

Lecture 11

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Sensitivity transfer functions

Outline

- Motivation
- Examples
- Sensitivity transfer functions
- Steady state errors
- Conclusions

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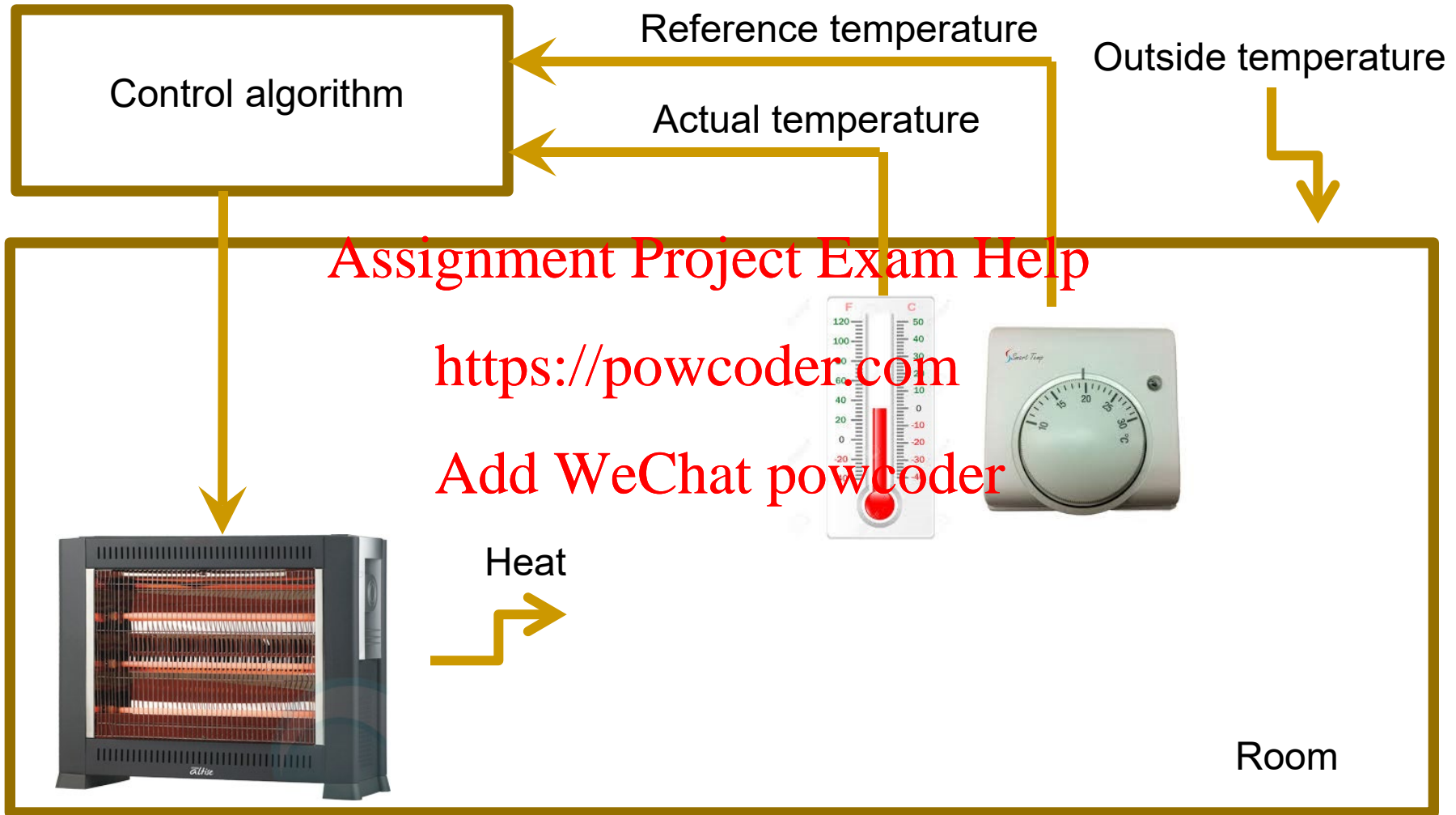
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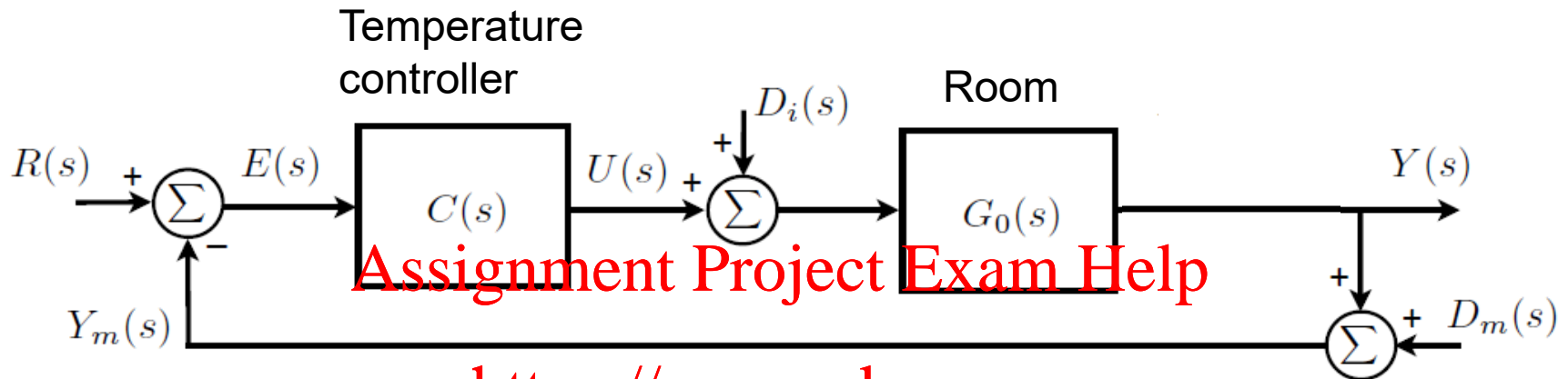
Motivation

