Econometric Methods

Submission deadline: 29/01/21, 14.00hrs GMT

Instructions:

- You must answer **all five questions in Section A** and **two out of the four questions in Section B**. If you answer more questions than are required and do not indicate which answers should be ignored, we will mark the requisite number of answers in the order in which they appear in your answer submission: answers beyond that number will not be considered.
- Your answers could be typed or hand-written (and scanned to a single pdf file that can be submitted) or a combination of a typed answer with included images of algebra or figures.
- Where relevant, questions include word limits. These are limits, not targets. Excellent answers can be shorter than the word limit. If you go beyond the word limit the additional text will be ignored. Where a question includes a word limit you HAVE to include a word count for your answer (excluding formulae). You could use https://wordcounter.net to obtain word counts.
- Candidates are advised that the examiners attach considerable importance to the clarity with which answers are expressed.
- **You must correctly enter your registration number and the course code on your answer.**

SECTION A

1. Suppose a researcher is interested in the following linear regression model

$$y_i \; = \; x_i'\beta_0 \, + \, u_i, \qquad i = 1, 2, \ldots N,$$

where $x_i = (1, x_{2,i}, x_{3,i}, x_{4,i})'$ and $\beta_0 = (\beta_{0,1}, \beta_{0,2}, \beta_{0,3}, \beta_{0,4})'$. Given the context, the researcher is able to assume that $\{u_i, x_i'\}_{i=1}^N$ form a sequence of independently and identically distributed random vectors with $E[x_i x_i'] = Q$, a finite, positive definite matrix of constants, $E[u_i|x_i] = 0$ and $Var[u_i|x_i] = \sigma_0^2$. Therefore, she estimates the model via Ordinary Least Squares (OLS), obtaining the following estimated equation

$$\hat{y}_i \; = \; \underset{(0.1416)}{1.0213} \, - \, \underset{(0.0005)}{0.0020}\, x_{2,i} \, - \, \underset{(0.0080)}{0.0208}\, x_{3,i} \, + \, \underset{(0.0088)}{0.0095}\, x_{4,i},$$

where the number in parenthesis is the conventional OLS standard error for the coefficient in question. The OLS estimator of σ_0^2 in this model is $\hat{\sigma}_N^2 = 0.0081$.

Given these results, the researcher concludes that $\beta_{0,4} = 0$ and so decides to estimate the model via OLS with $x_{4,i}$ excluded, obtaining the following estimated equation

$$\hat{y}_i = 1.1485 - 0.0020\, x_{2,i} - 0.0211\, x_{3,i}. \tag{1}$$

Given that the sample size is $N = 108$ in both estimations, calculate the OLS estimator of the error variance σ_0^2 from the estimation in (1). Be sure to carefully explain your calculations. *Hint: consider the F-statistic for testing $\beta_{0,4} = 0$.* **[8 marks]**

2. Suppose it is desired to predict z_t using $\hat{z}_t = w_t'\gamma$ where w_t is a vector of observable variables and γ is a vector of constants that needs to be specified. The choice of γ associated with the linear projection of z_t on w_t is γ_0 where $E[(z_t - w_t'\gamma_0)w_t] = 0$.
   (a) What optimality property does $\hat{z}_t^o = w_t'\gamma_0$ possess? *(Word limit: 50)* **[1 mark]**
   (b) Now consider the regression model $z_t = w_t'\gamma_0 + v_t$. Is w_t contemporaneously exogenous or strictly exogenous for estimation of γ_0 in this model? Justify your answer. *(Word limit: 150)* **[4 marks]**
   (c) Suppose the model in part (b) is dynamically complete. What optimality property does $\hat{z}_t^o$ possess? Briefly justify your answer. *(Word limit: 150)* **[3 marks]**

Continued over

SECTION A continued

3.(a) Let A be $n \times n$ nonsingular symmetric matrix. Show that if A is positive definite then A^{-1} is positive definite. *Hint:* $AA^{-1} = I_n$. **[4 marks]**

