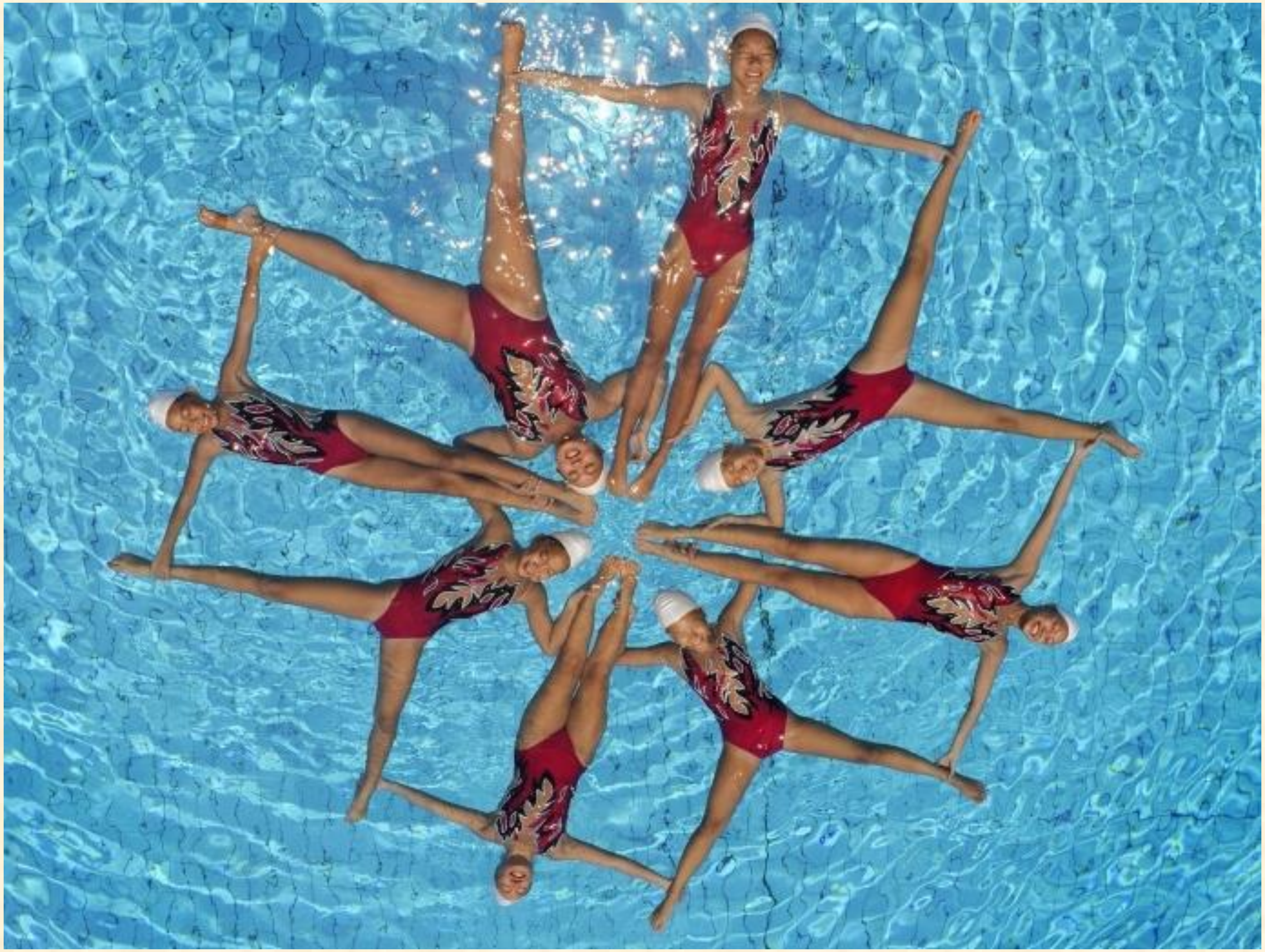
Complex Networks:

Lecture 7: Network Synchronization

EE 6605

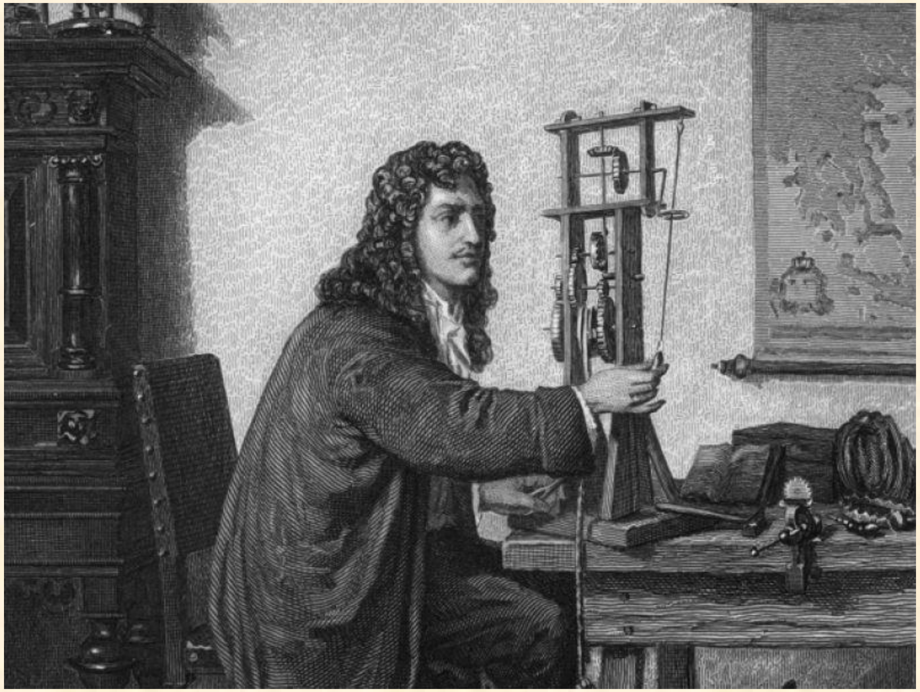
Instructor: G Ron Chen



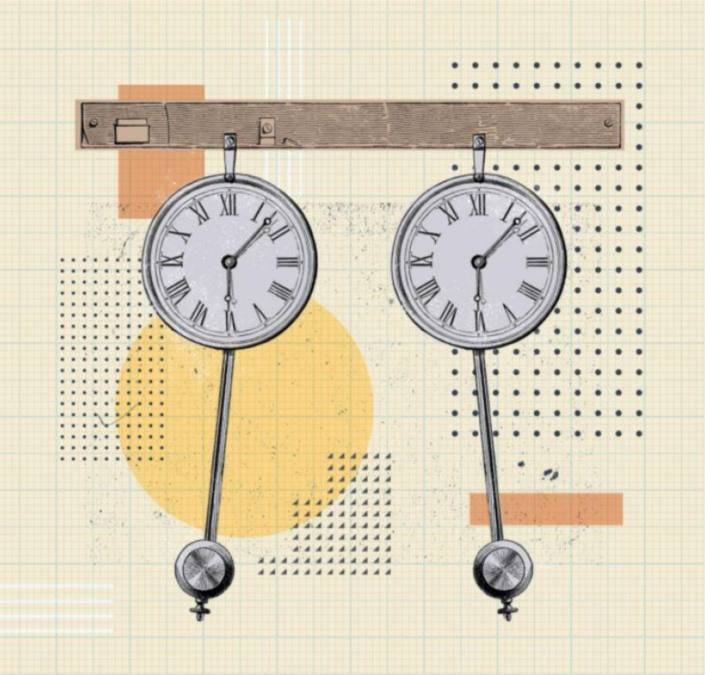
[Sync Video]

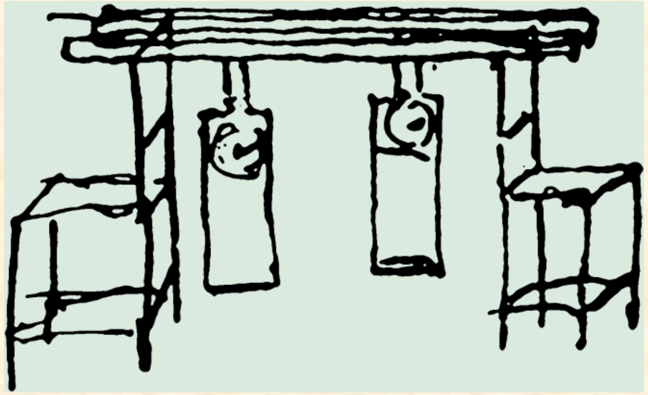
Synchronization

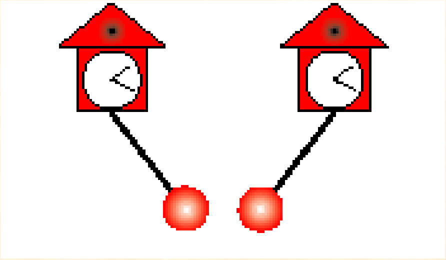
(Christiaan Huygens, 1665)



Huygens invented the pendulum clock
and observed their synchronization

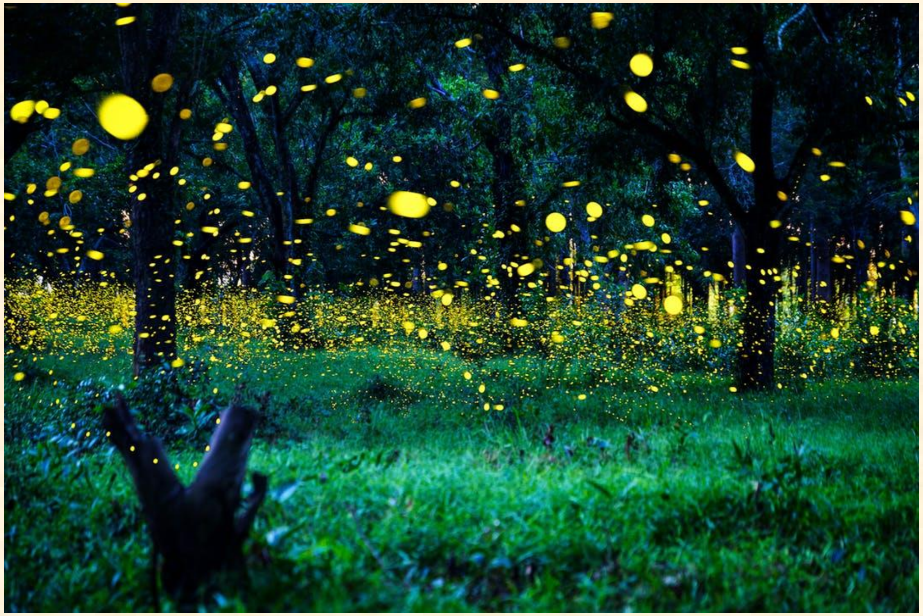




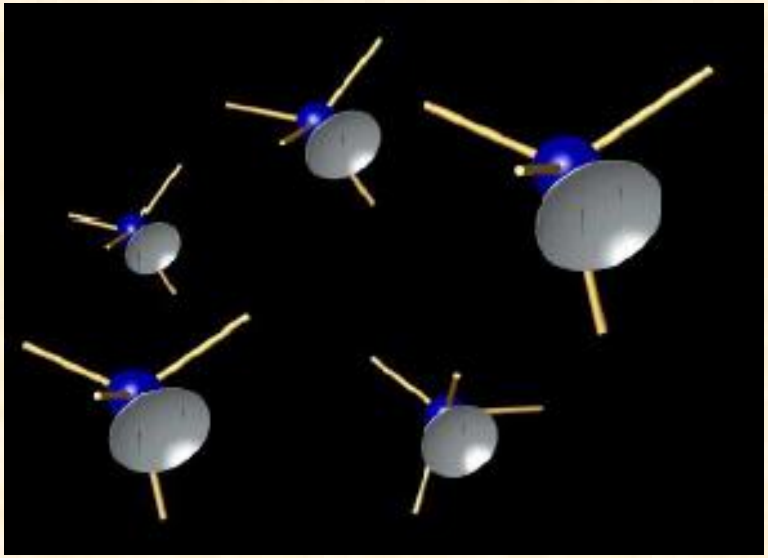


DEMO

Fireflies Synchronization



Attitude Alignment



The attitude of each spacecraft is synchronized with its two adjacent neighbors via a bi-directional communication channel

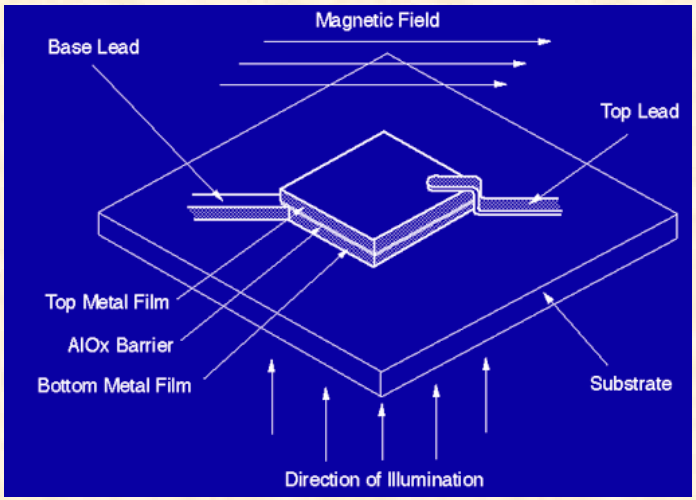
Synchronization-based Laser

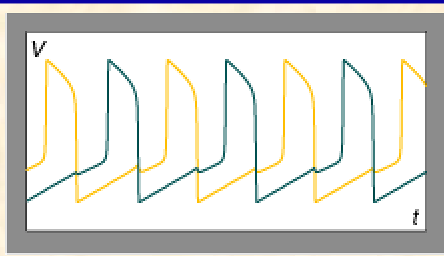
Astounding coherence of a laser beam comes from trillions of atoms pulsing in concert, all emitting photons of the same phase and frequency

Electric Currents

Through Josephson Junctions

Oscillate as One





Phase Sync

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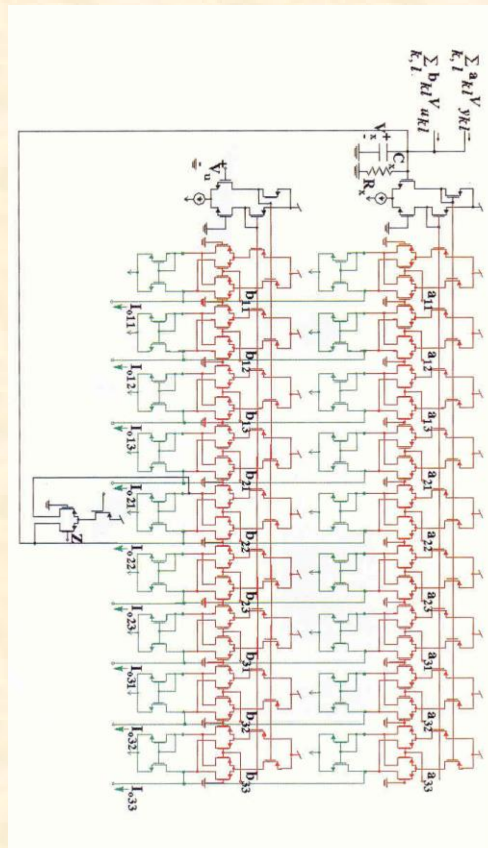
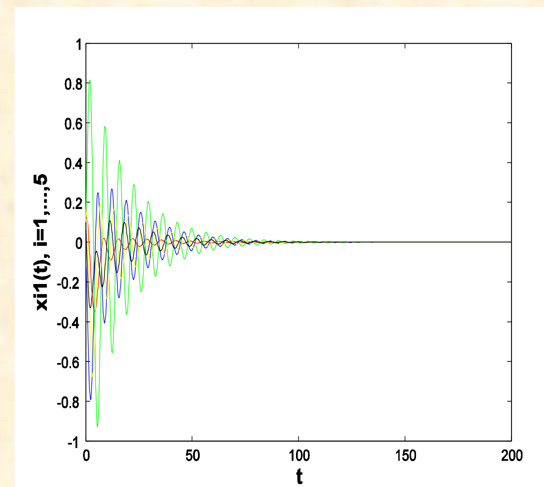




