

# Lecture 12

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**Root locus**

# Motivation

(performance via “dominant poles” design)

- Given a combined requirement on overshoot, rise time and settling time, we could get:

If we place dominant poles of the closed loop in the white region of the complex plane, the system would have the desired transients in step response

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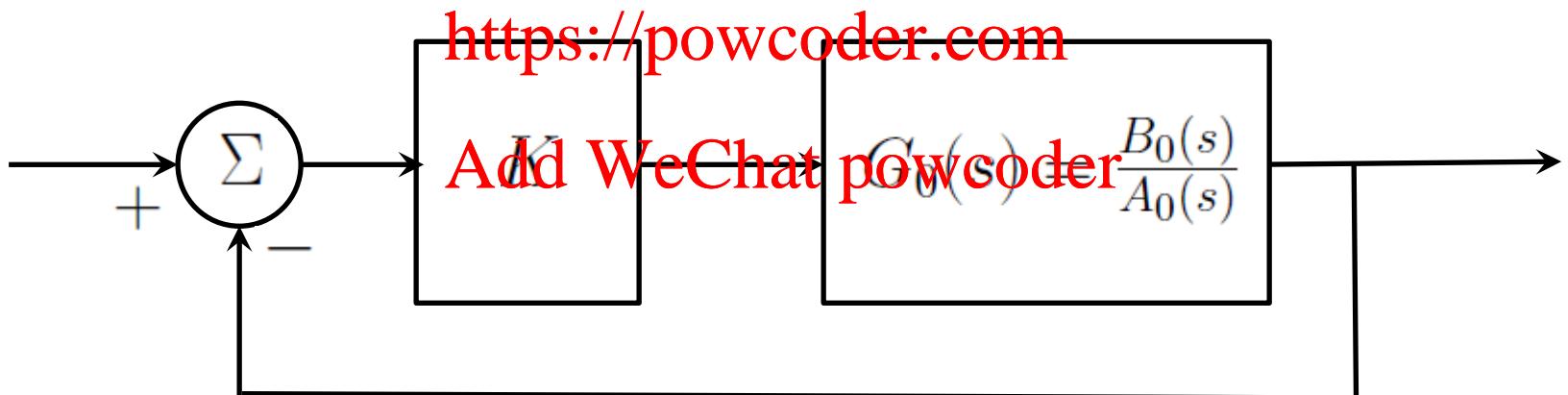
Brown colour represents a “forbidden region”.

# Motivation:

(simplest proportional controller)

- Consider the closed-loop system, where  $K$  is a parameter (i.e. “proportional” controller)

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**Design question:** Can we select  $K$  so that the poles are in the desired region?

# Motivation:

- Transfer function of the closed loop system:

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 $G(s) = \frac{KG_0(s)}{1+KG_0(s)} = \frac{KA_0(s)}{1+K\frac{B_0(s)}{A_0(s)}}$   
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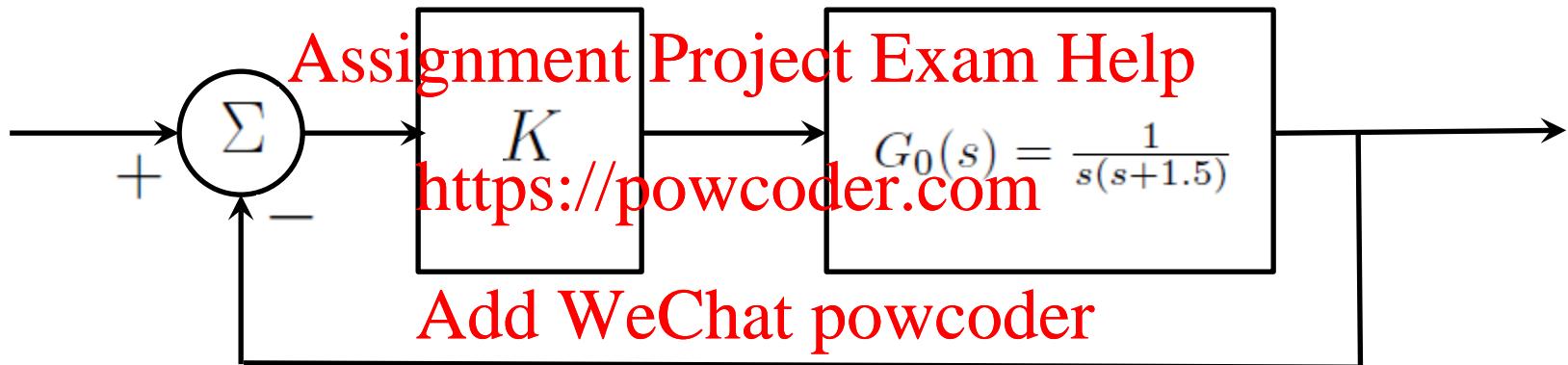
- Locations of closed loop poles depend on the parameter K:  
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$$1 + K \frac{B_0(s)}{A_0(s)} = 0 \quad \Leftrightarrow \quad A_0(s) + KB_0(s) = 0$$

# Motivation

(proportional controller)