3.(b) Consider the classical linear regression model

$$y = X\beta_0 + u, \tag{2}$$

where X is the $T \times k$ observable data matrix that is fixed in repeated samples with $rank(X) = k$, and u is a $T \times 1$ vector with $E[u] = 0$ and $Var[u] = \sigma_0^2 I_T$ where σ_0^2 is an unknown positive finite constant. Let $\hat{\beta}_T$ be the OLS estimator of β_0 based on (2) and $\hat{\beta}_{R,T}$ be the Restricted Least Squares (RLS) estimator of β_0 based on (2) subject to the restrictions $R\beta_0 = r$ where R is a $n_r \times k$ matrix of specified constants with $rank(R) = n_r$ and r is a specified $n_r \times 1$ vector of constants. Assuming the restrictions are correct, prove that $\hat{\beta}_{R,T}$ is at least as efficient as $\hat{\beta}_T$. *Hint: you may quote the formula for the variance-covariance matrix of the OLS and RLS estimators without proof; you may also take advantage of the stated result in part (a).* **[4 marks]**

4. Consider the linear regression model

$$y = X\beta_0 + u,$$

where y and u are $T \times 1$ vectors, X is $T \times k$ matrix, and β_0 is the $k \times 1$ vector of unknown regression coefficients. Assume that X is fixed in repeated samples with $rank(X) = k$, and $u \sim N(0, \sigma_0^2 I_T)$ where σ_0^2 is an unknown positive constant. Let $\hat{\theta}_T$ denote the maximum likelihood estimator of the unknown parameter vector $\theta_0 = (\beta_0', \sigma_0^2)'$. Derive the information matrix for this model. *Hint: you may state the form of the log likelihood function and score function for this model without proof.* **[8 marks]**

5. Consider the model

$$y_i = x_i'\beta_0 + u_i, \qquad i = 1, 2, \ldots, N,$$

where β_0 is the $k \times 1$ vector of unknown regression coefficients, $\{(x_i', u_i)\}_{i=1}^N$ is a sequence of independently and identically distributed random vectors with $E[u_i|x_i] = 0$, $Var[u_i|x_i] = \sigma_0^2$, an unknown finite positive constant and $E[x_i x_i'] = Q$, a finite positive definite matrix of constants. Let $\hat{\sigma}_N^2$ be the OLS estimator of σ_0^2. Show that $N^{1/2}(\hat{\sigma}_N^2 - \sigma_0^2) \xrightarrow{d} N(0, \mu_4 - \sigma_0^2)$ where $\mu_4 = E[u_i^4]$.
Hint: You may assume that: $N^{-1}\sum_{i=1}^N x_i x_i' \xrightarrow{p} Q$; (ii) $N^{-1/2}\sum_{i=1}^N v_i \xrightarrow{d} N(0, \Omega)$ where $v_i = (u_i^2 - \sigma_0^2, x_i' u_i)'$, and $\Omega = Var[v_i]$ is a finite, positive definite $(k+1) \times (k+1)$ matrix whose elements you must specify as needed to develop your answer. **[8 marks]**

Continued over

SECTION B

6. Consider the regression model

$$y_i \;=\; x_i'\beta_0 \,+\, u_i, \qquad i \;=\; 1, 2, \ldots, N,$$

where β_0 is the $k \times 1$ vector of unknown regression coefficients, $\{(x_i', u_i)\}_{i=1}^N$ is a sequence of independently and identically distributed random vectors with $E[u_i|x_i] = 0$, $Var[u_i|x_i] = \sigma_0^2$, an unknown finite positive constant and $E[x_i x_i'] = Q$, a finite positive definite matrix of constants. You may further assume that: (i) $N^{-1}\sum_{i=1}^N x_i x_i' \xrightarrow{p} Q$; (ii) $N^{-1/2}\sum_{i=1}^N x_i u_i \xrightarrow{d} N(0, \sigma_0^2 Q)$.

Let $\hat{\beta}_{R,N}$ denote the RLS estimator based on the linear restrictions $R\beta = r$ where R is a $n_r \times k$ matrix of pre-specified constants with rank equal to n_r and r is a $n_r \times 1$ vector of pre-specified constants, and let $\hat{\lambda}_N$ be the vector of Lagrange Multipliers associated with this RLS estimation. Assuming $R\beta_0 = r$, answer the following questions.