- Any control system typically has a number of different inputs (reference inputs, different disturbances)
- We want output to be sensitive to the reference, but also insensitive to disturbances entering the loop.
- Effects of references and disturbances on signals in the loop captured by a gang of four *sensitivity transfer functions*.

Example (room temperature control)



Example (room temperature)

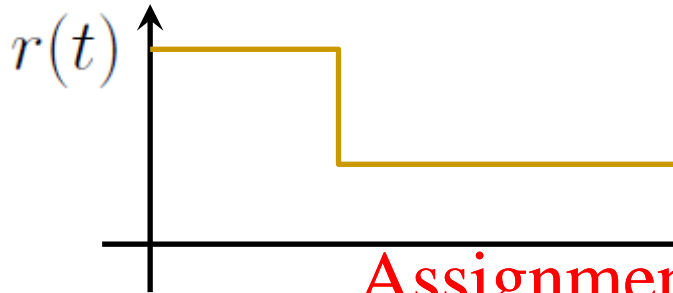


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- Reference signal = desired temperature $R(s)$
- Input disturbance = heat dissipation (outside temperature, opening doors & windows) $D_i(s)$
- Measurement noise $D_m(s)$

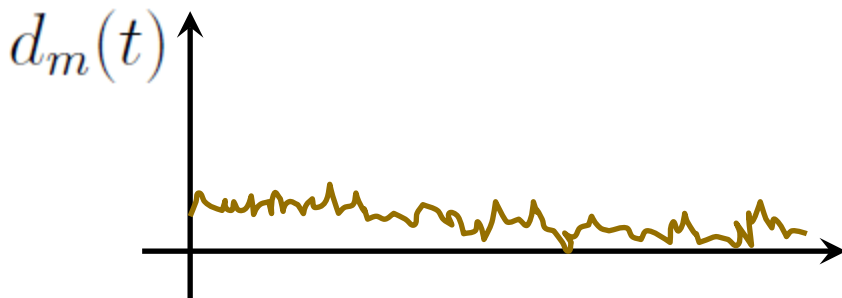
Example (room temperature)



Reference is typically a piecewise constant signal.



Input disturbance is typically a slowly varying large signal (low frequency, large amplitude).