Simulation

Pedestrians Make London's Millennium Bridge Wobble

June 10, 2000

Video

Out of tumultuous applause:
Synchronized claps

Self-organization

in the concert-hall:

the dynamics of rhythmic applause

--- ***Nature*** (2001)

Birds Flocking

Flocking: to congregate or travel in flock

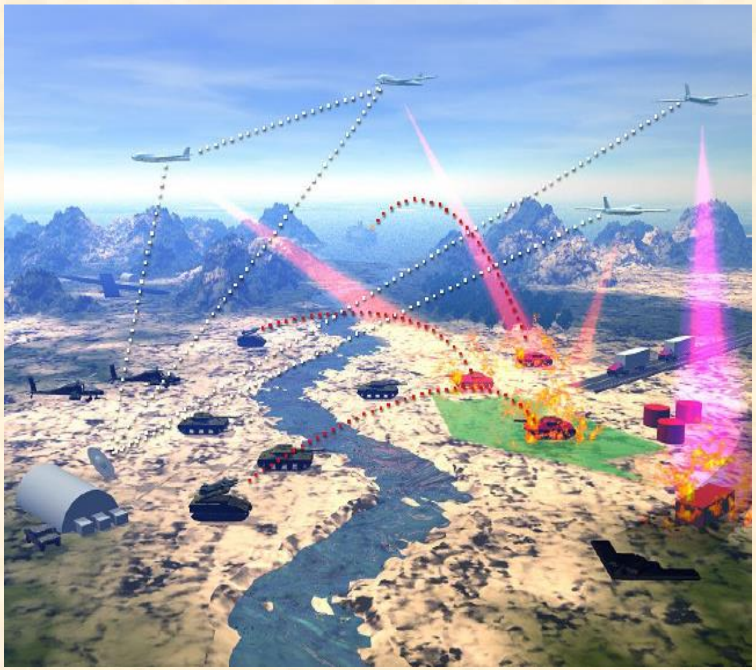
Fish Swarming

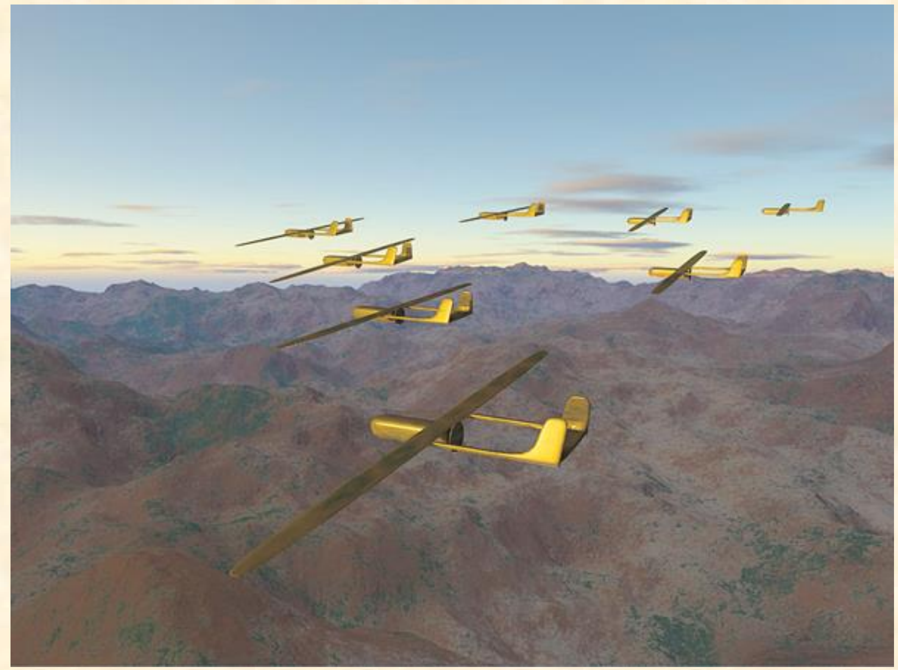
- Swarming:

  to move or gather in group

Consensus

A position reached by a group as a whole





Battle space management scenario

Useful Synchronization

Examples:

- **Secure communications**
- **Generation of harmonic oscillations**
- **Language emergence and development:**

  Synchronization in conversations

  → common vocabulary
- **Organization management:**

  Agents' synchronization

  → work efficiency
- **Biological systems (brain, heart)**
- **……**

Harmful Synchronization (in Internet)

- **TCP window increase/decrease cycles**

  Synchronization occurs when separate TCP connections share a common bottleneck router

- **Synchronization to an external clock**

  Two processes can become synchronized simply if they are both synchronized to the same external clock

- **Client-server models**

  Multiple clients can become synchronized as they wait for services from a busy (or recovering) server

- **Periodic message routing**
- **……**

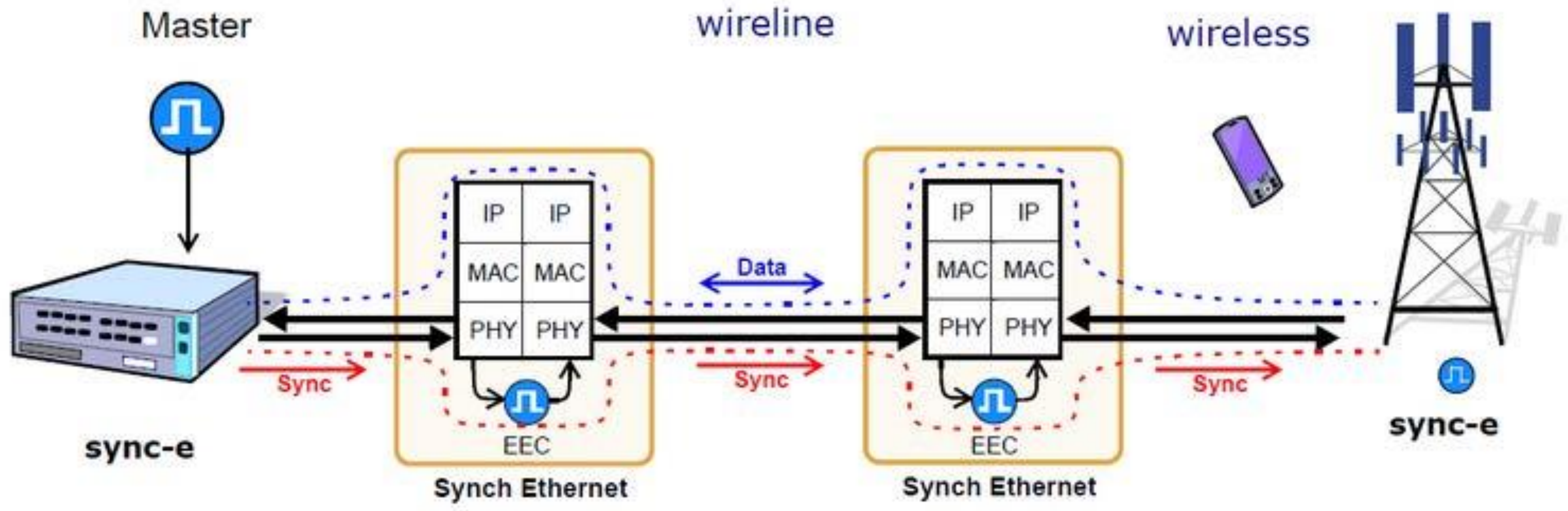
Yik-Chung Wu, Qasim Chaudhari, and Erchin Serpedin

Clock Synchronization of Wireless Sensor Networks

Message exchange mechanisms and statistical signal processing techniques

Clock synchronization is a critical component in the operation of wireless sensor networks (WSNs), as it provides a common time frame to different nodes. It supports functions such as fusing voice and video data from different sensor nodes, time-based channel sharing, and coordinated sleep wake-up node scheduling mechanisms. Early

Communication Networks



Mobile Networks require a kind of synchronization

FULL SCREEN

MIT neuroscientists found that brain waves originating from the striatum (red) and from the prefrontal cortex (blue) become synchronized when an animal learns to categorize different patterns of dots.

Illustration: Jose-Luis Olivares/MIT

Synchronized brain waves enable rapid learning

MIT study finds neurons that hum together encode new information.

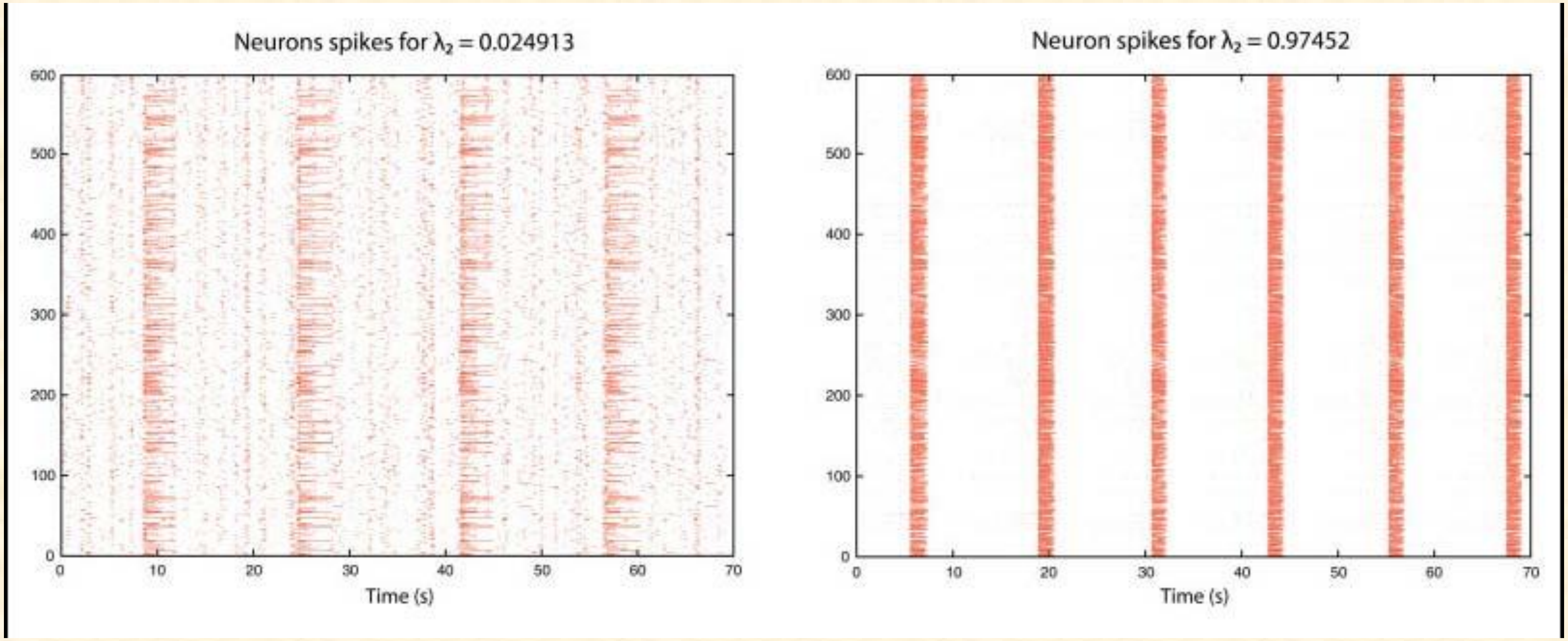
Anne Trafton | MIT News Office
June 12, 2014

Press Inquiries

PRESS MENTIONS

One Example

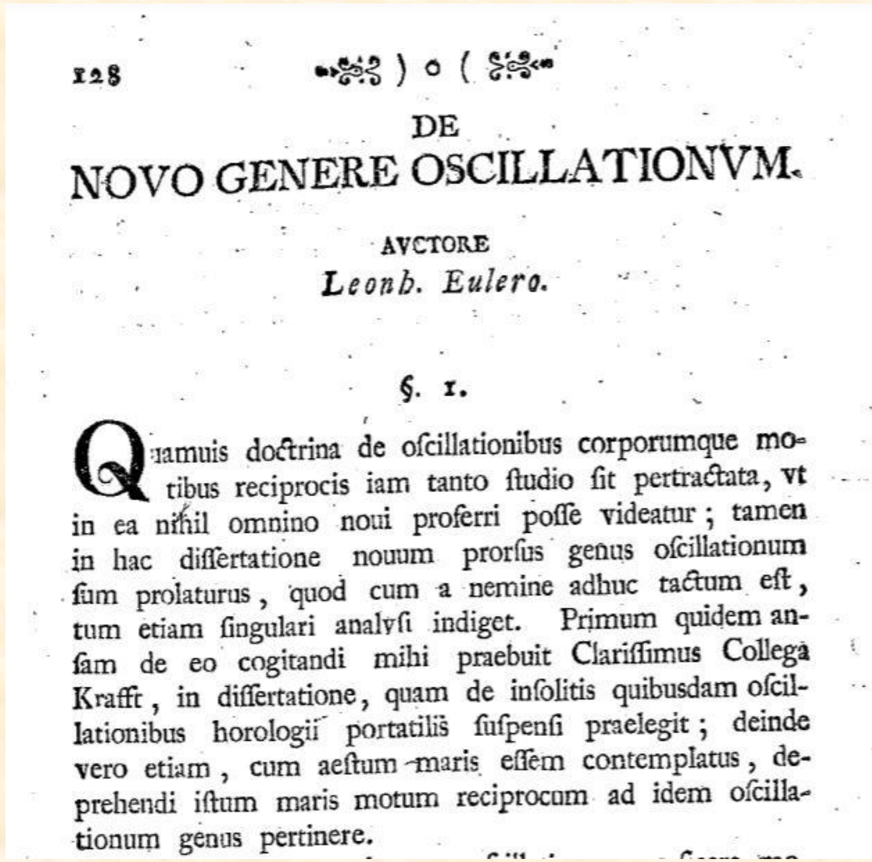
In mammals, a small group of neurons in the brain stem, named pre-Bötzinger complex, is responsible for generating a regular rhythmic output to motor calls that initiate a breath.



non-synchrous

synchrous

Euler discovered mechanical resonance in 1750



DE

NOVO GENERE OSCILLATIONVM.

AVCTORE

Leonh. Eulero.