- Plot the poles for this system for  $K>0$

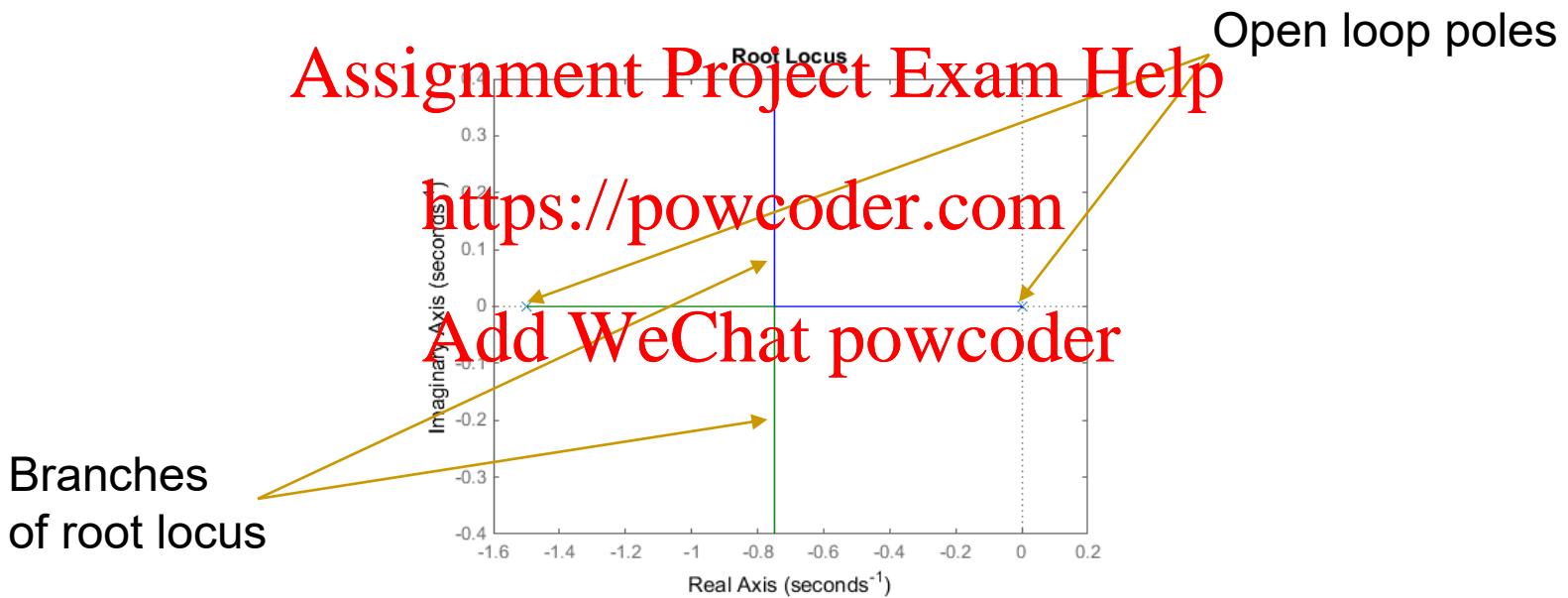


for all possible values of  $K>0$ . We have:

$$s^2 + 1.5s + K = 0 \quad \Rightarrow \quad s_{1,2} = \frac{-1.5 \pm \sqrt{1.5^2 - 4K}}{2}$$

# Motivation

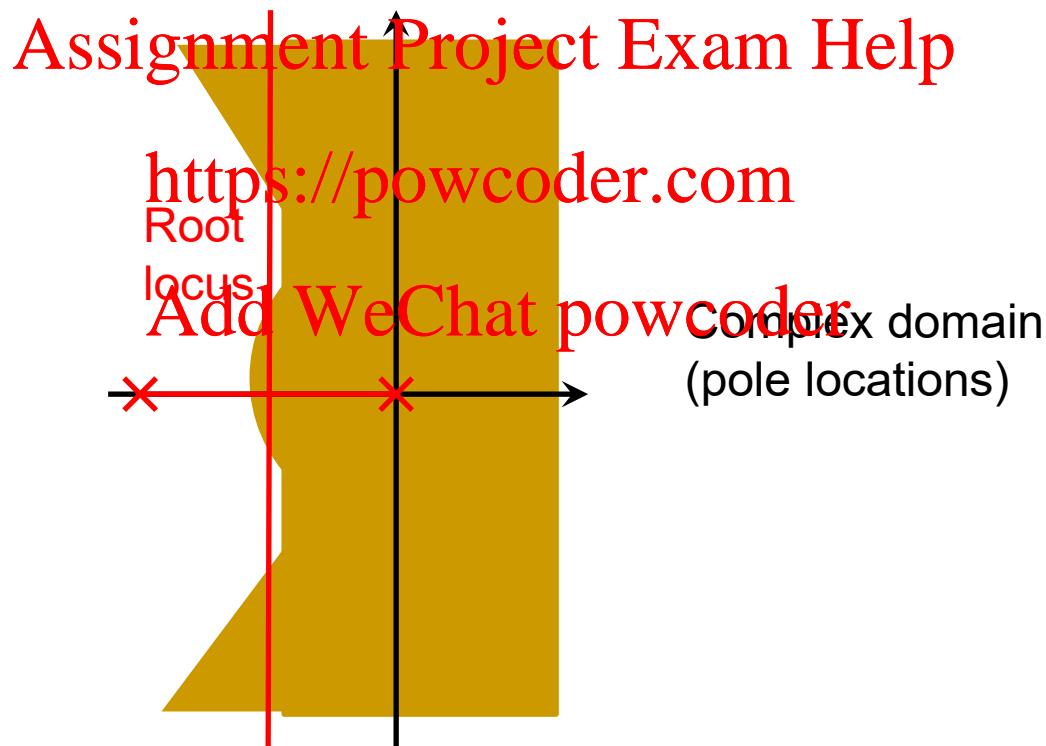
- We obtain a plot in complex plane



- This is a “root locus” for the given system.

