Lecture 13

Sketching root locus

Recap:
(Problem formulation)

- Plot in the complex s plane the locations of all roots of the equation

$$1 + K \cdot F(s) = 0 \quad \text{where} \quad F(s) = \frac{M(s)}{D(s)} = \frac{\prod_{k=1}^{m}(s-\beta_k)}{\prod_{k=1}^{n}(s-\alpha_k)}$$

as K varies from 0 to infinity.

- This plot is called the (positive) “root locus”.

Recap:
(phase and magnitude conditions)

- Note that if a point s_0 in the complex plane lays on the root locus, it has to satisfy

$$1 + KF(s_0) = 0 \quad \Leftrightarrow \quad KF(s_0) = -1$$

which implies that these conditions hold:

magnitude condition: $|K \cdot F(s_0)| = 1$

phase condition: $\angle K \cdot F(s_0) = (2l+1)\pi \quad \text{for } l = 0, \pm 1, \pm 2, \ldots$

Main features of root locus

- Number of branches
- Open loop poles (starting points for K=0)
- Open loop zeros (limiting points for K infinity)
- Parts of real line that belong to root locus
- Asymptotes
- Breakaway point (branches intersect)
- Intersections with imaginary axis
- Angles of departure or arrival at poles/zeroes

Number of branches

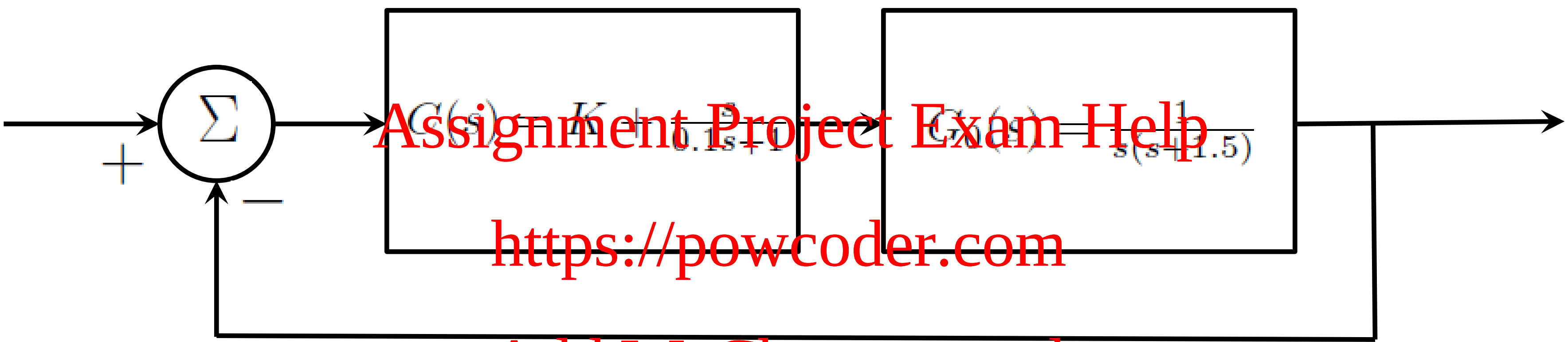
Summary:

- The root locus will have L branches, where L is the maximum between numbers of poles/zeros of F(s).

- If F is proper, then L is equal to the number of poles.

- F does not need to be proper in general, as it does not correspond to a model of physical system.

Example:



Characteristic equation is:

$$\underbrace{0.1s^3 + 1.15s^2 + 2.5s}_{D(s)} + K\underbrace{(0.1s+1)}_{M(s)} = 0$$

$$1 + K\frac{M(s)}{D(s)} = 0$$

We consider this example in detail.

Example:

- Consider the polynomial

$$0.1s^3 + 1.15s^2 + (0.1K + 2.5)s + K = 0 \Leftrightarrow 1 + K\frac{0.1s+1}{s(0.1s^2+1.15s+2.5)} = 0$$


```
>> z = [-10];
>> r = roots([0.1,1.15,2.5])
r =
   -8.5895
   -2.9105
>> p = [0 r(1) r(2)];
>> g = 1;
>> F = zpk(z,p,g)
Zero/pole/gain:
       (s+10)
---------------------
s (s+8.589) (s+2.911)
```

$$\Longrightarrow \quad F(s) = \frac{(s+10)}{s(s+8.5895)(s+2.9105)}$$

3 poles at: $s = 0, -2.9105, -8.5895$

1 zero at: $s = -10$

Root locus has max{3,1}=3 branches in this example.

