Complex Dynamical Networks:

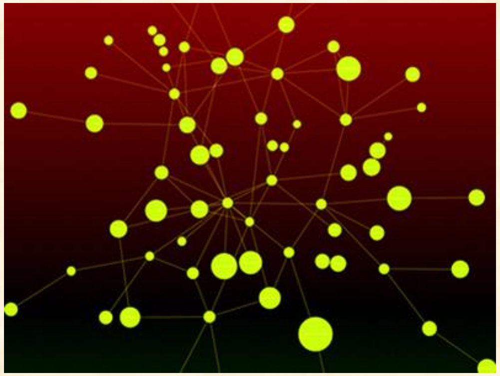
Lecture 8: Network Control

EE 6605

Instructor: G Ron Chen

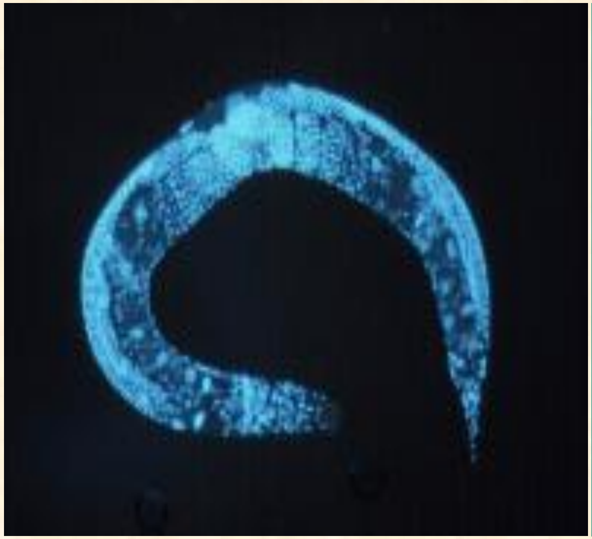
Most pictures on this ppt were taken from
un-copyrighted websites on the web with thanks

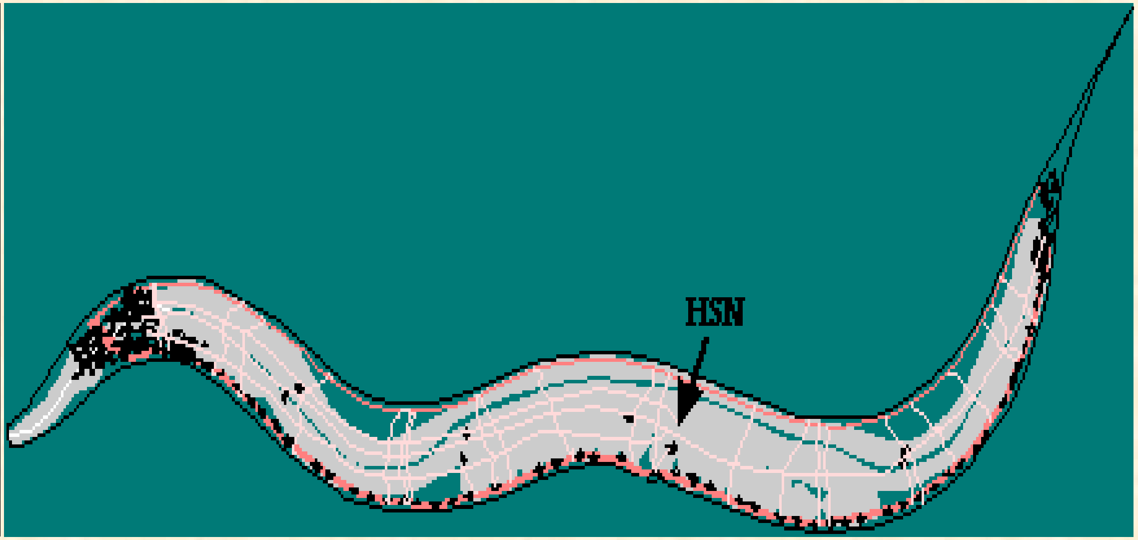
Motivational Examples



Motivation: Example

C. elegans





In its Neural Network:

Neurons: ~ 300   Synapses: ~ 3000

Excerpt

The worm Caenorhabditis elegans has 297 nerve cells. The neurons switch one another on or off, and, making 2345 connections among themselves. They form a network that stretches through the nematode's millimeter-long body.

How many neurons would you have to commandeer in order to control the network with complete precision?

The answer is: 49

-- Adrian Cho, *Science*, 13 May **2011**, vol. 332, p 777

Here, control = stimuli

Another Example

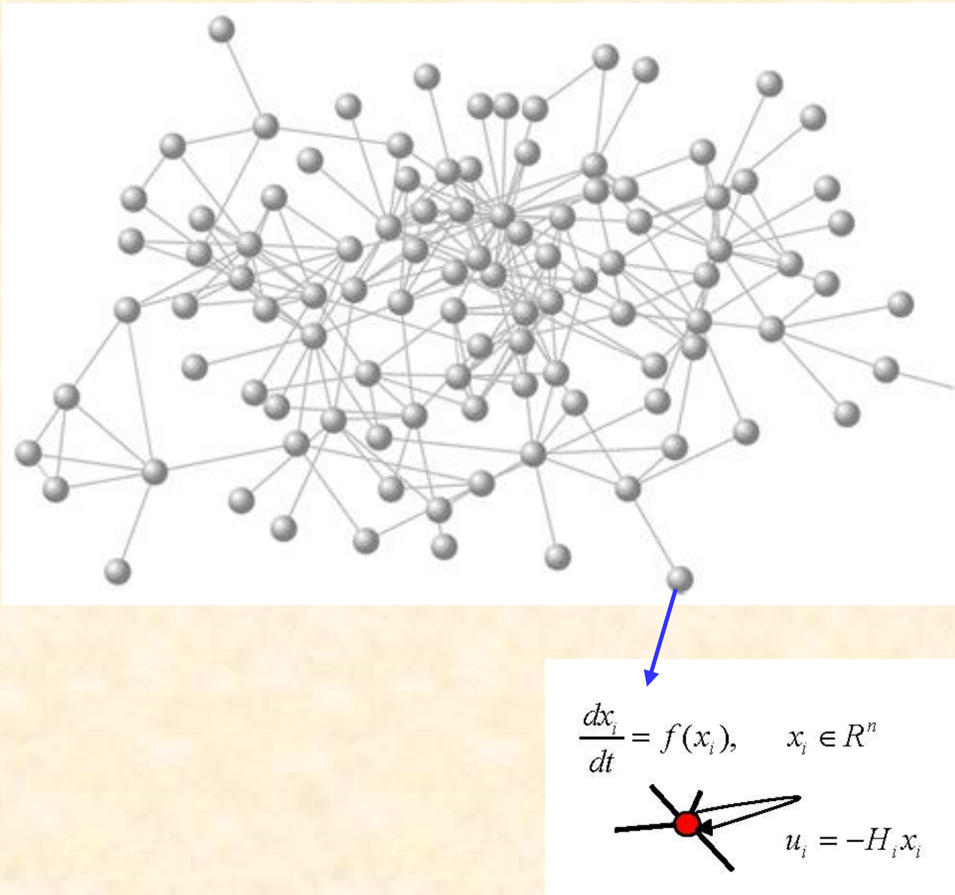
" … very few individuals (approximately **5**%) within honeybee swarms can guide the group to a new nest site."

I.D. Couzin et al., ***Nature***, 3 Feb 2005, vol. 433, p 513

These **5%** of bees can be considered as "**controlling**" or "**controlled**" **agents**

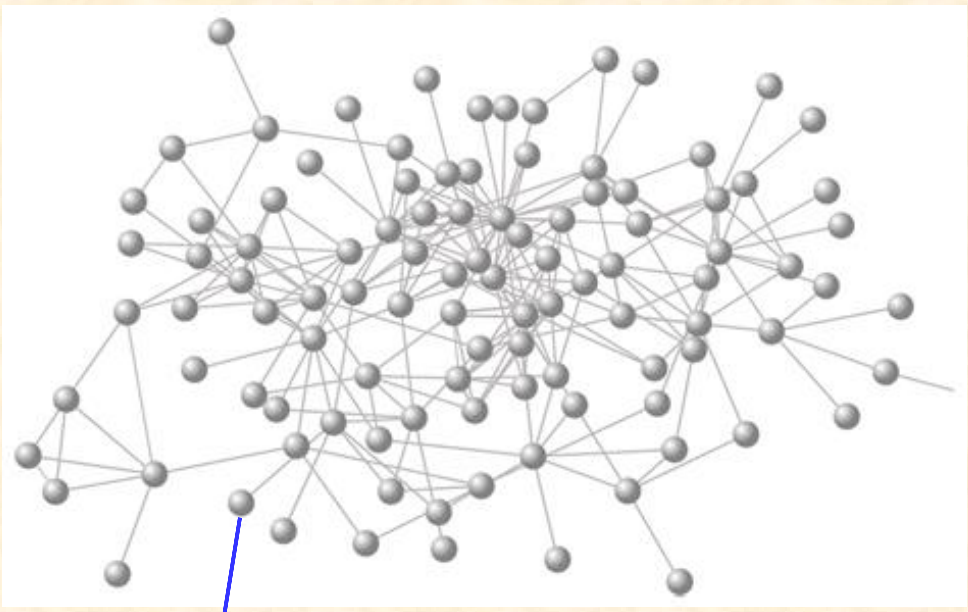
Leader-Follower networks

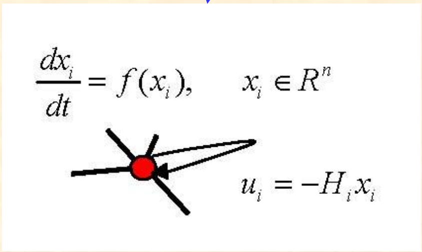
Now …



- Given a network of dynamical systems (e.g., ODEs)
- Given a specific control objective (e.g., synchronization)
- Assume: a certain class of controllers (e.g., local state-feedback controllers) are chosen to use

Control Problem:





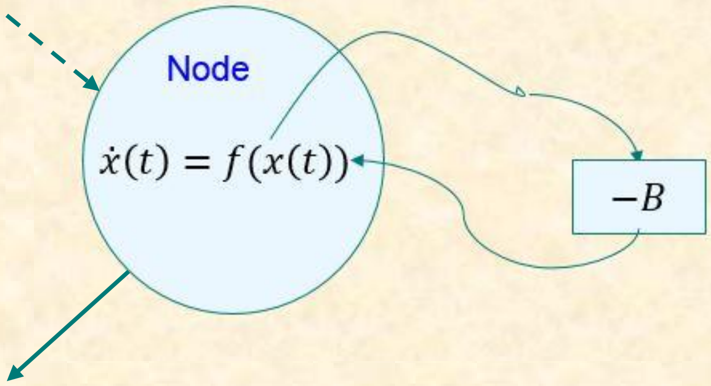
Pining Control:

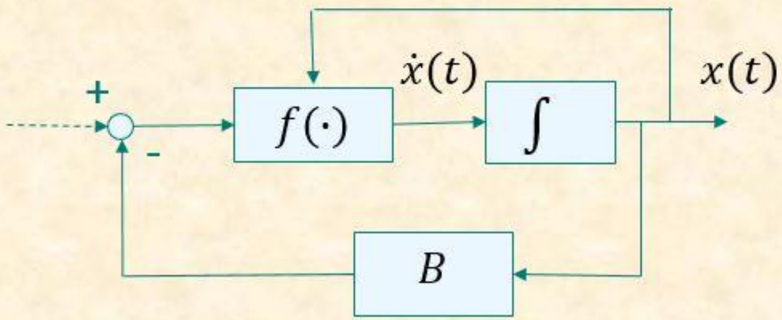
- How many controllers to use?
- Where to put them (where to "pin")?