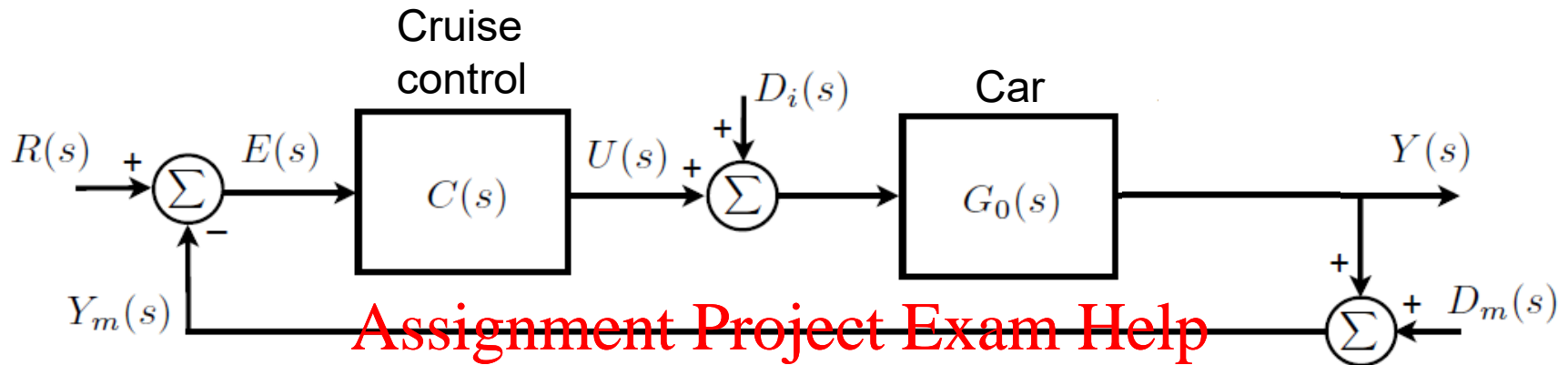
Measurement noise is typically a fast varying small signal (high frequency, small amplitude).

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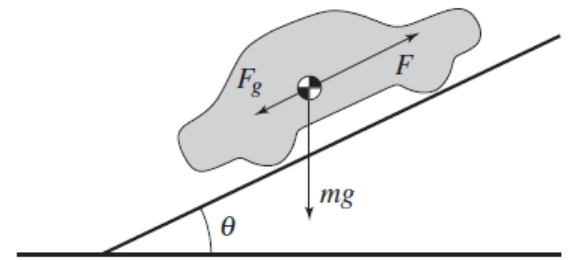
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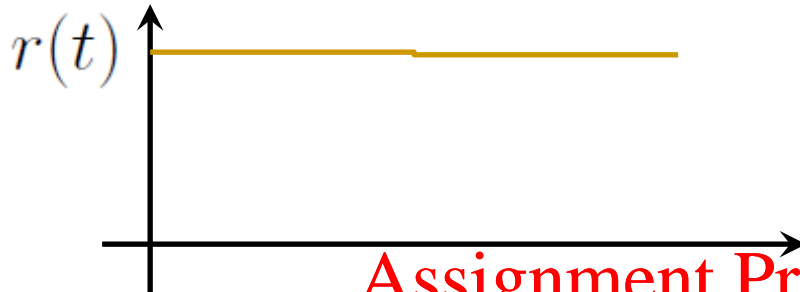
Example (cruise control)



- Reference signal = desired velocity $R(s)$
- Input disturbance = slope of the road $D_i(s)$
- Measurement noise $D_m(s)$

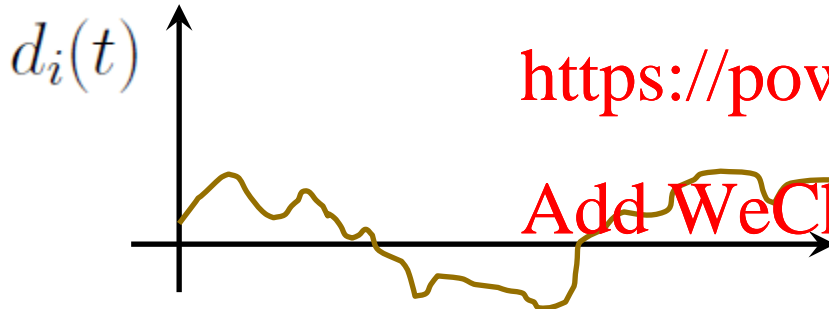


Example (cruise control)



Reference is typically a constant signal.

