City University of Hong Kong
Department of Economics and Finance
Course EF5213 Assignment #3 **(due March 18, 2018)**

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1. Consider the MVO problem that determines the optimal portfolio content $\mathbf{w}$ and w_0 by minimizing the portfolio risk subject to an expected return μ_P as

$$\textit{minimize} \quad \tfrac{1}{2}\,\mathbf{w}^T \boldsymbol{\sigma}\, \mathbf{w}$$

$$\textit{subject to} \quad \mathbf{w}^T \boldsymbol{\mu} + w_0\, \mu_0 = \mu_P\ ,\ \mathbf{u}^T\mathbf{w} + w_0 = 1\ ,\ \text{and}\ \ a_1 \geq w_1 \geq -b_1\ , \ldots ,\ a_n \geq w_n \geq -b_n\ ,$$

given riskfree rate μ_0, asset mean returns $\boldsymbol{\mu}$, and their variance-covariance $\boldsymbol{\sigma}$. There are buy and sell limits in the optimization according to the given positive quantities $\{a_1, \ldots , a_n\}$ and $\{b_1, \ldots , b_n\}$. It should be noted that the optimal portfolio content can be determined through the Kuhn-Tucker conditions as

$$\frac{\partial L}{\partial w_i} = (\boldsymbol{\sigma}\, \mathbf{w} - \lambda\, \boldsymbol{\mu} + \mu_0\, \lambda\, \mathbf{u})_i = 0 \ \text{ when } \ a_i \geq w_i \geq -b_i$$
$$< 0 \ \text{ when } \ w_i = a_i$$
$$> 0 \ \text{ when } \ w_i = -b_i \ \ , \ \text{ for } i = 0, 1, \ldots , n$$

Consider the following procedures in your implementation :

(1) Define an *OUT* subset Ω, and separate Ω into two disjoint subsets Ω_L and Ω_U.
Consider the MVO problem with $w_i = -b_i$ for $i \in \Omega_L$, and $w_i = a_i$ for $i \in \Omega_U$. The optimal solution of this MVO problem is given by

$$w_0 = 1 - \gamma - \lambda\,(A_R - \mu_0\, C_R) + \mathbf{u}_R^T\, \boldsymbol{\sigma}_R^{-1} \boldsymbol{\beta}\ , \quad \mathbf{w}_R = \lambda\,(\boldsymbol{\sigma}_R^{-1} \boldsymbol{\mu}_R - \mu_0\, \boldsymbol{\sigma}_R^{-1} \mathbf{u}_R) - \boldsymbol{\sigma}_R^{-1} \boldsymbol{\beta}\ ,$$

$$\lambda = \frac{\mu_P - \varphi - \mu_0(1-\gamma) - \mu_0 \mathbf{u}_R^T \boldsymbol{\sigma}_R^{-1} \boldsymbol{\beta} + \boldsymbol{\beta}^T \boldsymbol{\sigma}_R^{-1} \boldsymbol{\mu}_R}{(C_R \mu_0^2 - 2A_R \mu_0 + B_R)}$$

where $\beta_i = \sum_{j\in\Omega_L} \sigma_{ij}\,(-b_j) + \sum_{j\in\Omega_U} \sigma_{ij}\,(a_j)\ ,\ \ \varphi = \sum_{i\in\Omega_L} (-b_i)\mu_i + \sum_{i\in\Omega_U} (a_i)\mu_i\ ,$

$$\gamma = \sum_{i\in\Omega_L} u_i\,(-b_i) + \sum_{i\in\Omega_U} u_i\,(a_i)$$

Here, $\{\boldsymbol{\sigma}_R\ ,\ \boldsymbol{\mu}_R\ ,\ \mathbf{u}_R\ ,\ \mathbf{w}_R\}$ refer to the reduced forms of $\{\boldsymbol{\sigma}\ ,\ \boldsymbol{\mu}\ ,\ \mathbf{u}\ ,\ \mathbf{w}\}$ by ignoring the rows and columns corresponding to those assets in the *OUT* subset. The terms $\{\boldsymbol{\beta}\ ,\ \varphi\ ,\gamma\}$ are defined according to the contents in Ω_L and Ω_U using $\{\boldsymbol{\sigma}\ ,\ \boldsymbol{\mu}\ ,\ \mathbf{u}\}$.

(2) Check that all the entries of $\mathbf{w}_R$ satisfy both the buy and sell limits. If so, proceed to step (3). If this is not the case, return to step (1) and try another separation of Ω or another *OUT* subset.

(3) Check that KKT conditions have been satisfied. If so, w_0 and $\mathbf{w}$ defined in (1) will be an optimal solution given portfolio return μ_P. Otherwise, return to step (1) and try another separation of Ω or another *OUT* subset.

(80 points)