State Feedback Control

Node system: $\dot{x}(t) = f(x(t)) + u(t)$ ← Controller

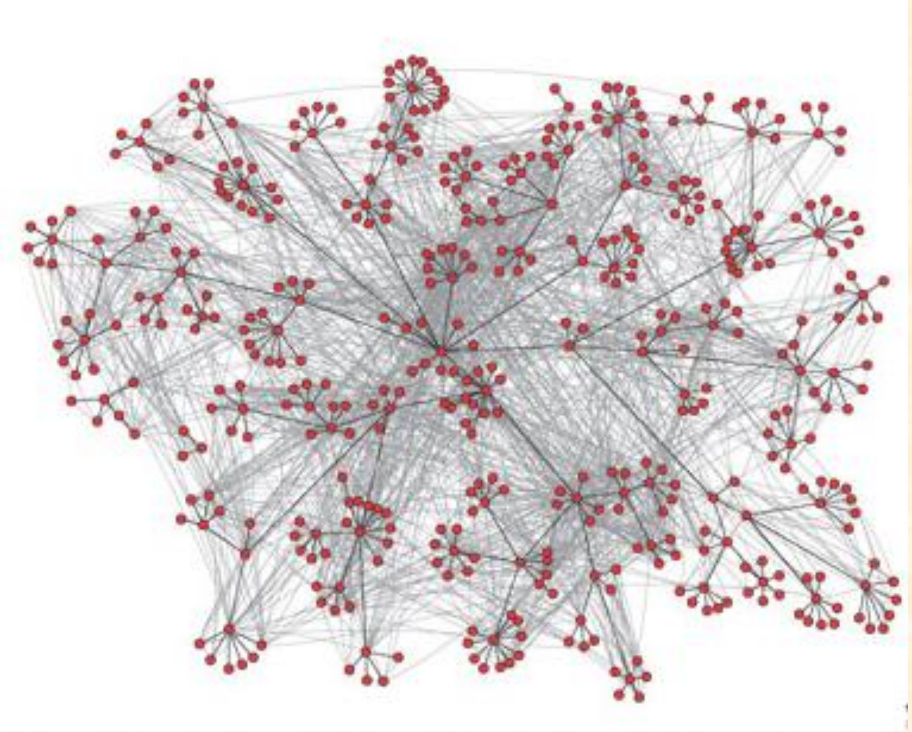
Linear State-Feedback Controller: $u(t) = -Bx(t)$





Attempts to Answer

First, consider undirected and unweighted networks



Each node is a higher-dimensional nonlinear dynamical system:

$$\frac{dx_i}{dt} = f(x_i), \qquad x \in R^n, \quad i = 1, 2, \ldots, N$$

- **Regular networks**
- **Random-graph networks**
- **Small-world networks**
- **Scale-free networks**
- **…….**

Network Model

Linearly coupled network:

$$\frac{dx_i}{dt} = f(x_i) - c\sum_{j=1}^{N} a_{ij} H x_j \qquad x_i \in R^n \qquad i = 1,2,\ldots,N$$

a general assumption is that $f(x)$ is Lipschitz (e.g., linear: Ax)

coupling strength $c > 0$ and

coupling matrices (undirected):

$$A = [a_{ij}]_{N\times N} \qquad H = \begin{bmatrix} r_{11} & & & 0 \\ & r_{22} & & \\ & & \ddots & \\ 0 & & & r_{nn} \end{bmatrix}$$

A: If node i connects to node j $(j \neq i)$, then $a_{ij} = a_{ji} = 1$; else, $a_{ij} = a_{ji} = 0$; $a_{ii} = 0$

Laplacian matrix: $L = D - A$ $\quad D = diag\{d_1, \ldots, d_N\}$ $\quad d_i$ - degree of node i

What kind of controllers? How many? Where?

$$\frac{dx_i}{dt} = f(x_i) - c\sum_{j=1}^{N} a_{ij} H x_j \leftarrow + \; u_i \qquad i = 1,2,\ldots,N$$

$$(u_i = -Hx_i)$$

$$\frac{dx_i}{dt} = f(x_i) - c\sum_{j=1}^{N} a_{ij} H x_j + \delta_i H x_i \qquad i = 1,2,\ldots,N$$

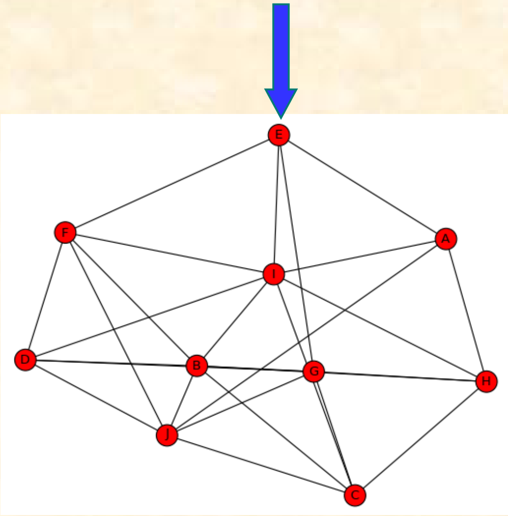
$$\delta_i = \begin{cases} 1 & if \;\; to - control \\ 0 & if \;\; not - control \end{cases}$$

Q: How many $\delta_i = 1$? Which i ?

How Many? – One, or a few, if … c is large enough

$$\frac{dx_1}{dt} = f(x_1) - c\sum_{j=1}^{N} a_{1j} H x_j + u_1 \qquad u_1 = H x_1$$

$$\frac{dx_i}{dt} = f(x_i) - c\sum_{j=1}^{N} a_{ij} H x_j \qquad i = 2,3,\ldots,N$$



X.F. Wang, G.R. Chen, Physica A (2002): Let l =1 and s = 0 therein

Still yet,

where to apply controllers makes a difference

Pinning Control: (How many and which nodes to pin?)

Only a small fraction of nodes are selected for control:

1. Selective pinning scheme
2. Random pinning scheme

X.F. Wang, G.R. Chen, Physica A (2002)

Example

Consider a scale-free CNN, which has a zero equilibrium ($s = 0$):

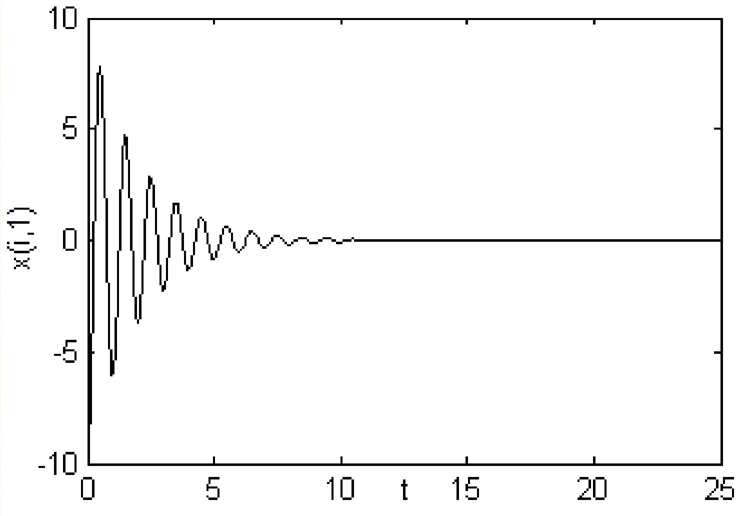
$$\frac{dx_i}{dt} = \begin{pmatrix} \frac{dx_{i1}}{dt} \\ \frac{dx_{i2}}{dt} \\ \frac{dx_{i3}}{dt} \\ \frac{dx_{i4}}{dt} \end{pmatrix} = \begin{pmatrix} -x_{i3} - x_{i4} + c\sum_{j=1}^{N} a_{ij} x_{j1} \\ 2x_{i2} + x_{i3} + c\sum_{j=1}^{N} a_{ij} x_{j2} \\ 14x_{i1} - 14x_{i2} + c\sum_{j=1}^{N} a_{ij} x_{j3} \\ 100x_{i1} - 100x_{i4} \\ \quad +100(|x_{i4}+1| - |x_{i4}-1|) \\ \quad + c\sum_{j=1}^{N} a_{ij} x_{j4} \end{pmatrix}$$

$$i = 1,2,\dots,N = 60$$

1 Selective Pinning Control

Here, network size $N = 60$, coupling strength $c = 8$ and number of controlled nodes is $m = 15$, by $u_i = -hx_i$

Pin the first 15 largest nodes:



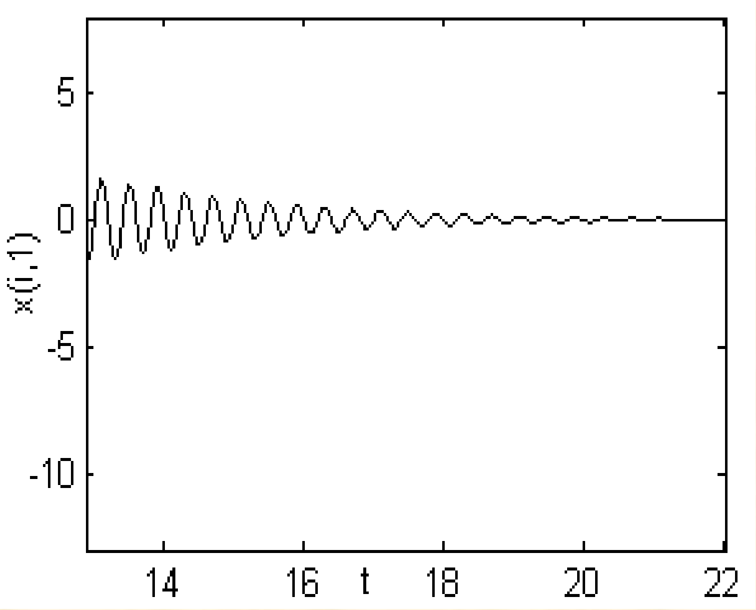
The controlled state x_1

Control gain: $h \approx 30$

Settling time = 10

2 Random Pinning Control

Randomly pin 15 nodes.



The controlled state x_1

Comparison:

1. Control gain is much larger:

$$h \approx 513$$

Recall the last one:

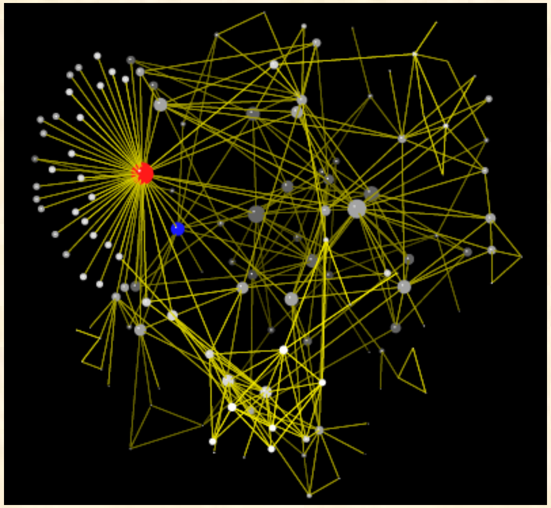
$$h \approx 30$$

2. It takes twice longer time to synchronize the network:

Settling time = 20

Recall the last one: 10

Network Controllability Theory



In retrospect, …

J.S.I.A.M. Control
Ser. A, Vol. 1, No. 2
Printed in U.S.A., 1963

(1930-2016)

MATHEMATICAL DESCRIPTION OF LINEAR DYNAMICAL SYSTEMS*

R. E. KALMAN†

Abstract. There are two different ways of describing dynamical systems: (i) by means of state variables and (ii) by input/output relations. The first method may be regarded as an axiomatization of Newton's laws of mechanics and is taken to be the basic definition of a system.

It is then shown (in the linear case) that the input/output relations determine only one part of a system, that which is completely observable and completely controllable. Using the theory of controllability and observability, methods are given for calculating irreducible realizations of a given impulse-response matrix. In particular, an explicit procedure is given to determine the minimal number of state variables necessary to realize a given transfer-function matrix. Difficulties arising from the use of reducible realizations are discussed briefly.

System Controllability

Linear Time-Invariant (LTI) system

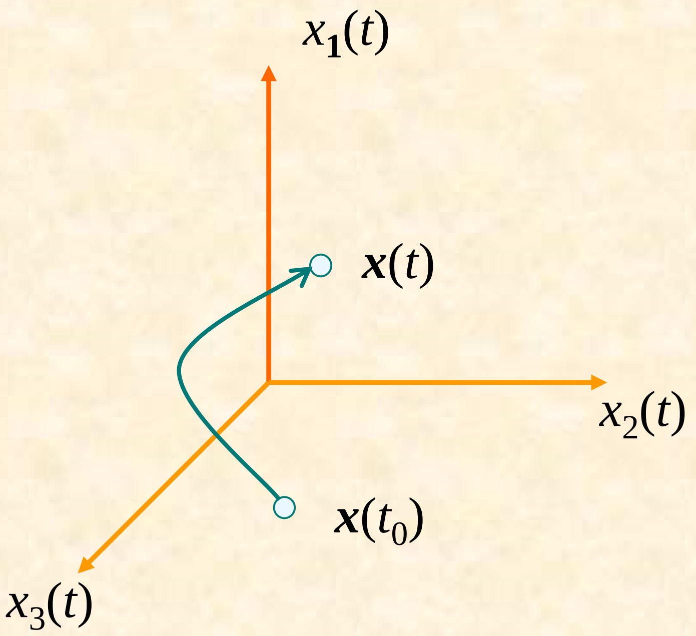
$$\dot{x}(t) = Ax(t) + Bu(t)$$

$x \in R^n$: state vector

$u \in R^p$: control input

$A \in R^{n \times n}$: system matrix

$B \in R^{n \times p}$: control matrix



Controllable: The system orbit can be driven by an input from any initial state to any target state in finite time.