(a) Show that $N^{1/2}(\hat{\beta}_{R,N} - \beta_0) \xrightarrow{d} N(0, V_R)$ where

$$V_R = \sigma_0^2 \left(Q^{-1} - Q^{-1}R'(RQ^{-1}R')^{-1}RQ^{-1} \right).$$

Hint: you may quote the formulae for $\hat{\beta}_{R,N}$ and $\hat{\beta}_N$, the Ordinary Least Squares estimator of β_0, without proof. **[10 marks]**

(b) A colleague proposes testing $H_0: R\beta_0 = r$ versus $H_1: R\beta_0 \neq r$ using the decision rule of the form: reject H_0 at the (approximate) $100\alpha\%$ significance level if

$$\hat{\lambda}_N' M_N \hat{\lambda}_N \;>\; c_{n_r}(1-\alpha)$$

where $c_{n_r}(1-\alpha)$ is the $100(1-\alpha)^{th}$ percentile of the $\chi^2_{n_r}$ distribution. However, your colleague is unsure what the matrix M_N should be in order that this decision rule has the properties implied by the stated significance level. Provide a suitable choice of M_N, being sure to justify your choice carefully. *Hint: you may quote without proof: (i) the formulae for $\hat{\lambda}_N$ and $\hat{\beta}_N$; (ii) that both the OLS and RLS estimators of σ_0^2 are consistent under the conditions of the question.* **[20 marks]**

Continued over

<u>SECTION B continued</u>

7.(a) Let $\{v_t\}_{t=-3}^{T}$ be a weakly stationary time series process. Consider the following statistic,

$$\hat{\rho}_{4,T} \;=\; \frac{\sum_{t=1}^{T} v_t v_{t-4}}{\sum_{t=1}^{T} v_t^2}.$$

Let $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ denote a sequence of independently and identically distributed (i.i.d.) random variables with mean zero and variance σ_ε^2.

(i) Assume that $v_t = \varepsilon_t$. Show that $T^{1/2}\hat{\rho}_{4,T} \xrightarrow{d} N(0,1)$. **[6 marks]**

(ii) Assume that

$$v_t \;=\; \theta_4 v_{t-4} \,+\, \varepsilon_t,$$

where $|\theta_4| < 1$. What is the probability limit of $\hat{\rho}_{4,T}$ as $T \to \infty$? Be sure to justify your answer carefully. *Hint: v_t has the following representation, $v_t = \sum_{i=0}^{\infty} \theta_4^i \varepsilon_{t-4i}$.* **[9 marks]**

7.(b) A researcher wishes to test the simple efficient-markets hypothesis in the foreign exchange market. Let $s_t = ln(S_t)$ and $f_{t,n} = ln[F_{t,n}]$, where S_t and $F_{t,n}$ are the levels of the spot exchange rate at time t and the n-period forward exchange rate at time t. The simple efficient-markets hypothesis is that $f_{t,n} = E[s_{t+n}\,|\,\mathcal{I}_t]$ where $\mathcal{I}_t$ is the information set at time t which for the purposes of this question can be taken to be $\mathcal{I}_t = \{s_t, f_{t,n}, s_{t-1}, f_{t-1,n}, s_{t-2}, f_{t-2,n}, \ldots\}$. Using daily spot and thirty-day forward exchange rate data for the US dollar UK pound exchange rate, the researcher estimates the model,

$$y_{t+n} \;=\; x_t'\beta_0 \,+\, u_{t,n}, \qquad (3)$$

where $y_{t+n} = s_{t+n} - f_{t,n}$, x_t is the 3×1 vector given by

$$x_t' \;=\; (\,1,\; s_t - f_{t-n,n},\; s_{t-1} - f_{t-1-n,n}\,)\,,$$

$n = 30$ and $u_{t,n}$ is the error term. If the simple efficient markets hypothesis holds in this foreign exchange market then $E[u_{t,n}\,|\,\mathcal{I}_t] = 0$ and the regression coefficients in (3) satisfy a set of restrictions denoted here by $g(\beta_0) = 0$ where $g(\,\cdot\,)$ is $n_g \times 1$ vector.

(i) What is $g(\beta_0)$? Briefly justify your answer. *(Word limit: 75)* **[4 marks]**

Continued over

7.(b) continued

(ii) The researcher tests $H_0 : g(\beta_0) = 0$ versus $H_1 : g(\beta_0) \neq 0$ using the test statistic

$$S_T = T g(\hat{\beta}_T)' \left(G(\hat{\beta}_T) \hat{V}_\beta G(\hat{\beta}_T)' \right)^{-1} g(\hat{\beta}_T), \tag{4}$$

where $G(\hat{\beta}_T) = \partial g(\beta)/\partial \beta \big|_{\beta=\hat{\beta}_T}$. Assuming $T^{1/2}(\hat{\beta}_T - \beta_0) \xrightarrow{d} N(0, V_\beta)$ and $\hat{V}_\beta \xrightarrow{p} V_\beta$, write down a suitable decision rule for this test. If $S_T = 8.2$ then what is the outcome of the test? **[3 marks]**