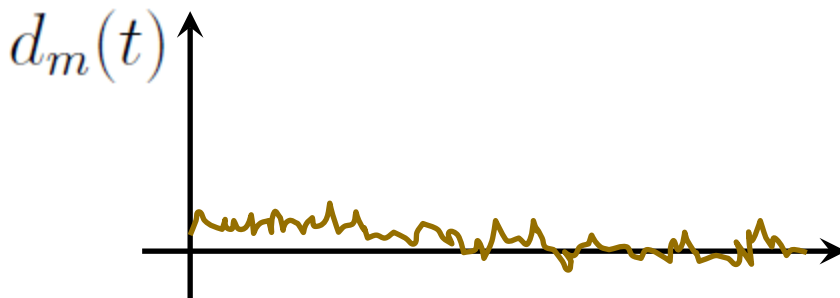
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Input disturbance is typically a slowly varying large signal (low frequency, large amplitude).

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Measurement noise is typically a fast varying small signal (high frequency, small amplitude).

Comments

- Examples suggest that different control systems share similar structure and inputs possess similar properties.

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- We now consider a general closed loop and analyse the effect of various inputs on the output.

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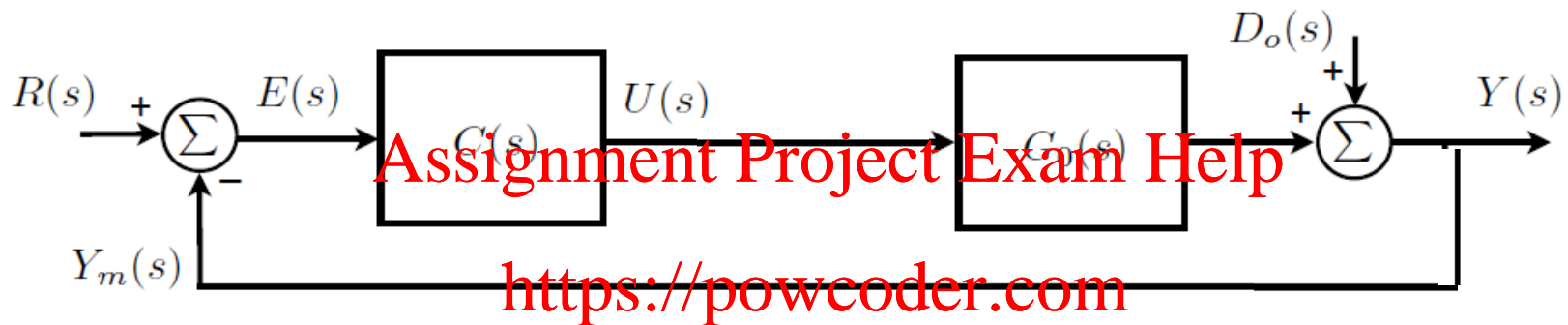
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Sensitivity transfer functions

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Recall:

Turn off one input and compute the transfer function in the usual manner for the other input.