Example

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \qquad\qquad + bu, \\
\dot{x}_3 &= \qquad\qquad a_{32}x_2 + a_{33}x_3
\end{aligned}$$

$$x(t) = [x_1(t) \quad x_2(t) \quad x_3(t)]^T$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

Classical Controllability Theorem

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Rank Criterion:

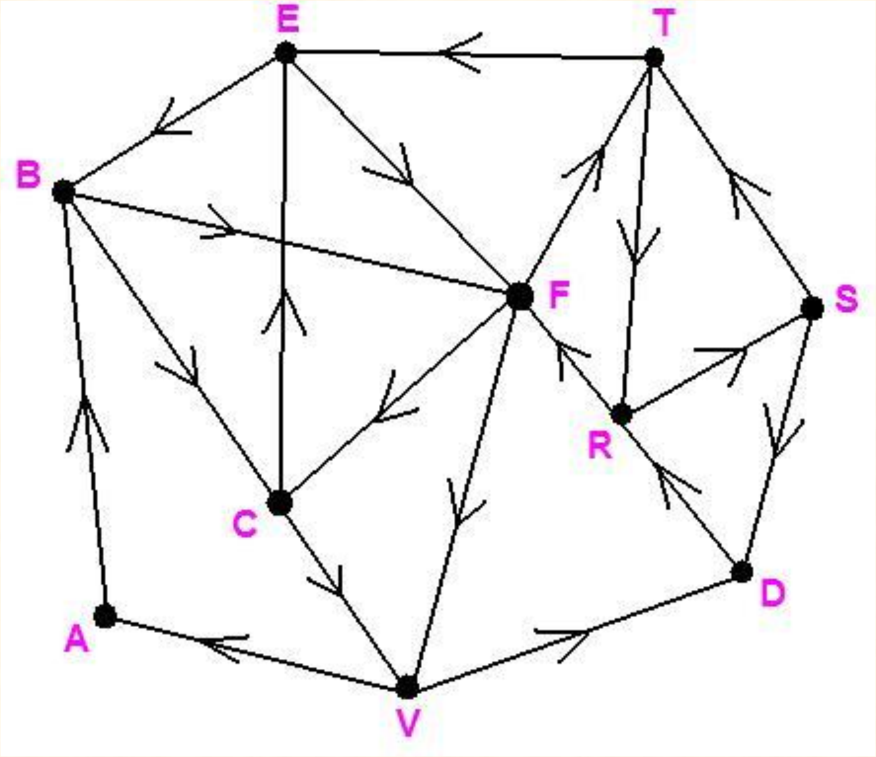
The linear system is controllable if and only if the controllability matrix $\boldsymbol{C}$ has full row rank:

$$\boldsymbol{C} = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}]$$

Previous Example
uncontrollable

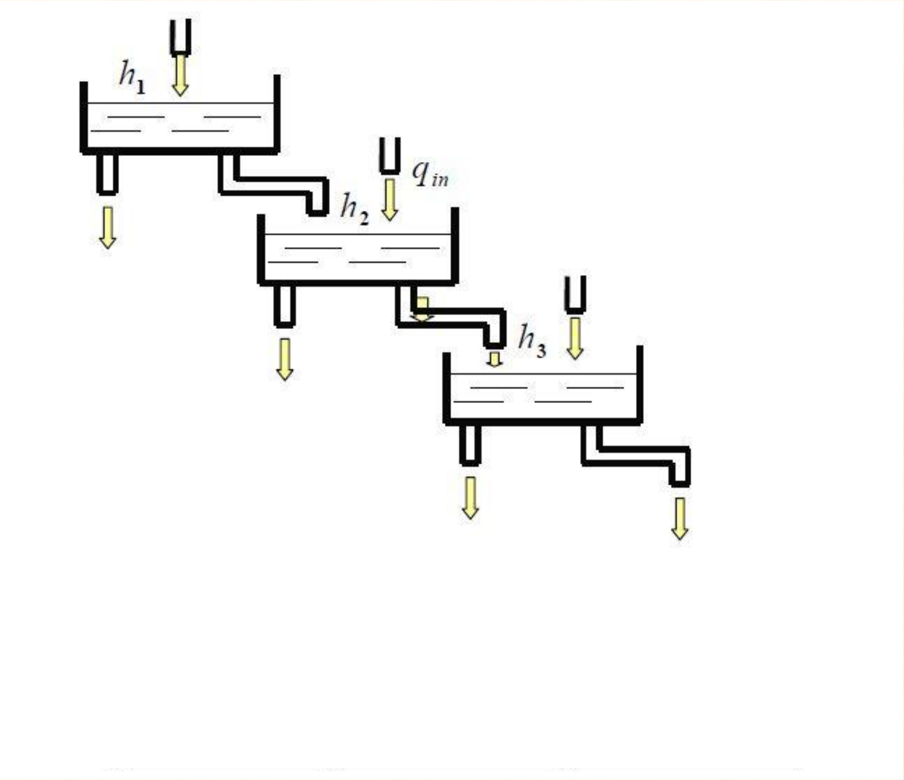
$$[B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 0 & 0 \\ b & a_{22}b & a_{22}^2 b \\ 0 & a_{32}b & a_{32}(a_{22}+a_{33})b \end{bmatrix}$$

What About Directed Networks?



In retrospect: large-scale systems theory

Structural Analysis of Dynamical Systems



Q:

Is this kind of structures controllable?

Structural Controllability

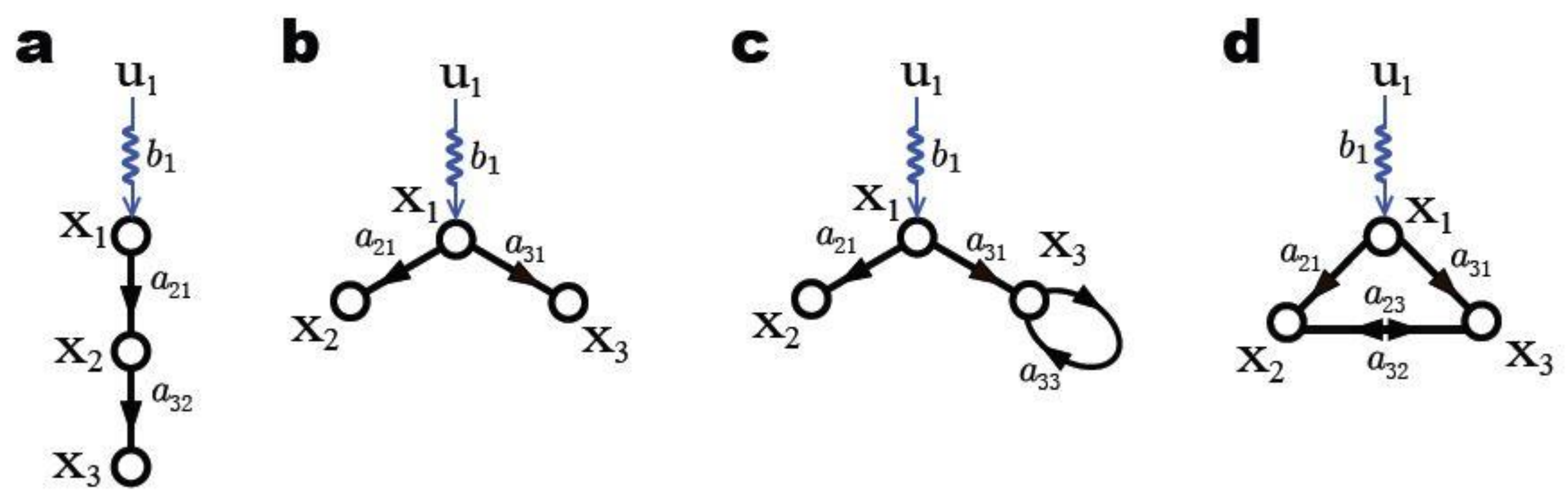
Corresponding linearized system has the following general form:

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \qquad\qquad + bu, \\ \dot{x}_3 &= \qquad\qquad a_{32}x_2 + a_{33}x_3\end{aligned} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

$$\text{Rank}\,[B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 0 & 0 \\ b & a_{22}b & a_{22}^2 b \\ 0 & a_{32}b & a_{32}(a_{22}+a_{33})b \end{bmatrix} \leq 2$$

$\rightarrow$ $\boxed{\text{Uncontrollable}}$

Examples: Structure matters



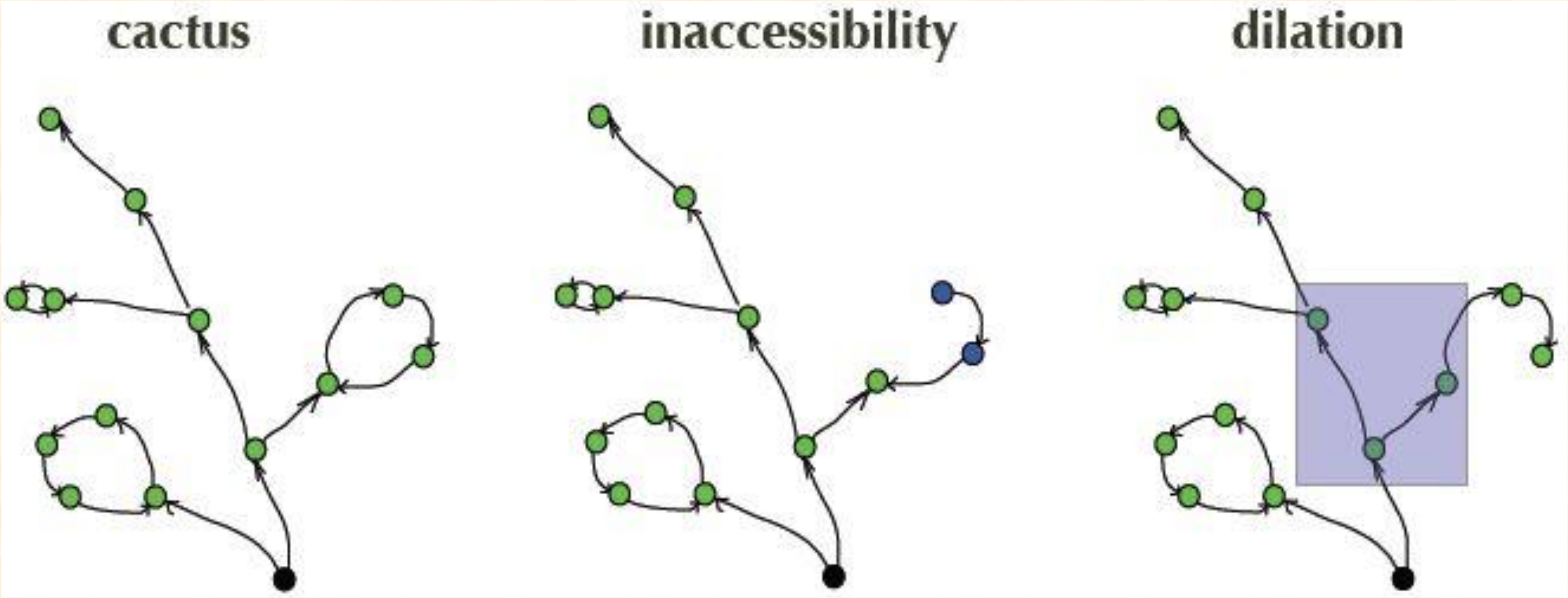
$$\mathbf{C} = [\mathbf{B}, \mathbf{A}\cdot\mathbf{B}, \mathbf{A}^2\cdot\mathbf{B}]$$

$$b_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & 0 & a_{32}a_{21} \end{bmatrix}, \quad b_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{bmatrix}, \quad b_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & a_{33}a_{31} \end{bmatrix}, \quad b_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23}a_{31} \\ 0 & a_{31} & a_{32}a_{21} \end{bmatrix}$$

rank $\mathbf{C} = 3 = n$	rank $\mathbf{C} = 2 < n = 3$	rank $\mathbf{C} = 3 = n$	rank $\mathbf{C} = ?$
controllable	**uncontrollable**	**controllable**	**controllable?**

See also: Y.Y. Liu, J.J. Slotine, A.L. Barabási, *Nature* (2011)

Building Blocks



Cactus is the minimum structure which contains no inaccessible nodes and no dilations

Structural Controllability Theorem

The following statements are equivalent:

Algebra:

1. An LTI control system (A,B) is controllable

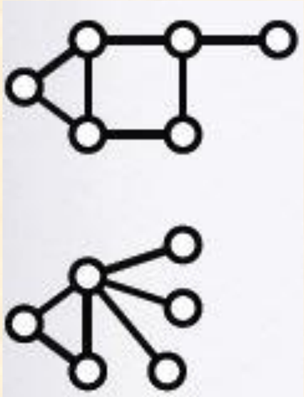
Geometry:

2. The digraph $G\,(A,B)$ contains neither inaccessible nodes nor dilations, but is spanned by a cactus

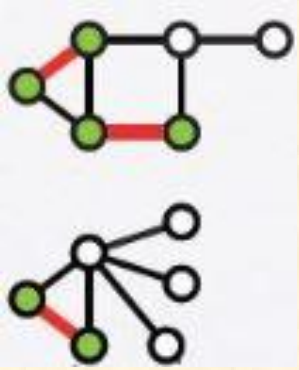
C.T. Lin, *IEEE Trans. Auto. Contr.*, 1977

Matching in Undirected Networks

Undirected network



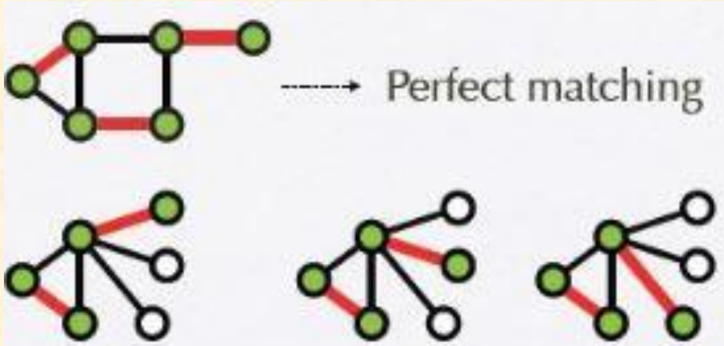
Matching:
a set of edges without common nodes



