

Lecture 8

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Grandfather of control and telecommunications

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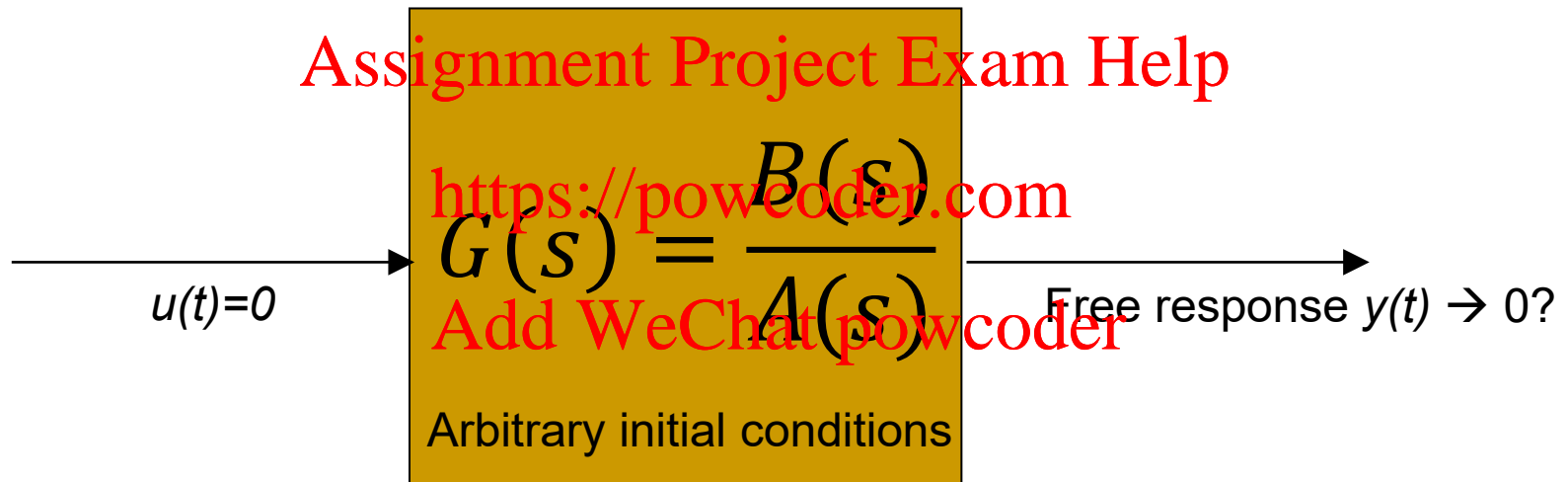
Two Flavours of Stability

Routh-Hurwitz Criterion

Motivation

- Control systems must be designed to meet performance criteria – e.g. step response specs.
- If the control system will be operated over a long period of time, stability is usually also necessary to be able to meet these criteria
- We present two definitions of stability.
- For LTI systems, both conditions turn out to be very closely related

Stability w.r.t. Initial Conditions



Say $A(s)$, $B(s)$ are given polynomials, e.g. from a diff. eq. model

If $A(s)$ has order n , then *initial conditions* are $\frac{d^j y}{dt^j}(0^-)$, $j = 0, \dots, n - 1$.

Definition

(input=zero)

- **Stable** if output converges to zero for any initial condition.