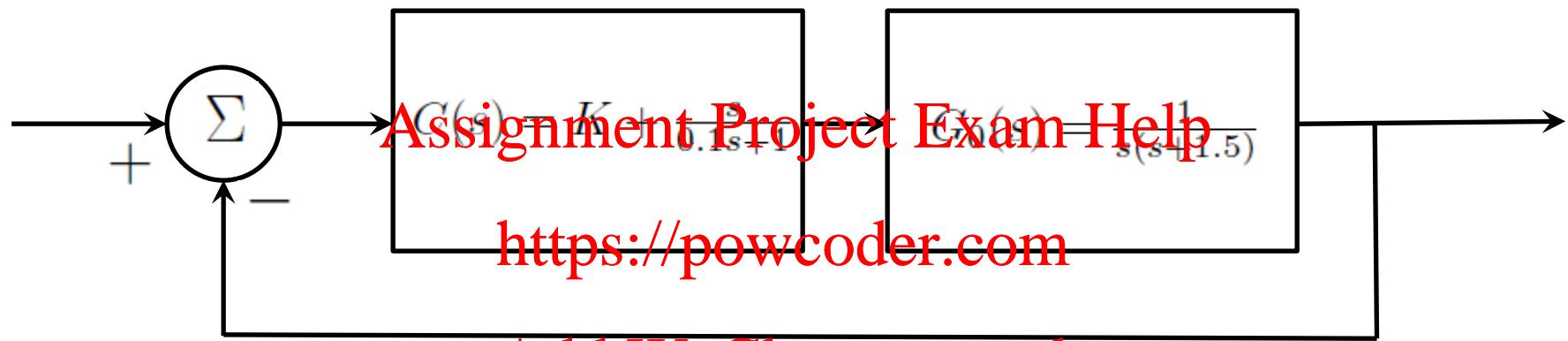
# A typical design question:

- Can all poles be “placed” into the desired region of complex plane by choosing K?



# Motivation:

arbitrary parameter in the controller



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Characteristic equation is:

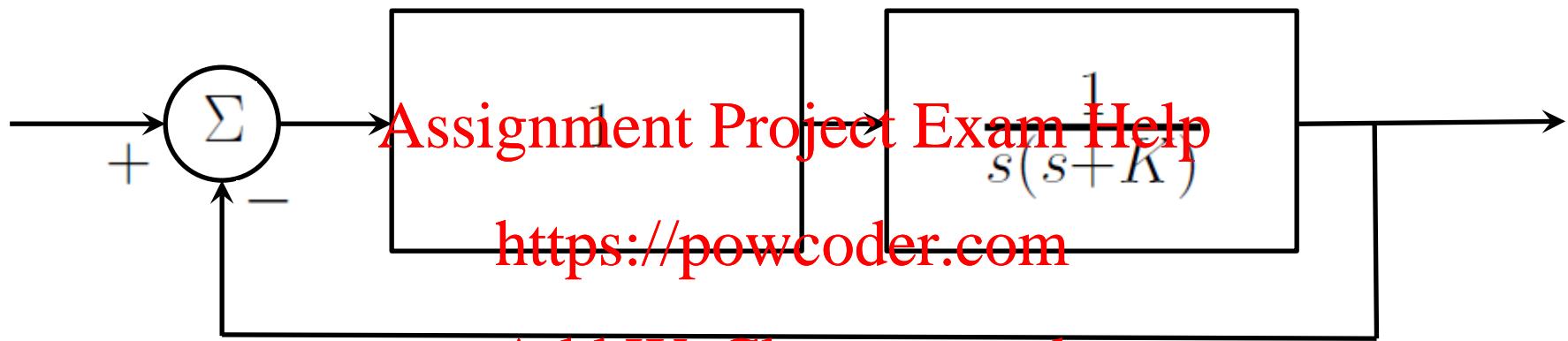
$$\underbrace{0.1s^3 + 1.15s^2 + 2.5s}_{D(s)} + K \underbrace{(0.1s + 1)}_{M(s)} = 0$$

$$1 + K \frac{M(s)}{D(s)} = 0$$

We will consider this example in detail later.

# Motivation:

## robustness analysis



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Characteristic equation is:

$$s^2 + Ks + 1 = \underbrace{(s^2 + 1)}_{D(s)} + K \underbrace{s}_{M(s)} = 0$$

$$1 + K \frac{M(s)}{D(s)} = 0$$

# Problem formulation:

- Plot in the complex s plane the locations of all roots of the equation

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$$1 + K \cdot F(s) = 0 \quad \text{where} \quad F(s) = \frac{M(s)}{D(s)} = \frac{\prod_{k=1}^m (s - \beta_k)}{\prod_{k=1}^n (s - \alpha_k)}$$

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as K varies from 0 to infinity.

- This plot is called the (positive) “root locus”.

# Phase and magnitude conditions

- Note that if a point  $s_0$  in the complex plane lays on the root locus, it has to satisfy

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$$1 + KF(s_0) = 0 \Leftrightarrow KF(s_0) = -1$$

which implies ~~Add WeChat powcoder~~ these conditions hold:

magnitude condition:  $|K \cdot F(s_0)| = 1$

phase condition:  $\angle K \cdot F(s_0) = (2l + 1)\pi \quad \text{for } l = 0, \pm 1, \pm 2, \dots$

# Phase condition

- Since K is positive, the phase depends only on poles and zeros of F(s). In other words, for any point  $s_0$  on the root locus, we have:

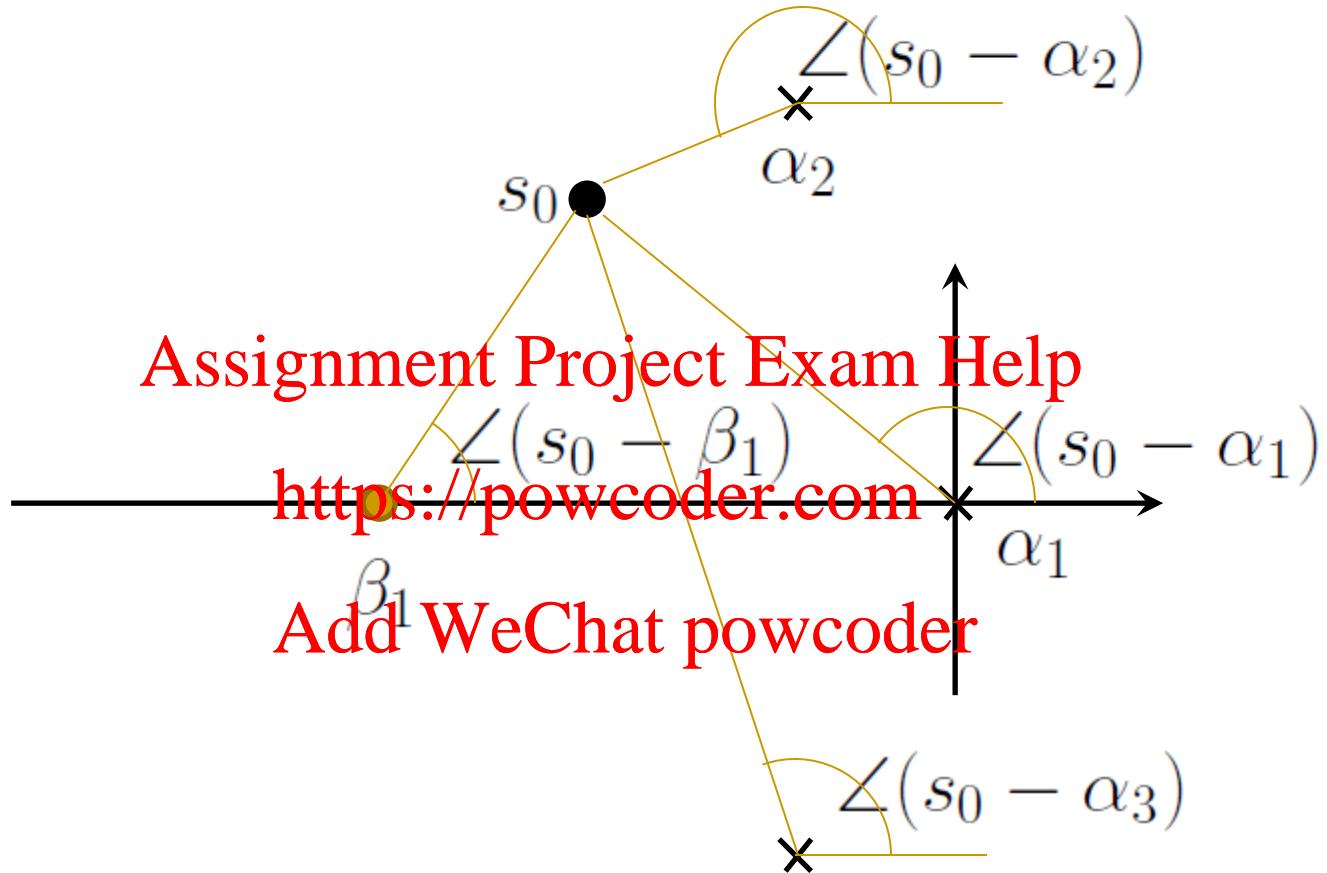
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$$(2l+1)\pi = \angle F(s_0) = \sum_{k=1}^m \angle(s_0 - \beta_k) - \sum_{k=1}^n \angle(s_0 - \alpha_k) \text{ for } l = 0, \pm 1, \pm 2, \dots$$

where

$$F(s) = \frac{M(s)}{D(s)} = \frac{\prod_{k=1}^m (s - \beta_k)}{\prod_{k=1}^n (s - \alpha_k)}$$

# Graphical interpretation



$$\begin{aligned}\angle(s_0 - \beta_1) - \angle(s_0 - \alpha_1) - \angle(s_0 - \alpha_2) - \angle(s_0 - \alpha_3) = \\(2l + 1)\pi, l = 0, \pm 1, \pm 2, \dots\end{aligned}$$

# Calculate K for a point on root locus

- The root locus is parameterized with the gain  $K > 0$ .  
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- If we want to calculate the value of K that corresponds to a specific point on the root locus, we can use the gain condition:

$$K = \frac{1}{|F(s_0)|}$$

# Sketching root locus via Matlab

- User can define a transfer function and then use the command “rlocus” to plot its root locus. [Assignment](#) [Project](#) [Exam](#) [Help](#)
- To understand <https://powcoder.com>, it is useful to learn how to sketch root locus by hand.  
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- It is useful to use the command “sgrid” to get a grid of lines with constant damping and constant natural frequencies.

