$w(t)$ is the fraction of the bond's present value paid in year t:

$$\underset{(1\times \text{Years})}{\mathbf{w}} = \begin{pmatrix} \frac{df(1)\cdot cf(1)}{\mathbf{df}\cdot\mathbf{cf}} & \frac{df(2)\cdot cf(2)}{\mathbf{df}\cdot\mathbf{cf}} & \frac{df(3)\cdot cf(3)}{\mathbf{df}\cdot\mathbf{cf}} \end{pmatrix}$$
$$= \begin{pmatrix} 5.64/97.84 & 5.28/97.84 & 86.92/97.84 \end{pmatrix}$$
$$= \begin{pmatrix} 0.0576 & 0.0540 & 0.8884 \end{pmatrix}$$

Duration

The bond's duration is the average waiting time for (coupon) payments:

$$D = \sum_{t=1}^{3} t \cdot w(t) = 1 \cdot w(1) + 2 \cdot w(2) + 3 \cdot w(3)$$
$$= 1 \cdot 0.0576 + 2 \cdot 0.0540 + 3 \cdot 0.8884 = 2.8308$$

Duration using Bond yield

Duration of a Bond is often calculated using yield-to-maturity.

$$D = \sum_{t=1}^{3} t \cdot \tilde{w}(t) = \sum_{t=1}^{3} t \cdot \frac{cf(t)/\left(1+y\right)^t}{\sum_{t=1}^{3} cf(t)/\left(1+y\right)^t}$$

$$= \sum_{t=1}^{3} t \cdot \frac{cf(t)/\left(1+y\right)^t}{P_{bond}}$$

In our example: $y = 0.0682$, $P_{bond} = 97.84$ therefore

$$D = \left(1 \cdot \frac{6}{1.0682} + 2 \cdot \frac{6}{(1.0682)^2} + 3 \cdot \frac{106}{(1.0682)^3}\right) / 97.84$$

$$= 2.8315$$

Modified Duration

Let $v(t) = cf(t)/(1+y)^t$ and note

$$\frac{dv(t)}{dy} = -t \cdot cf(t) \cdot (1+y)^{-t-1} \Rightarrow$$

$$dv(t) = -t \cdot v(t) \cdot \frac{dy}{(1+y)} \Rightarrow$$

$$\sum_{t=1}^{N} dv(t) = -\sum_{t=1}^{N} t \cdot v(t) \cdot \frac{dy}{(1+y)} \Rightarrow$$

$$\sum_{t=1}^{N} \frac{dv(t)}{v} = -\left(\sum_{t=1}^{N} t \cdot \frac{v(t)}{v}\right) \cdot \frac{dy}{(1+y)} \Rightarrow$$

$$\frac{dv}{v} = -md \cdot dy$$

Modified Duration

- Modified Duration, **md**: $md = \frac{D}{(1+y)}$;
- It measures the (negative) relative change in the value of the bond per marginal change in its own yield-to-maturity.

- Or, in short, the *interest rate risk* of the bond.