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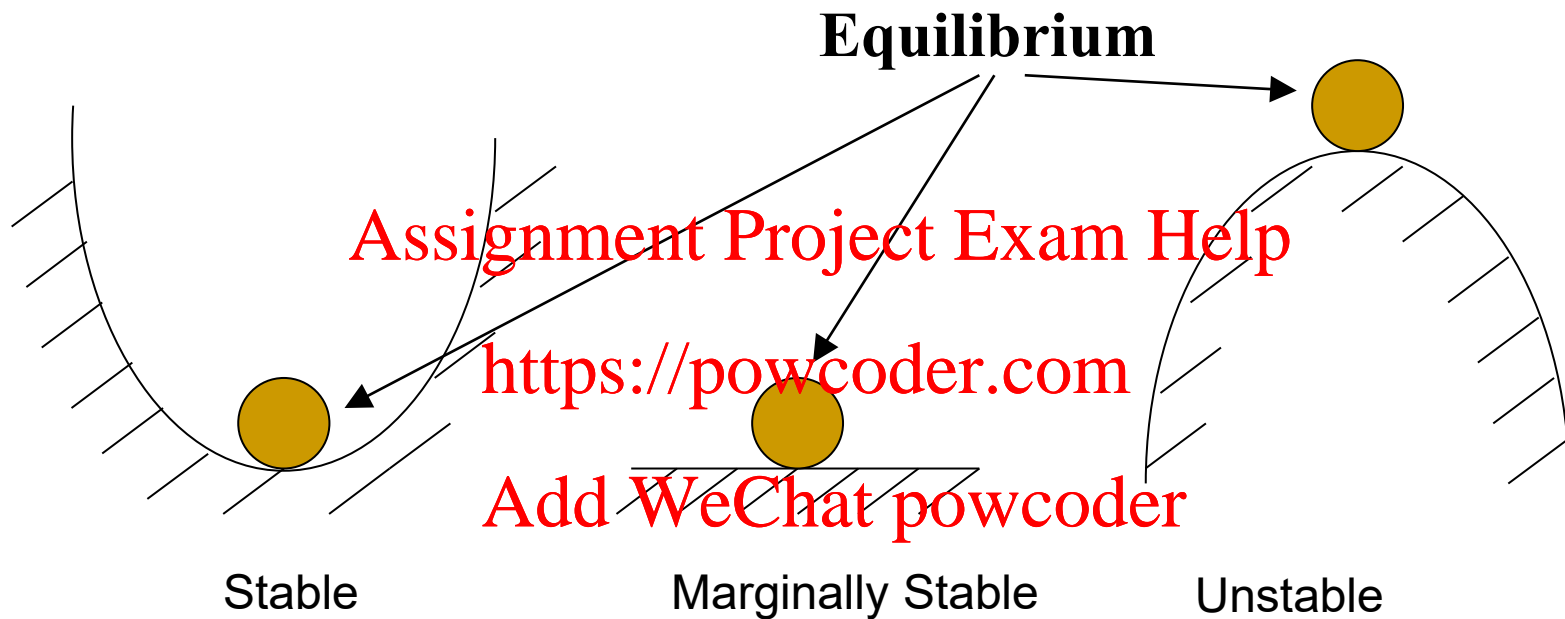
- **Unstable** if output diverges for at least some initial conditions

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- **Marginally stable** if output stays bounded from all initial conditions and may converge to zero for some initial conditions.

Visualisation



Free Response

- Free response

$$\sum_{i=0}^n a_i \frac{d^i y}{dt^i} = \sum_{i=0}^m b_i \frac{d^i u}{dt^i} = 0, \quad t \geq 0$$

$$\Leftrightarrow \sum_{i=0}^n a_i \left(s^i Y(s) - \sum_{j=0}^{i-1} s^j \frac{d^j y}{dt^j}(0^-) \right) = 0$$

$$\Leftrightarrow Y(s) = \frac{\sum_{j=0}^{n-1} s^j \left(\frac{d^j y}{dt^j}(0^-) \sum_{i=j+1}^n a_i \right)}{\sum_{i=0}^n a_i s^i} = A(s)$$

LTI Stability w.r.t. Initial Conditions

■ Stable:

*All poles of $A(s)$ **strictly** in the left-half plane (LHP).*

■ Marginally stable:

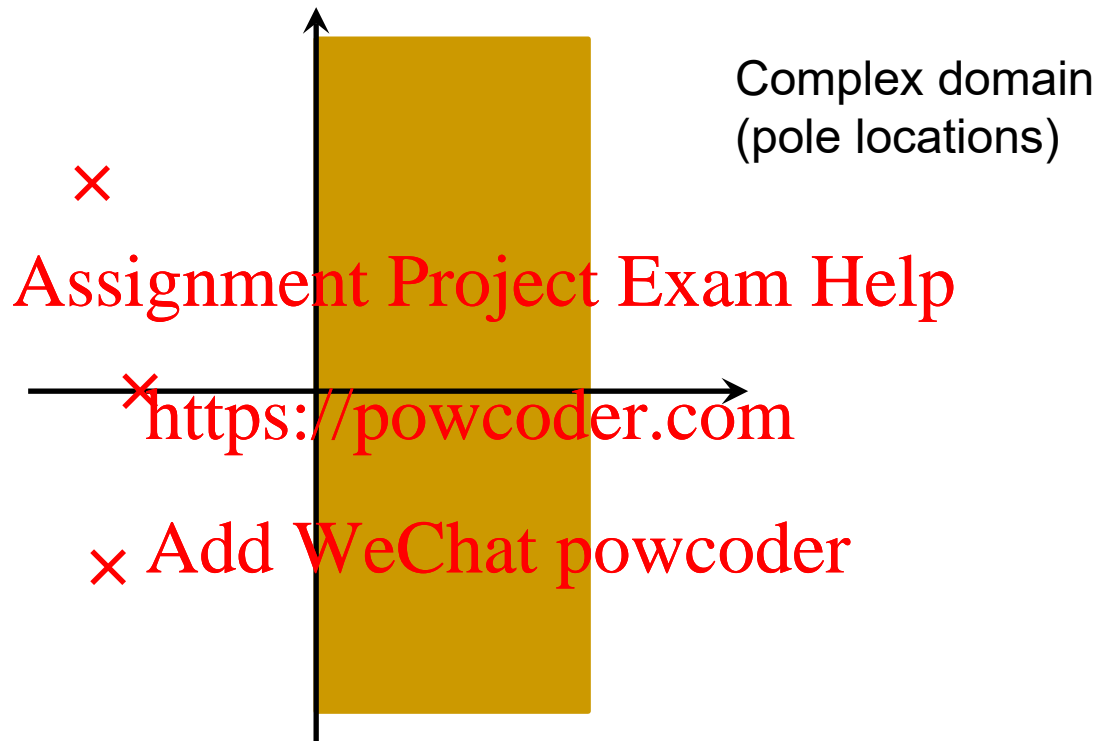
Non-repeated pole(s) on imaginary axis, all other poles strictly in the LHP

■ Unstable:

*Some pole(s) strictly in the right-half plane
OR repeated poles on the imaginary axis*

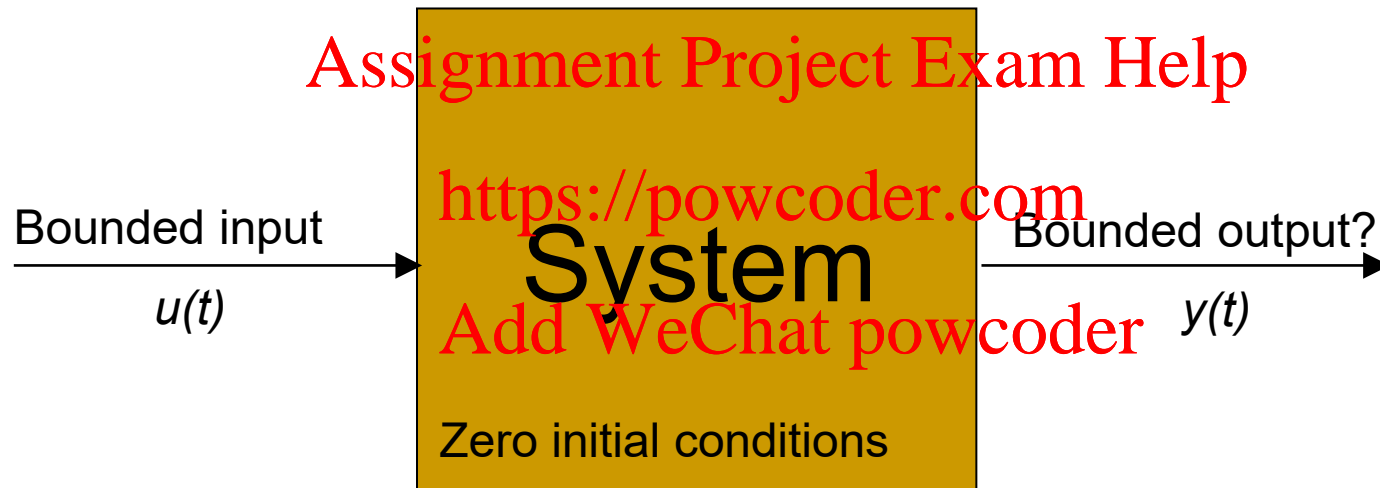
You can understand this if you remember how terms in the partial fraction expansion affect the impulse response.