Open loop poles/zeroes

Summary:

- For very small values of K, root locus contains points close to the poles of F(s):

$$1 + KF(s) = \frac{D(s)+KM(s)}{D(s)} = 0 \overset{K\approx 0}{\Longleftrightarrow} D(s) = 0$$

- Zeroes of characteristic polynomial are the poles of the transfer function!
- We can say that branches “emanate from” open loop poles (poles as “sources”).

Summary:

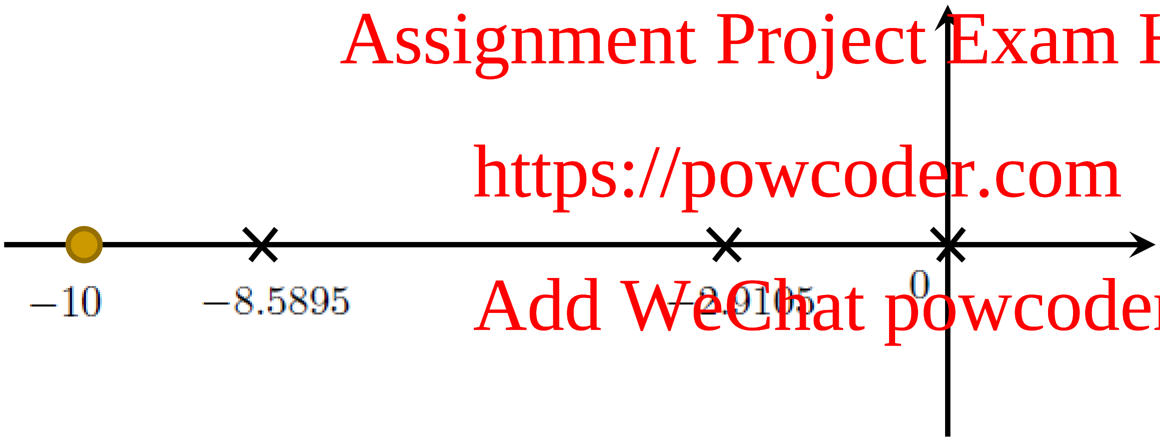
- For large values of K, root locus is close to zeros of F(s):

$$1 + KF(s) = K\frac{\frac{1}{K}D(s)+M(s)}{D(s)} = 0 \overset{K\to\infty}{\Leftrightarrow} M(s) = 0$$

- We can say that zeroes are limits of branches of root locus as K grows to infinity (zeroes as “sinks”).

Example:

- We enter the poles and zeros of F(s)



Poles are denoted as crosses and zeros with circles.

Real line segments on the root locus

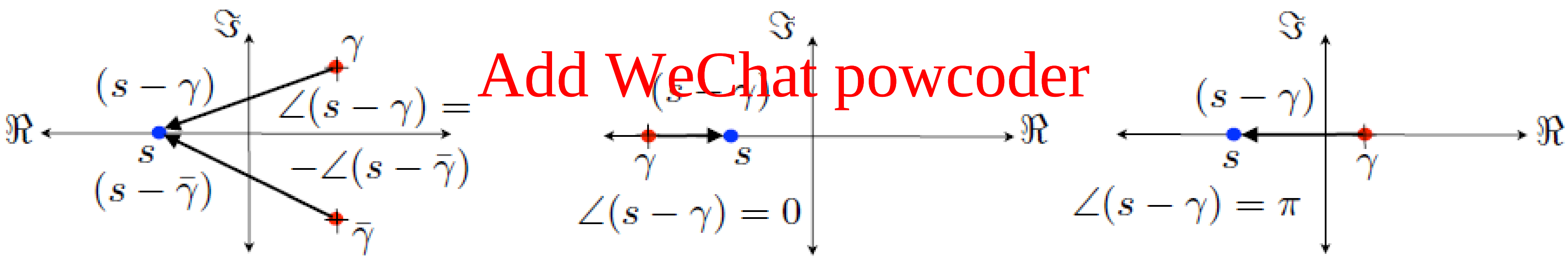
Summary:

- We can quickly determine which parts of real axis belong to the root locus because of the phase condition:

phase condition: $\angle K \cdot F(s_0) = (2l+1)\pi$ for $l = 0, \pm 1, \pm 2, \ldots$

Summary:

- A point on the real axis is a part of root locus iff it is to the left of an odd number of real poles and zeros



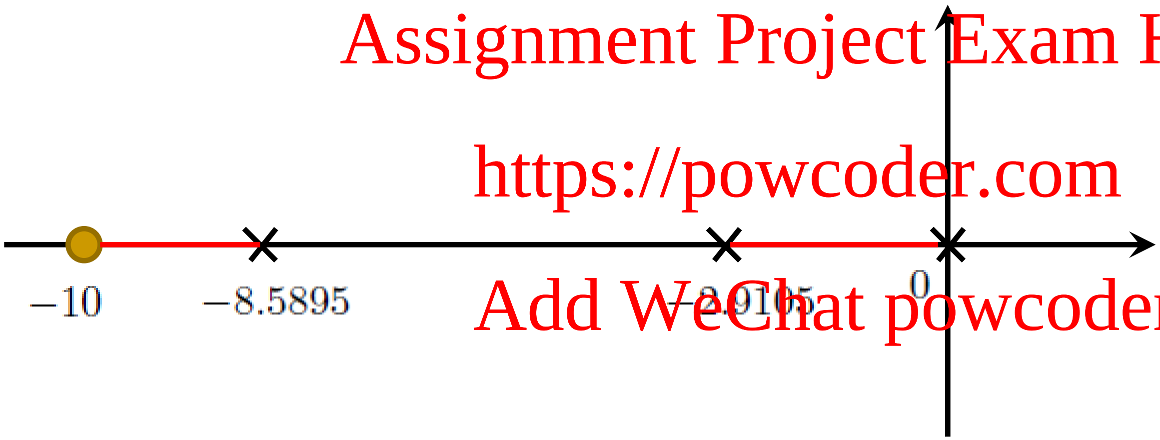
Complex conjugate poles/zeros are irrelevant.

Phase condition does not hold when we are on the right

Phase condition holds when we are on the left

Example:

- Red lines belong to root locus



Asymptotes

Asymptotes

- If n>m then the root locus has n-m branches that approach infinity along asymptotes that intersect the real axis at

poles zeros

$$\sigma \doteq \frac{\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{m} \beta_k}{n-m}$$

this is a real number because poles and zeros occur in conjugate pairs pairs

and with angles $\eta_k \doteq \frac{(2k-1)\pi}{n-m}$ for $k = 1, \ldots, n-m$

the number of distinct asymptotes depends on the relative degree

Situation m>n is discussed in lecture notes.

Sketch of proof (see Ogata):

- As $K \rightarrow \infty$, $F(s) \rightarrow 0$ on each branch. Of the n branches, m terminate at zeros of numerator M(s). The remaining n-m branches must therefore stretch indefinitely. Since $n>m$, $F(s) \rightarrow 0$ as $|s| \rightarrow \infty$.
- For large s, root locus approaches root locus of $1 + K\dfrac{1}{s^{n-m}}$ but with origin shifted to σ:

$$F(s) = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0} = \frac{\prod_{k=1}^{m}(s-\beta_k)}{\prod_{k=1}^{n}(s-\alpha_k)}$$

$$= \frac{1}{\boxed{s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1}} + \cdots + d_1 s + d_0 + \frac{c_{m-1}s^{m-1} + \cdots + c_1 s + c_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}}$$

For large s: $$\approx \frac{1}{\boxed{(s-\sigma)^{n-m}}} = \frac{1}{s^{n-m} - \sigma(n-m)s^{n-m-1} + \cdots + (n-m)(-\sigma)^{n-m-1}s + (-\sigma)^{n-m}}$$

$$\sigma = -\frac{(a_{n-1} - b_{m-1})}{n-m} = \frac{\sum_{k=1}^{n}\alpha_k - \sum_{k=1}^{m}\beta_k}{n-m}$$