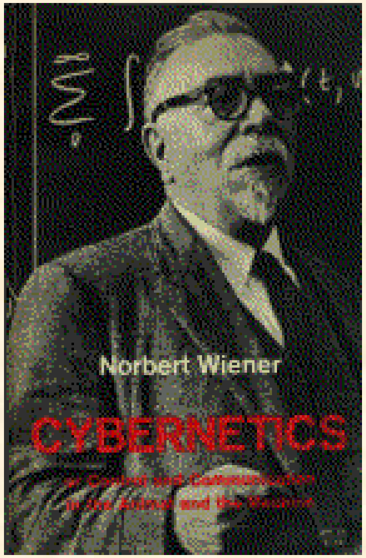
§. 1.

Quamuis doctrina de oſcillationibus corporumque motibus reciprocis iam tanto ſtudio ſit pertractata, vt in ea nihil omnino noui proferri poſſe videatur; tamen in hac diſſertatione nouum prorſus genus oſcillationum ſum prolaturus, quod cum a nemine adhuc tactum eſt, tum etiam ſingulari analyſi indiget. Primum quidem anſam de eo cogitandi mihi praebuit Clariſſimus Collega Krafft, in diſſertatione, quam de inſolitis quibusdam oſcillationibus horologii portatilis ſuſpenſi praelegit; deinde vero etiam, cum aeſtum maris eſſem contemplatus, deprehendi iſtum maris motum reciprocum ad idem oſcillationum genus pertinere.

Leonhard Euler, "De novo genere oscillationvm", Comm. Acad. Sci. Petrop, 11 (1139), 128-149, 1750.

Norbert Wiener

(1894-1964)



- **Collective synchronization was first studied mathematically by Wiener, who recognized its ubiquity in the natural world, and speculated that it was involved in the generation of alpha rhythms in the brain.**
- **Unfortunately, Wiener's mathematical approach based on Fourier integrals has turned out to be a dead end.**

- N. Wiener, Nonlinear Problems in Random Theory, MIT Press, 1958
- N. Wiener, Cybernetics, MIT Press, 1st ed., 1948, 2nd ed., 1961.

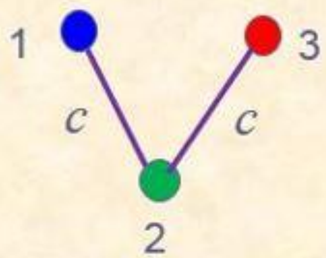
Network Synchronization

- Network Synchronization versus Individual Dynamics
  - Typical networks: coupled map lattices, cellular neural networks, complex networks
  - Typical dynamics: Turing patterns, autowaves, spiral waves, spatiotemporal chaos
- Network topology and the dynamics of individual nodes determine the network dynamical behaviors

  → Synchronization

Dynamical Network Model

Example: (undirected)



$$\dot{x}_1 = f(x_1) \qquad\qquad + c x_2$$

$$\dot{x}_2 = f(x_2) + c x_1 \qquad\qquad + c x_3$$

$$\dot{x}_3 = f(x_3) \qquad\qquad + c x_2$$

$$\Rightarrow \quad \dot{x}_i = f(x_i) + c \sum_{j=1}^{3} a_{ij} x_j \qquad i = 1,2,3$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \dot{x}_i = f(x_i) - c \sum_{j=1}^{3} a_{ij} H x_j \qquad i = 1,2,3 \quad (H = -1)$$

A General Dynamical Network Model

Linearly and diffusively coupled:

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i = \begin{bmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^n \end{bmatrix} \in R^n \qquad i = 1,2,\ldots,N$$

dynamics: $f(.)$ – Lipschitz function

coupling strength $c > 0$

coupling matrices $A = [a_{ij}]_{N\times N}$ $\quad H(x_j) = \begin{bmatrix} H_1(x_j) \\ H_2(x_j) \\ \vdots \\ H_n(x_j) \end{bmatrix}$ e.g. linear: Hx_j

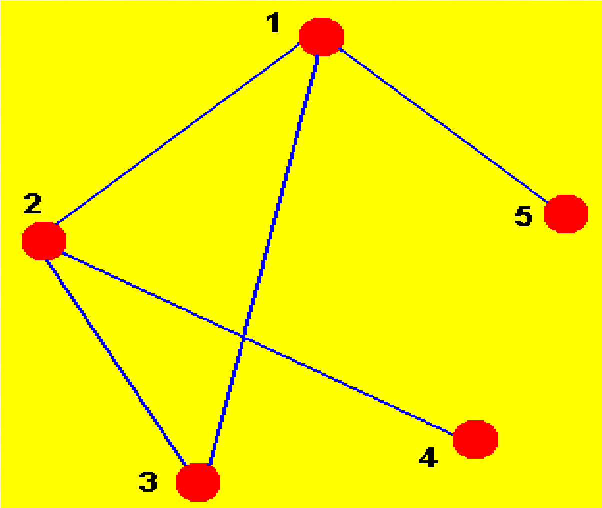
If there is a connection between node i and node j $(j \neq i)$, then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$; And $a_{ii} = 0$

$$D = [d_{ii}] = diag\{k_1,\ldots,k_N\}$$

Laplacian matrix $L = D - A$ (zero row sum: diffusive coupling)

Examples

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j)$$



$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

symmetrical, diffusive, irreducible, zero row-sum

eigenvalues of L:

$$\lambda_1 = 0 < \lambda_{2,3} = \frac{5}{2} \pm \frac{\sqrt{5}}{2}, \lambda_{4,5} = \frac{5}{2} \pm \frac{\sqrt{13}}{2}$$

$$L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

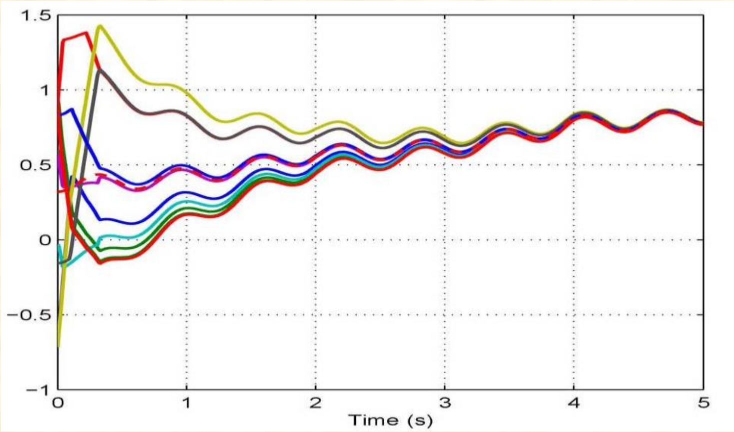
eigenvalues of L: $\lambda_1 = 0 \quad \lambda_2 = 2$

Network Synchronization

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1,2,\ldots,N$$

Complete state synchronization: $\lim_{t\to\infty} \| x_i(t) - x_j(t) \|_2 = 0, \quad i, j = 1,2,\cdots,N$

State tracking: $x_i(t) \to s(t) \qquad i = 1,2,\ldots,N$



Network Synchronization

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1,2,\ldots,N$$

Put all equations together with $\mathrm{x} = [x_1^T, x_2^T, \ldots, x_N^T]^T$

Then linearize it at equilibrium s :

$$\dot{\mathrm{x}} = [I_N \otimes [\nabla f(s)]] - c[A \otimes [\nabla H(s)]]\mathrm{x}$$

Only

$f(.)$ Lipschitz, or assume: $\|\nabla f(s)\| \le M$ ⇨ cA or $\{c\lambda_i\}$

is important

After linearization, perform local analysis

Recall that for a linear system

$$\dot{x} = -Lx$$

with L being a constant matrix (here, the Laplacian matrix), its solution can be expressed as:

$$x(t) = \sum_{k=1}^{N} c_k e^{-Re(\lambda_k)\,t}\left[\cos(Im(\lambda_k))t - \mathrm{i}\sin(Im(\lambda_k))t\,\right]$$

For directed networks, $\{\lambda_k\}$ are complex

For undirected networks, $\{\lambda_k\}$ are real →

$$x(t) = \sum_{k=1}^{N} c_k e^{-\lambda_k t}$$

→ Eigenvalues of Laplacian determine the solution

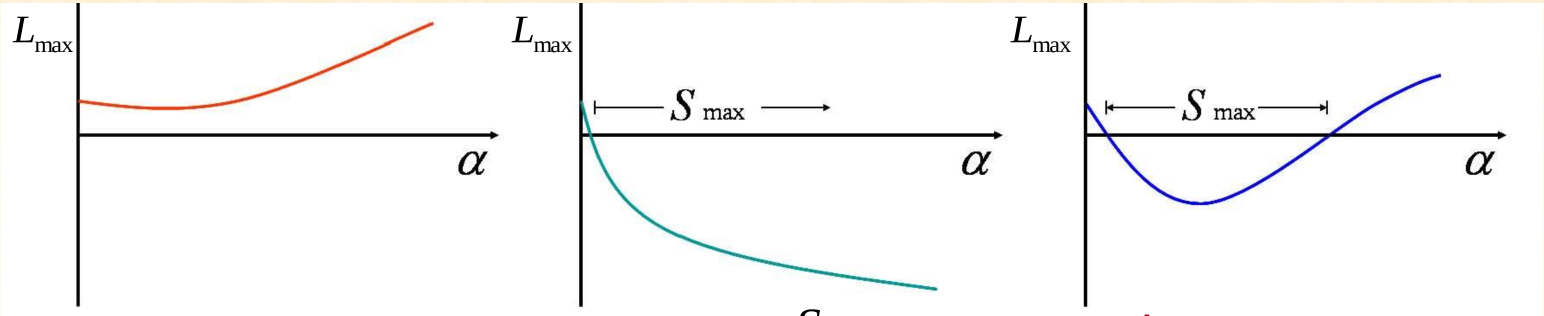
Network Synchronization: Criteria

$$\dot{\mathrm{x}} = [I_N \otimes [\nabla f(s)]] - c[A \otimes [\nabla H(s)]]\mathrm{x} \qquad 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$$

Master stability equation: (L.M. Pecora and T. Carroll, 1998)

$$\dot{\mathrm{y}} = [[\nabla f(s)] - \alpha[\nabla H(s)]]\mathrm{y}, \qquad \alpha = \inf\{c\lambda_i(s), i = 2,3,\ldots, N\}$$

Maximum Lyapunov exponent $L_{\max}$ is a function of α



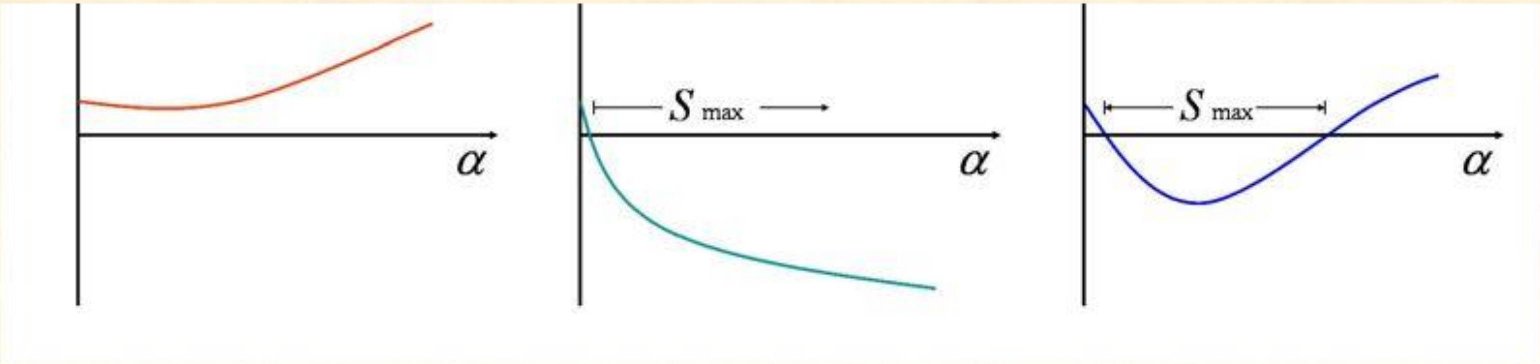
$S_{\max}$ -- sync region

Synchronizing: if $S_1 = (\alpha_1, \infty)$ $\quad 0 \le \alpha_1 < \alpha < \infty$

or if $S_2 = (\alpha_2, \alpha_3)$ $\quad 0 \le \alpha_2 < \alpha < \alpha_3 < \infty$

A Linearization Approach

Recall: Eigenvalues of $L = D - A$ $\quad 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$



Synchronizing: if $0 \le \alpha_1 < c\lambda_2 < \infty$ or if $0 < \alpha_2 < \dfrac{\lambda_2}{\lambda_N} < \alpha_3$

Case I:
No sync

Case II:
Sync region
$S_1 = (\alpha_1, \infty)$

Case III:
Sync region
$S_2 = (\alpha_2, \alpha_3)$

Case IV:
Union of intervals

λ_2 bigger is better

$\dfrac{\lambda_2}{\lambda_N}$ bigger is better

(X. F. Wang and G. Chen, 2002)

(M. Banahona and L. M. Pecora, 2002)

Summary: Synchronizability

1. Bounded region (M. Banahona and L. M. Pecora, 2002)

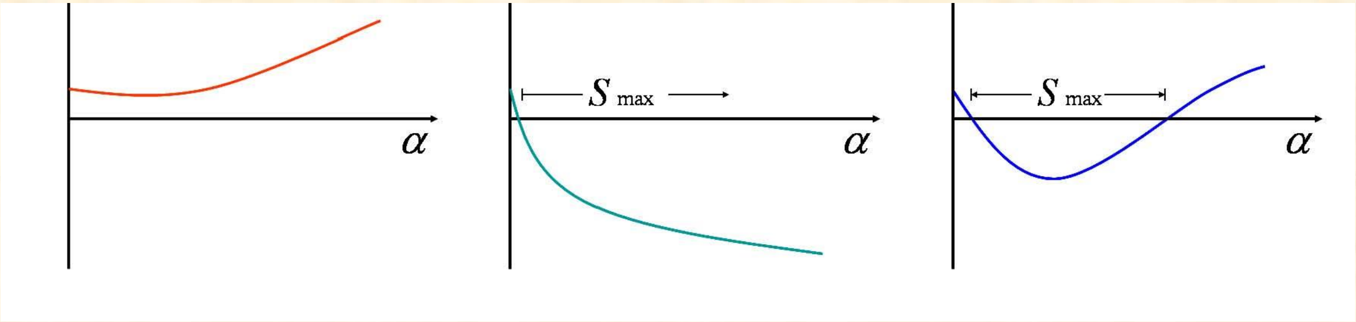
$\lambda_2 / \lambda_N , \quad 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N \quad \left(\alpha_1 < \frac{\lambda_2}{\lambda_N} < \alpha_2\right)$ bigger better

2. Unbounded region (X. F. Wang and G. Chen, 2002)

$c\lambda_2 , \quad 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N \quad (0 < c\lambda_2 < \infty)$ bigger better

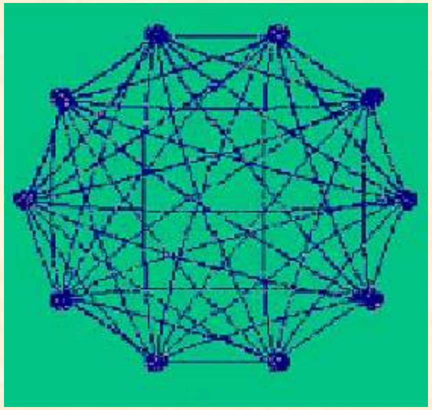
3. Union of several disconnected regions $(\alpha_1, \alpha_2) \cup \cdots \cup (\alpha_m, \alpha_n)$