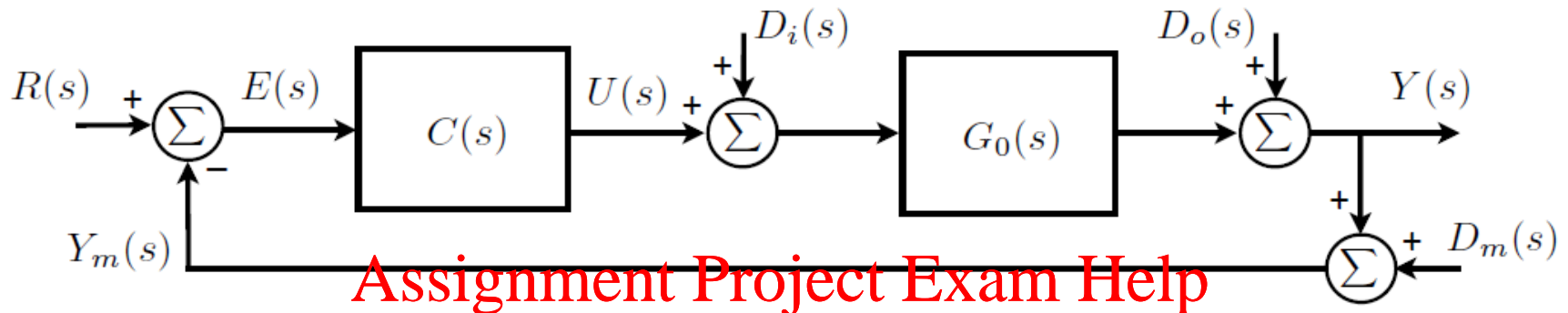
$$Y(s) = T_0(s)R(s) + S_0(s)D_o(s)$$

$$T_0(s) = \frac{Y(s)}{R(s)} = \frac{G_0(s)C(s)}{1 + G_0(s)C(s)} \quad \text{with only } r \neq 0$$

$$S_0(s) = \frac{Y(s)}{D_o(s)} = \frac{1}{1 + G_0(s)C(s)} \quad \text{with only } d_o \neq 0$$

This is how we will compute “sensitivity functions”.

Typical block diagram



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$$G_0(s) = \frac{B_0(s)}{A_0(s)} \text{ nominal plant transfer function,}$$

$$C(s) = \frac{P(s)}{L(s)} \text{ controller transfer function}$$

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$\{B_0(s), A_0(s)\}$ and $\{P(s), L(s)\}$ coprime (no common roots) polynomial pairs

$D_i(s)$, $D_o(s)$, $D_m(s)$ plant input, plant output and measurement noise uncertainty

$R(s)$, $U(s)$, $Y(s)$ reference, control, and plant output signals

Coprime polynomials

- Transfer function with coprime polynomials

$$A_0(s), B_0(s)$$

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$$G_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{s-3}{(s+1)(s+2)} = \frac{s-3}{s^2+3s+2}$$

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- Transfer function with polynomials $A_0(s), B_0(s)$ that are NOT coprime

$$G_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{s+1}{(s+1)(s+2)} = \frac{s+1}{s^2+3s+2}$$

Possible performance specifications:

- It would be good if the output follows the reference (despite disturbances); i.e. the error should be very sensitive to the reference input.

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- It would be good if disturbance not to affect the error for any value of reference; i.e. the error should be insensitive to the disturbance input.

Transfer functions:

- From the block diagram we have

$$Y(s) = D_0(s) + G_0(s)(U(s) + D_i(s)) \quad (1)$$

$$U(s) = C(s)(R(s) - D_m(s) - Y(s)) \quad (2)$$

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- Next we will show that there exist “sensitivity” transfer functions so that we can write:

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$$Y(s) = T_0(s)R(s) + S_0(s)D_o(s) + S_{i0}D_i(s) - T_0(s)D_m(s)$$

$$U(s) = S_{u0}(s)(R(s) - D_m(s) - D_o(s) - G_0(s)D_i(s))$$

Transfer functions with Y as output

- We substitute (2) into (1) – i.e. eliminate U:

$$Y = D_0 + G_0 C(R - D_m - Y) + G_0 D_i$$

$$\underbrace{Y}_{\text{Assignment Project Exam Help}} = D_0 + G_0 C R - G_0 C D_m - G_0 C Y + G_0 D_i$$

$$Y(1 + G_0 C) = \underbrace{G_0 C R}_{\text{https://powcoder.com}} + D_0 + G_0 D_i - G_0 C D_m$$

$$Y(s) = \underbrace{\frac{G_0(s)C(s)}{1 + G_0(s)C(s)}}_{T_0(s)} R(s) + \underbrace{\frac{1}{1 + G_0(s)C(s)}}_{S_0(s)} D_0(s) + \underbrace{\frac{G_0(s)}{1 + G_0(s)C(s)}}_{S_{i0}(s)} D_i(s) - \underbrace{\frac{G_0(s)C(s)}{1 + G_0(s)C(s)}}_{T_0(s)} D_m(s)$$

Transfer functions with U as output

- We substitute (1) into (2) – i.e. eliminate Y. Please do this as an exercise! We obtain:

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- $$U(s) = \frac{C(s)}{1+C(s)G_0(s)} (R(s) - D_m(s) - D_o(s)) - \frac{C(s)G_0(s)}{1+C(s)G_0(s)} D_i(s)$$

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- NB: Another approach to getting Y(s) or U(s)
 1. Suppress all inputs except one
 2. Redraw the loop in unity or general feedback form. Use the corresponding transfer function to get Y(s) or U(s) in terms of that input.
 3. Use linear superposition to sum up the contributions from all inputs.