(iii) Since the y_{t+n} is a financial variable, the researcher is concerned that the errors may exhibit autoregressive conditional heteroscedasticity and so has calculated $\hat{V}_\beta$ in (4) using White's heteroscedasticity robust estimator. Given this information, do you have any concerns about the test in part (ii)? If so then explain your concerns and how you would modify the test to address these concerns. *(Word limit: 350)* **[8 marks]**

Continued over

SECTION B continued

8. Consider the linear regression model

$$y_{1,i} \;=\; \gamma_0 y_{2,i} \,+\, z_{1,i}'\delta_0 \,+\, u_{1,i} \;=\; x_i'\beta_0 \,+\, u_{1,i},$$

where $x_i' = (y_{2,i}, z_{1,i}')$, $\beta_0 = (\gamma_0, \delta_0')'$ and assume that

$$y_{2,i} \;=\; z_i'\eta_0 \,+\, u_{2,i}, \tag{5}$$

where $y_{1,i}$ and $y_{2,i}$ are observable random variables, $z_i = (z_{1,i}', z_{2,i}')'$ is a random vector of observable variables, $u_{1,i}$ and $u_{2,i}$ are the error terms (unobservable scalar random variables), γ_0 is an unknown scalar parameter, and δ_0, and η_0 are vectors of unknown parameters. Suppose there is a sample of N observations, and let $\hat{y}_{2,i}$ denote the predicted value of $y_{2,i}$ based on Ordinary Least Squares (OLS) estimation of (5). Define $\hat{x}_i = (\hat{y}_{2,i}, z_{1,i}')'$. Let X be the $N \times k$ matrix with i^{th} row x_i', $\hat{X}$ be the $N \times k$ matrix with i^{th} row $\hat{x}_i'$, Z be the $N \times q$ matrix with i^{th} row z_i' and y_1 be the $N \times 1$ vector with i^{th} element $y_{1,i}$. Consider the following three estimators of β_0:

- $\hat{\beta}_1 \;=\; (\hat{X}'\hat{X})^{-1}\hat{X}'y_1$;
- $\hat{\beta}_2 \;=\; (\hat{X}'X)^{-1}\hat{X}'y_1$;
- $\hat{\beta}_3 \;=\; \{\, X'Z(Z'Z)^{-1}Z'X \,\}^{-1}\, X'Z(Z'Z)^{-1}Z'y_1$.

(a) Show that $\hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3$. **[15 marks]**

(b) Let $\hat{\phi} = (\hat{\beta}', \hat{\theta})'$ be the OLS estimator of $\phi_0 = (\beta_0', \theta_0)'$ based on the model

$$y_{1,i} \;=\; x_i'\beta_0 \,+\, \theta\hat{u}_{2,i} \,+\, \text{“error”}$$

where $\hat{u}_{2,i}$ is the i^{th} element of $\hat{u}_2$, the $N \times 1$ vector of residual from OLS estimation of (5). Via an application of the Frisch-Waugh-Lovell Theorem or otherwise, show that $\hat{\beta} = \hat{\beta}_1$.

[15 marks]

Continued over

SECTION B continued

9.(a) Let $\{(y_i, x_i')\}_{i=1}^{N}$ be a sequence of independently and identically distributed (i.i.d.) random vectors. Suppose that y_i is a dummy variable and so has a sample space of $\{0, 1\}$. Consider the model

$$y_i = x_i'\beta_0 + u_i.$$

(i) Assume that $E[u_i|x_i] = 0$. Derive the Generalized Least Squares (GLS) estimator of β_0 in this model? *Hint: you may quote the generic formula for the GLS estimator that is, $\hat{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$, but you must derive Σ for this model.* **[6 marks]**

(ii) Is your answer to part (i) a feasible or infeasible GLS estimator? If infeasible then suggest a feasible GLS estimator. Do you foresee any potential problems in implementing your proposed Feasible GLS estimator? **[4 marks]**

(b) Let $\{Y_i\}_{i=1}^{N}$ be a sequence of i.i.d. Bernoulli random variables with $P(Y_i = 1) = \theta_0$. We assume here that $\theta_0 \in (0, 1)$ and that our sample size is large enough for both outcomes to occur.

