ELEN90055 Control Systems
Worksheet 1
Semester 2

Instructions

This worksheet covers materials about *Equilibria and Linearisation* from lectures 5 and 6.

Review of key concepts

Consider a system described by its model:

$$\ell\left(\frac{d^n y}{dt^n}, \ldots, \frac{dy}{dt}, y, \frac{d^n u}{dt^n}, \ldots, \frac{du}{dt}, u\right) = 0 \ . \tag{1}$$

Equilibria are calculated by considering:

$$\ell\left(0, \ldots, 0, \bar{y}, 0, \ldots, 0, \bar{u}\right) = 0 \ . \tag{2}$$

To perform linearisation, we find it convenient to consider the multivariate function:

$$\ell(y_n, \ldots, y_0, u_n \ldots, u_0) \ .$$

We obtain the linearisation of the above equation around the chosen equilibrium $(0, \ldots, 0, \bar{y}, 0, \ldots, 0, \bar{u})$:

$$\begin{aligned}
&\ell_{lin}\left(y_n, \ldots y_1, y_0 - \bar{y}, u_n \ldots, u_1, u_0 - \bar{u}\right) \\
&= \left.\frac{\partial \ell}{\partial y_n}\right|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})} \cdot (y_n - 0) + \ldots + \left.\frac{\partial \ell}{\partial y_1}\right|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})} \cdot (y_1 - 0) \\
&\quad + \left.\frac{\partial \ell}{\partial y_0}\right|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})} \cdot (y_0 - \bar{y}) + \left.\frac{\partial \ell}{\partial u_n}\right|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})} \cdot (u_n - 0) + \ldots \\
&\quad + \left.\frac{\partial \ell}{\partial u_1}\right|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})} \cdot (u_1 - 0) + \left.\frac{\partial \ell}{\partial u_0}\right|_{(0,\ldots,0,\bar{y},0,\ldots,0,\bar{u})} \cdot (u_0 - \bar{u}).
\end{aligned} \tag{3}$$

Introducing the incremental variables $\delta_y := y - \bar{y}, \delta_u := u - \bar{u}$, and noting that $\frac{d^i \delta_y}{dt^i} = \frac{d^i y}{dt^i}$ and $\frac{d^i \delta_u}{dt^i} = \frac{d^i u}{dt^i}$, the linearised model is given by:

$$\ell_{lin}\left(\frac{d^n \delta_y}{dt^n}, \ldots, \frac{d\delta_y}{dt}, \delta_y, \frac{d^n \delta_u}{dt^n}, \ldots, \frac{d\delta_u}{dt}, \delta_u\right) = 0 \ .$$

1 Tutorial problems

1. Consider the water tank from Lecture 5. The model of the change of water level in the tank can be written as:

$$\frac{dV}{dt} = \frac{d(Ax)}{dt} = A\frac{dx}{dt} = q_i - q_o \ , \tag{4}$$

   where x is the level of water in the tank, A is the area of the cross-section of the tank (assumed constant) and q_i and q_o are the volumetric flows into and out of the tank respectively. q_i is assumed to be the input to the system and x the output. Moreover, suppose that the outflow is given by the following relation:

$$q_o = k\sqrt{x} \ . \tag{5}$$

   Find all equilibria and linearise the system around the equilibrium where $x = 1$ $[m]$; assume that $A = 1$ $[m^2]$ and $k = 2$ $[m^{5/2}/s]$.

2. Consider a mass m sliding on a horizontal surface and attached to a vertical surface through a spring, see Figure 1. The mass is subjected to an external force F (the input to the system), the spring force and a friction force F_f. We define y as the displacement from a reference position (the output of the system). We assume that $F_f = c\dot{y}$ and we will consider two models for the spring force:

   - softening spring model $F_{sp} = k(1 - a^2y^2)y, |ay| \leq 1$;
   - hardening spring model $F_{sp} = k(1 + a^2y^2)y$,

   which are typically valid for large displacements y and for different types of springs. Newton's law of motion is given by:

$$m\ddot{y} + F_f + F_{sp} = F \ .$$

   Assume that $m = 1kg, c = 1kg/s, k = 1N/m, a = 10$.

   - Find all equilibria for the system with softening spring. Linearise the system around equilibrium for which $y = 0.01m$;
   - Find all equilibria for the system with hardening spring. Linearise the system around equilibrium for which $y = -0.01m$.

3. Consider the *Lotka-Volterra* or *predator-prey* equations, which model the change in predator and prey populations over time in a closed ecology:

$$\begin{aligned}\dot{x} &= \alpha x - \beta xy \\ \dot{y} &= \delta xy - \gamma y \ ,\end{aligned}$$

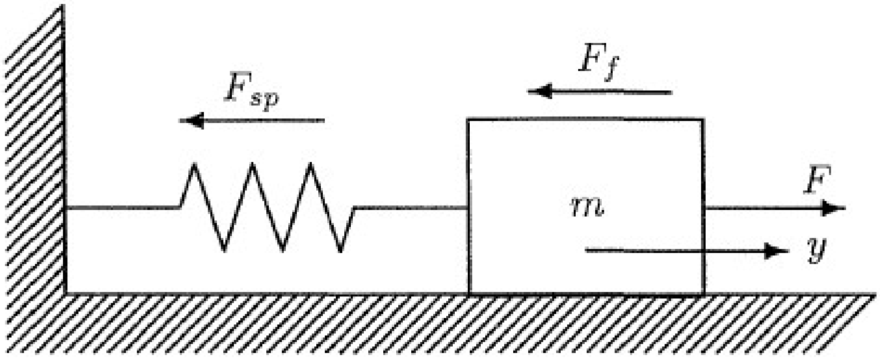

Figure 1: A mass-spring system.

where $\alpha, \beta, \gamma, \delta$ are constants that depend on the particular predator-prey system, x denotes the number of prey (e.g. rabbits) and y denotes the number of predators (e.g. foxes). Find all equilibria of this system and discuss the solution. Suppose that $\alpha = 2/3, \beta = 4/2, \gamma = \delta = 1$. Linearise the system around each equilibrium.