“Sensitivity” transfer functions:

- For nominal plant $G_0(s) = \frac{B_0(s)}{A_0(s)}$ and controller

$C(s) = \frac{P(s)}{L(s)}$ we can define:

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$$T_0(s) \doteq \frac{G_0(s)C(s)}{1 + G_0(s)C(s)} = \frac{B_0(s)P(s)}{A_0(s)L(s) + B_0(s)P(s)} \quad \text{complementary sensitivity}$$

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$$S_0(s) \doteq \frac{1}{1 + G_0(s)C(s)} = \frac{A_0(s)L(s)}{A_0(s)L(s) + B_0(s)P(s)} \quad \text{(output) sensitivity}$$

$$S_{i0}(s) \doteq \frac{G_0(s)}{1 + G_0(s)C(s)} = \frac{B_0(s)L(s)}{A_0(s)L(s) + B_0(s)P(s)} \quad \text{input-disturbance sensitivity}$$

$$S_{u0}(s) \doteq \frac{C(s)}{1 + G_0(s)C(s)} = \frac{A_0(s)P(s)}{A_0(s)L(s) + B_0(s)P(s)} \quad \text{control sensitivity}$$

Algebraic relationships:

- Sensitivities are algebraically related:

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$$S_0(s) + T_0(s) = 1, \quad S_{i0}(s) = G_0(s)S_0(s) = T_0(s)/C(s),$$

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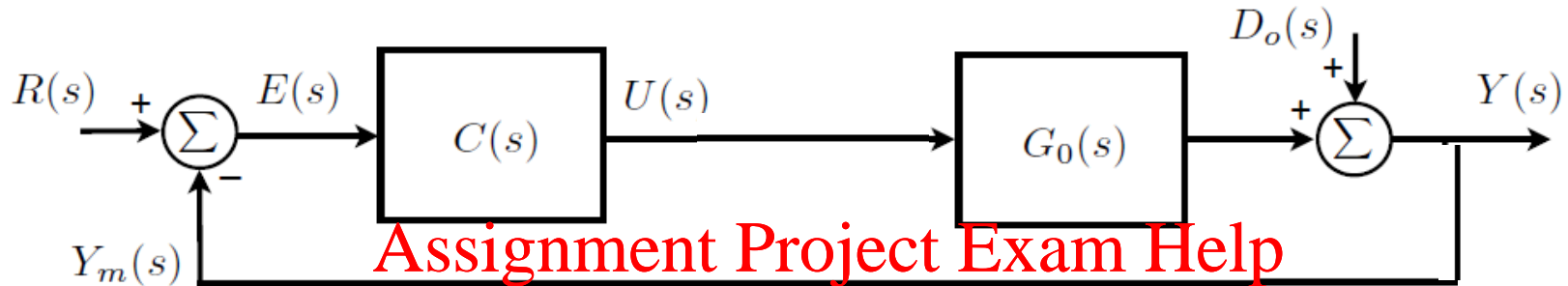
$$S_{u0}(s) = S_0(s)C(s) = T_0(s)/G_0(s)$$

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Special case: two inputs

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Two inputs:



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$$Y(s) = T_0(s)R(s) + S_0(s)D_o(s)$$

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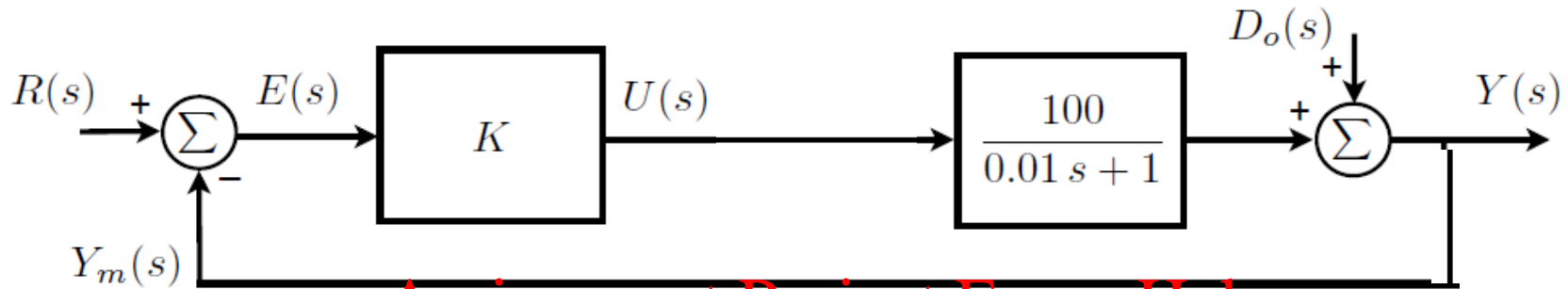
$$T_0(s) = \frac{Y(s)}{R(s)} = \frac{G_0(s)C(s)}{1 + G_0(s)C(s)} \quad \text{with only } r \neq 0$$