Matched node

Unmatched node

Maximum matching
(Perfect matching):
matching of largest size

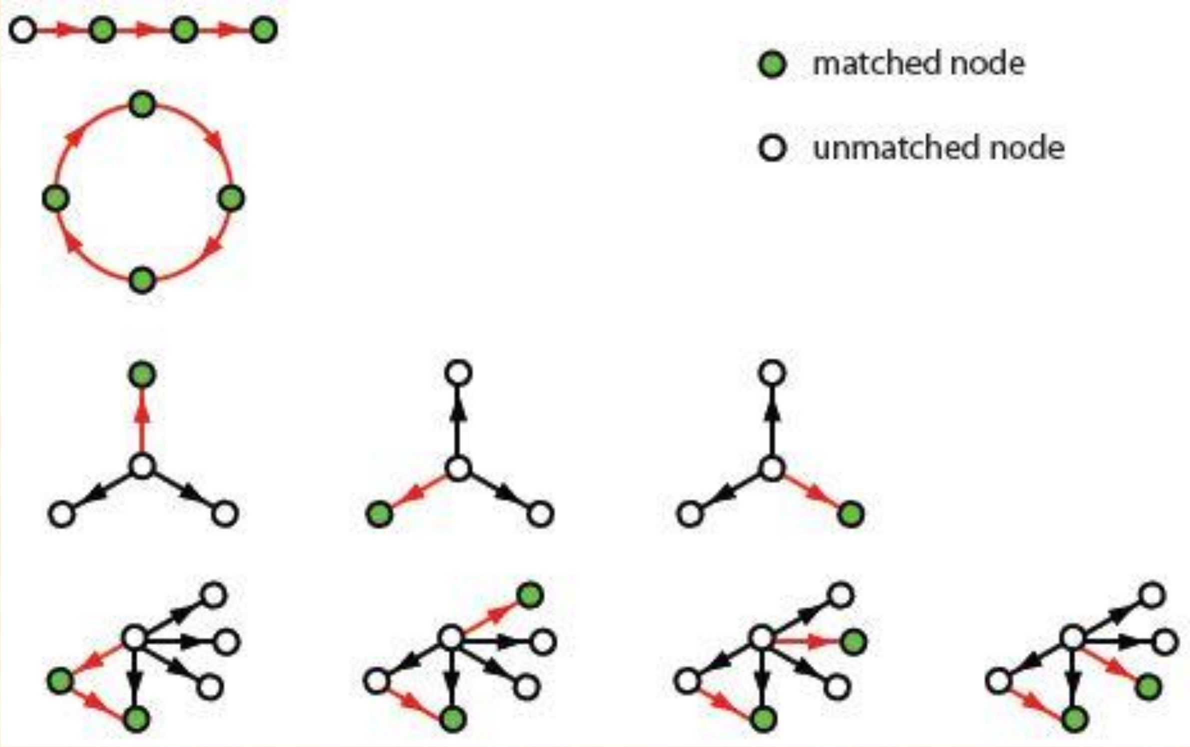


Matchings may not be unique

L. Lovász, M.D. Plummer, Matching Theory (2009)

Matching in Directed Networks

- **Matching**: a set of directed edges without common heads and common tails
- **Matched node**: the head node of a matching edge



Minimum Inputs Theorem

Q: How many?

A: The minimum number of inputs N_D needed is:

Case 1: If there is a perfect matching, then

$$N_D = 1$$

Case 2: If there is no perfect matching, then

$$N_D = \text{number of unmatched nodes}$$

Q: Where to put them?

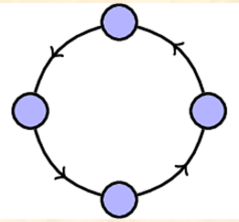
A: **Case 1:** Anywhere

Case 2: At unmatched nodes

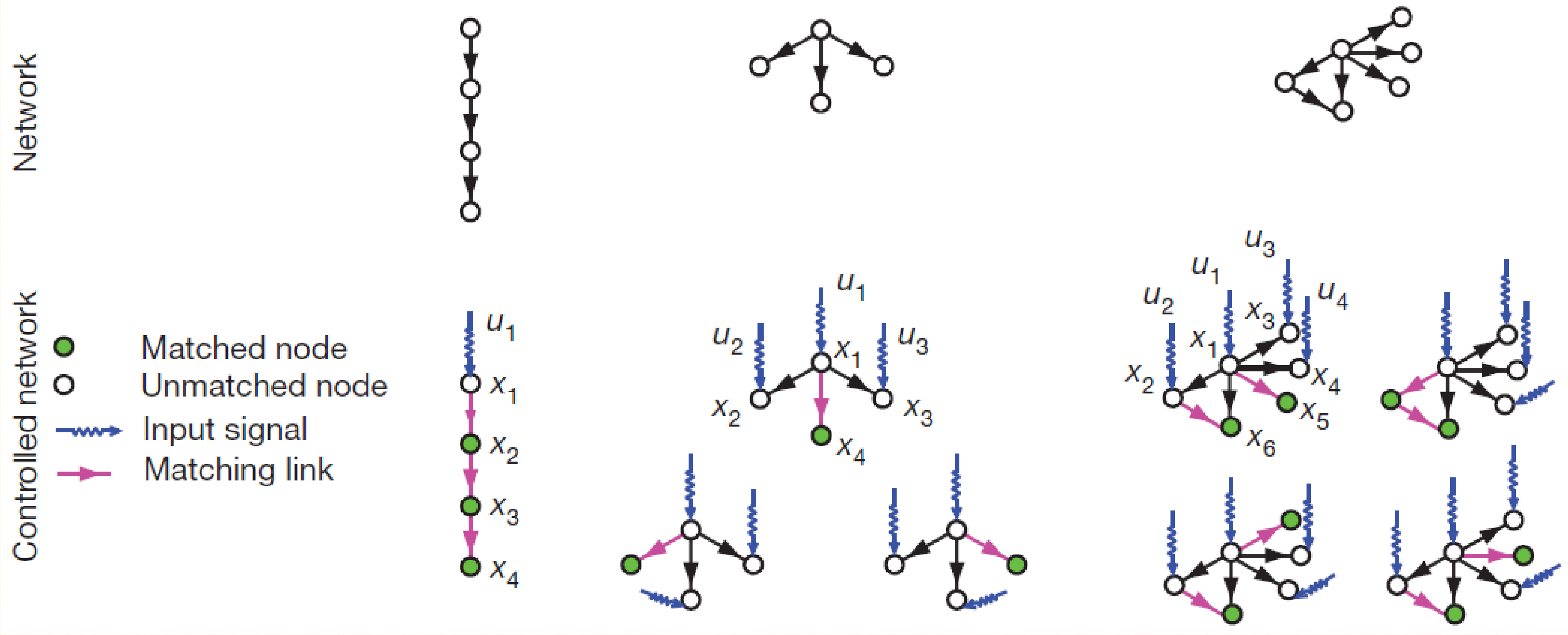
Y. Y. Liu, J. J. Slotine, A. L. Barabási, *Nature* (2011)

Examples

Perfect Matching



← 1 input; anywhere



A Network of MIMO LTI Systems

Node system $\dot{x}_i = Ax_i + Bu_i \qquad y_i = Cx_i \qquad x_i \in R^n \quad y_i \in R^m \quad u_i \in R^p$

Networked system $\dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} H y_j, \quad y_i = Cx_i, \qquad i = 1, 2, \cdots, N$

Networked system with external control $\dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} HCx_j + \delta_i Bu_i, \quad i = 1, 2, \cdots, N$

$\delta_i = 1:$ **with** external control

$\delta_i = 0:$ **without** external control

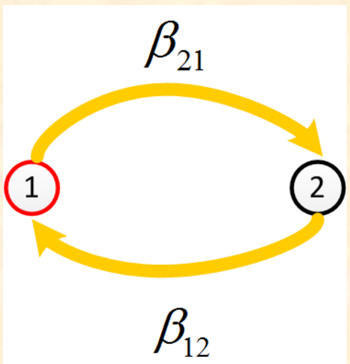
Some notations

Node system (A,B,C) Network structure $L = [\beta_{ij}] \in R^{N \times N}$

Coupling matrix H External control inputs $\Delta = diag(\delta_1, \cdots, \delta_N)$

Some counter-intuitive examples

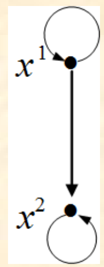
Network structure



$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

structurally controllable

Node system



$$H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

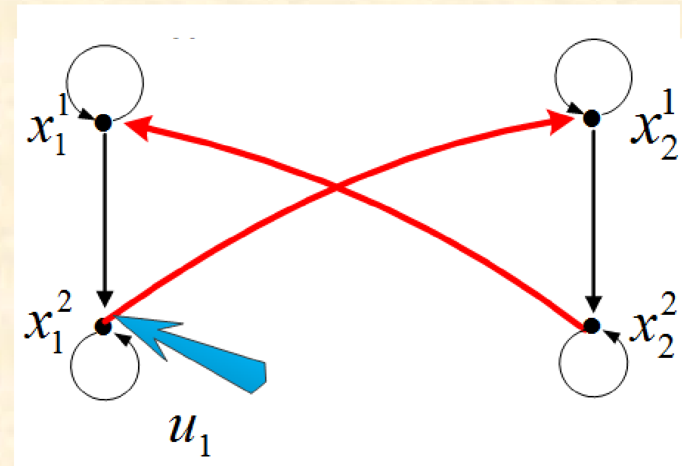
$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

(A,B) is uncontrollable

(A,C) is observable

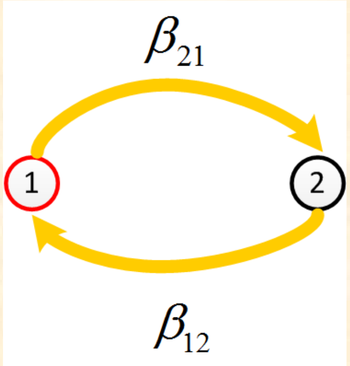
Networked MIMO system



state controllable

Some counter-intuitive examples

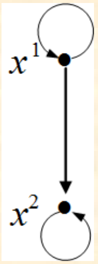
Network structure



$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

structurally controllable

Node system



$$H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

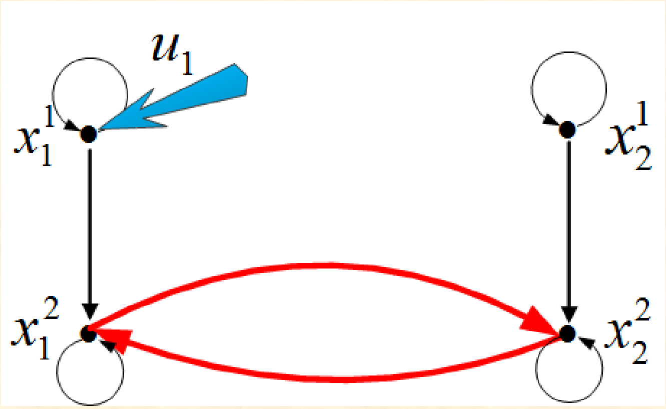
$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

(A,B) is controllable

(A,C) is observable

Networked MIMO system



state uncontrollable

Now, in general …

A Network of MIMO LTI Systems

$$\dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} H y_j + \delta_i B u_i, \quad y_i = Cx_i, \quad i = 1, 2, \cdots, N$$

$\delta_i = 1$: **with** external control $\delta_i = 0$: **without** external control

Question:

Given the MIMO node system (A, B, C), network structure L, and coupling matrix H, how to determine external inputs Δ to guarantee the controllability?

$$\Delta = diag\{\delta_1, \delta_2, \ldots, \delta_N\}$$

A Network of MIMO LTI Systems

$$\dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} HCx_j + \sum_{k=1}^{s} \delta_{ik} Bu_k,$$

$$y_l = \sum_{j=1}^{N} m_{lj} Dx_j$$

$$x_i \in R^n, \quad i = 1, \cdots N$$

$$u_k \in R^p, \quad k = 1, \cdots s$$

$$y_l \in R^q, \quad l = 1, \cdots r$$

$$L = [\beta_{ij}] \in R^{N \times N} \qquad \Delta = [\delta_{ij}] \in R^{N \times s}$$

Necessary and Sufficient Condition

Controllability $\longleftrightarrow$ If and only if

Matrix equation

$$\Delta^T XB = 0, \; L^T XHC = X(\lambda \mathrm{I} - \mathrm{A})$$

Has a unique solution $X = 0$

λ is any complex number

L. Wang, X.F. Wang, G.R. Chen and W.K.S. Tang, *Automatica*, 2016

Pinning Control of MIMO Networks

State Controllable $\longleftrightarrow$ **If and only if**

Matrix equations

$\Delta^T XB = 0, L^T XHC = X(\lambda \mathrm{I} - \mathrm{A})$ $\forall \lambda \in$ ¢

have a unique solution $X = 0$

Solution to Pinning Control: How many? Where to pin?