Sketch of proof

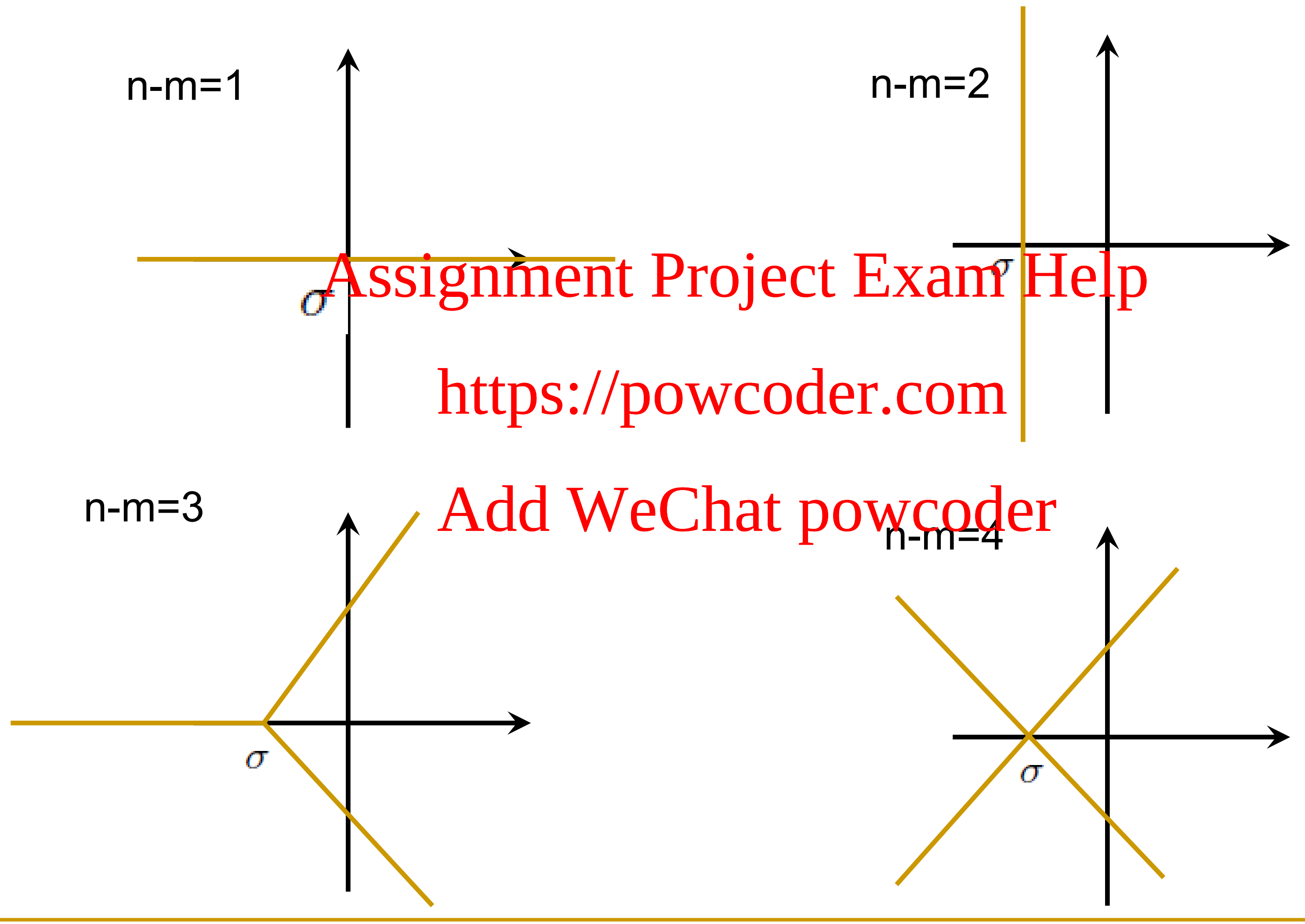
- The phase condition for $1 + K\frac{1}{s^{n-m}}$ is

