CS 70 Discrete Mathematics and Probability Theory
Spring 2019 Ayazifar and Rao Midterm 1

PRINT Your Name: _____, _____
(last) (first)

SIGN Your Name: _____

PRINT Your Student ID: _____

PRINT Your Exam Room: _____

Name of the person sitting to your left: _____

Name of the person sitting to your right: _____

- After the exam starts, please *write your student ID on every odd page* (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.

- The questions vary in difficulty, so if you get stuck on any question, it might help to leave it and try another one.

- On questions 1-2: You need only give the answer in the format requested (e.g., true/false, an expression, a statement, a short argument.) Note that an expression may simply be a number or an expression with a relevant variable in it. **For short answer questions, correct clearly identified answers will receive full credit with no justification. Incorrect answers may receive partial credit.**
- On question 3-6, do give arguments, proofs or clear descriptions if requested. If there is a box do use it for your answer.
- You may consult only *one sheet of notes*. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are NOT permitted.
- There are **14** single sided pages including the cover sheet on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture.**
- **You have 120 minutes: there are 6 questions (with 47 parts) on this exam worth a total of 140 points. The first is true/false and has 24 points, the second is short answer and has 57 points, the third has 15 points, the fourth has 12 points, the fifth has 14 points, and the sixth has 18 points. They are not necessarily in order of difficulty.**
- Graphs are simple and undirected unless we say otherwise.

Do not turn this page until your instructor tells you to do so.

1. TRUE or FALSE?: 2pts each

For each of the questions below, answer TRUE or FALSE. No need to justify answer.

Please fill in the appropriate bubble!

1. $(\neg P \vee Q) \vee \neg(P \implies Q)$ is true for all P and Q.

   ◯ True

   ◯ False

2. $\exists n \in \mathbb{N}, \forall y \in \mathbb{Z}, n > y$.

   ◯ True

   ◯ False

3. $\neg(\forall n \in \mathbb{N}, P(n)) \implies \exists n \in \mathbb{N}, \neg P(n)$.

   ◯ True

   ◯ False

4. $(\neg A \implies \neg B) \equiv (A \implies B)$.

   ◯ True

   ◯ False

5. For $n > 2$, there is a stable marriage instance of n men and n women in which the traditional marriage algorithm takes **at least** n^2 days.

   ◯ True

   ◯ False

6. If a stable pairing P has a pair (m, w) where P is optimal for both m and w, then every stable pairing is optimal for both m and w.

   ◯ True

   ◯ False

7. If at any point in the traditional marriage algorithm a woman's optimal partner proposes to her, then every stable pairing is optimal for her.

   ◯ True

   ◯ False

8. There is an n-edge, n-vertex connected graph where each pair of vertices is connected by 2 disjoint paths. (Paths are disjoint if they do not share an edge.)

   ◯ True

   ◯ False

9. Consider a function $f : A \to B$, where $|A| = |B|$. An inverse function for f is a function $g : B \to A$ where $\forall x \in A, (g(f(x)) = x \wedge f(g(x)) = x)$. $f(\cdot)$ is a bijection if there is an inverse function.

◯ True

◯ False

10. $f(x) = ax \pmod{m}$ is a bijection **if and only if** m is prime.

◯ True

◯ False

11. Any graph that is a simple cycle can be vertex colored with 2 colors.

◯ True

◯ False

12. For a hypercube of dimension 3, there is an edge between vertex 000 and 111.

◯ True

◯ False

2. Short Answer. 3 pts each.

Write your answer in the simplest form possible. You should use only the variables in the question unless otherwise specified.

1. Write a logical formula that describes the proposition: *the square of any natural number is a natural number.*

2. What is the number of edges in an n-vertex acyclic graph having k connected components?

3. What is the number of edges in a planar graph where every face has five sides? (Answer in terms of v, the number of vertices.)

4. If there are $n/2$ leaves in an n-vertex tree, what is the average degree of the other $n/2$ vertices? (Assume n is even.)

5. For a hypercube of dimension 4, how many edges (u, v) are there where u has a leading 0 and v has a leading 1?

6. What is the longest simple cycle in a d-dimensional hypercube, for $d > 1$?

7. For positive x, y, $2^x = 1 \pmod{n}$ and $2^y = 1 \pmod{n}$ what is $2^{gcd(x,y)} \pmod{n}$?

8. If $x - y < x/2$, then $y >$ _____. (The answer should be in terms of x.)

9. The least common multiple of two positive numbers m and n is the smallest positive number that is a multiple of both. What is the least common multiple of m and n in terms of m, n and $d = gcd(m,n)$?

10. What is the maximum number of solutions in $\{0,1,\ldots,n-1\}$ to the equation $ax = b \pmod{n}$ if $gcd(n,a) = d$? (Answer in terms of n and d.)

11. What is the size of the range of the function $f(x) = ax \pmod{n}$ where $x \in \{0,1,\ldots,n-1\}$ if $gcd(n,a) = d$?