→ Select $\Delta = diag[\delta_i]$ such that the above algebraic matrix equations has a unique zero solution X

→ How many $\delta_i = 1$ and which $\delta_i = 1$

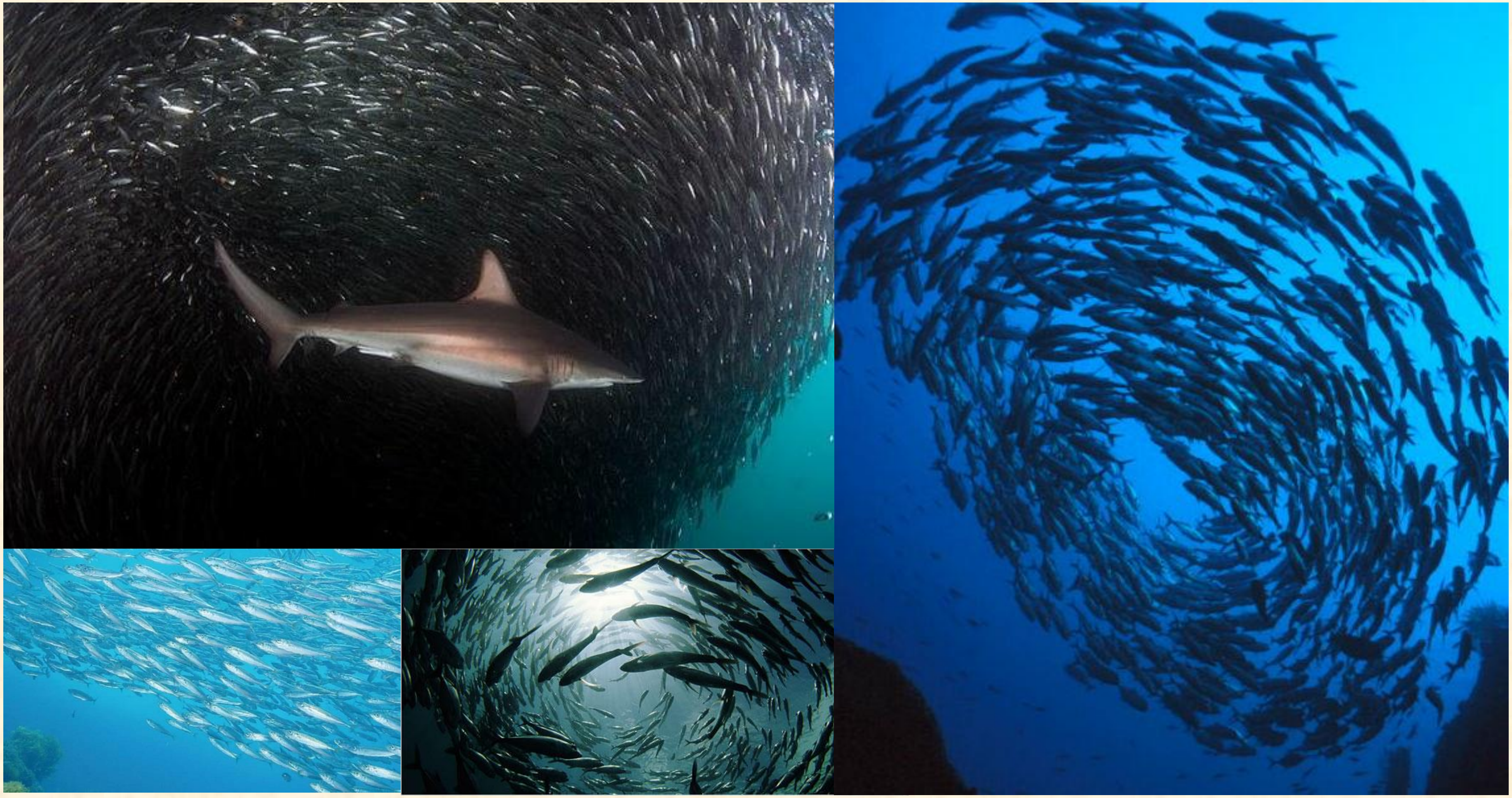
BREAK

10 minutes

Consensus and Control over Complex Networks

- **Swarm Dynamics / Modeling**
- **Consensus Protocols / Analysis**
- **Flocking Algorithms / Control**
- **DEMO**

Fish Schooling



DEMO

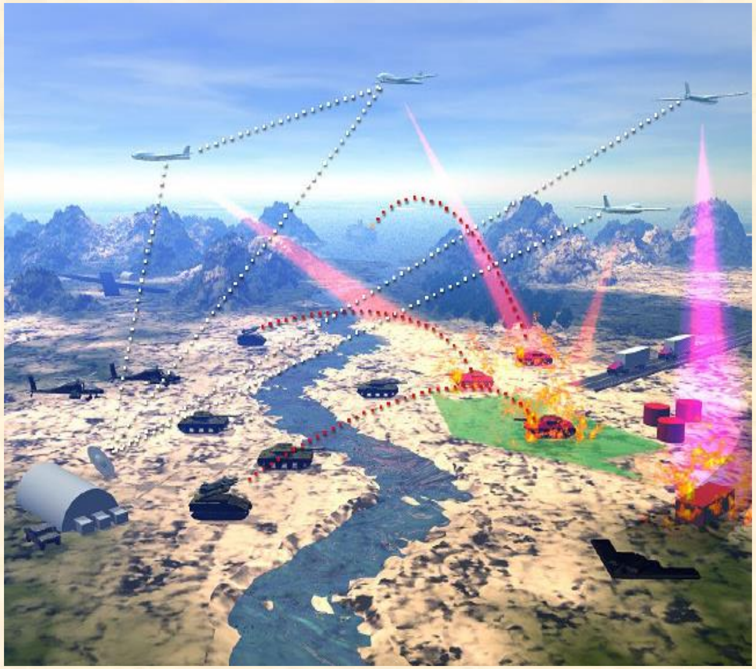
Birds Flocking

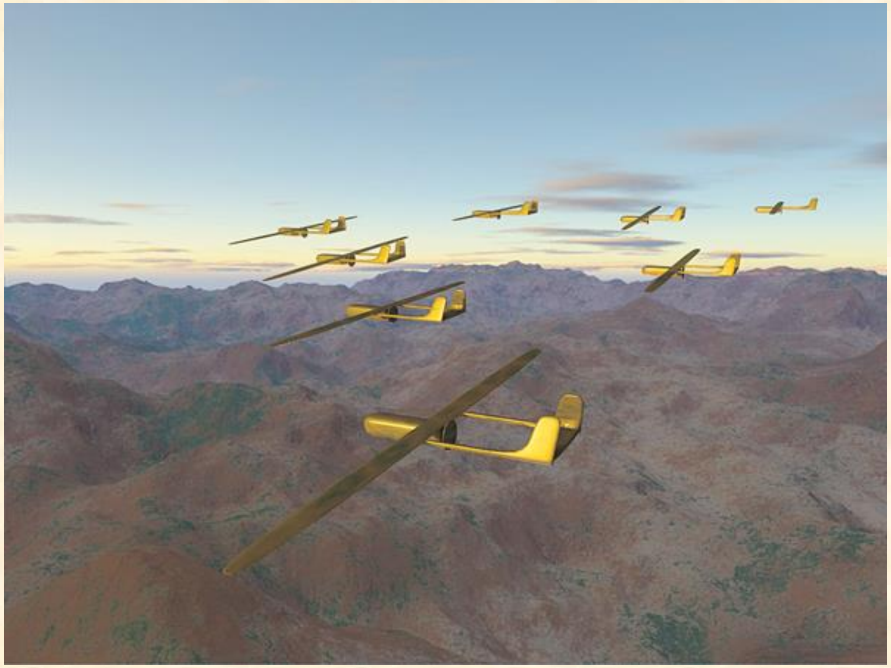
Flocking: to congregate or travel in flock

DEMO

Consensus

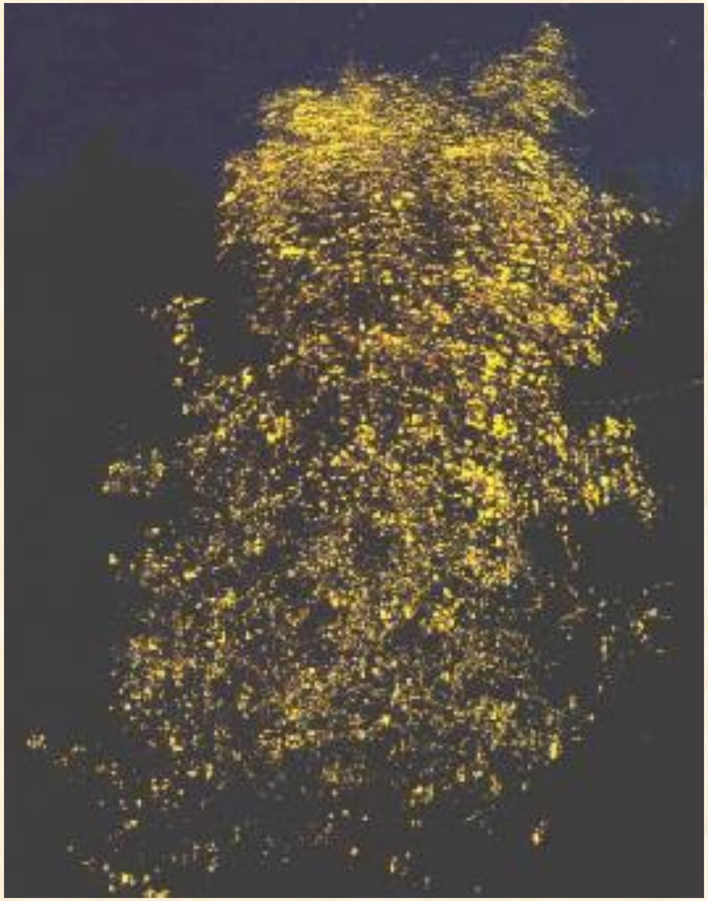
A position reached by a group as a whole



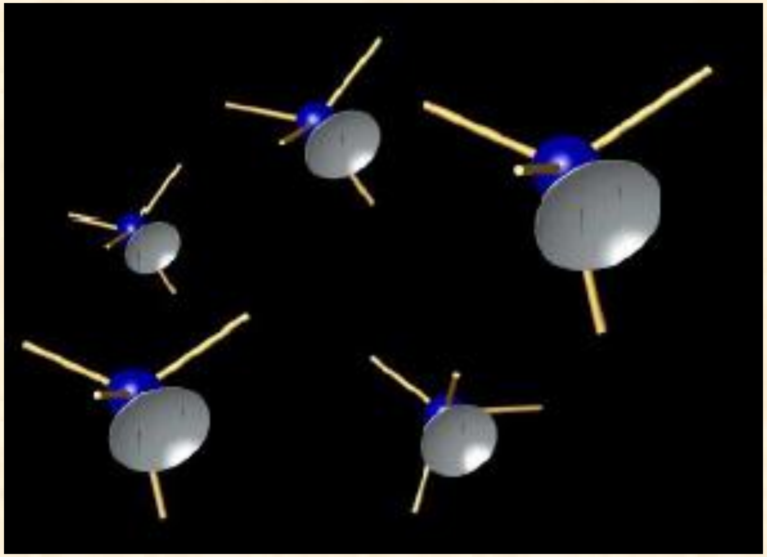


Battle space management scenario

Fireflies Synchronization



Attitude Alignment



The attitude of each spacecraft is synchronized with its two adjacent neighbors via a bi-directional communication channel

What are in common ?

- Swarming
- Flocking
- Rendezvous
- Consensus
- Synchrony
- Cooperation
- ......

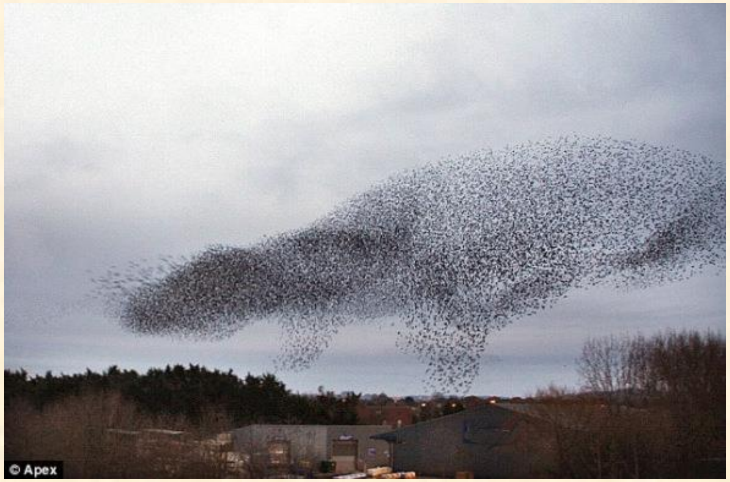
⇨

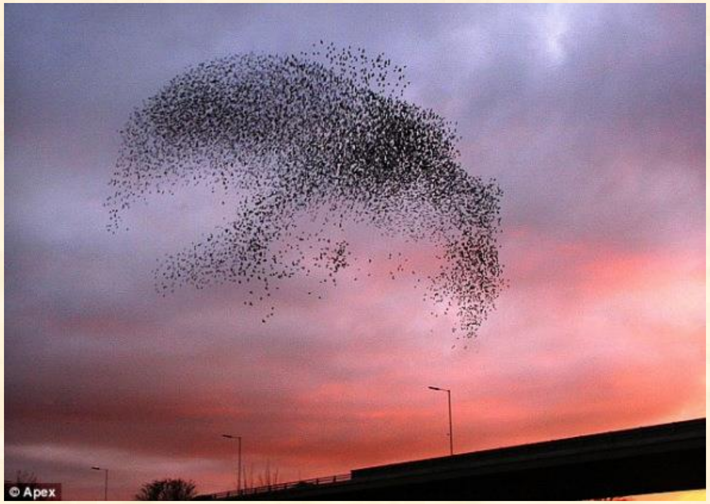
Distributed coordination of a network of agents:

- ✓ Agents
- ✓ Network
- ✓ Distributed local control
- ✓ Global consensus

Flocking

Flocking (Some Real Photos)





DEMO

Vicsek Model

Randomly place N agents into a lattice, with initial positions $x_i(0)$ and initial heading $\theta_i(0)$, $i = 1,2, \ldots, N$