$$(n-m)\angle\frac{1}{s_0} = \angle -1 = (2l+1)\pi,\ l = 0, \pm 1, \ldots,$$

or equivalently

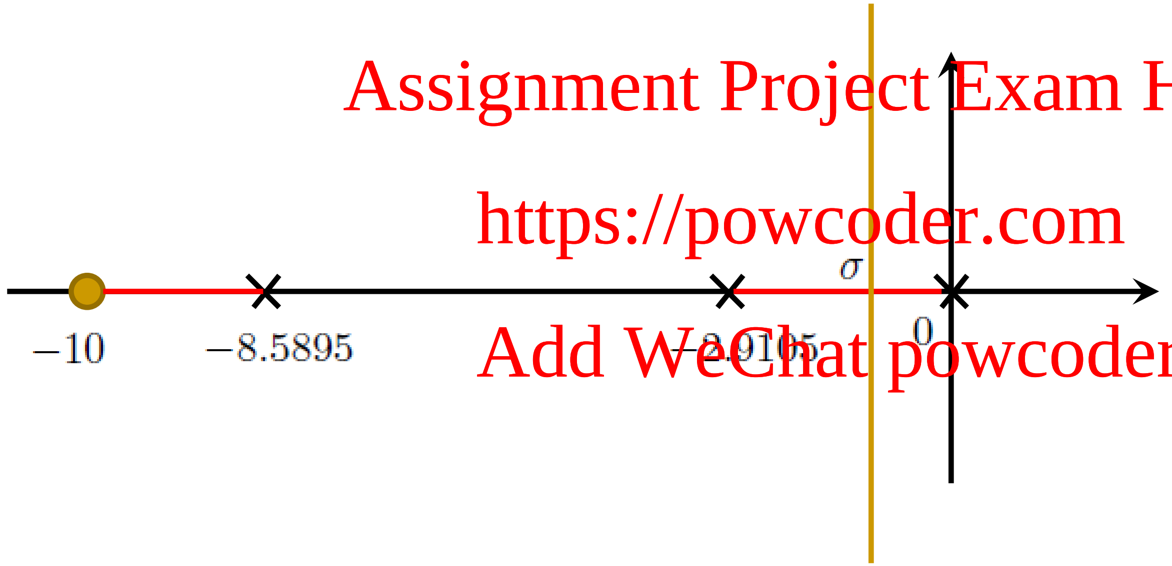
$$\angle s_0 = (2k-1)\pi/(n-m),\ k = 1, 2, \ldots, n-m$$

Several typical cases



Example:

- Since n-m=2, asymptotes are:



$$\sigma = \frac{0-8.5895-2.9105+10}{2} = -0.75 \qquad \eta_1 = \frac{\pi}{2},\ \eta_2 = \frac{3\pi}{2}$$

Points where branches intersect

(repeated roots of characteristic equation)

Summary

- Consider a function f and suppose that

$$f(s) = (s-\alpha)^2 \tilde{f}(s) = 0$$

- Then, we have

$$\frac{df}{ds}(s) = 2(s-\alpha)\tilde{f}(s) + (s-\alpha)^2 \frac{d\tilde{f}}{ds}(s)$$

$$\begin{aligned} f(\alpha) &= 0 \\ \frac{df}{ds}(\alpha) &= 0 \end{aligned}$$

Formula for repeated roots:

- Consider

$$f(s) = D(s) + KM(s) = 0$$

- K at which repeated roots occur:

$$\frac{dD}{ds} + K\frac{dM}{ds} = 0 \;\Rightarrow\; K = -\frac{\frac{dD}{ds}}{\frac{dM}{ds}}$$

$$D(s) - \frac{\frac{dD}{ds}}{\frac{dM}{ds}}M(s) = 0 \;\Leftrightarrow\; D(s)\frac{dM}{ds} - \frac{dD}{ds}M(s) = 0$$

Alternative approach:

- We can alternatively consider:

$$K(s) := -\frac{D(s)}{M(s)}$$

- Points where branches intersect can be obtained alternatively from

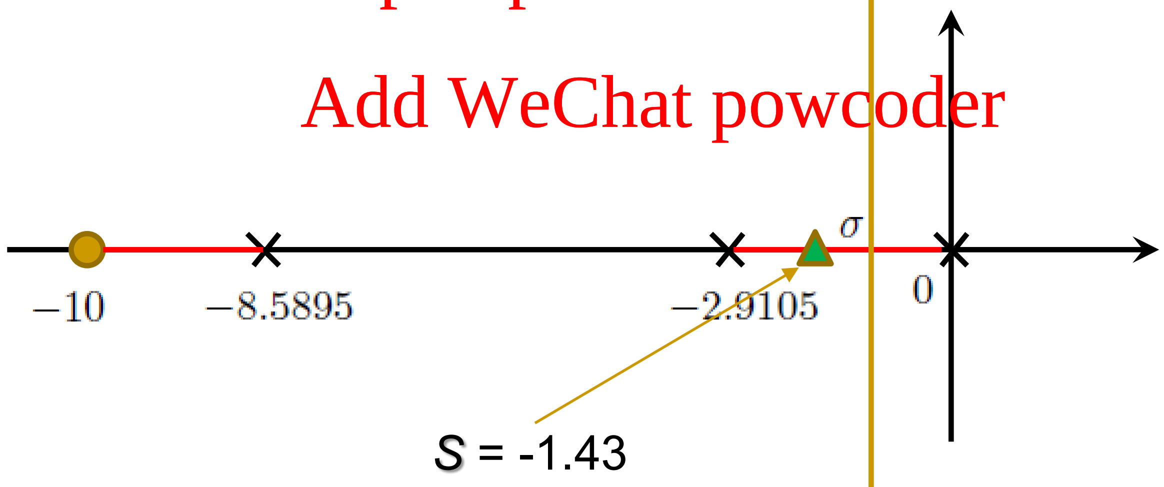
$$\frac{dK}{ds} = -\frac{\frac{dD}{ds}M(s) - D(s)\frac{dM}{ds}}{M(s)^2} = 0$$

NOTE: these points have to be on the root locus!

Example:

- Solve

$$\frac{dK}{ds}=0 \Leftrightarrow -0.02s^3 - 0.41s^2 - 2.3s - 2.5 = 0$$



Intersections with imaginary axis

Intersections with imaginary axis

- We can first compute the Routh array as a function of K.
- Then, we look for values of K for which some elements in the first column are equal to zero.
- Those values of K yield poles with zero real parts.
- With those values of K, we can find the purely imaginary poles of the closed loop, i.e. intersections with the imaginary axis.

Example

- We computed the Routh array for the example in the last lecture:

$$\begin{array}{c|cc} s^3 & 0.1 & (0.1K+2.5) \\ s^2 & 1.15 & K \\ s^1 & \frac{3K}{230}+2.5 & 0 \\ s^0 & K & \end{array}$$