(A. Stefanski, P. Perlikowski, and T. Kapitaniak, 2007)
(Z. S. Duan, C. Liu, G. Chen, and L. Huang, 2007)



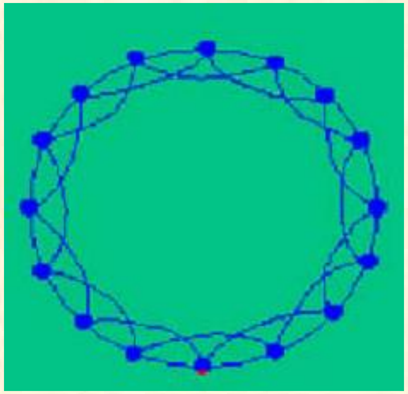
Synchronization in Regularly Coupled Networks

Globally coupled network: $\lambda_2 = \cdots = \lambda_N = N$



Usually, no matter how small the coupling strength c is, a global coupled network will synchronize if its size is sufficiently large

Locally coupled network: $\lambda_2 = 4\sum_{j=1}^{K/2} \sin^2\left(j\pi / N\right)$



Usually, no matter how large the coupling strength c is, a locally coupled network will not synchronize if its size is too large

Wang and Chen (2002)

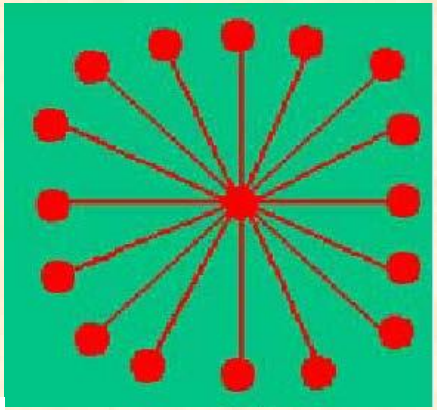
Synchronization in Regularly Coupled Networks

Star-shaped network:

Usually, the size of the network does not matter:

$$\lambda_2 = \cdots = \lambda_{N-1} = 1, \ \lambda_N = N$$

When the coupling strength c is strong enough, the coupled network will synchronize.

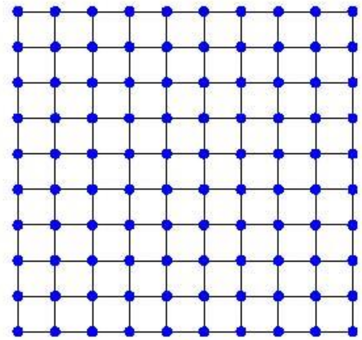


Lattice:

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & 0 & \dots & a_L \\ a_L & a_1 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_L & 0 & \dots & \dots & a_1 \end{pmatrix}$$

with $a_1 = \rho(1-\epsilon)$ and $a_2 = a_L = \rho\epsilon/2$. The eigenvalues of $\mathbf{A}$ are

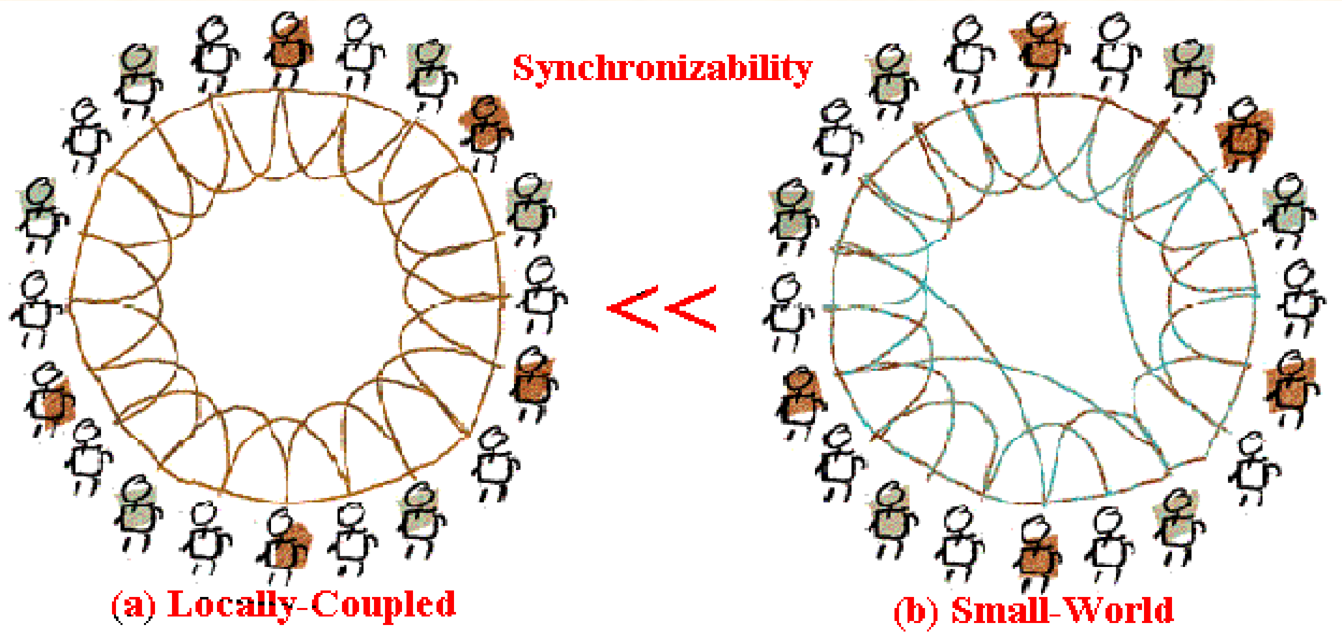
$$\begin{aligned} \lambda_{k_1} &= \sum_{j=1}^{L} a_j \ e^{i2\pi k_1 j/L} \qquad k_1 = 1, 2, ..L. \\ &= e^{i2\pi k_1/L}\left[a_1 + 2a_2\cos(2\pi k_1/L)\right] \end{aligned}$$



S. Arora and M.S. Santhanam (J. Applied Nonl. Dynamics, 2014)

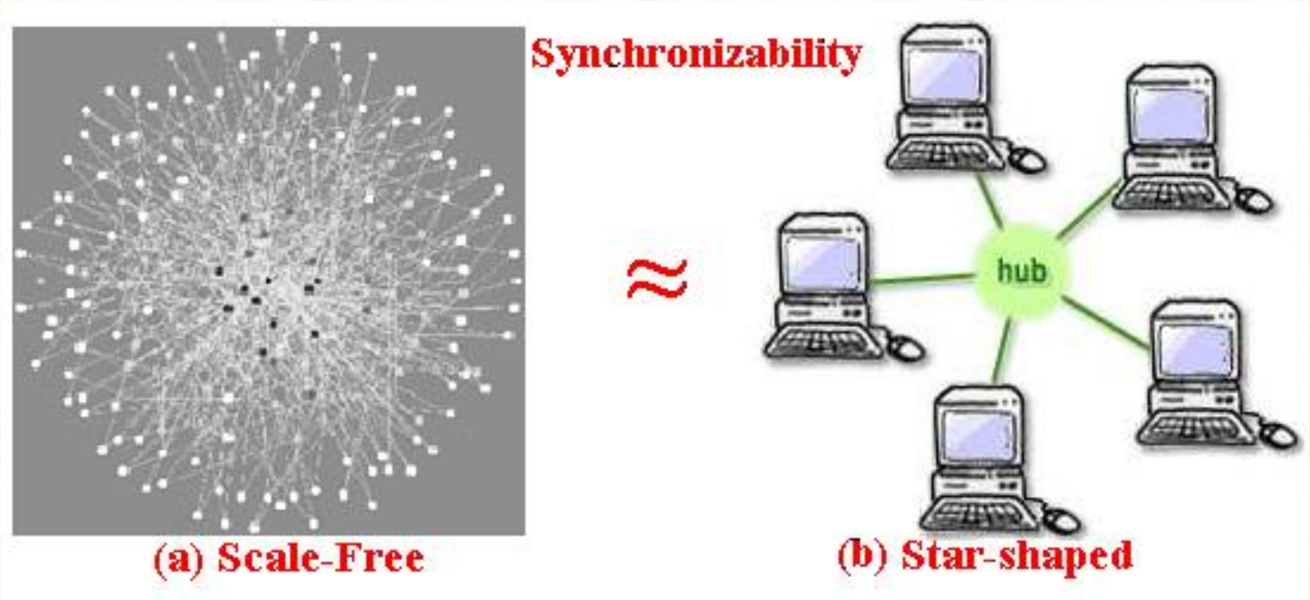
Synchronization in Small-World Networks

Synchronizability can be greatly enhanced by adding a tiny fraction of long-range connections, revealing an advantage of small-world network for synchronization



Wang and Chen (2002)

Synchronization in Scale-Free Networks

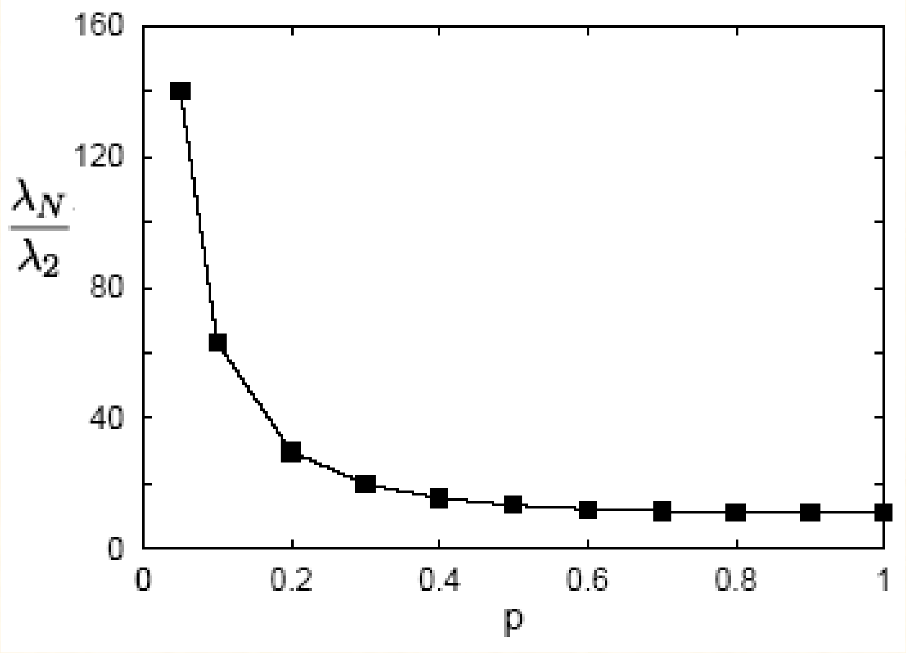


The synchronizability of an assortative SF network is about the same as that of a star-shaped network, mainly determined by the (one-to-one) relations between the central node and the neighboring nodes

Network Topology versus Synchronizability

Connection Probability:

Synchronizability of small-world networks will increase as the small-world feature increases



Hong, et al. (2004)

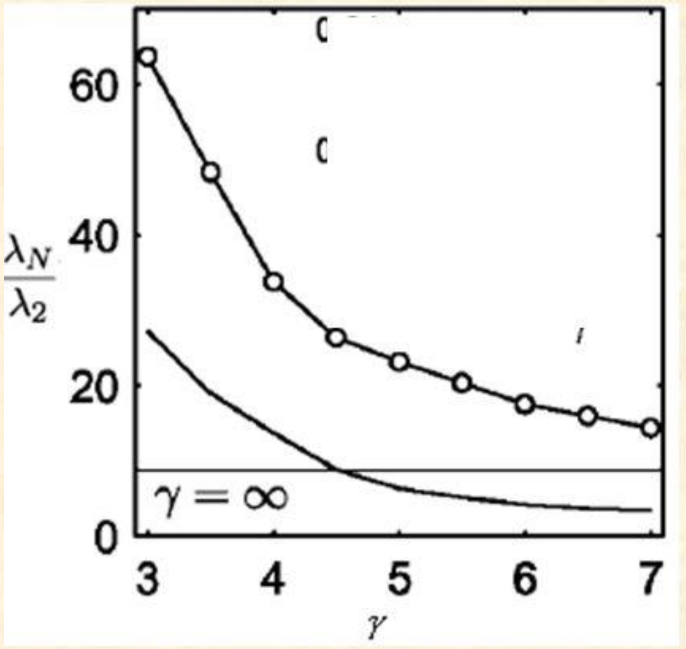
Larger p
→
Smaller λ_N / λ_2
(Criterion II)
→
Better synchrony

$p = 1$ → **fully connected**

Network Topology versus **Synchronizability**

Power-Law Exponent:

Synchronizability of scale-free networks will increase as the power-law exponent increases



Nishikawa, et al. (2003)

Larger γ

→

Smaller λ_N / λ_2 (Criterion II)

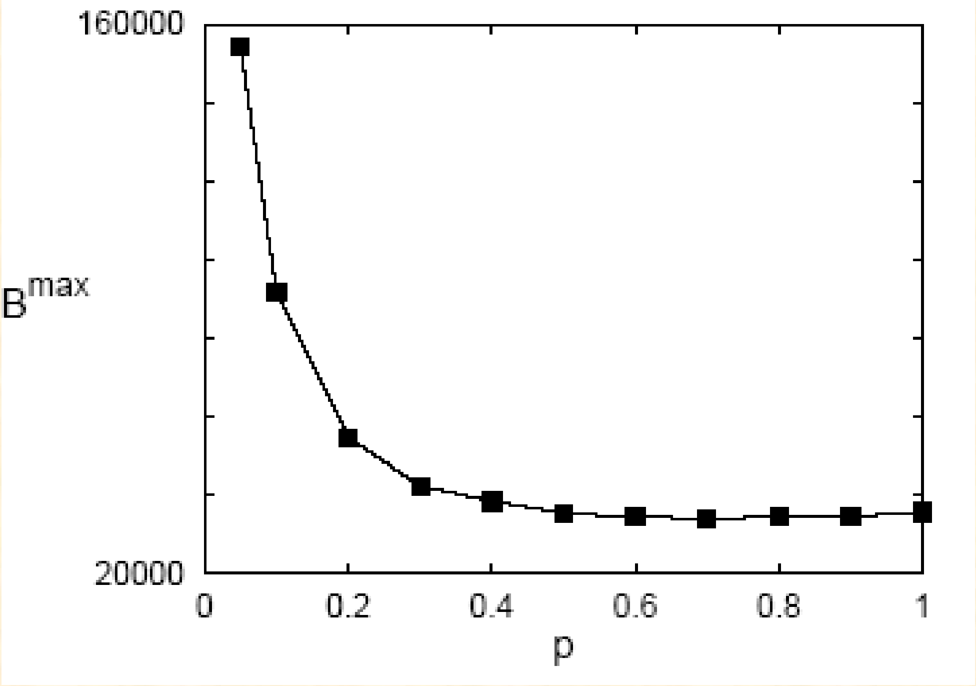
→

Better synchrony

Network Topology versus Synchronizability

Betweenness Centrality:

Synchronizability of small-world will increase as the node betweenness decreases



Hong et al. (2004)

Larger p

→

Smaller

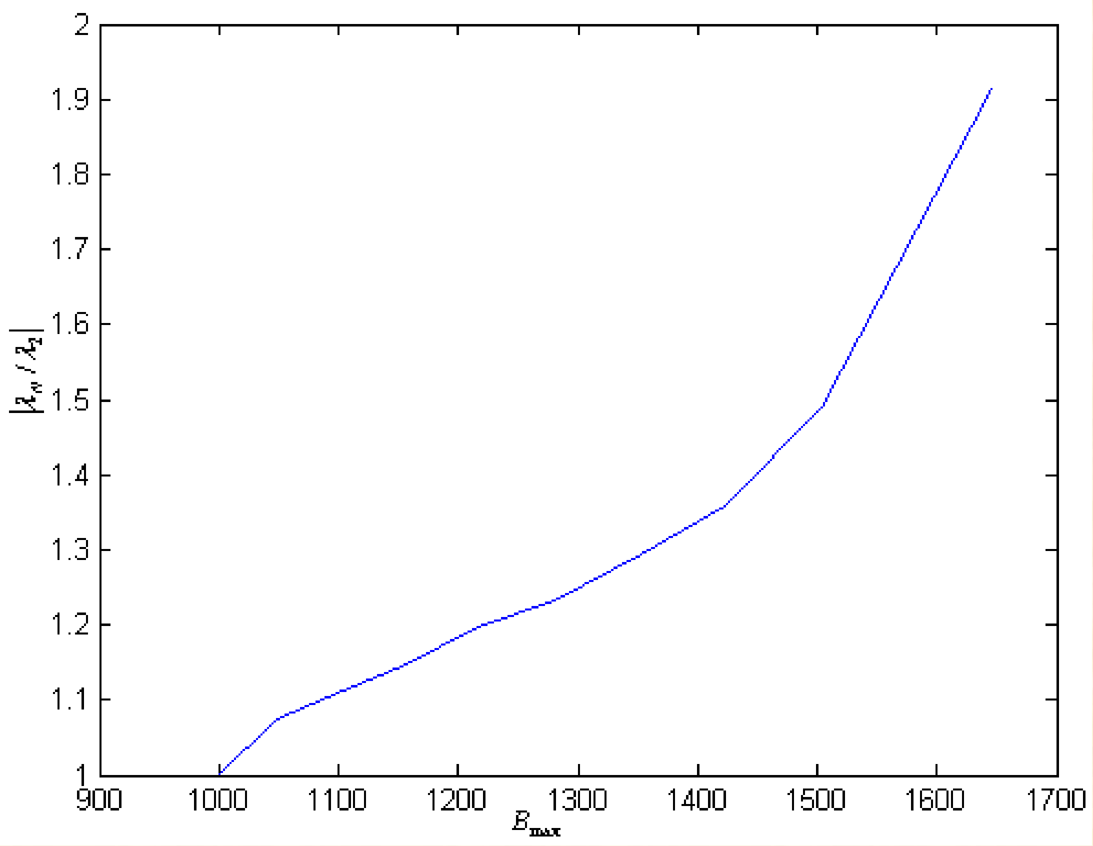
node betweenness

→

Better synchrony

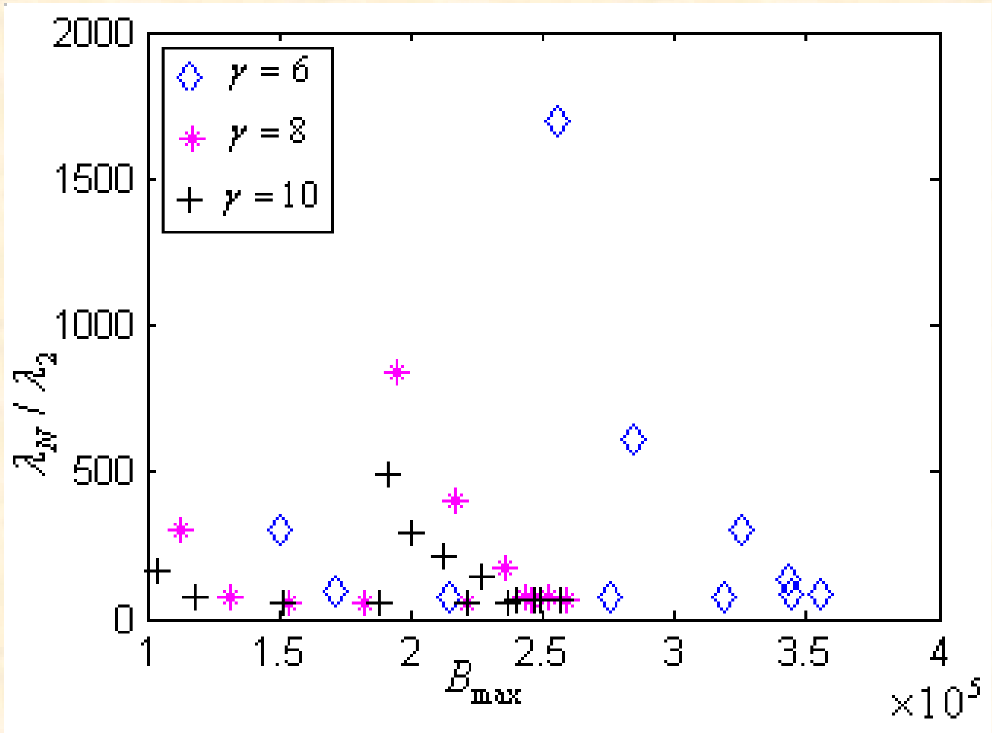
(p = 1 → fully connected)

Betweenness versus **Synchronizability**



Homogeneous Networks – there is a clear **correlation** between **betweenness** and **synchronizability**

Betweenness versus **Synchronizability**



Heterogeneous Hetworks – there is no clear correlation between betweenness and synchronizability for scale-free networks with large exponents in power-law

Enhancing Network Synchronizability

Two examples:

- Perturbing the network structure

  To eliminate the maximal betweenness

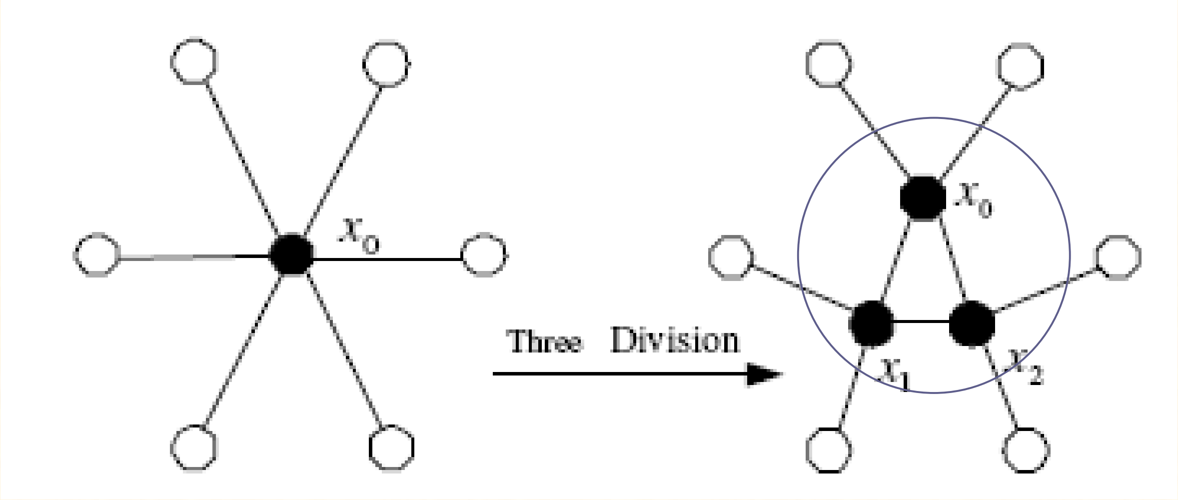
- Modifying the coupling structure

  To reduce the impact of the heterogeneity of degree and betweenness distributions

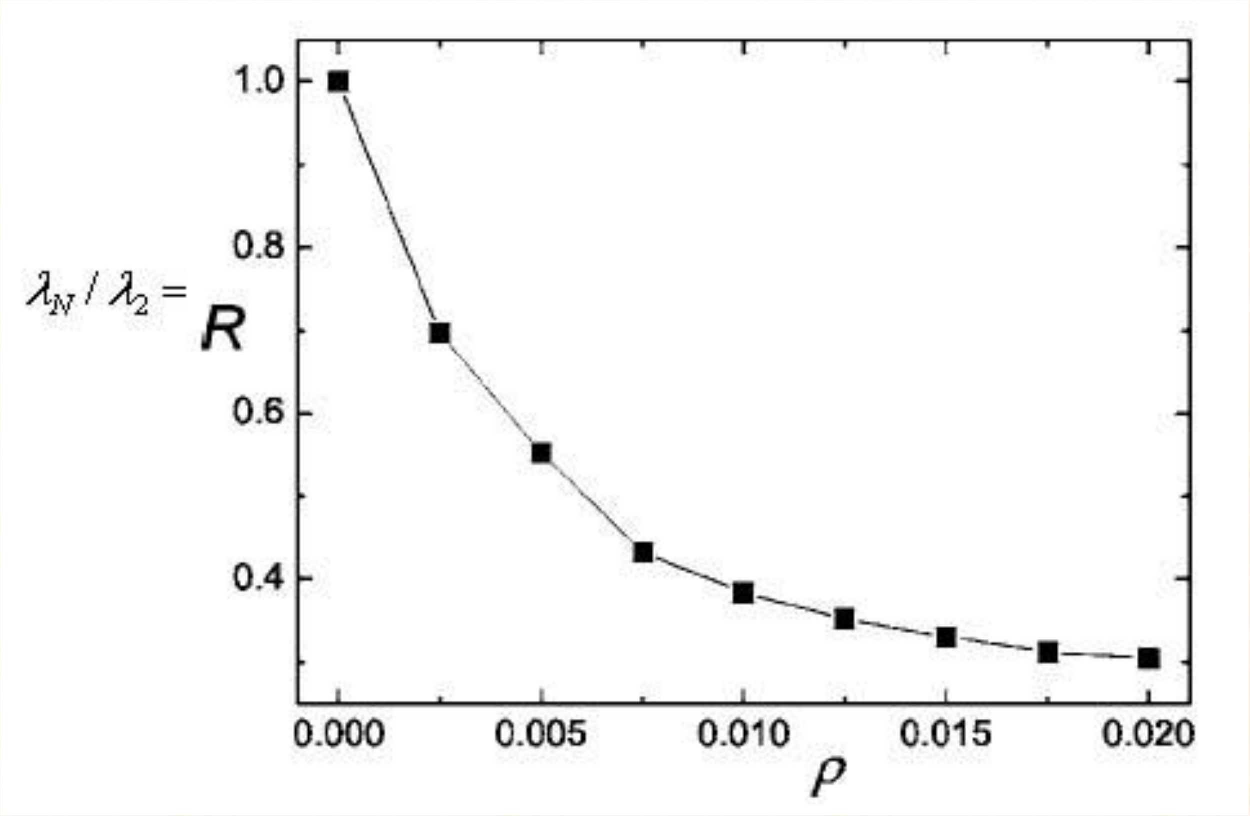
Perturbing Network Topology

(Zhao, et al. Phys. Rev. E 2005, 72: 057102)

The node with the largest betweenness is replaced by several connected nodes, so that the shortest paths that passed through the original node will now only pass one or two new nodes, which will reduce the maximal betweenness dramatically



Simulation Results



Fraction of divided nodes

Decoupling Method
(scale-free networks)

(CY Yin, et al., Phys. Rev. E 2006, 74: 047102)

Not only the nodes with large betweenness, but also the edges with large loads can cause data-traffic congestion

Hence, if such heavily-loaded edges are decoupled, the data-traffic will be redistributed so become more efficient

Algorithm:

1. Calculate the significance of all edges (i,j) by $\sqrt{k_i k_j}$
2. Rank the edges from large to small
3. At each time step, cut an edge with the highest rank (This will decouple two nodes of higher degrees)

Simulation Results

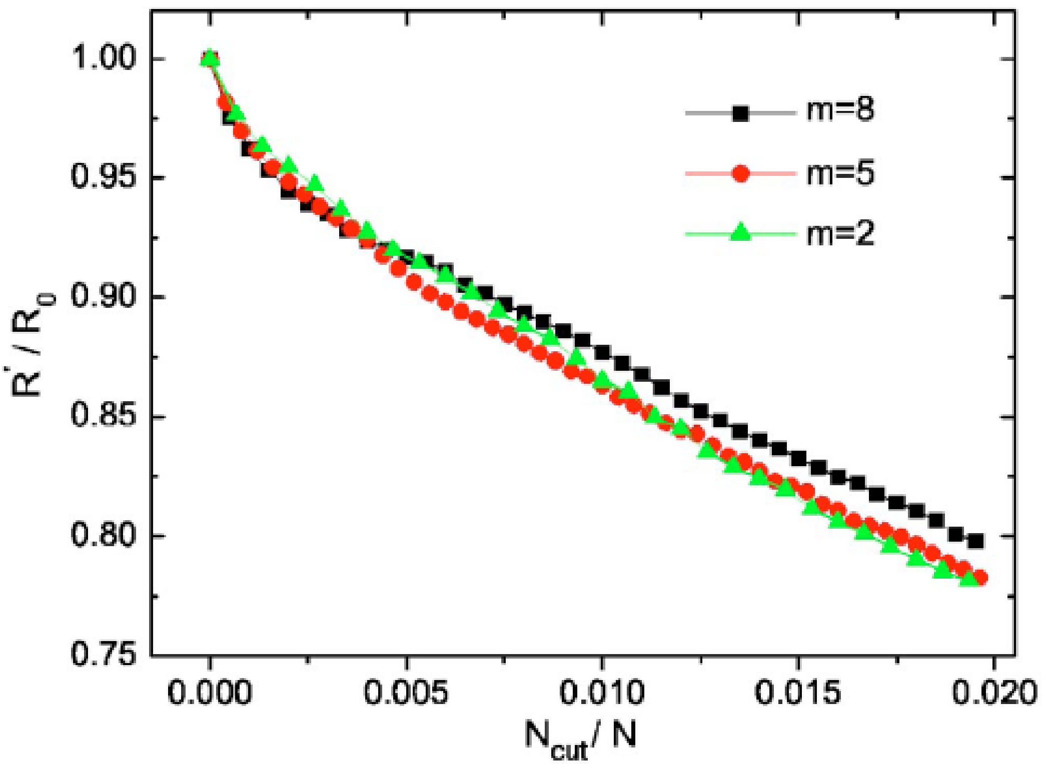


FIG. 1. (Color online) The change of synchronizability as a function of the proportion of cut edges N_{cut}/N for different values of m. Network size $n=2000$.