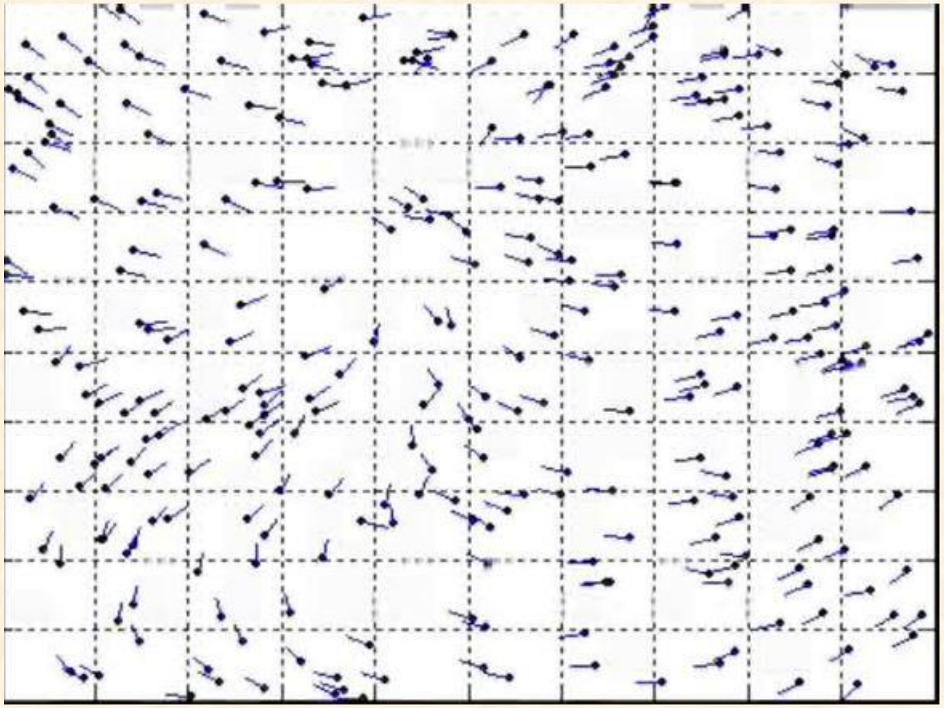
Position:

$$x_i(t+1) = x_i(t) + v\Delta t$$

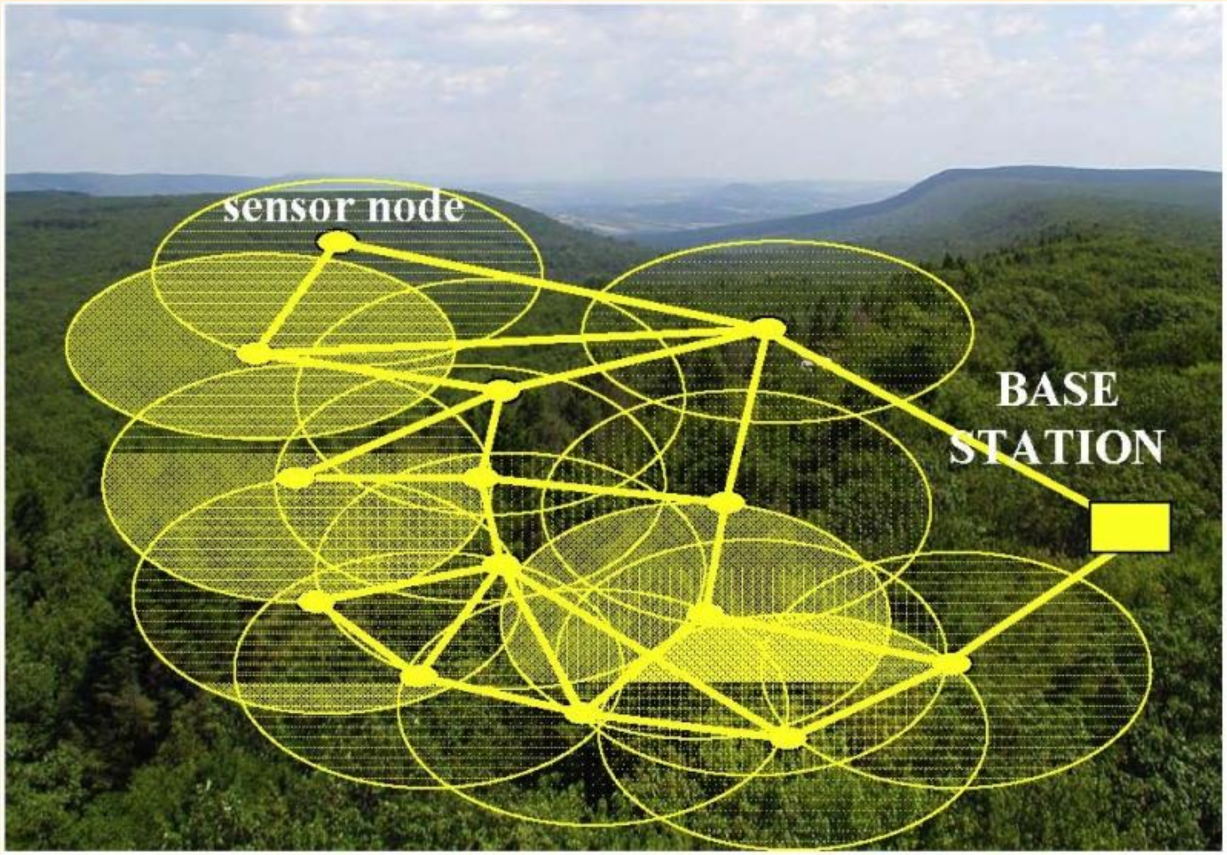
Heading:

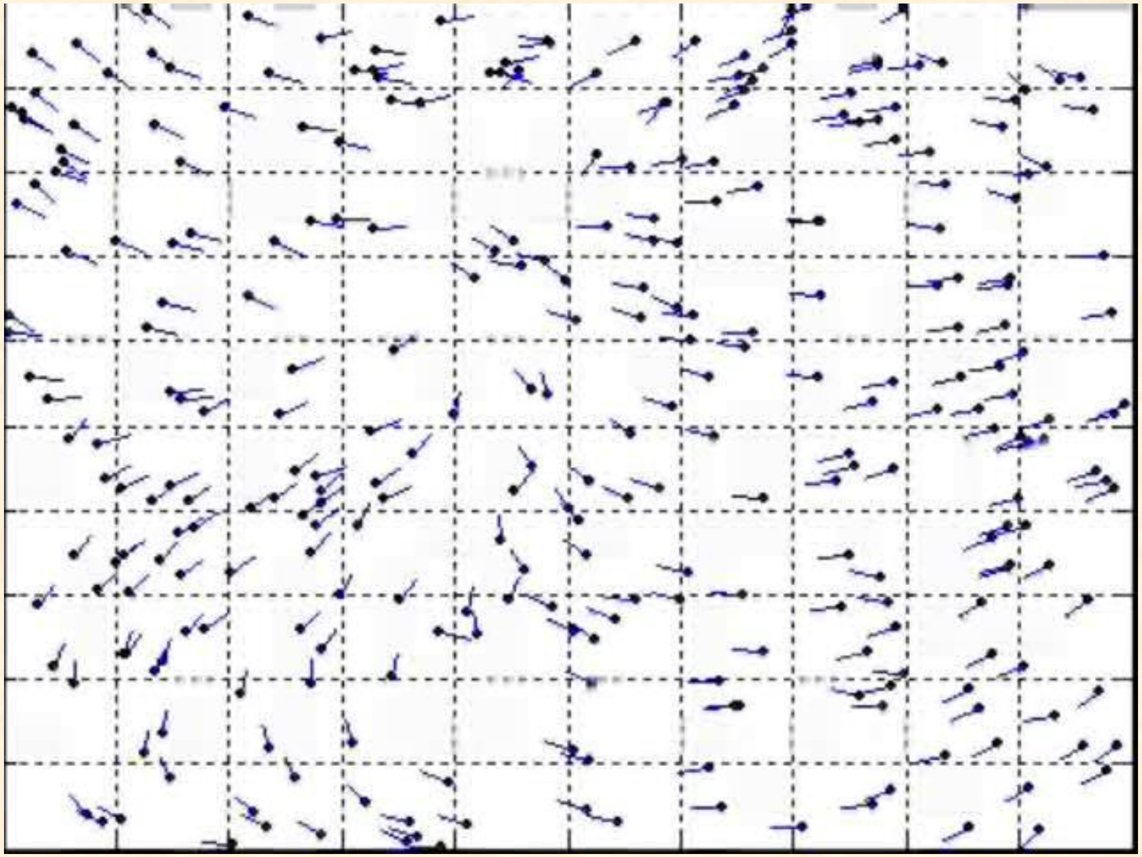
$$\theta(t+1) = \langle \theta(t) \rangle_R + \Delta\theta$$

Vicsek, et al, Phys. Rev. Lett. (1995)



$$\theta(t+1) = \langle \theta(t) \rangle_R + \Delta\theta$$





Converging to average direction of initial headings:

$$\rightarrow \quad \theta_{\infty} = \frac{1}{N}\sum_{i=1}^{N}\theta_i(0)$$

Vicsek Model

DEMO

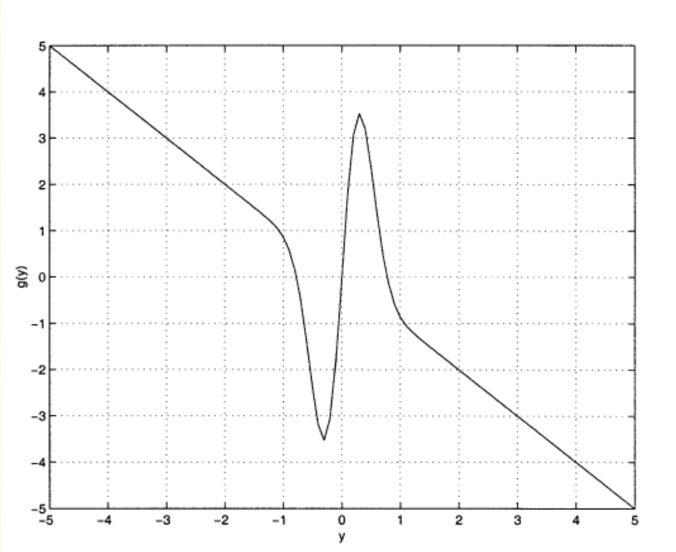
Stability Analysis

- A group of M globally nonlinearly coupled individuals:

$$\frac{dx_i}{dt} = \sum_{j=1}^{M} g(x_i - x_j), \qquad i = 1,2,\ldots,M$$

Attraction/Repulsion function:

$$g(y) = -y\left[a - b\exp\left(-\frac{\|y\|^2}{c}\right)\right]$$



- All the agents will converge to a spherical region:

$$B_\varepsilon = \left\{x : \|x - \bar{x}\| \le \varepsilon\right\} \qquad \bar{x} = \frac{1}{M}\sum_{i=1}^{M} x_i \qquad \varepsilon = \frac{b}{a}\sqrt{c/2}\exp(-1/2)$$

Gazi, Passino, IEEE Trans. Auto. Control, 2003, 48: 692–697

Stability Analysis

$$\dot{x}^i = -\nabla_{x^i}\sigma(x^i) + \sum_{j=1, j\neq i}^{M} g(x^i - x^j), i = 1,\dots, M$$

A/R function: $g(y) = -y[g_a(\|y\|) - g_r(\|y\|)]$

Linear attraction

$$g_a(\|x^i - x^j\|) = a,$$

Bounded repulsion

$$g_r(\|x^i - x^j\|)\|x^i - x^j\| \leq b,$$

$$\|\nabla_y \sigma(y)\| \leq \bar{\sigma} \quad \Rightarrow \quad x^i(t) \to B_\varepsilon(\bar{x}(t))$$

$$M \to \infty \quad \Rightarrow \quad \varepsilon = b/a \qquad \varepsilon = \frac{M-1}{aM}\left[b + \frac{2\bar{\sigma}}{M}\right]$$