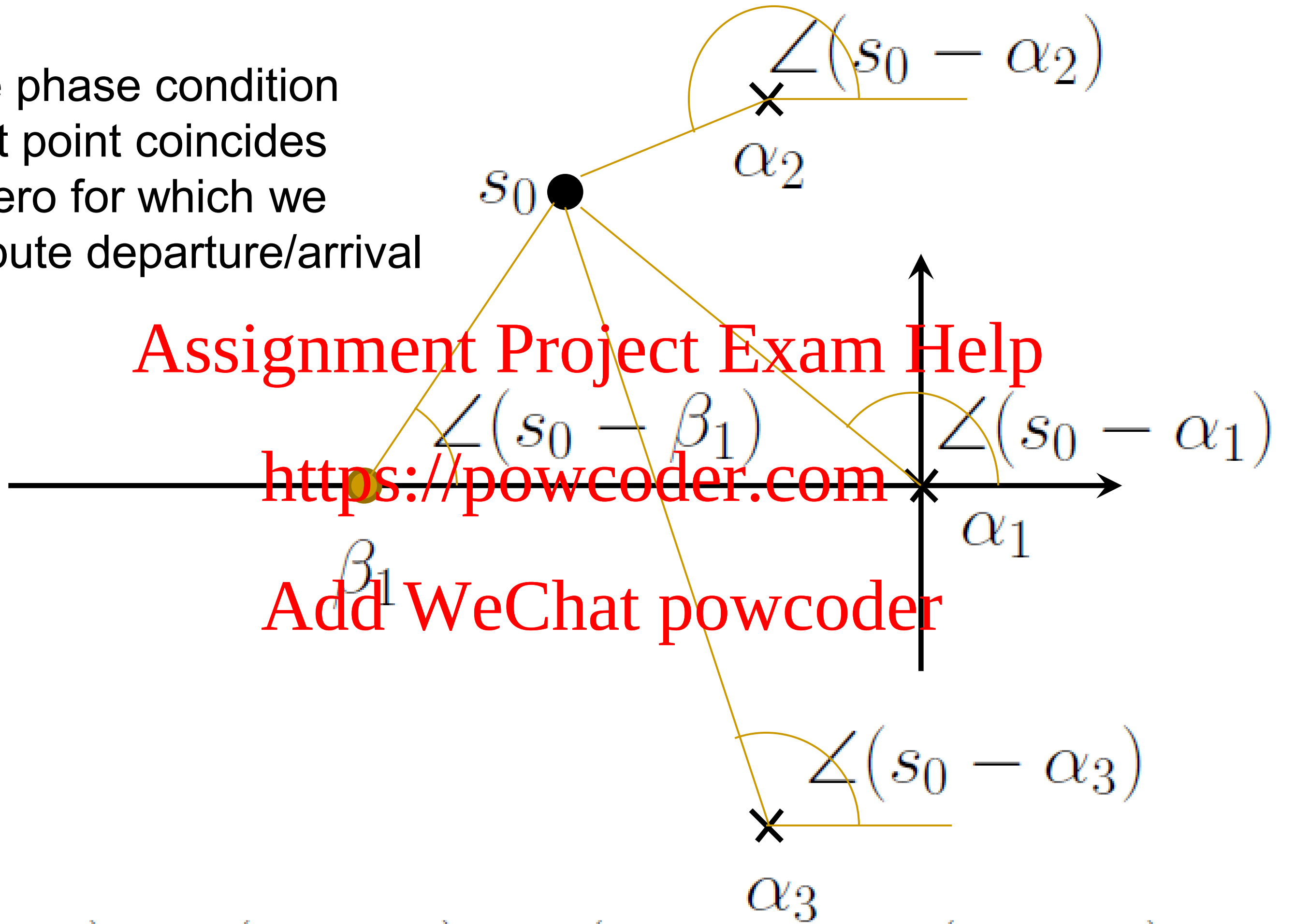
so stable if $K > 0$ and $\frac{3K}{230} + 2.5 > 0$
(i.e. for all $K > 0$)

- Hence, no intersections with the imaginary axis.

Arrival/departure angles

Phase condition

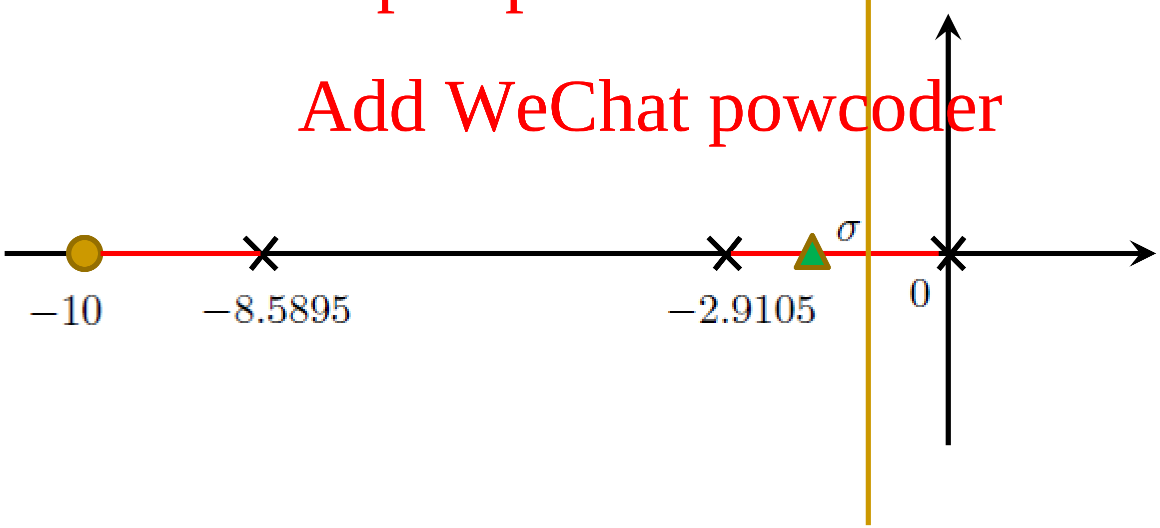
We apply the phase condition when the test point coincides with a pole/zero for which we want to compute departure/arrival angle.