# Root locus via Matlab

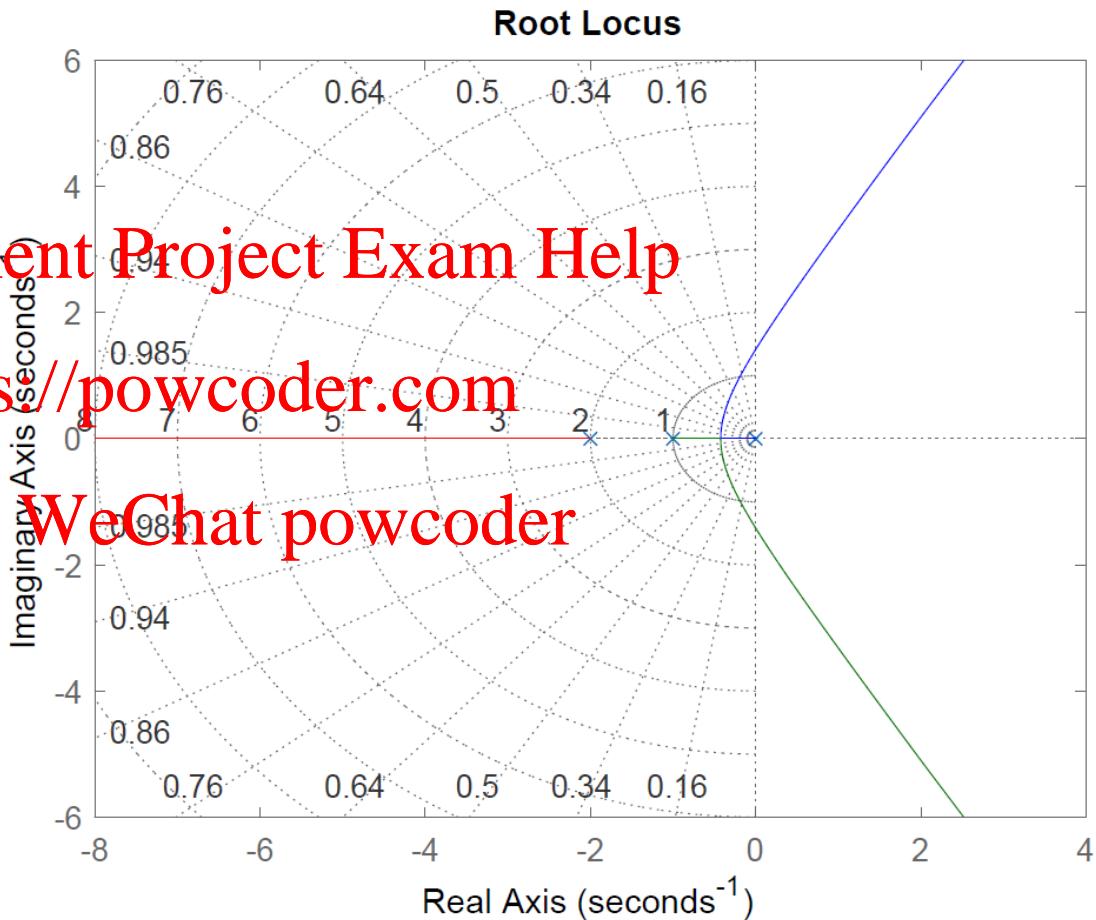
$$F(s) = \frac{1}{s^3 + 3s^2 + 2s} = \frac{1}{s(s+1)(s+2)}$$

```
>> rlocus([1],[1 3 2 0])  
>> sgrid
```

Lines of constant damping  
and constant natural frequency  
are plotted using “sgrid”

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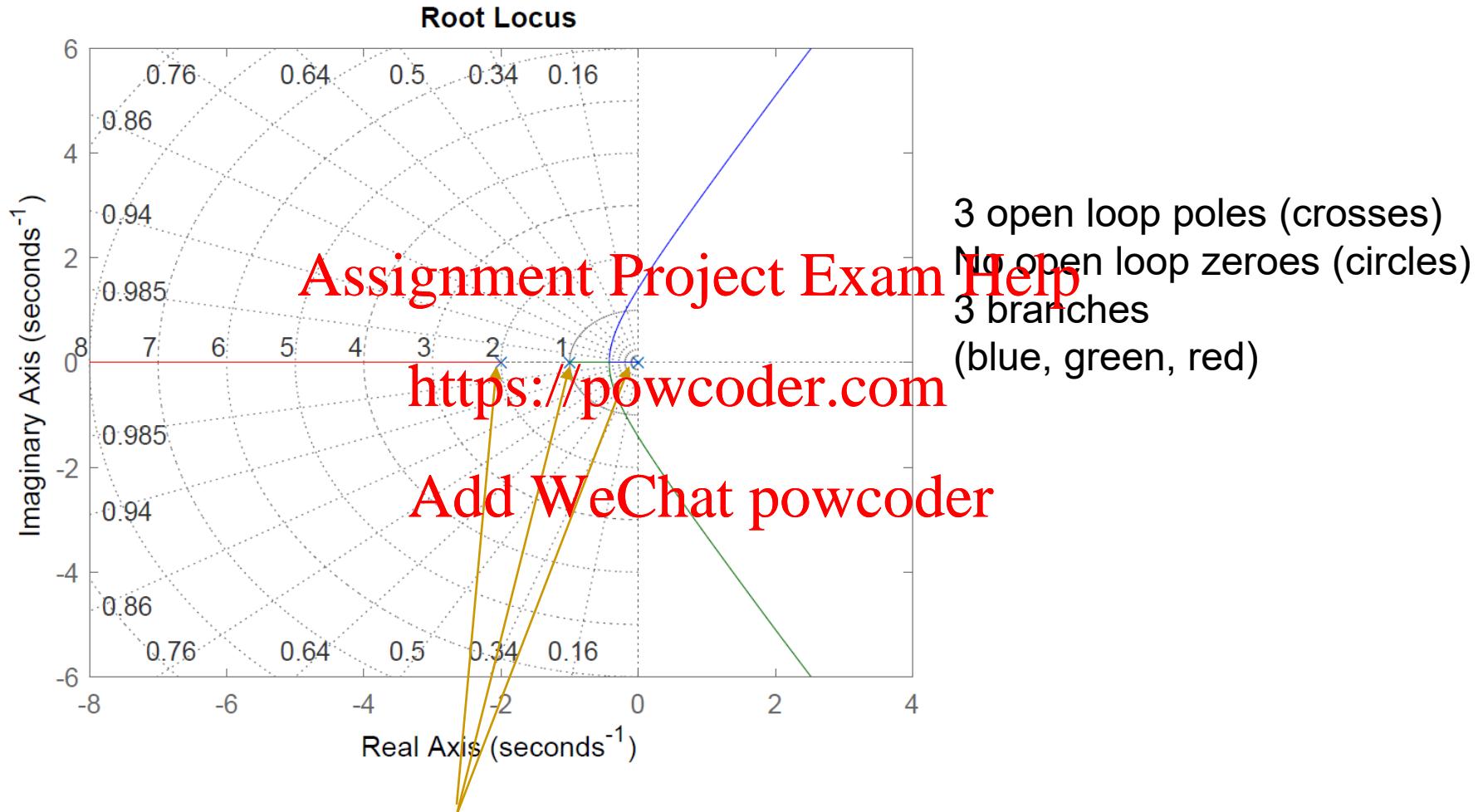
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# Main features of root locus

- Number of branches
- Open loop poles (starting points for K=0)
- Open loop zeros (limiting points for K infinity)
- Parts of real line that belong to root locus
- Asymptotes <https://powcoder.com> Add WeChat powcoder
- Breakaway point (branches intersect)
- Intersections with imaginary axis
- Angles of departure or arrival at poles/zeroes