Gazi, Passino, IEEE Trans. Sys. Man Cybern., B: 2004, 34(1): 539-557

Convergence Analysis

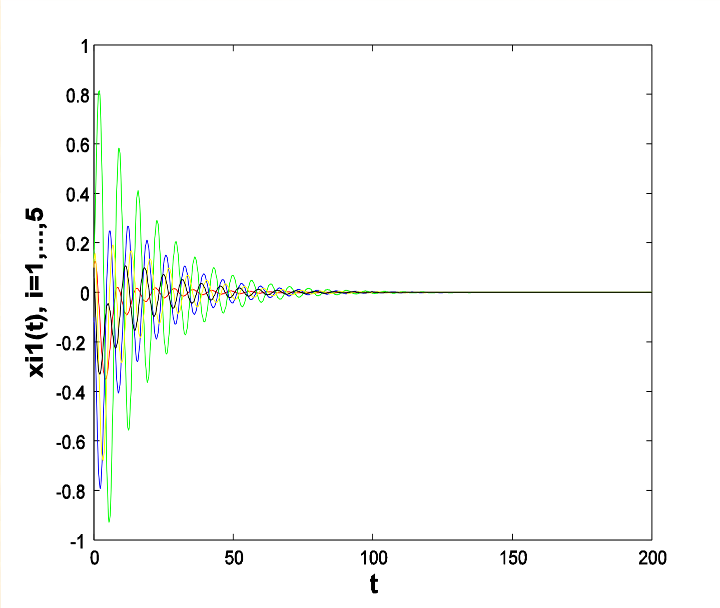
$$\theta(t+1) = F\theta(t)$$

$$\lim_{t\to\infty} \theta(t) = \theta_{ss}\mathbf{1}$$

$$F - random\ matrix$$

$$\theta_{ss} - steady\ state$$

$$\mathbf{1} = [1,1,\ldots,1]^T$$



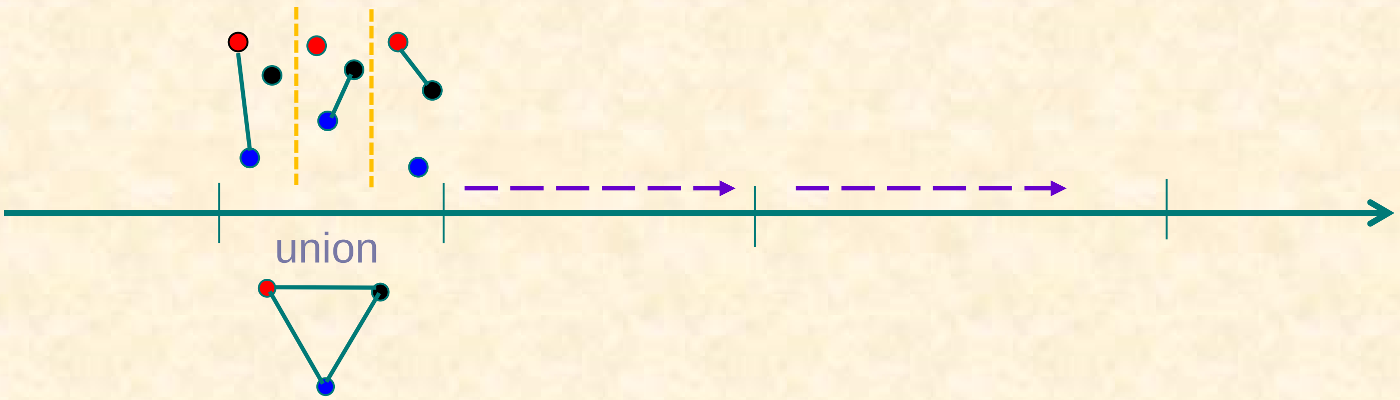
Contraction Mapping Principle

Jadbabaie, Lin, Morse, IEEE Trans. Auto. Control, 2003, 48(6): 988-1001
Moreau, IEEE Trans. Auto. Control, 2005, 50(2): 169-182

Consensus

Design a network connection topology, or design local control law, so that $\| x_i - x_j \| \to 0$ (here, consensus = synchronization)

Consensus is achieved asymptotically **if** there exists an infinite sequence of bounded intervals such that the union of the graphs over such intervals is totally connected.

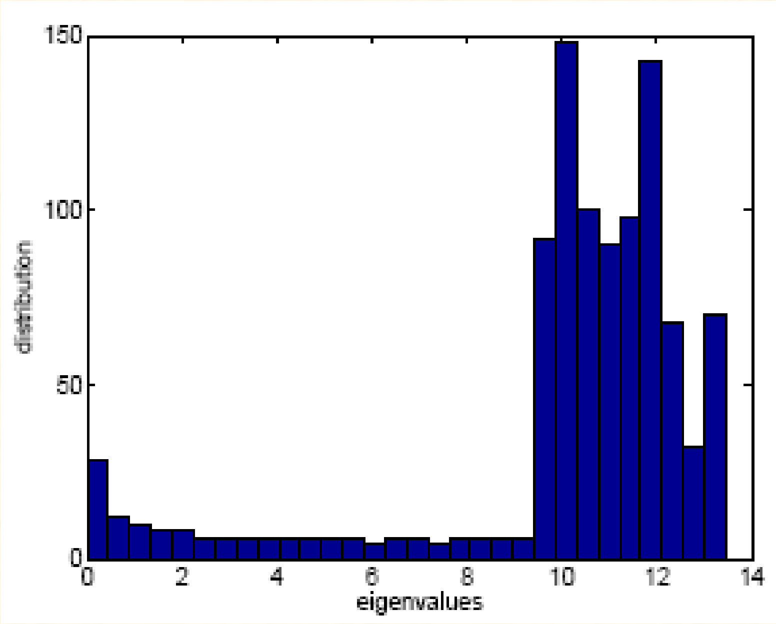


Olfati-Saber, Murray, IEEE Trans. Auto. Control, 2004, 49(9): 1520-1533

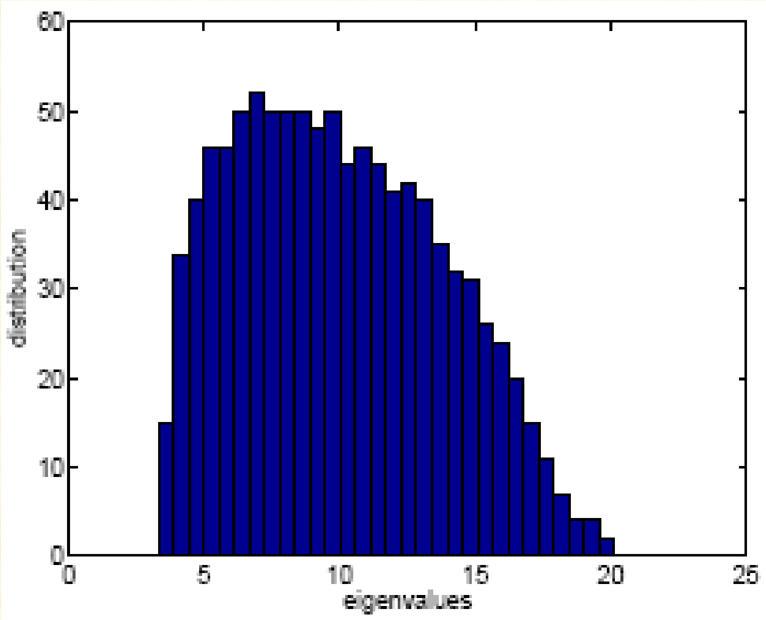
Small-World Networks are better for Consensus

λ_N / λ_2 Condition number -- the smaller, the better

Regular networks



Small-world networks



1000 times average

Olfati-Saber, Amer. Control Conf. (2005)

Flocking

DEMO


AIR CANADA

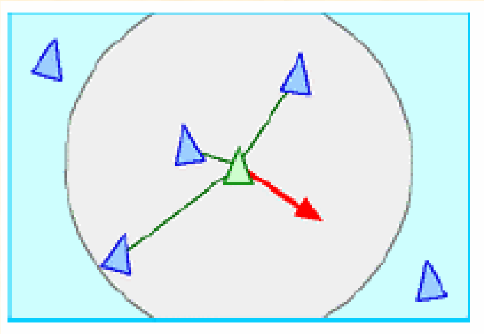
Boids Flocking Model

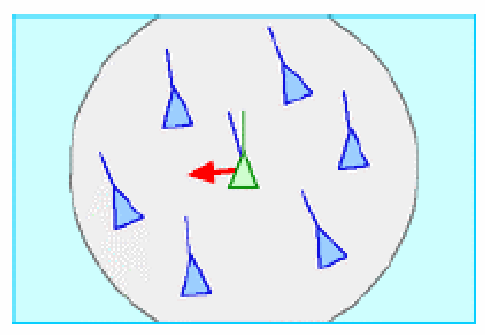
Three Rules:

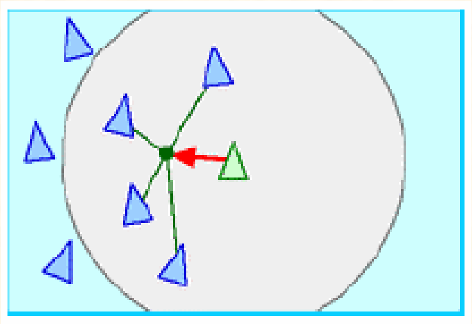
Separation: Steer to avoid crowding local flockmates

Alignment: Steer to move toward the average heading of local flockmates

Cohesion: Steer to move toward the average position of local flockmates



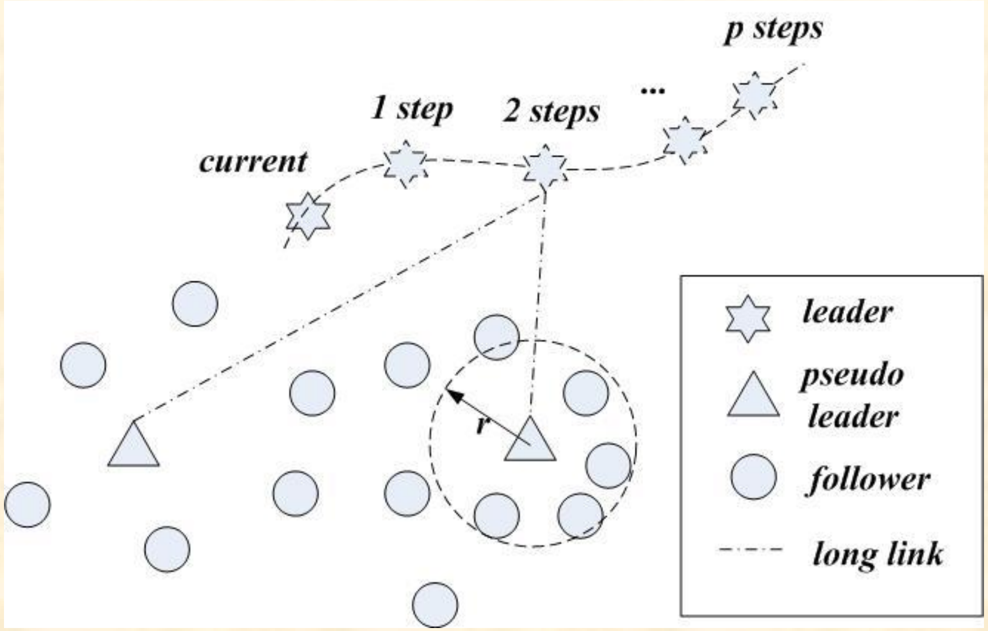




Reynolds, "Flocks, herd, and schools: A distributed behavioral model,"
Computer Graphics, 1987, 21(4): 15-24
http://www.red3d.com/cwr/boids/

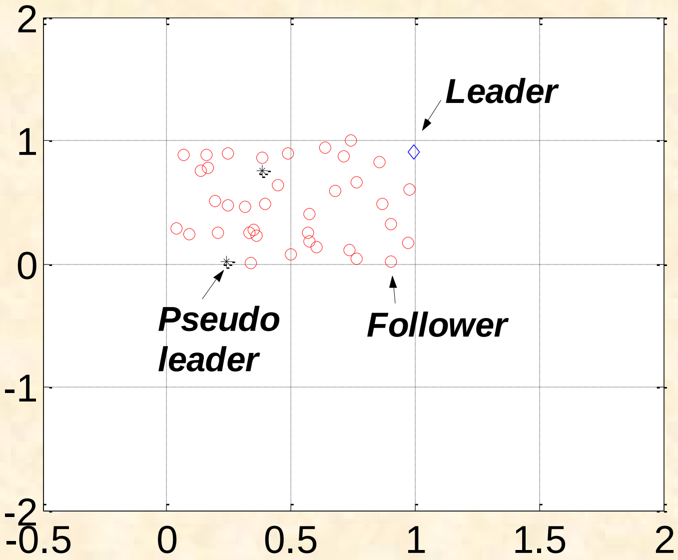
Flocking: Small-World Organization

Small-world communication generates "pseudo-leaders" who control their neighbors

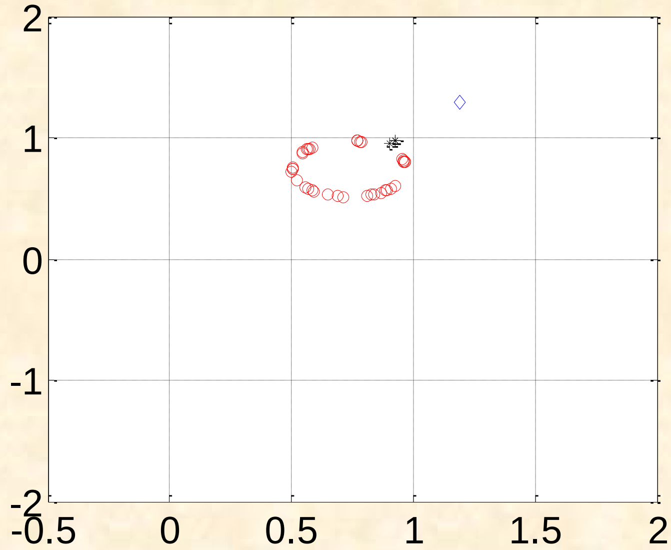


H.T. Zhang, G.R. Chen, PhysCon, Germany (2007)

