City University of Hong Kong
Department of Economics and Finance
Course EF5213 Assignment #3 **(due March 28, 2021)**

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1. Consider the MVO problem that determines the optimal portfolio content $\mathbf{w}$ and w_0 by minimizing the portfolio risk as

$$\textit{minimize} \quad \tfrac{1}{2}\,\mathbf{w}^T \boldsymbol{\sigma}\, \mathbf{w}$$

$$\textit{subject to} \quad \mathbf{w}^T \boldsymbol{\mu} + w_0\, \mu_0 = \mu_p\,, \quad \mathbf{u}^T \mathbf{w} + w_0 = 1\,, \text{ and } \; a_1 \geq w_1 \geq -b_1\,, \ldots, a_n \geq w_n \geq -b_n$$

given portfolio mean return μ_p, riskfree rate μ_0, asset mean returns $\boldsymbol{\mu}$, and their variance-covariance $\boldsymbol{\sigma}$. There are buy and sell limits in the optimization according to the given positive quantities $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$. It should be noted that the optimal portfolio content can be determined through the Kuhn-Tucker conditions as

$$\frac{\partial L}{\partial w_i} = (\boldsymbol{\sigma}\, \mathbf{w} - \lambda_1 \boldsymbol{\mu} + \mu_0\, \lambda_1\, \mathbf{u})_i \begin{cases} = 0 & \text{when } a_i \geq w_i \geq -b_i \\ < 0 & \text{when } w_i = a_i \\ > 0 & \text{when } w_i = -b_i \end{cases} \quad , \text{ for } i = 0, 1, \ldots, n$$

Modify the Markowitz algorithm in the lecture and develop a VBA implementation for the current MVO problem.

(80 points)

Consider the following procedures in your implementation:

(1) Define an *OUT* subset Ω, and separate Ω into two disjoint subsets A and B. Consider the MVO problem with $w_i = -b_i$ for $i \in B$, and $w_i = a_i$ for $i \in A$. The optimal solution of this MVO problem is given by

$$\mathbf{w} = \lambda_1 (\boldsymbol{\sigma}_m^{-1} \boldsymbol{\mu}_m - \mu_0\, \boldsymbol{\sigma}_m^{-1} \mathbf{u}_m) + \boldsymbol{\sigma}_m^{-1} \boldsymbol{h}\,, \quad \text{where } \boldsymbol{h}_i = \begin{cases} a_i & , i \in A \\ -b_i & , i \in B \\ -\beta_i & , i \notin \Omega \end{cases} \quad , \text{ and } \; \beta_i = \Sigma_{j\in A}\, \sigma_{ij}\, a_j + \Sigma_{j\in B}\, \sigma_{ij} \left(-b_j\right)$$

$$w_0 = 1 - \mathbf{u}^T \mathbf{w}$$

$$\lambda_1 = \frac{\mu_p - \mu_0 + \mu_0 \mathbf{u}^T \boldsymbol{\sigma}_m^{-1} \boldsymbol{h} - \boldsymbol{\mu}^T \boldsymbol{\sigma}_m^{-1} \boldsymbol{h}}{C_m \mu_0^2 - 2 A_m \mu_0 + B_m}$$

Here, $\{\boldsymbol{\sigma}_m, \boldsymbol{\mu}_m, \mathbf{u}_m\}$ refer to the modified versions of $\{\boldsymbol{\sigma}, \boldsymbol{\mu}, \mathbf{u}\}$ according to the assets in the *OUT* subset Ω.

(2) Check that all the entries of $\mathbf{w}$ satisfy both the buy and sell limits. If so, proceed to step (3). If this is not the case, return to step (1) and try another separation of Ω or another *OUT* subset.

(3) Check that KKT conditions have been satisfied. If so, w_0 and $\mathbf{w}$ defined in (1) will be an optimal solution. Otherwise, return to step (1) and try another separation of Ω or another *OUT* subset.

Some useful comments:

Suppose *OUT* subset Ω is given by $I(\,1 : N_c,\ 1 : N_{out}\,)$.

Referring to label L with content $I(\,L,\ 1 : N_{out}\,)$, we want to generate the disjoint subsets as

$$I_A(\,1 : M_c,\ 1 : N_{out} - M_{out}\,) \text{ and } I_B(\,1 : M_c,\ 1 : M_{out}\,) \text{ with label from 1 to } M_c$$

where M_{out} runs from 0 to N_{out} .

Consider the following algorithm.

If($M_{out} = 0$) then

I_B is empty and I_A has the same content as I.
There is $M_c = 1$ possibility with label $H = 1$ only

elseif($M_{out} = 1$) then

I_B takes one element in I, and I_A keeps the rest.
There are $M_c = N_{out}$ possibilities with different labels $H = 1$ to M_c

elseif($M_{out} \geq 2$) then

From previous $\bar{I}_B(\,1 : \bar{M}_c,\ 1 : M_{out} - 1)$ and $\bar{I}_A(\,1 : \bar{M}_c,\ 1 : N_{out} - (M_{out} - 1)\,)$
Generate current $I_B(\,1 : M_c,\ 1 : M_{out}\,)$ and $I_A(\,1 : M_c,\ 1 : N_{out} - M_{out}\,)$
$H = 0$
For (Running through previous label $\bar{H} = 1$ to $\bar{M}_c$) {
For (Running through the content of previous $\bar{I}_A$, $i = 1$ to $N_{out} - (M_{out} - 1)$){
$H = H + 1$
$I_B(\,H,\ 1 : M_{out}\,) = \bar{I}_B(\bar{H},\ 1 : M_{out} - 1)$ plus the i-th entry in $\bar{I}_A\,(\bar{H},\ 1 : N_{out} - (M_{out} - 1)\,)$
$I_A(\,H,\ 1 : N_{out} - M_{out}\,) = \bar{I}_A\,(\bar{H},\ 1 : N_{out} - (M_{out} - 1)\,)$ excluding the i-th entry
}
}
$M_c = H$

endif