(i) Derive the Wald, Likelihood Ratio and Lagrange Multiplier statistics for testing $H_0 : \theta_0 = \theta_*$ against $H_1 : \theta_0 \neq \theta_*$. *Hint: you may quote the form of the log likelihood function, the score equation and the formula for the maximum likelihood estimator for this model without proof.* **[16 marks]**

(ii) Given that $N = 100$ and the sample contains 55 outcomes that are one, use your statistics in part (i) to test the hypothesis $H_0 : \theta_0 = 0.5$ against $H_1 : \theta_0 \neq 0.5$ at the 5% significance level. **[4 marks]**

END OF EXAMINATION

1 Table 1: Percentage Points for the t distribution

Student's t Distribution Function for Selected Probabilities The table provides values of $t_{\alpha,v}$ where $\Pr(T \leq t_{\alpha,v}) = \alpha$ and $T \sim t_v$										
α	0.750	0.800	0.900	0.950	0.975	0.990	0.995	0.9975	0.999	0.9995
ν	Values of $t_{\alpha,v}$									
1	1.000	1.376	3.078	6.314	12.706	31.821	63.657			
2	0.816	1.061	1.886	2.920	4.303	6.965	9.925			
3	0.765	0.978	1.638	2.353	3.182	4.541	5.841			
4	0.741	0.941	1.533	2.132	2.776	3.747	4.604			
5	0.727	0.920	1.476	2.015	2.571	3.365	4.032	4.773		
6	0.718	0.906	1.440	1.943	2.447	3.143	3.707	4.317	5.208	
7	0.711	0.896	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
70	0.678	0.847	1.294	1.667	1.994	2.381	2.648	2.899	3.211	3.435
80	0.678	0.846	1.292	1.664	1.990	2.374	2.639	2.887	3.195	3.416
90	0.677	0.846	1.291	1.662	1.987	2.368	2.632	2.878	3.183	3.402
100	0.677	0.845	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390
110	0.677	0.845	1.289	1.659	1.982	2.361	2.621	2.865	3.166	3.381
120	0.677	0.845	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	0.842	1.282	1.645	1.960	2.326	2.576	2.808	3.090	3.297