Flocking: Small-World Organization

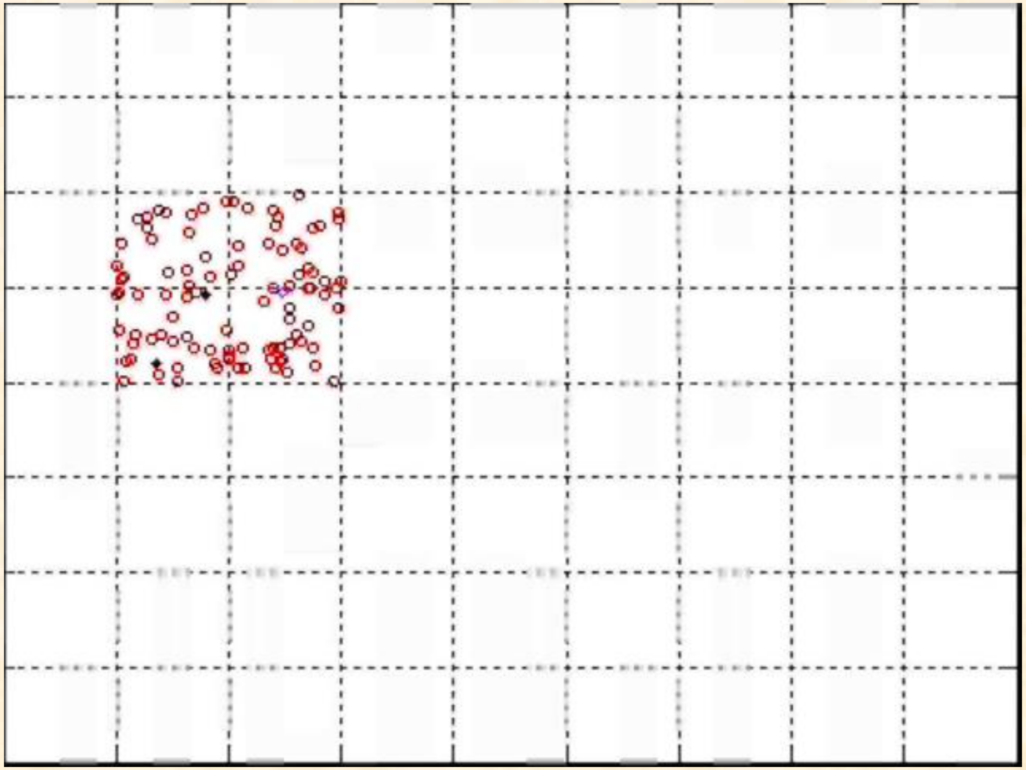


Initial position of the flock



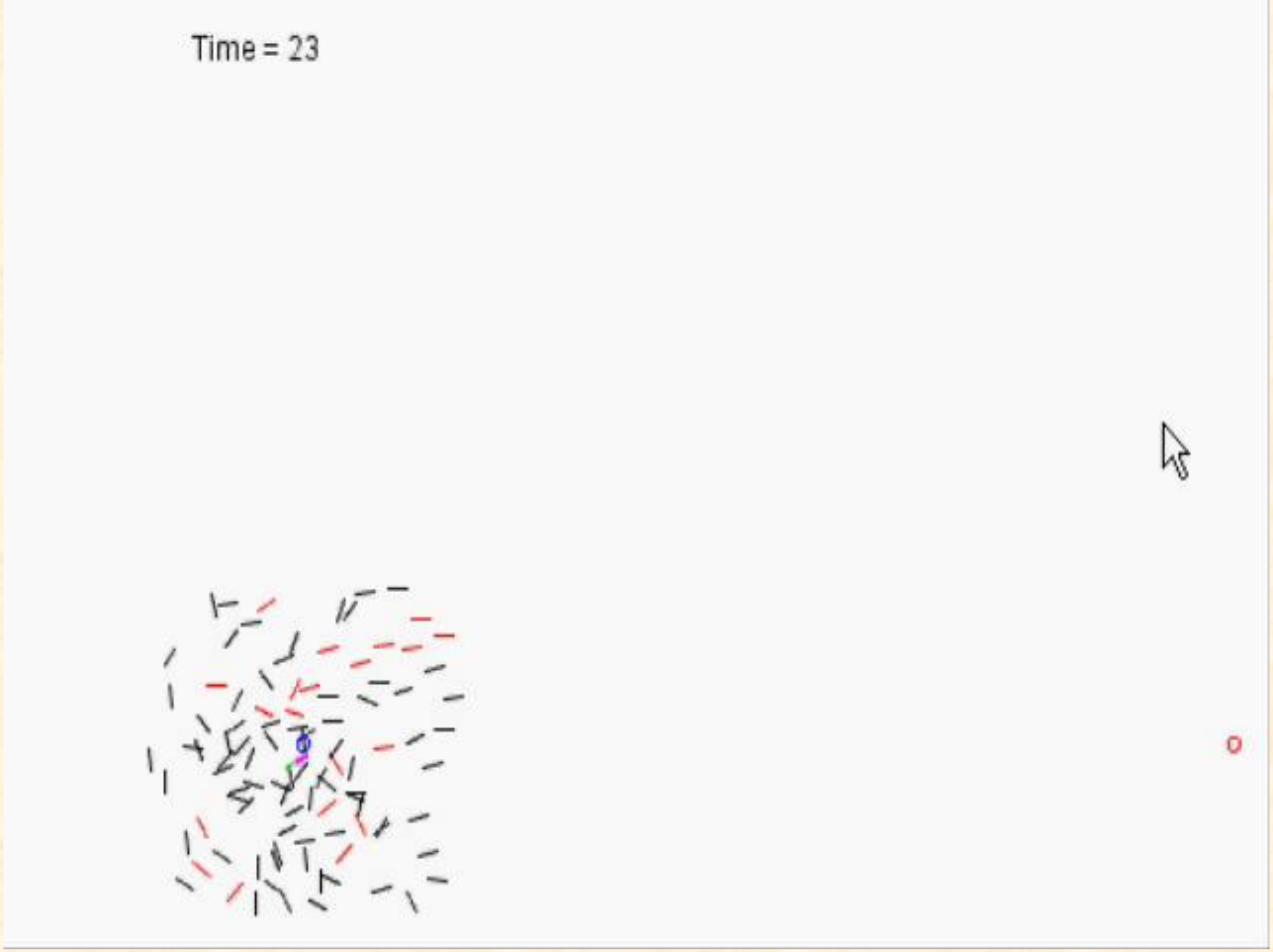
Flock position after 40 iterations

Flocking: DEMO



H.T. Zhang, G.R. Chen, PhysCon, Germany (2007)

Another Flocking Algorithm: DEMO

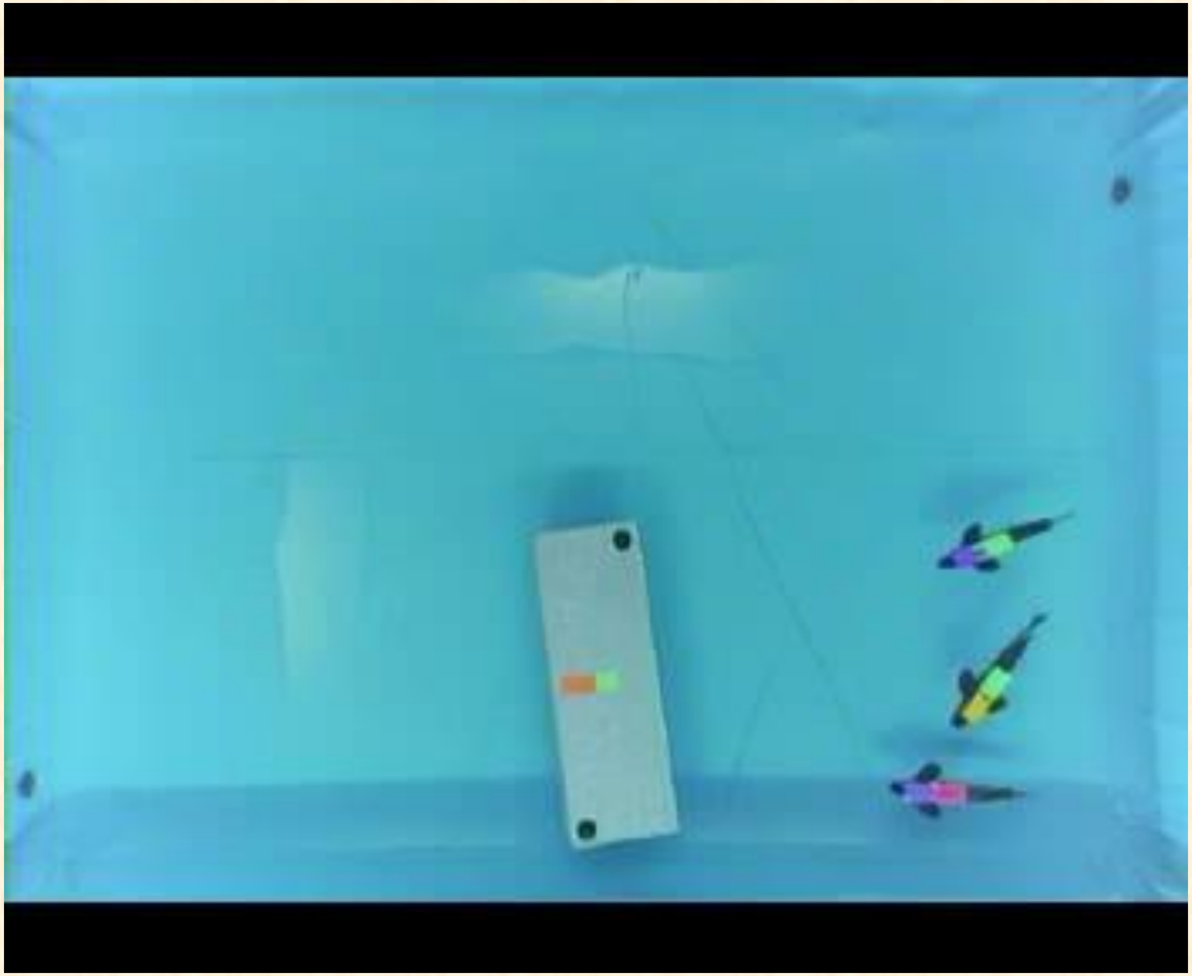


J. Lu and G. Chen (simulation)

Movies: Robot Fish

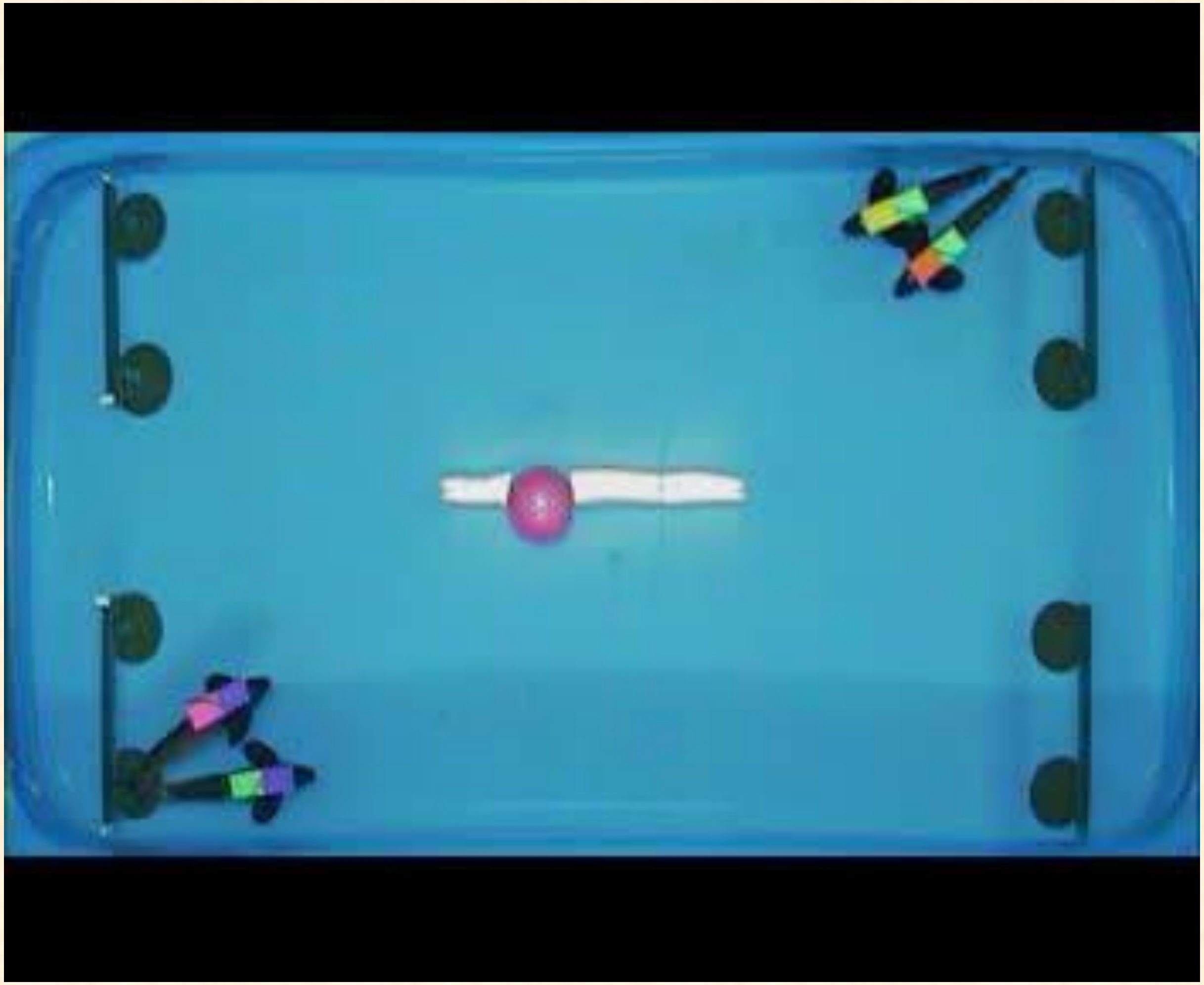
Acknowledgement: College of Engineering, Peking University

Movie 1: Coordination



Acknowledgement: College of Engineering, Peking University

Movie 2: Cooperation



Acknowledgement: College of Engineering, Peking University

End