ELEN90055 Control Systems
Worksheet 2
Semester 2

Laplace transforms - key concepts

1. Calculating Laplace transform and inverse Laplace transform:

   Method 1. Using the following equations:

$$\mathscr{L}\{y(t)\} = Y(s) = \int_{0^-}^{\infty} e^{-st} y(t)dt$$
$$\mathscr{L}^{-1}\{Y(s)\} = y(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} e^{st} Y(s)ds$$

   Method 2. (*Preferred method*) Using the Laplace transform table and the *properties of Laplace transforms* table.[1]

   We might need to first perform a *partial fraction expansion* and then we find expressions that can be found in the Laplace transform table. There are two different ways to perform partial fraction expansion: (*i*) cross multiplication, (*ii*) the following direct formulas:

   i. *Partial fraction expansion for systems with distinct poles:*
   A transfer function $G(s)$ of the following general form where $m < n$ (which means $G(s)$ is strictly proper)

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = \frac{\sum\limits_{i=0}^{m} b_i s^i}{\prod\limits_{k=1}^{n} (s - \alpha_k)}$$

   can be expanded as

$$G(s) = \sum_{k=1}^{n} \frac{B_k}{s - \alpha_k},$$

   where the residues B_k are computed as

$$B_k = \lim_{s \to \alpha_k} (s - \alpha_k) G(s).$$

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[1] These tables can be found in almost all linear control textbooks. For example, see the attached tables from "Modern Control Engineering" by K. Ogata, 5th edition.

ii. *Partial fraction expansion (general formula)*:
If the transfer function has repeated poles, $G(s)$ can be written as (we continue to assume $m < n$)

$$G(s) = \frac{\sum\limits_{i=0}^{m} b_i s^i}{\prod\limits_{k=1}^{n} (s - \alpha_k)^{r_k}}.$$

Then the partial fraction expansion representation is

$$G(s) = \sum_{k=1}^{n} \sum_{\ell=1}^{r_k} \frac{B_{k\ell}}{(s - \alpha_k)^{\ell}},$$

where the residues $B_{k\ell}$ are computed as

$$B_{k\ell} = \frac{1}{(r_k - \ell)!} \lim_{s \to \alpha_k} \left(\frac{d^{r_k - \ell}}{ds^{r_k - \ell}} \left[(s - \alpha_k)^{r_k} G(s) \right] \right).$$

1 Tutorial problems

1. Find the Laplace transform of the following functions:

   (a*) $f(t) = 1 + 2t + t^2 + \delta(t)$

   (b*) $f(t) = (t+3)^2$

   (c*) $f(t) = t\sin t$
   Hint: use the *multiplication by time property* of Laplace transform.

   (d) $f(t) = \sin^2 t$

   (e) $f(t) = e^{-t}\sin 2t$

   You can check your answers to the above problems using the 'laplace' MATLAB command. For example, for part (d) you can use the following MATLAB code:

   $>>$ syms t
   $>>$ laplace$(sin(t)^2)$

   You may also need to use the 'simplify' command in MATLAB.

2. Find the time signal $f(t)$ corresponding to the following Laplace transforms:

   (a*) $F(s) = \frac{1}{s(s+1)}$

   (b*) $F(s) = \frac{s^2}{s^2+1}$

   (c*) $F(s) = \frac{3s+2}{s^2+4s+20} = \frac{3s+2}{(s+2)^2+16}$

   (d*) $F(s) = \frac{s^2+2s+3}{(s+1)^3}$

   (e) $F(s) = \frac{1}{s^4-1} = \frac{1}{(s-1)(s+1)(s^2+1)}$
   Hint: Perform partial fraction expansion.

   (f) $F(s) = \frac{s^2-1}{(s^2+1)^2}$
   Hint: First find the inverse Laplace transform of $G(s) = \frac{s}{s^2+1}$ and then use the *multiplication by time property of Laplace transform*: $\frac{d}{ds}G(s) = -\mathscr{L}\{tg(t)\}$. Verify your answer using MATLAB:

   $>>$ syms s
   $>>$ ilaplace$((s^2-1)/(s^2+1)^2)$

3. Solve the following differential equations using Laplace transforms:

   (a*) $\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; \qquad y(0) = 1, \quad \dot{y}(0) = 2$

   (b) $\ddot{y}(t) + \dot{y}(t) = \sin t; \qquad y(0) = 1, \quad \dot{y}(0) = 2$
   Hint: make sure the transfer function is strictly proper before doing partial fraction expansion.

4. Find $\lim_{t\to\infty} f(t)$ for functions with the following Laplace transforms:

   (a) $F(s) = \frac{10}{s(s+1)(s+2)}$

   (b) $F(s) = \frac{2}{s(s-2)(s+2)}$
   Hint: You may solve this problem using two different methods: (*i*) first find $f(t)$ and then find $\lim_{t\to\infty} f(t)$. (*ii*) use the final value theorem, but make sure the conditions of the theorem are satisfied.