For scale-free networks, smaller eigen-ratio R'/R_0 implies better synchronizability, where *m* is the number of new edges in the scale-free networks generation

BREAK

10 minutes

Graph-Theoretic Approach

- Relationships between structural parameters and the synchronizability of complex networks
- Relationships between graph theory and the synchronizability of complex networks
- Enhancing network synchronizability

Structure versus **Synchronizability**

- Network Model:

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j), \quad i = 1, 2, \cdots, N \qquad (1)$$

- Laplacian matrix L: symmetric, diffusive and irreducible, with eigenvalues

$$0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N \qquad (2)$$

- Synchronization of network (1) :

$$\lim_{t\to\infty} \| x_i(t) - x_j(t) \|_2 = 0, \qquad i, j = 1,2,\cdots,N \qquad (3)$$

- Remark 2: Key factors influencing the synchronizability:

  (i) inner-coupling matrix H

  (ii) eigenvalues of outer-coupling matrix A

- Synchronizability in terms of (ii):

1. bounded region (M. Banahona and L. M. Pecora, 2002)

$$\lambda_2 / \lambda_N , \quad 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$$

2. unbounded region (X. F. Wang and G. Chen, 2002)

$$c\lambda_2 , \qquad 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$$

3. union of several disconnected regions

   (A. Stefanski, P. Perlikowski, and T. Kapitaniak, 2007)

   (Z. S. Duan, C. Liu, G. Chen, and L. Huang, 2007)

Recall:

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j), \quad i = 1, 2, \cdots, N \qquad (1)$$

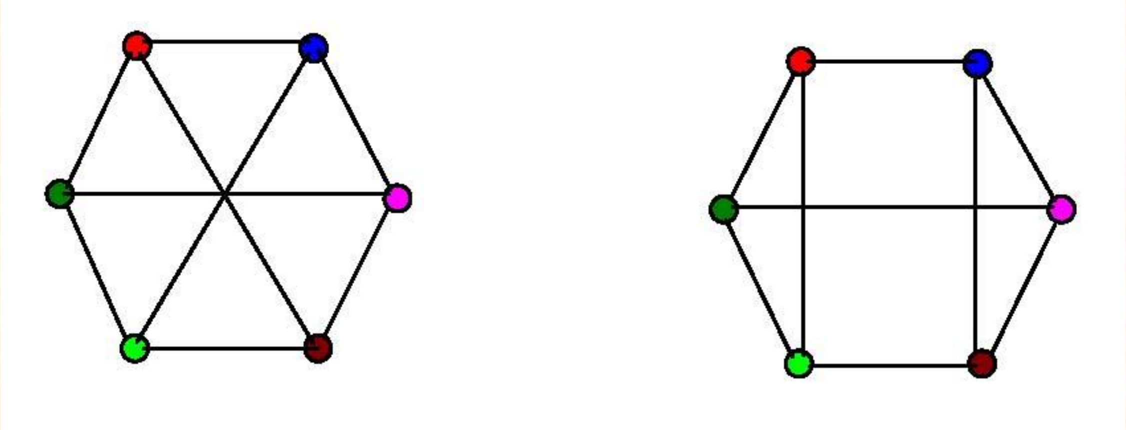
- Structural parameters that may affect the synchronizability:

average distance, degree distribution, clustering coefficient, node betweenness, ...

- Question:

How does synchronizability depend on various structural parameters?

Two simple graphs



Graph G_1 Graph G_2

- They have the same structural characteristics: Graphs G_1 and G_2 have
- the same degree sequence: all node degrees: 3
- the same average distance: 7/5
- the same node-betweenness centrality: 2

But they have different synchronizabilities:

eigenvalues of G_1 are $0, 3, 3, 3, 3$ and 6
eigenvalues of G_2 are $0, 2, 3, 3, 5$ and 5

$$\lambda_2(G_1) = 3 > \lambda_2(G_2) = 2$$

Let $\gamma = \lambda_2 / \lambda_N$ Then

$$\gamma(G_1) = 0.5 > \gamma(G_2) = 0.4$$

About edge-adding:

Lemma 1: For any given connected undirected graph G, all its nonzero eigenvalues will not decrease with the number of added edges, i.e., by adding any edge e, one has $\lambda_i(G+e) \geq \lambda_i(G)$.

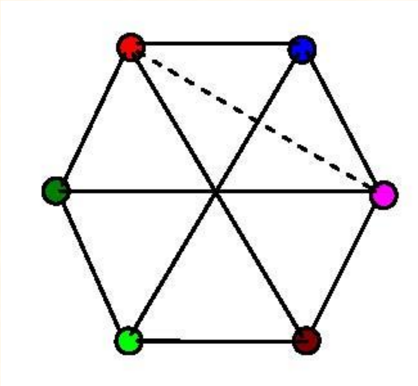
Z. Duan, G. Chen, and L. Huang (2007)

Lemma 1 → If the synchronized region is unbounded, then adding edges never decrease the synchronizability.
Main reason: $\lambda_i(G+e) \geq \lambda_i(G)$

However, for bounded synchronized regions, this is not necessarily true because the eigen-ratio can either increase or decrease

$$\frac{\lambda_2}{\lambda_N} \qquad\qquad \frac{\lambda_2}{\lambda_N}$$

Example: Adding an edge between red node and pink node in the graph G_1 : $e\{red, pink\}$ leads to a new graph $G_1 + e\{red, pink\}$, whose eigenvalues are 0, 2.2679, 3, 4, 5 and 5.7321, with $\gamma(G_1 + e\{red, pink\}) = 0.3956$, smaller than $\gamma(G_1) = 0.5$



❖ **Example:** what about betweeness?

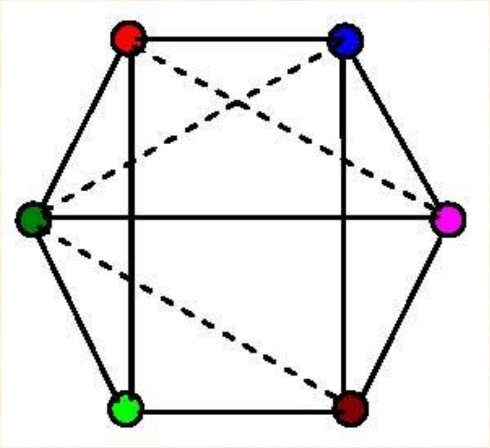
Consider the new graph

$$G = G_2 + e\{red, pink\} + e\{blue, green\} + e\{green, brown\}$$

Maximum node betweenness centrality of G is $11/6$

→

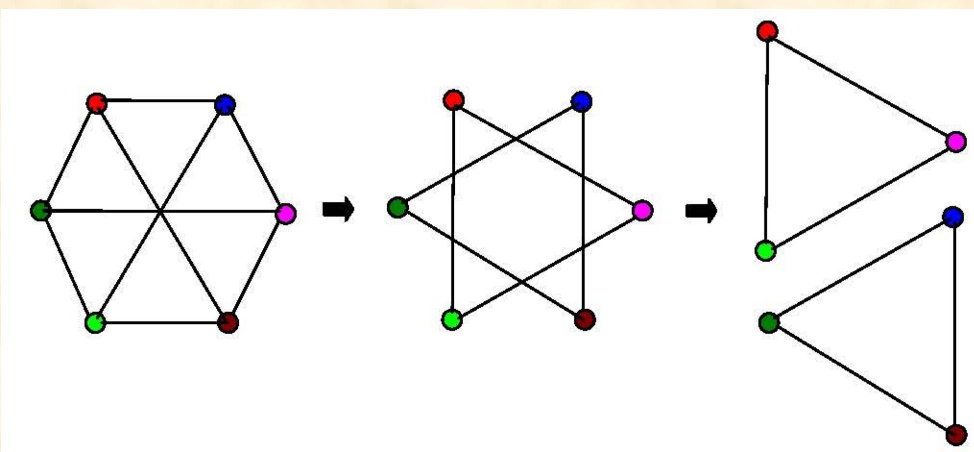
The synchronizability may have other (yet unknown) relationships to the common network structural parameters

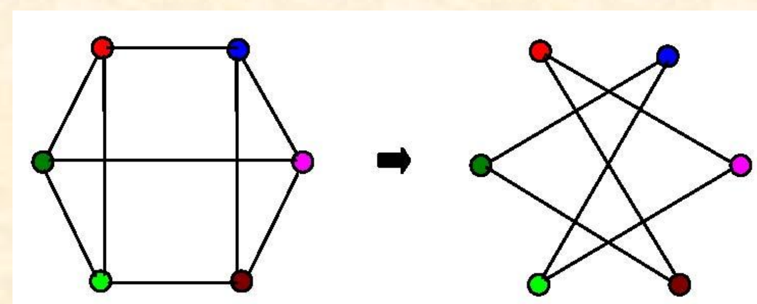


Complementary Graphs:

particularly when they are disconnected

- For a given graph G, the complement of G is the graph containing all the nodes of G, and the edges that are <u>not</u> in G, denoted by G^c
- The complements of G_1 and G_2 :





G_1 G_1^c G_2 G_2^c

❖ **Lemma 2:** For any given graph G :

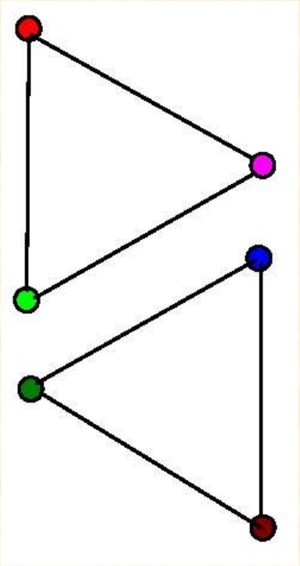
(i) the largest eigenvalue of G, $\lambda_N(G)$, satisfies $\lambda_N(G) \le N$

(ii) $\lambda_N(G) = N$ if and only if G^c is disconnected. Moreover, if G^c has (exactly) q connected components, then the multiplicity of $\lambda_N(G) = N$ as an eigenvalue of G is $q-1$

(iii) $\lambda_i(G^c) = N - \lambda_{N-i+2}(G), \quad 2 \le i \le N$

G_1^c is disconnected. The largest eigenvalue of G_1 is 6, which remains the same for the graph with any more edges being added (Lemma 1).

→ the synchronizability of all the networks built on graph G_1 never decrease with edge-adding.



Recall: G_1 has eigenvalues: 0, 3, 3, 3, 3, 6

- Remark 3: With the same number of edges, generally the synchronizability of a network built on graph G is better when G^c has three (or more) disconnected components than the case that G^c has only two disconnected components, since in the former case, the multiplicity of the largest eigenvalue of G is larger than 1. This implies that some edges there have no contributions to the synchronizability.

- Remark 4: Better understanding and careful manipulation of complementary graphs are effective for enhancing network synchronizability

Disconnected Synchronized Regions

Typical types of synchronized regions:

- Type I: empty
- Type II: unbounded $S_1 = (\alpha_1, \infty)$
- Type III: bounded $S_2 = (\alpha_1, \alpha_2)$
- Type IV: union of some of the above

$(\alpha_1, \alpha_2) \cup (\alpha_3, \infty)$ $\quad (\alpha_1, \alpha_2) \cup (\alpha_3, \alpha_4) \cup (\alpha_5, \alpha_6)$ and so on

A. Stefanski, P. Perlikowski, and T. Kapitaniak, (2007)

Z. Duan, G. Chen, and L. Huang, (2007)

- When the synchronous state is an equilibrium, the synchronized region problem reduces to a stability problem of the matrix pencil $F+\alpha H$ with respect to parameter α.

  (F is the Jacobian and H is the inner-linking matrix)

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H x_j, \quad i = 1, 2, \cdots, N \qquad (1)$$

- **Theorem:** For any natural number n, there are two matrices F and H of order $2(n-1)$ such that $F+\alpha H$ has n disconnected stable regions with respect to parameter α.

- Recall: $\dot{x}_i = \left[[Df(x_i)]_{x_i=s} - c\sum_{j=1}^{N} a_{ij} [DH]_{x_i=s} \right] x_i$

Z. Duan, G. Chen, and L. Huang (2007)

Examples of networks with disconnected synchronized regions

Example: Consider network

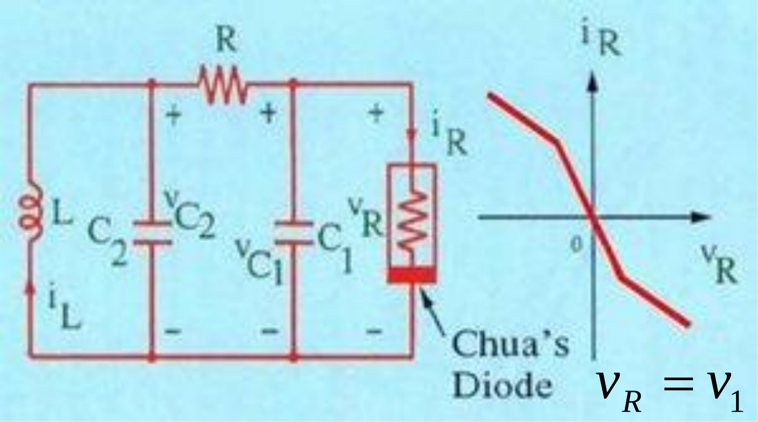
$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H x_j, \quad i = 1, 2, \cdots, N \qquad (1)$$

with third-order smooth Chua's circuits, namely, each node is a Chua's circuit described by

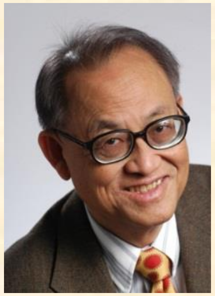
$$\begin{aligned} \dot{x}_{i1} &= -k\alpha\, x_{i1} + k\alpha x_{i2} - k\alpha(a x_{i1}^3 + b x_{i1}) \\ \dot{x}_{i2} &= k x_{i1} - k x_{i2} + k x_{i3} \qquad (2) \\ \dot{x}_{i3} &= -k\beta\, x_{i2} - k\gamma\, x_{i3} \end{aligned}$$

Chua's Circuit:

$$\begin{cases} \dfrac{dv_1}{dt} = \dfrac{1}{C_1}\left[G(v_2 - v_1) - f(v_1)\right] \\ \dfrac{dv_2}{dt} = \dfrac{1}{C_2}\left[G(v_1 - v_2) + i_3\right] \\ \dfrac{di_3}{dt} = -\dfrac{1}{L}\left[v_2 + R_0 i_3\right] \end{cases}$$