$$\angle(s_0 - \beta_1) - \angle(s_0 - \alpha_1) - \angle(s_0 - \alpha_2) - \angle(s_0 - \alpha_3) = (2l+1)\pi, l = 0, \pm 1, \pm 2, \ldots$$

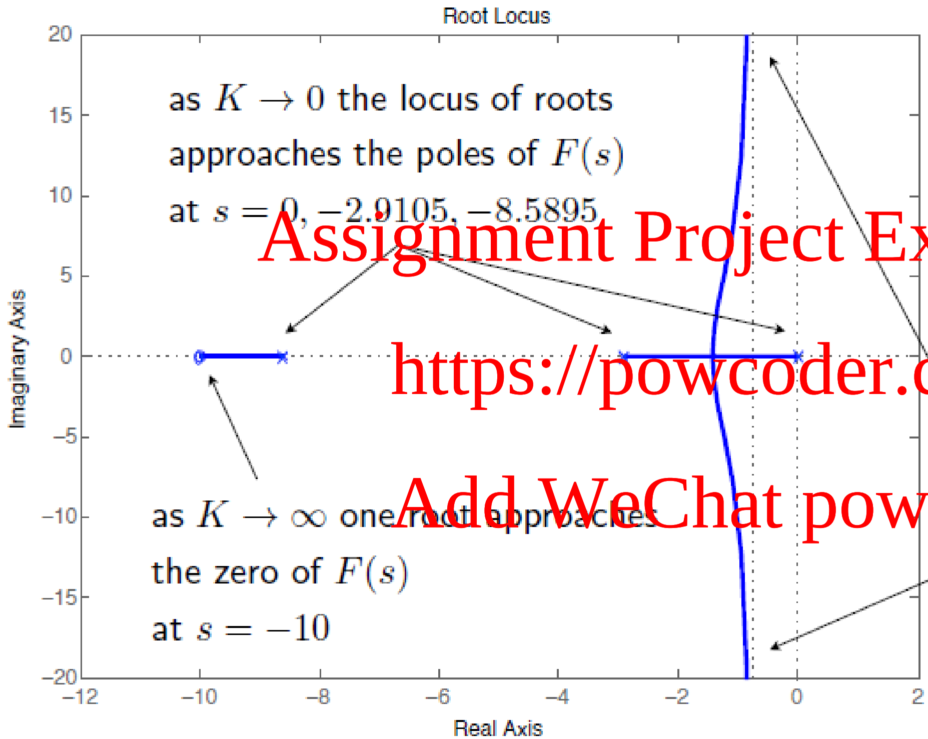
Example

- In this example, since all poles and zeroes are on the real axis it is trivial to compute departure/arrival angles.



Example

(completed root locus)

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Zero/pole/gain:
        (s+10)
---------------------
s (s+8.589) (s+2.911)
>> rlocus(F);
```

as $K \to \infty$ two $(n - m = 2)$ roots approach ∞ along asymptotes that intersect the real axis at

$$\sigma = \frac{0 - 8.5895 - 2.9105 + 10}{2}$$

with angles $\eta_1 = \frac{\pi}{2}, \ \eta_2 = \frac{3\pi}{2}$

The plot shows that, starting from K=0, increasing K initially 'improves' the stability and performance properties of the closed-loop; these properties then 'degrade' as K increases beyond a certain point

Exercise:

- Plot the root locus for:

$$1 + KF(s) = 0, \qquad F(s) = \frac{(s-1)(s+2)}{s(s+3)(s+10)}$$

Conclusions

- We showed how to construct root locus of arbitrary systems via its main features:
  - Number of branches; open loop poles/zeroes.
  - Segments of real line on root locus.
  - Asymptotes.
  - Intersections of branches.
  - Intersections with imaginary axis.
  - Multiple roots.
  - Angles of arrival/departure.

Thank you for your attention.