12. For a prime p, how many numbers in $\{0,1,\ldots,p^2-1\}$ have an inverse modulo p^2? (Answer in terms of p.)

13. Given x and m with $gcd(x,m) = d$, and $d = ax + bm$, what is a value of z where $zx = 5d \pmod{m}$? (In terms of some subset of the variables x, m, a, d and b.)

14. What is $2^{75} \pmod{73}$?

15. Let $p > 2$ be prime. What is $2^{k(p-1)} \pmod{p}$?

16. Find $x \pmod{20}$ that satisfies the equations: $x = 2 \pmod{4}$ and $x = 4 \pmod{5}$?

17. If $gcd(m,n) = 1$, let $x = a + km$, what should k be to satisfy $x = b \pmod{n}$?

18. What are the last two digits of 49^{19}? (Hint: $gcd(4,25) = 1$ and $24 \equiv -1 \pmod{25}$.)

19. A fixed point of a function is a value x where $f(x) = x$. Consider the function $f(x) = x^3 \pmod{mn}$ for relatively prime $m > 2$ and $n > 2$. Note that $x = -1$, $x = 0$, and $x = 1$ are fixed points for x^3. Find another fixed point of $f(\cdot)$. (Your answer can include m, n and their inverses.)

3. Short Proofs. 5 pts each.

1. Prove that if $d \nmid n^2$ then $d \nmid n$. ($d \nmid n$ means n is not a multiple of d.)

2. Prove by induction that $(1-x)^n \geq 1-nx$ for any natural number $n \geq 1$ and $0 < x < 1$.

3. Prove that $x^2 = 7$ has no rational solutions.

4. Sleepiness.

1. (5 pts) Consider a set of intervals $[s_1, e_1], \ldots, [s_n, e_n]$ where s_i is the start time and e_i is the end time of an interval. The associated interval graph has a vertex for each interval and an edge between any pair of intervals that overlap; for example, if $i_1 = [3,5]$ and $i_2 = [4,6]$ and $i_3 = [6,7]$, there are edges (i_1, i_2), and (i_2, i_3) but no edge between i_1 and i_3.
   Prove that if there is a cycle in an interval graph then there is a point in time where at least 3 intervals overlap.

2. (3 pts) Argue that for a graph $G = (V, E)$, if $|E| \geq |V|$, then G has a cycle.

3. (4 pts) Seven students each fall asleep three times during CS 70 lecture. Furthermore, for every two of these seven students, there exists a time when both of them are sleeping. Prove that there must be some time when at least three students were sleeping at once.

5. Colorings.

Define *exploding* a graph G as making a copy of it called G', and then adding an edge between each vertex $v \in G$ and its copy $v' \in G'$. Note that exploding a graph doubles the number of vertices. So, exploding an $(n-1)$-dimensional hypercube gets us an n-dimensional hypercube.

Jonathan has a graph G and wants to destroy all edges. At each step, he can choose to perform one of the following operations:

- Remove an odd-degree vertex and all its incident edges.
- Explode the graph.

Prove to Jonathan that no matter what graph G he starts with, he can get rid of all *edges* in a finite number of operations.

To help you out, we will break this down into parts. Define $f(G)$ as the smallest number of colors needed to vertex-color G.

1. (4 pts) Prove that exploding a graph where $f(G) > 1$ does not increase $f(G)$.

2. (4 pts) Suppose every vertex in G has even degree, and let G_{boom} be its exploded graph. Given a coloring of G_{boom}, prove that you can remove all vertices of a particular color in G_{boom}. (Potentially in multiple steps.)

3. (2 pts) Prove that if $f(G) = 1$, then Jonathan is done.

4. (4 pts) Now finish the proof: prove that there exists a finite sequence of operations for Jonathan to destroy all edges. (You may use results from previous parts.)

6. Chicken Nuggets.

In this problem, we explore the conundrum that Jonathan and Emaan face when they visit McDonald's. The chicken nuggets are sold in boxes of two different quantities, and they're interested in which quantities of chicken nuggets can be bought.

1. (3 pts) Find (x,y) that satisfy the equation $7x+11y=53$, where x and y have to be non-negative integers, and y is as small as it can be. *Hint: try taking mods of both sides to eliminate a variable first*

2. (3 pts) Explain why $7x+11y=59$ has no non-negative integer solutions for (x,y).

3. Consider a store where chicken nuggets are sold in boxes of m and n with $\gcd(m,n)=1$. Buying x boxes of m chicken nuggets and y boxes of n chicken nuggets yields $xm+yn$ chicken nuggets.

   (a) (2 pts) For any solution to $xm+yn=mn-m-n$, what is $x \pmod n$?

   (b) (4 pts) Argue that there is no solution for $xm+yn=mn-m-n$ where x and y are both non-negative integers.

4. The following two parts will prove that $mn - m - n$ is the largest number of chicken nuggets that **cannot** be bought.

   (a) (3 pts) Prove that for any integer z, there is a solution to $xm + yn = z$ where $0 \leq x \leq n - 1$.

   (b) (3 pts) Argue that for any $z > mn - m - n$, there is a solution to $xm + yn = z$ where x and y are non-negative.