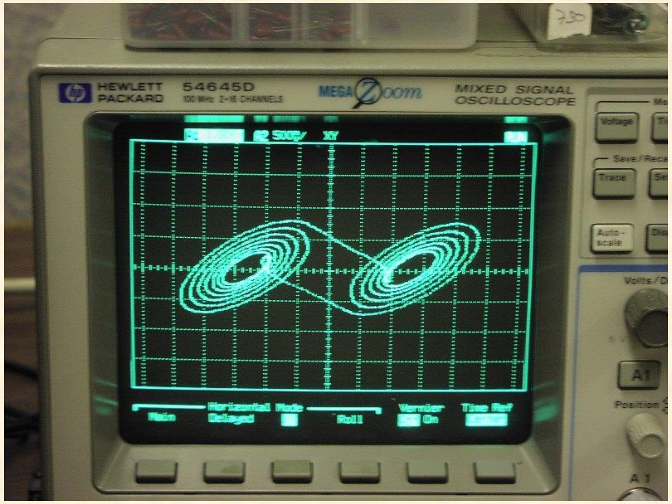


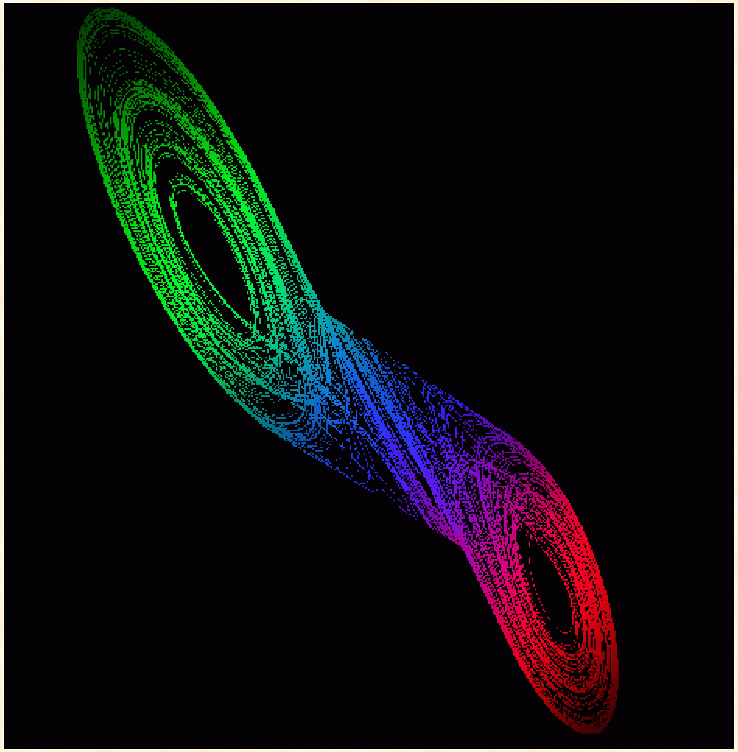
$$f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)\left\{|v_1 + E| - |v_1 - E|\right\}$$



Leon O Chua
(1936 -)

Chua’s attractor





Linearizing (2) at the zero equilibrium yields

$$dx_i/dt = Fx_i, \quad \mathrm{F} = \begin{pmatrix} -k\alpha - k\alpha b & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{pmatrix}$$

Take $k=1, \alpha=-0.1, \beta=-1, \gamma=1, a=1, b=-25,$ and

$$H = \begin{pmatrix} 0.8348 & 9.6619 & 2.6591 \\ 0.1002 & 0.0694 & 0.1005 \\ -0.3254 & -8.5837 & -0.9042 \end{pmatrix}$$

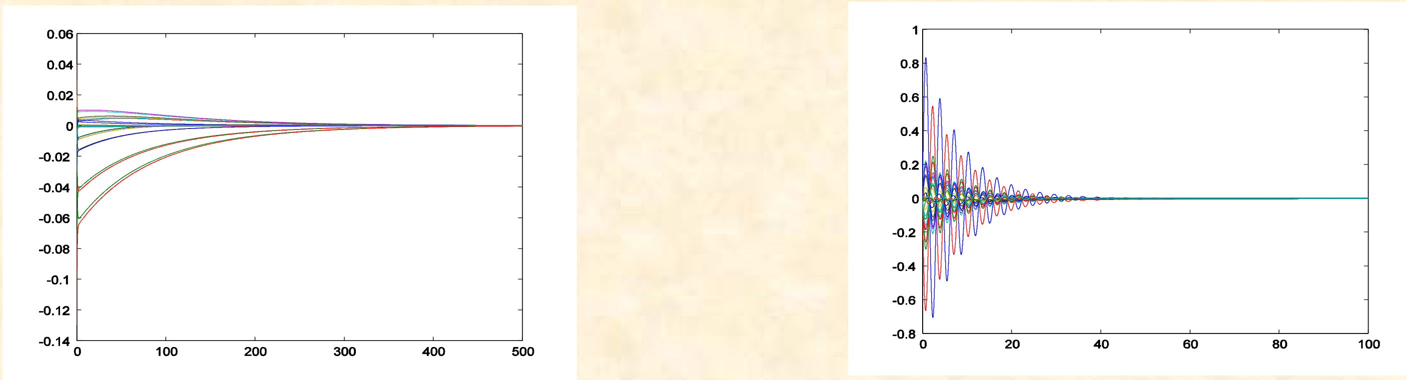
Then $F+\alpha H$ has a disconnected stable region:

$$S = [-0.0099, 0] \cup [-2.225, -1)$$

- Suppose that the number of nodes of network (1) is $N=10$ and the outer coupled matrix A is a globally coupled matrix. Then network (1) with the above data achieves local synchronization when the coupling strength c satisfies

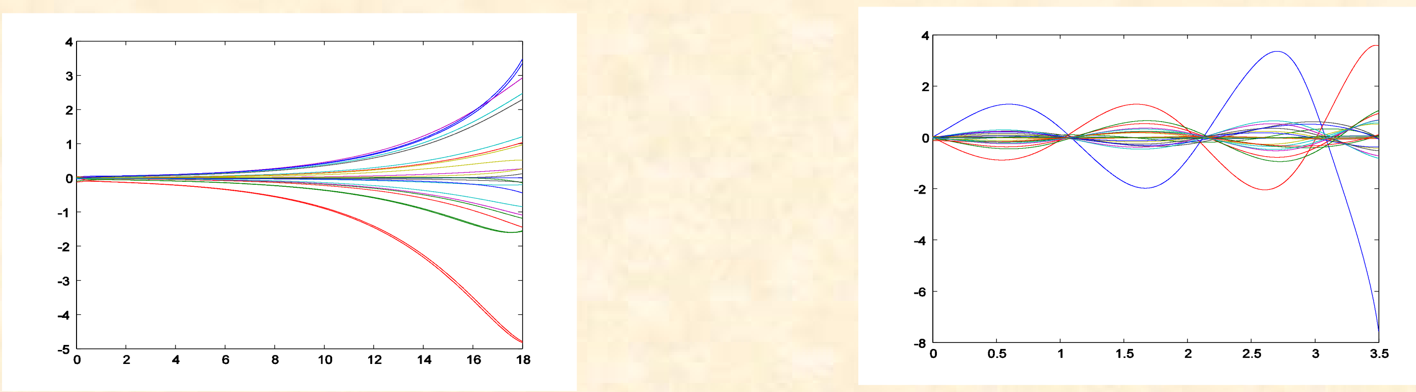
$$c \in [0, 0.00099] \quad \text{or} \quad c \in (0.1, 0.2225]$$

- Figures below visualize the synchronization and non-synchronization behaviors of this network



$(a)\ \ c = 0.0005 \in [0, 0.00099].$ $(b)\ \ c = 0.2 \in (0.1, 0.2225].$

Fig. 1 Synchronization of network (1) with different coupling strengths.



$(c)\ \ c = 0.02 \in (0.001, 0.1).$ $(d)\ \ c = 0.3 \in (0.2225, +\infty).$

Fig. 2 Non-synchronization of network (1) with different coupling strengths.

❖ **Example:** An interesting example of the coexistence of unbounded and bounded synchronization regions in the form of $S = (-\infty, -\alpha_1) \cup (-\alpha_2, -\alpha_3)$

Consider a linearly and diffusively coupled dynamical network with the state equations

$$\dot{x}_i = Fx_i - c\sum_{j=1}^{N} a_{ij} H x_j, \qquad i = 1, 2, \cdots, N, \qquad (3)$$

where

$$F = \begin{bmatrix} \frac{1}{4} & 0 & 1 \\ 0 & \frac{7}{16} & 1 \\ 2 & -3 - \frac{7}{16}\cdot\frac{23}{16} & -\frac{23}{16} \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

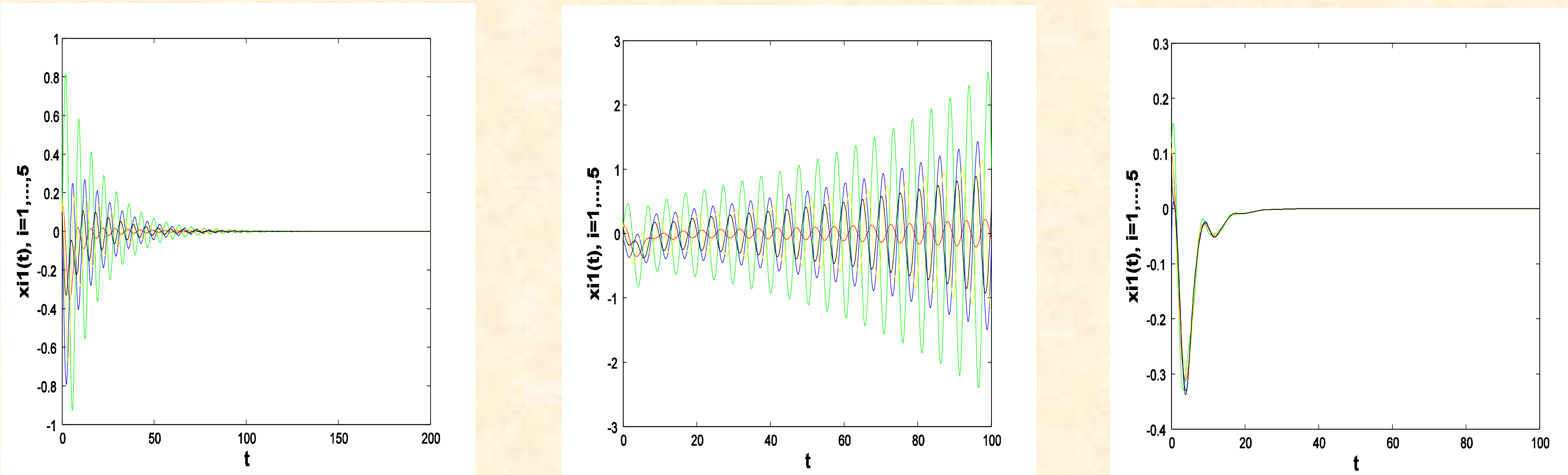
- The synchronization region of network (3) with is

$$S = \left(-\infty, \tfrac{1}{4}(-3-\sqrt{2})\right) \cup \left(\tfrac{1}{4}(-3+\sqrt{2}), 0\right]$$

- Suppose that the number of nodes of network (3) is $N = 5$ and the outer coupled matrix A is a globally coupled matrix. Then, network (3) achieves local synchronization when the coupling strength c satisfies:

  $c \in [0, 0.07929)$ or $c \in (0.22071, +\infty)$.

- Figures below visualize the synchronization and non-synchronization behaviors of this network



$(a)\ \ c = 0.04 \in [0, 0.07929).$ $(b)\ \ c = 0.14 \in (0.07929, 0.22071).$ $(c)\ \ c = 1 \in (0.22071, +\infty).$

Fig. 3 Synchronization and non-synchronization of the first state variables of the five nodes in network (3) in the global coupling configuration: $x_{i1}(t), i = 1, \cdots, 5.$

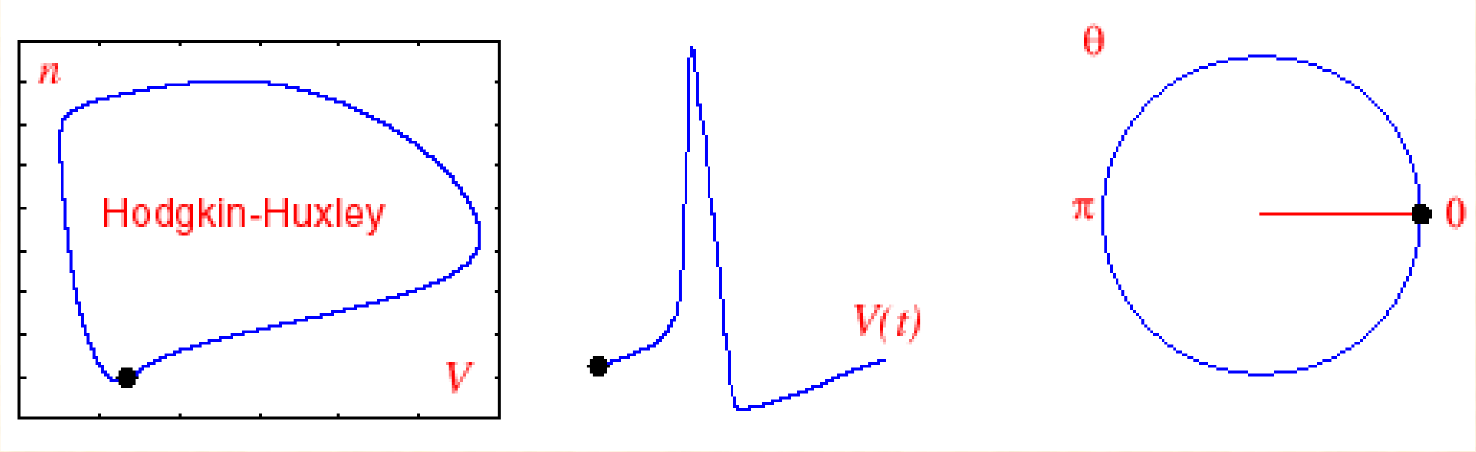
More …

Phase Synchronization

Assume each node is a periodic oscillator

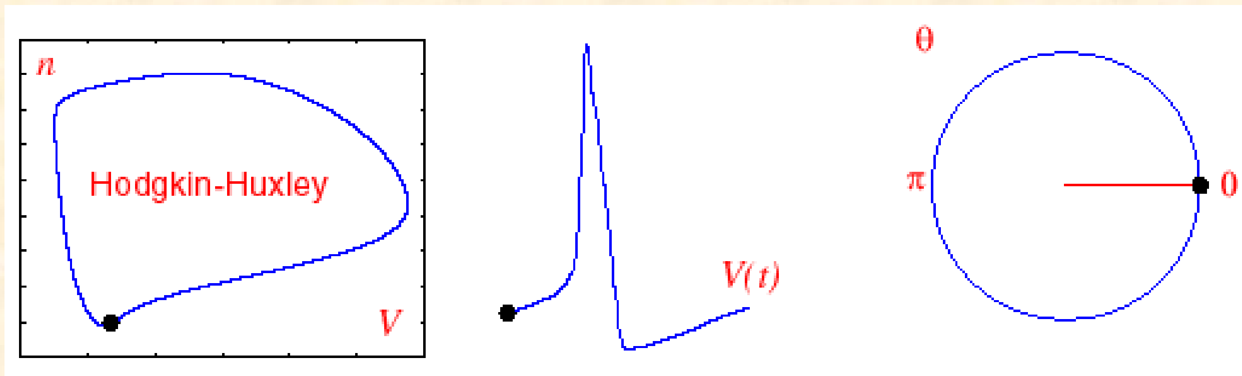
$$\dot{x} = f(x)$$

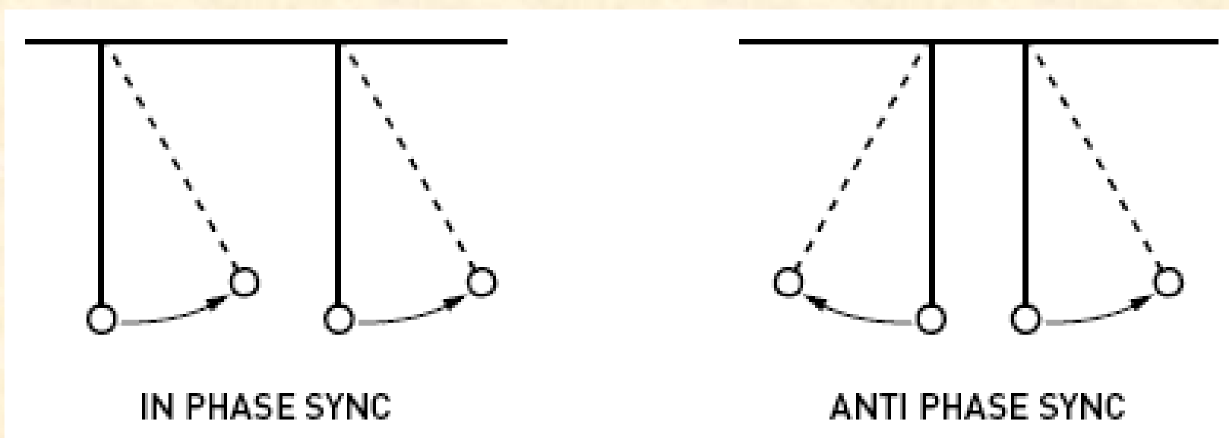
Examples: Van der Pol, Hodgkin-Huxley, Duffing oscillator…