$$S_0(s) = \frac{Y(s)}{D_o(s)} = \frac{1}{1 + G_0(s)C(s)} \quad \text{with only } d_o \neq 0$$

Important constraint:

- $C(s)$ is the only degree of freedom for “shaping” the two transfer functions.
- If we shape one transfer function (e.g. for good tracking), this fixes the other (e.g. for disturbance rejection).
- This may lead to a trade-off between:
 - Good tracking;
 - Good disturbance rejection.

Example:



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$$G_0(s) = \frac{100}{0.01s+1} \quad C(s) = K$$

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$$Y(s) = T_0(s)R(s) + S_0(s)D_o(s)$$

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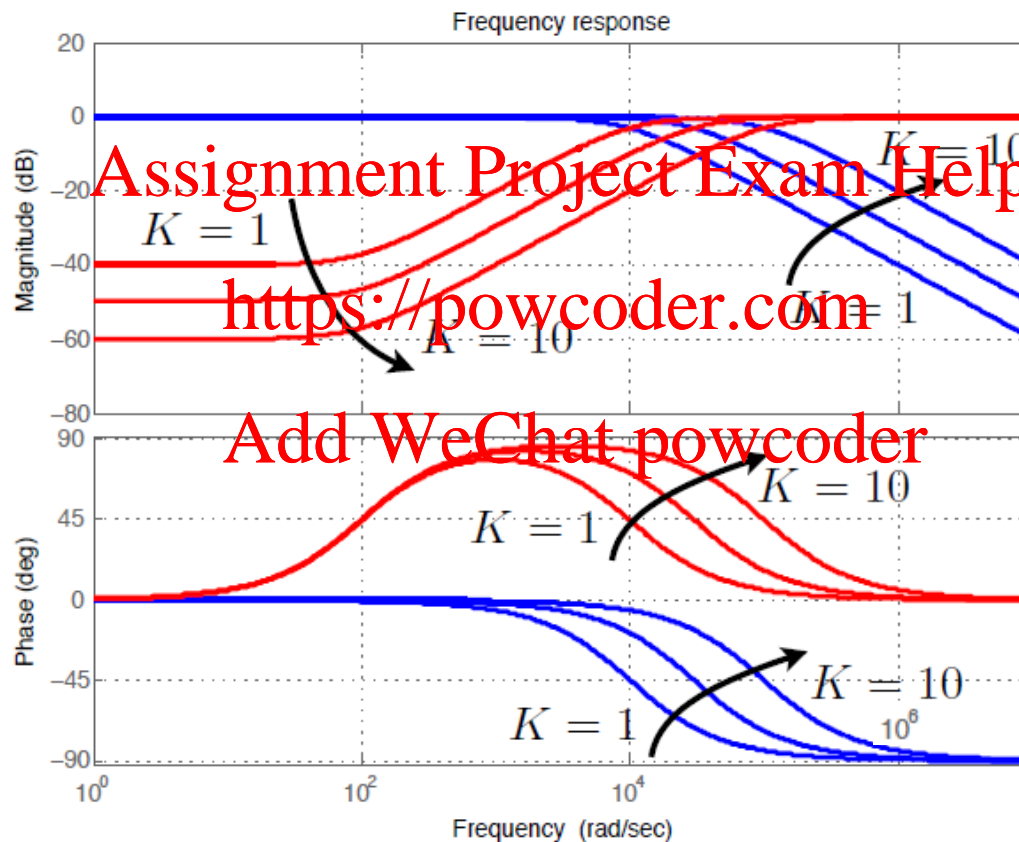
$$S_0(s) = \frac{1}{1+G_0(s)C(s)} = \frac{1}{1+\frac{100K}{0.01s+1}} = \frac{0.01s+1}{0.01s+1+100K}$$

$$T_0(s) = \frac{G_0(s)C(s)}{1+G_0(s)C(s)} = \frac{\frac{100K}{0.01s+1}}{1+\frac{100K}{0.01s+1}} = \frac{100K}{0.01s+1+100K}$$