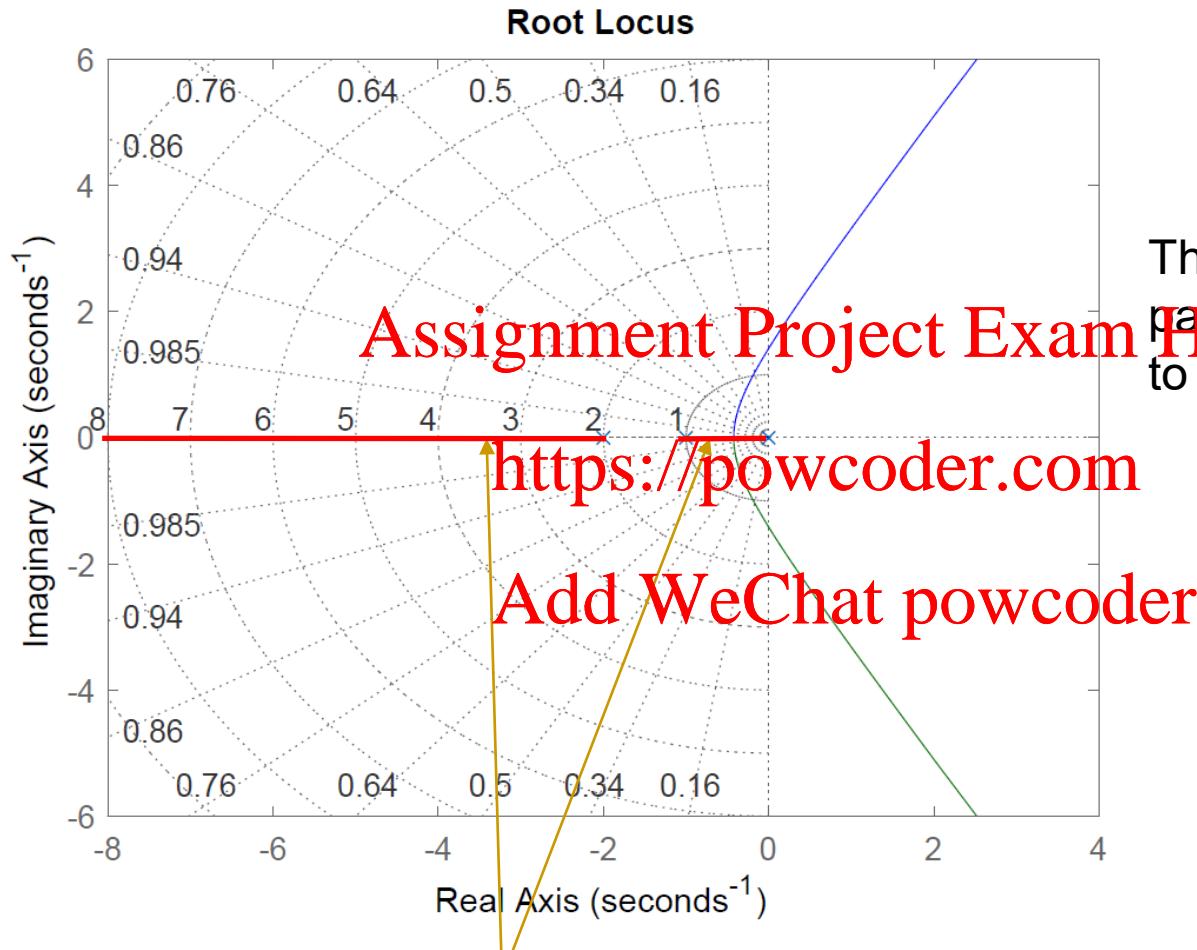
# Open loop poles/zeroes



Open loop poles are points where branches of root locus start from (small K).

Open loop zeroes are points where some branches of root locus converge to (large K).

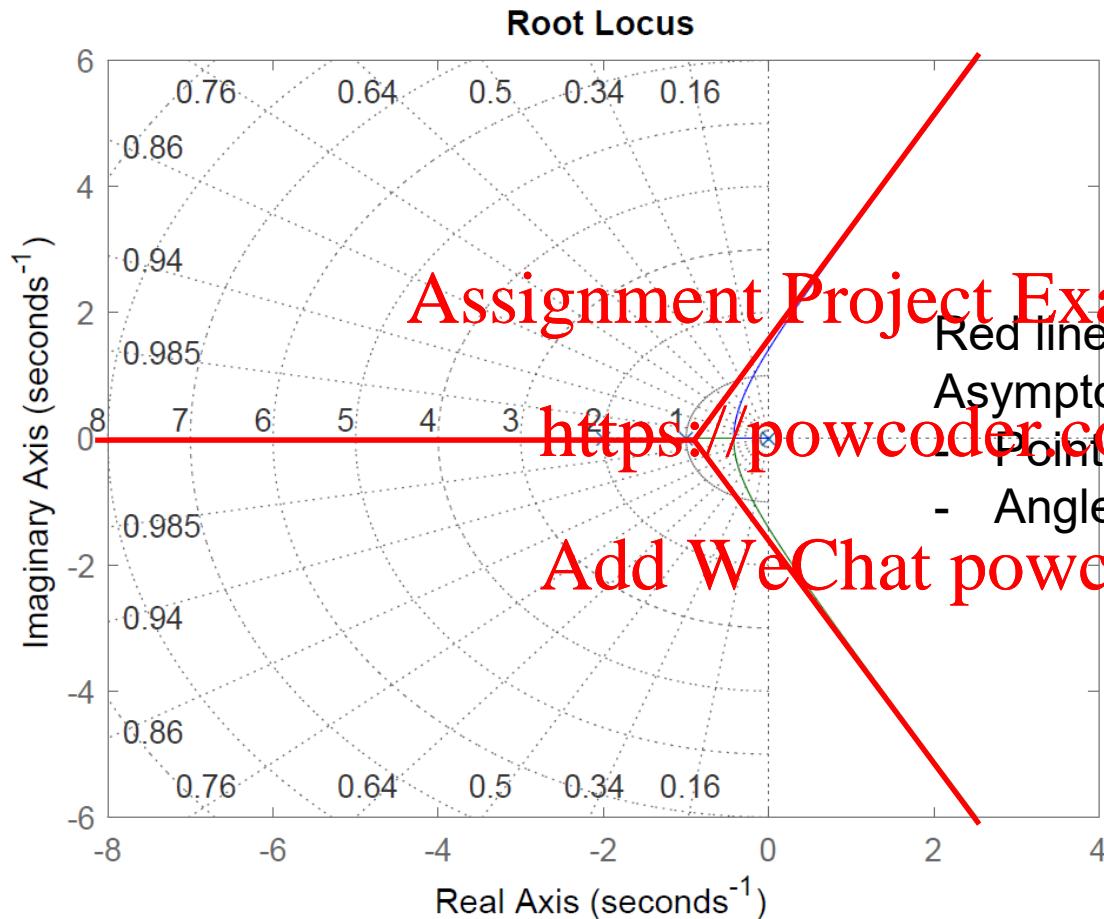
# Parts of real axis that belong to locus