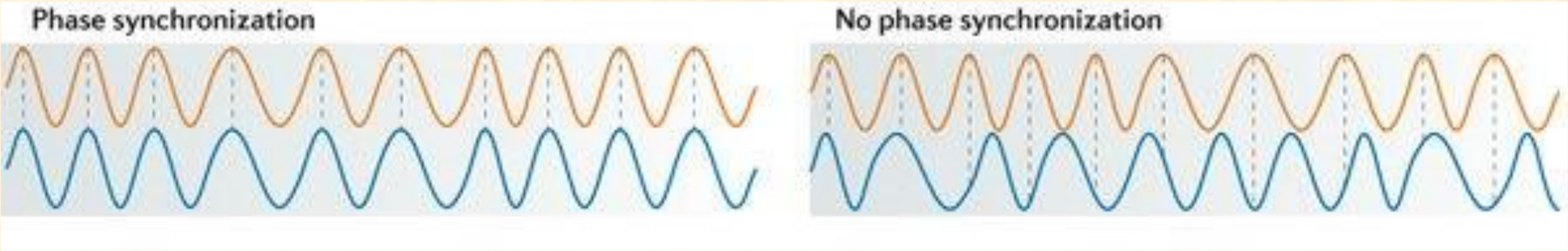
A canonical model for periodic oscillator

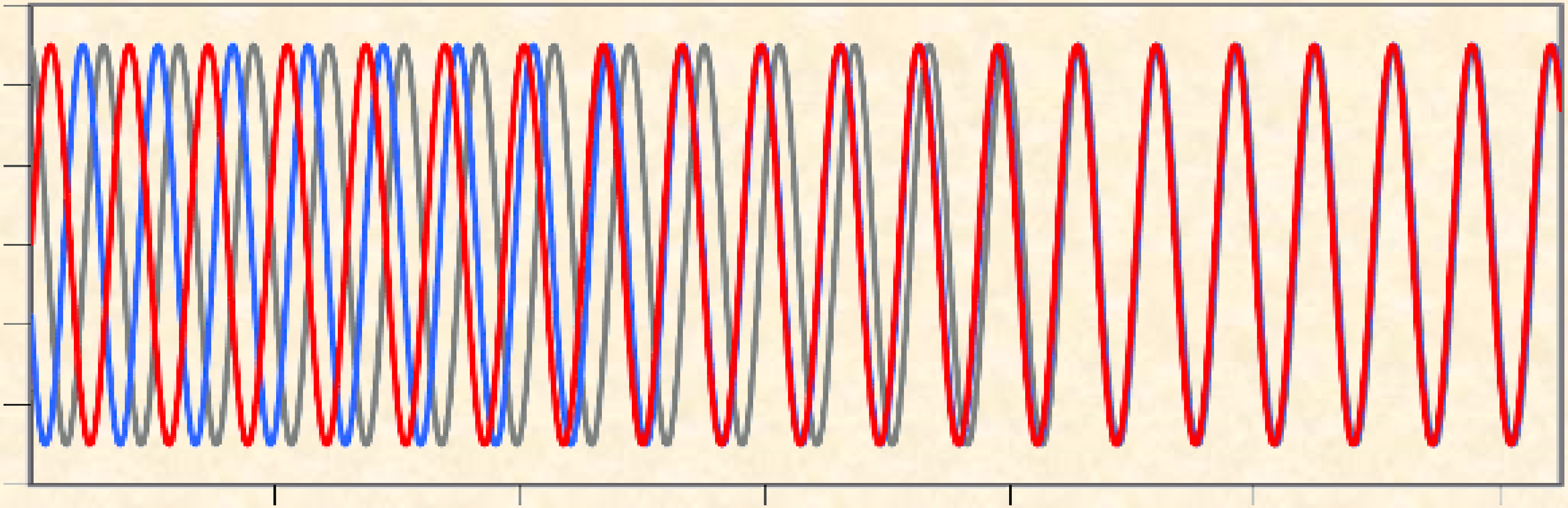
$\dot{\theta} = \omega$ where θ is the phase and ω is the frequency



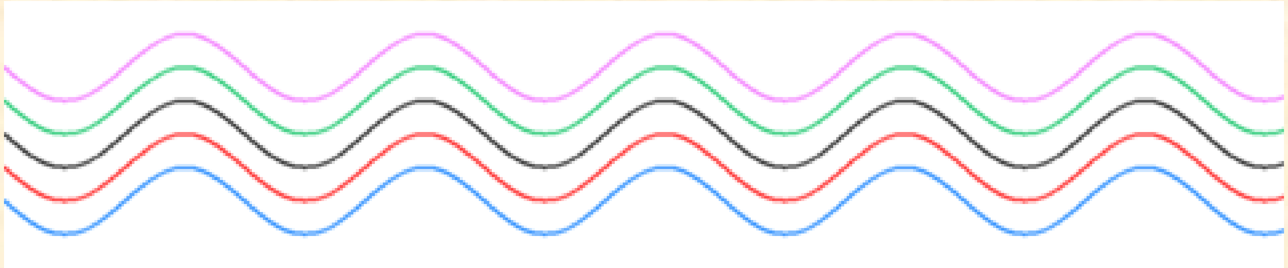


Phase Synchronizations

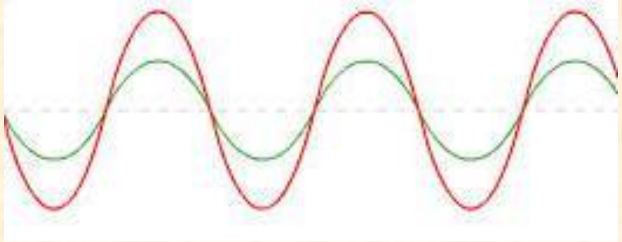




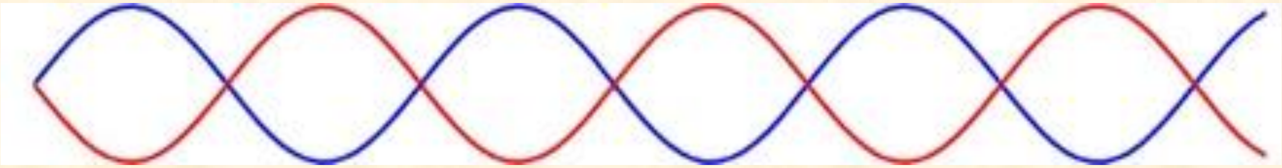
Phase Synchronization



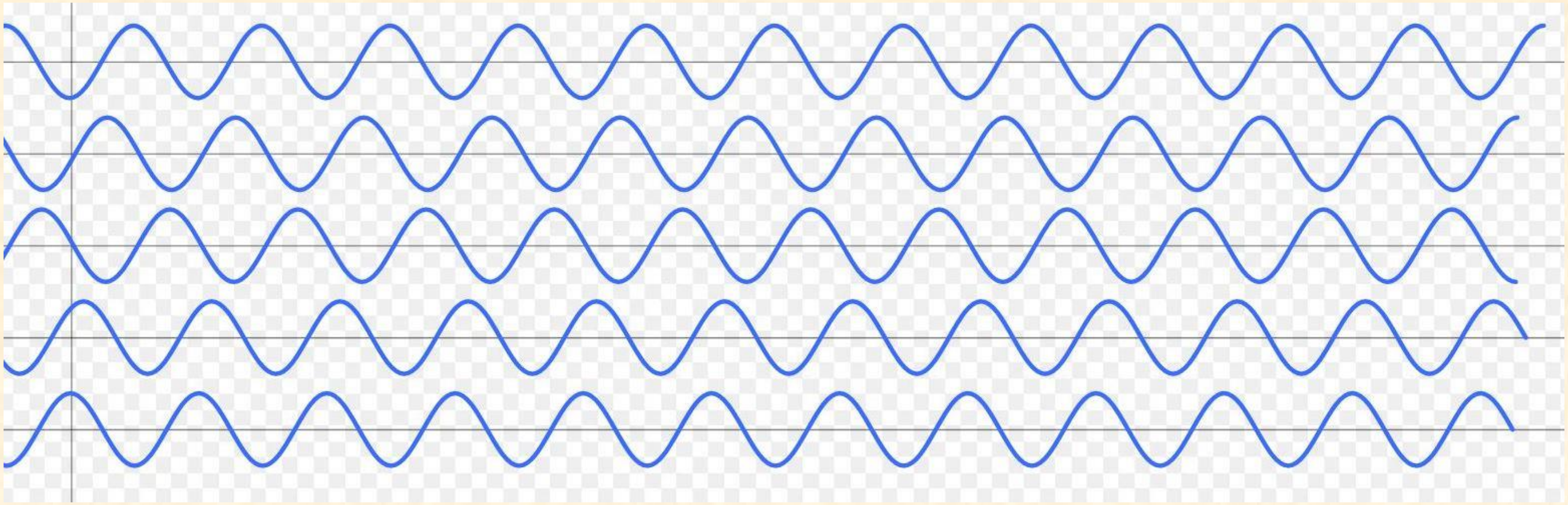
In-phase



Anti-phase



Phase Synchronization



Shifted-phase

Winfree's Pioneering Work

$$\dot{\theta}_i = \omega_i + \left(\sum_{j=1}^{N} X(\theta_j) \right) Z(\theta_i)$$

- When the spread of natural frequencies is large compared to the coupling, the system behaves incoherently, with each oscillator running at its natural frequency.
- As the spread is decreased, the incoherence persists until a certain threshold is crossed —then a small cluster of oscillators suddenly freezes into synchrony.

A. T. Winfree, *The Geometry of Biological Time*, Springer, NY, 1980.

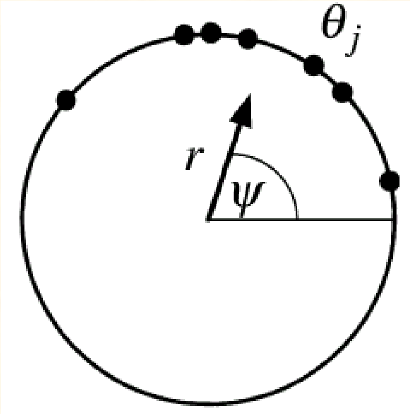
Kuramoto Model

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

- Here, K is the coupling strength and the factor $1/N$ ensures that the model is well behaved as $N \to \infty$
- The frequencies ω_i are distributed according to some probability density $g(\omega)$
- For simplicity, Kuramoto assumed that $g(\omega)$ is unimodal and symmetric about its mean frequency

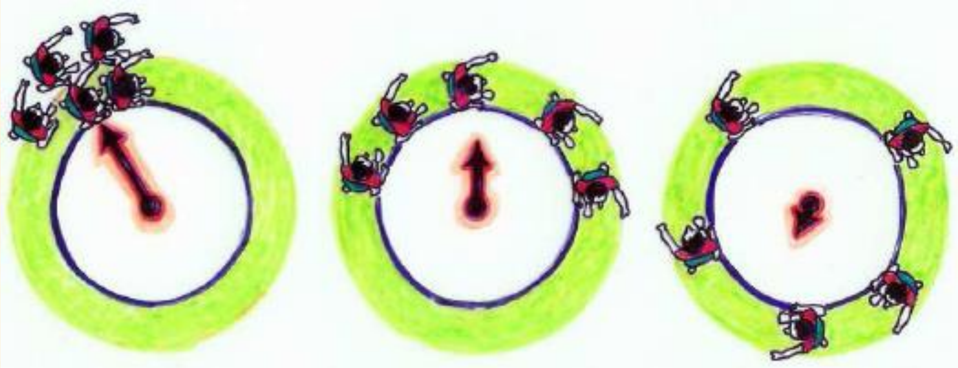
Order Parameter

- To visualize the dynamics of the phases, it is convenient to imagine a group of points running around the unit circle in the complex plane:



$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

r~1

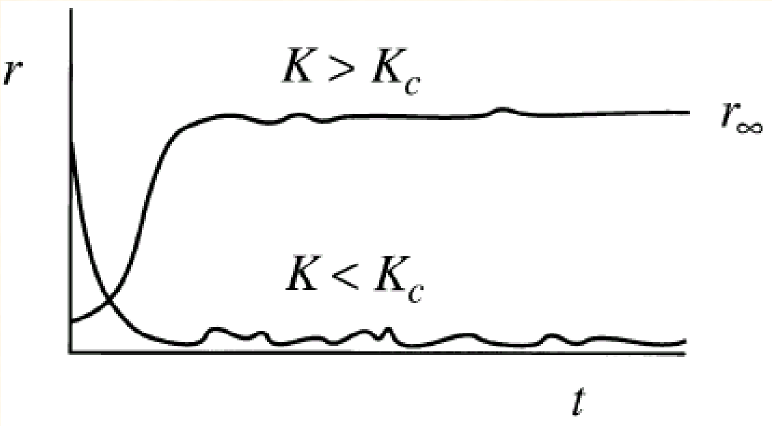


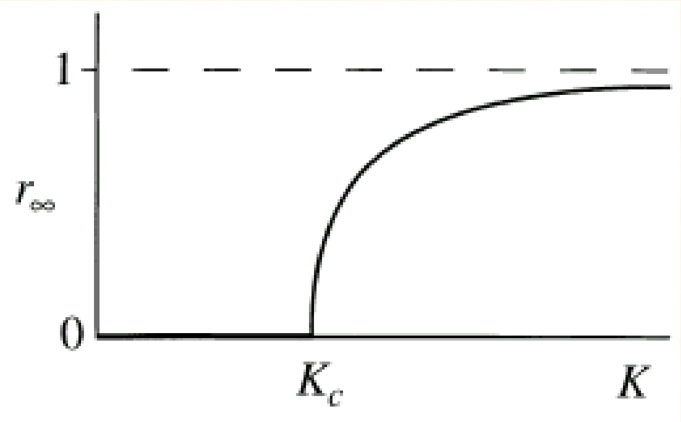
r~0

Synchronization Threshold

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$





J. A. Acebrón, et al: Rev. Mod. Phys. 77 (2005) 137

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