2 Table 2: Percentage Points for the χ^2 distribution

The χ^2 Distribution Function for Selected Probabilities											
The table provides values of $\chi^2_{\alpha,v}$ where $\Pr(\chi^2 \le \chi^2_{\alpha,v}) = \alpha$ and $\chi^2 \sim \chi^2_v$											
α	0.005	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99	0.995
v	Values of $\chi^2_{\alpha,v}$										
1	0.000	0.000	0.001	0.004	0.016	0.455	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	1.386	4.605	5.991	7.378	9.210	10.60
3	0.072	0.115	0.216	0.352	0.584	2.366	6.251	7.815	9.348	11.34	12.84
4	0.207	0.297	0.484	0.711	1.064	3.357	7.779	9.488	11.14	13.28	14.86
5	0.412	0.554	0.831	1.145	1.610	4.351	9.236	11.07	12.83	15.09	16.75
6	0.676	0.872	1.237	1.635	2.204	5.348	10.64	12.59	14.45	16.81	18.55
7	0.989	1.239	1.690	2.167	2.833	6.346	12.02	14.07	16.01	18.48	20.28
8	1.344	1.646	2.180	2.733	3.490	7.344	13.36	15.51	17.53	20.09	21.95
9	1.735	2.088	2.700	3.325	4.168	8.343	14.68	16.92	19.02	21.67	23.59
10	2.156	2.558	3.247	3.940	4.865	9.342	15.99	18.31	20.48	23.21	25.19
11	2.603	3.053	3.816	4.575	5.578	10.34	17.28	19.68	21.92	24.72	26.76
12	3.074	3.571	4.404	5.226	6.304	11.34	18.55	21.03	23.34	26.22	28.30
13	3.565	4.107	5.009	5.892	7.042	12.34	19.81	22.36	24.74	27.69	29.82
14	4.075	4.660	5.629	6.571	7.790	13.34	21.06	23.68	26.12	29.14	31.32
15	4.601	5.229	6.262	7.261	8.547	14.34	22.31	25.00	27.49	30.58	32.80
16	5.142	5.812	6.908	7.962	9.312	15.34	23.54	26.30	28.85	32.00	34.27
17	5.697	6.408	7.564	8.672	10.09	16.34	24.77	27.59	30.19	33.41	35.72
18	6.265	7.015	8.231	9.390	10.86	17.34	25.99	28.87	31.53	34.81	37.16
19	6.844	7.633	8.907	10.12	11.65	18.34	27.20	30.14	32.85	36.19	38.58
20	7.434	8.260	9.591	10.85	12.44	19.34	28.41	31.41	34.17	37.57	40.00
21	8.034	8.897	10.28	11.59	13.24	20.34	29.62	32.67	35.48	38.93	41.40
22	8.643	9.542	10.98	12.34	14.04	21.34	30.81	33.92	36.78	40.29	42.80
23	9.260	10.20	11.69	13.09	14.85	22.34	32.01	35.17	38.08	41.64	44.18
24	9.886	10.86	12.40	13.85	15.66	23.34	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	24.34	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	25.34	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	26.34	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	27.34	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	28.34	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	29.34	40.26	43.77	46.98	50.89	53.67
35	17.19	18.51	20.57	22.47	24.80	34.34	46.06	49.80	53.20	57.34	60.27
40	20.71	22.16	24.43	26.51	29.05	39.34	51.81	55.76	59.34	63.69	66.77
45	24.31	25.90	28.37	30.61	33.35	44.34	57.51	61.66	65.41	69.96	73.17
50	27.99	29.71	32.36	34.76	37.69	49.33	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	59.33	74.40	79.08	83.30	88.30	91.95
70	43.28	45.44	48.76	51.74	55.33	69.33	85.53	90.53	95.02	100.4	104.2
80	51.17	53.54	57.15	60.39	64.28	79.33	96.58	101.9	106.6	112.3	116.3
90	59.20	61.75	65.65	69.13	73.29	89.33	107.6	113.1	118.1	124.1	128.3
100	67.33	70.06	74.22	77.93	82.36	99.33	118.5	124.3	129.6	135.8	140.2
150	109.1	112.7	118.0	122.7	128.3	149.3	172.6	179.6	185.8	193.2	198.4
200	152.2	156.4	162.7	168.3	174.8	199.3	226.0	234.0	241.1	249.4	255.3

3 Table 3: Upper 5% percentage points for the F distribution

The F Distribution Function for $\alpha = 0.05$

The table provides values of F_{α,v_1,v_2} where $\Pr(F \geq F_{\alpha,v_1,v_2}) = 0.05$ and $F \sim F(v_1, v_2)$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01
35	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11	2.04	1.96
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92
45	4.06	3.20	2.81	2.58	2.42	2.31	2.22	2.15	2.10	2.05	1.97	1.89
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.87
55	4.02	3.16	2.77	2.54	2.38	2.27	2.18	2.11	2.06	2.01	1.93	1.85
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.89	1.81
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.88	1.79
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.86	1.78
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.85	1.77
110	3.93	3.08	2.69	2.45	2.30	2.18	2.09	2.02	1.97	1.92	1.84	1.76
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.82	1.73

4 Table 4: Upper 1% percentage points for the F distribution

The F Distribution Function for $\alpha = 0.01$												
The table provides values of F_{α,v_1,v_2} where $\Pr(F \geq F_{\alpha,v_1,v_2}) = 0.01$ and $F \sim F(v_1, v_2)$												
	$v_1 \rightarrow$											
$v_2 \downarrow$	1	2	3	4	5	6	7	8	9	10	12	15
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70
35	7.42	5.27	4.40	3.91	3.59	3.37	3.20	3.07	2.96	2.88	2.74	2.60
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52
45	7.23	5.11	4.25	3.77	3.45	3.23	3.07	2.94	2.83	2.74	2.61	2.46
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.56	2.42
55	7.12	5.01	4.16	3.68	3.37	3.15	2.98	2.85	2.75	2.66	2.53	2.38
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35
70	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67	2.59	2.45	2.31
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.42	2.27
90	6.93	4.85	4.01	3.53	3.23	3.01	2.84	2.72	2.61	2.52	2.39	2.24
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.37	2.22
110	6.87	4.80	3.96	3.49	3.19	2.97	2.81	2.68	2.57	2.49	2.35	2.21
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19
150	6.81	4.75	3.91	3.45	3.14	2.92	2.76	2.63	2.53	2.44	2.31	2.16