Bode diagram

$$\frac{1}{1 + G_0 C}$$

increasing the controller gain K reduces the 'sensitivity' of the controlled output to *low* frequency disturbances at the plant output



$$\frac{G_0 C}{1 + G_0 C}$$

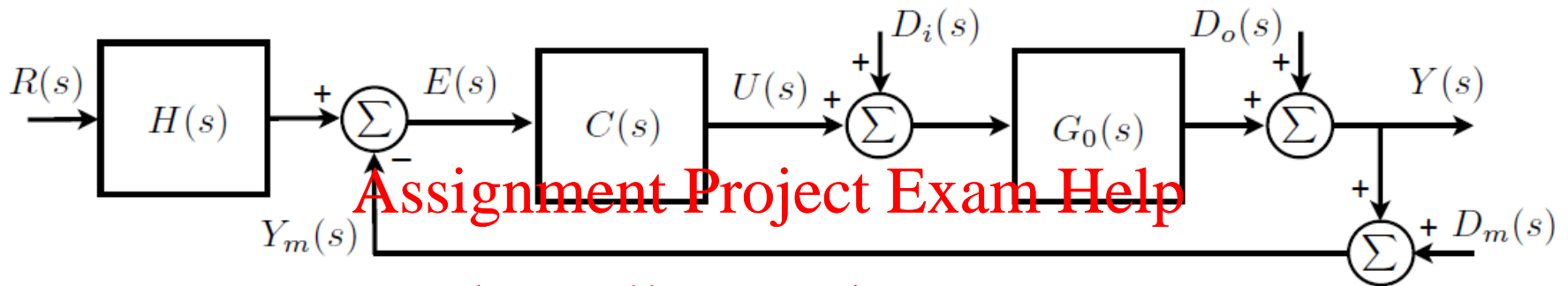
increasing the controller gain K increases the range of frequencies (i.e. oscillating signals) that can be tracked by the output without attenuation or lag in steady-state

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Two degree of freedom controllers

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Input-output relationships



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 $H(s)$ is a stable transfer function (reference filter)

provides a second degree-of-freedom for 'shaping' reference response
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$$Y(s) = \frac{G_0(s)C(s)H(s)}{1 + G_0(s)C(s)}R(s) + \frac{1}{1 + G_0(s)C(s)}D_o(s) + \frac{G_0(s)}{1 + G_0(s)C(s)}D_i(s) - \frac{G_0(s)C(s)}{1 + G_0(s)C(s)}D_m(s)$$

can only 'shape' one of these closed-loop transfer functions using $C(s)$...
 setting one determines the other two!!!

Disturbance-input relationships

- We are also interested in how disturbances affect the inputs so it is sometimes of interest to consider.

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$$U(s) = \frac{C(s)H(s)}{1 + G_0(s)C(s)}R(s) - \frac{C(s)}{1 + G_0(s)C(s)}D_o(s) - \frac{G_0(s)C(s)}{1 + G_0(s)C(s)}D_i(s) - \frac{C(s)}{1 + G_0(s)C(s)}D_m(s)$$

Input and output relationships

- We can rewrite the original relationships as

$$Y(s) = T_0(s)H(s)G(s) + T_0(s)L_j(s) + S_{u0}(s)D_i(s) + T_0(s)D_m(s)$$
$$U(s) = S_{u0}\left(H(s)R(s) - D_m(s) - D_i(s) - G(s)D_i(s)\right)$$

- We can obtain sensitivities in the same manner as before using the block diagram. Please do this as an exercise.

Conclusions

- We have introduced one and two degree of freedom controllers and have introduced different sensitivity functions.
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- Sensitivities are algebraically related! This typically leads to trade-offs in design.
- We will study in more detail these sensitivities in the next lecture.