Thick red line denotes parts of real axis that belong to root locus

These parts of real axis belong to root locus.

# Asymptotes of root locus



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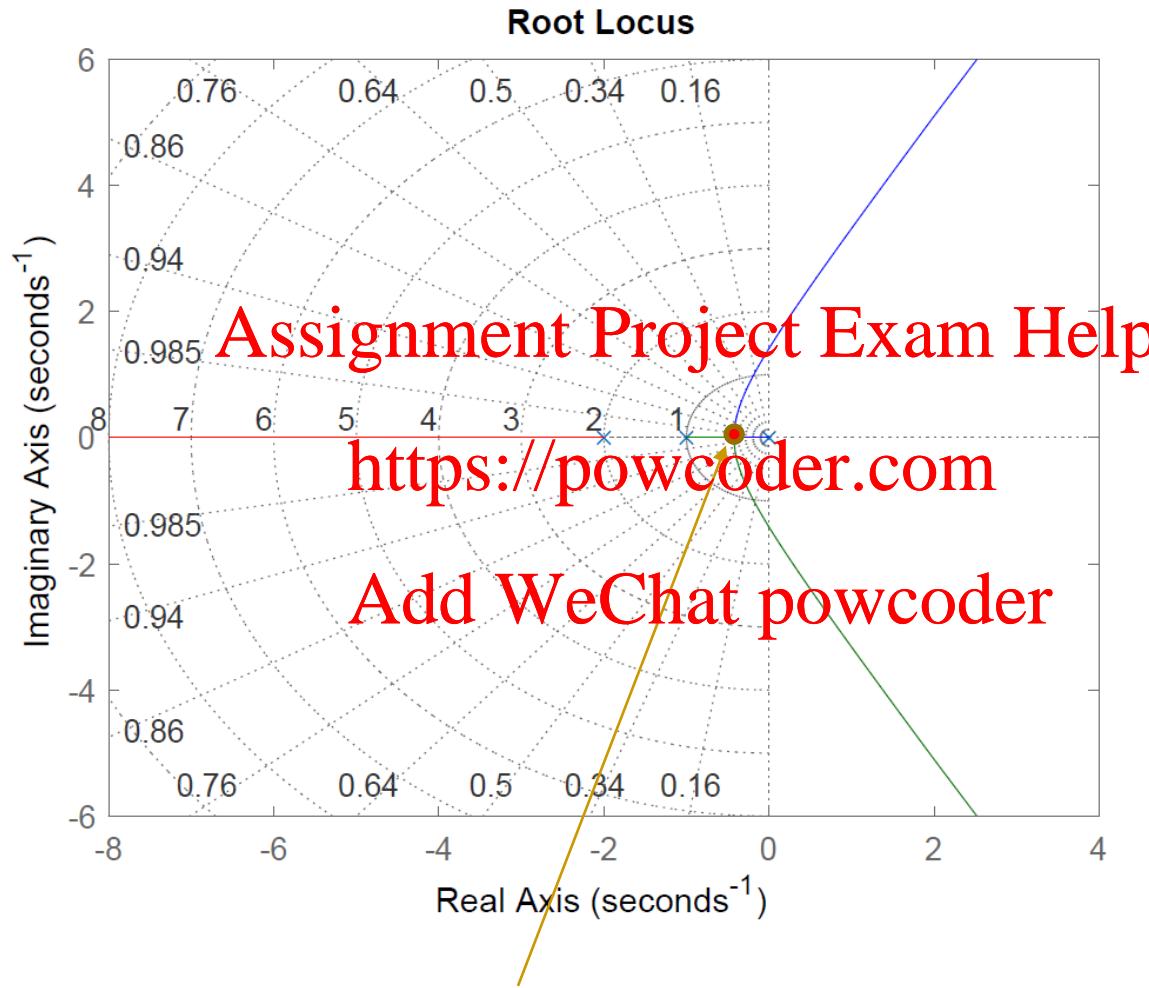
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Red lines denote 3 asymptotes

Asymptotes are determined by:

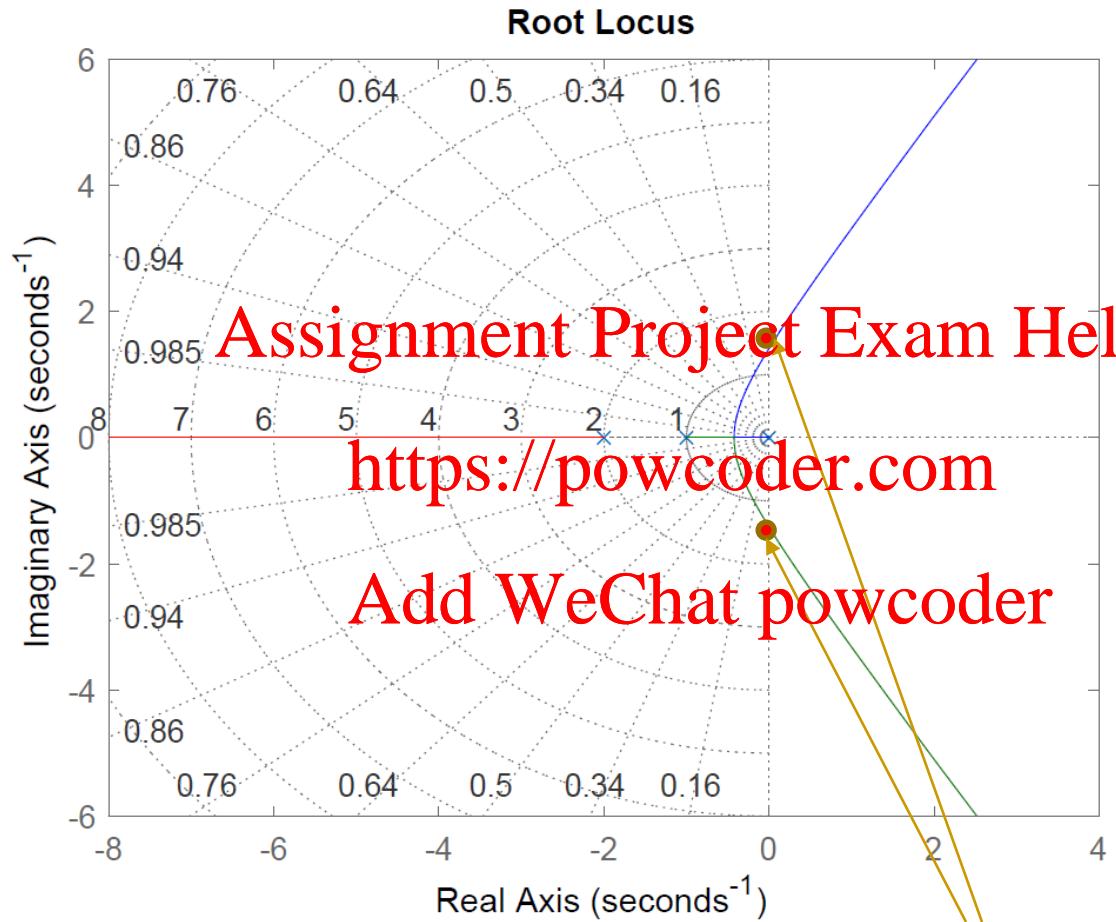
- Point where they intersect the real axis
- Angle with the positive real axis

# Breakaway point



Sometimes we have repeated complex poles. This is where several branches intersect.

# Intersections with imaginary axis



Intersections with imaginary axis.

# Angles of departure/arrival

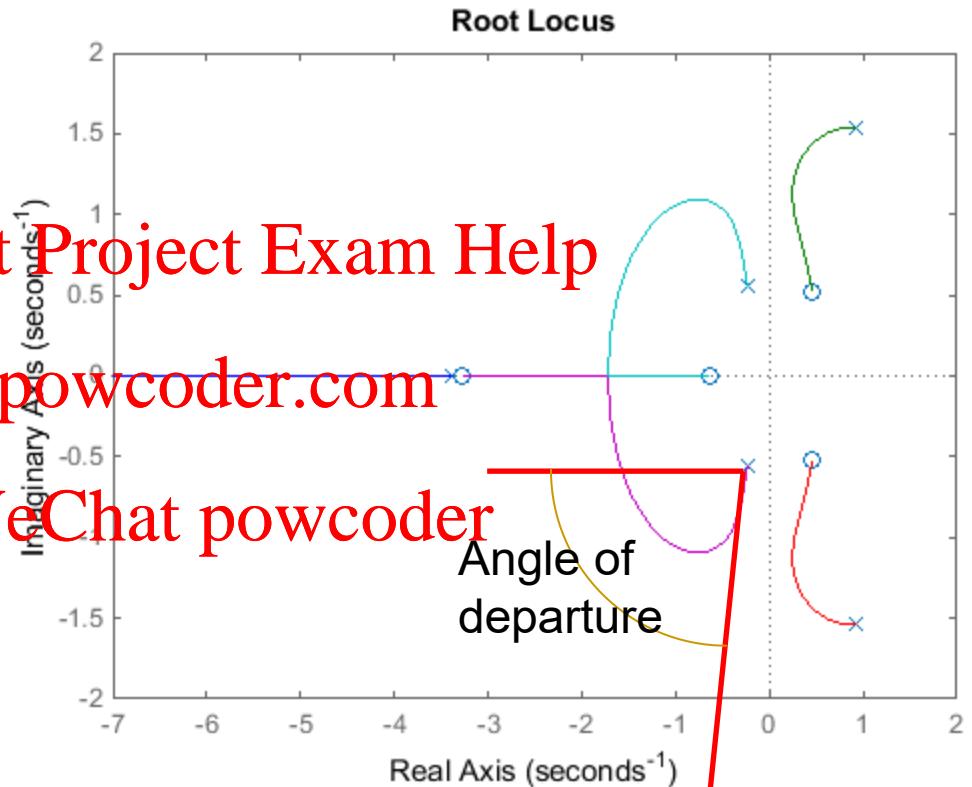
rlocus([1 3 -1 0 1],[1 2 -2 10 4 4])

$$F(s) = \frac{s^4 + 3s^3 - s^2 + 1}{s^5 + 2s^4 - 2s^3 + 10s^2 + 4s + 4}$$

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This system can not be stabilized by choosing K!