Graphical Interpretation of Stability

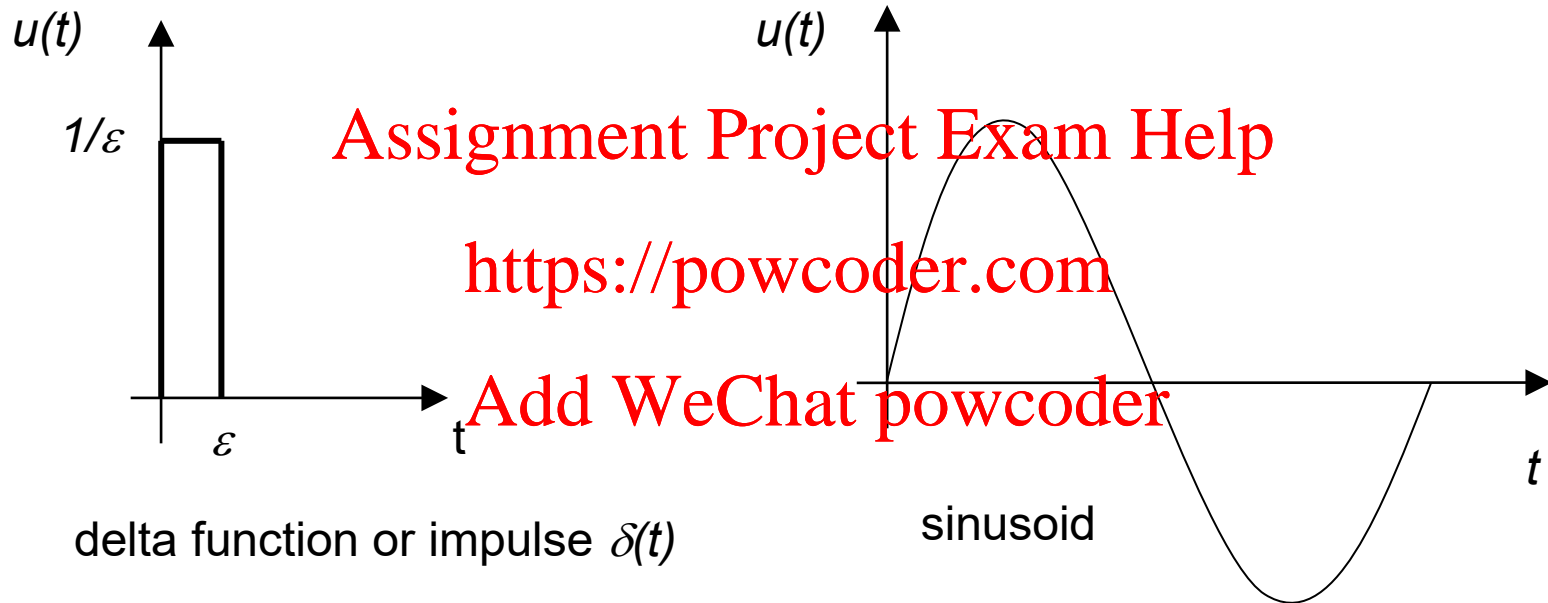


Brown colour represents a “forbidden (unstable) region”.

Bounded-Input Bounded-Output (BIBO) Stability



Signals as functions of time



We measure the size of signal u using: $\bar{u} = \max_{t \geq 0} |u(t)| < \infty$

BIBO Stability Definition

(zero initial conditions)

- Formally: For any $\bar{u} > 0$, there exists a finite $\bar{y} > 0$ such that for any input u with $u(t) \leq \bar{u}$,
 $|y(t)| \leq \bar{y}$, for all $t \geq 0$.

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- Informally: Bounded inputs yield bounded outputs.

BIBO stability for LTI systems

- BIBO stability holds *if and only if (iff)* the impulse response g is *absolutely integrable*, i.e.

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$$\int_{-\infty}^{\infty} |g(t)| dt < \infty$$

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- Proof of sufficiency (←):

$$\begin{aligned} |y(t)| &= |(g * u)(t)| \leq \int_0^t |g(t - \tau)| \cdot |u(\tau)| d\tau \\ &= \int_0^t |g(\nu)| \cdot |u(t - \nu)| d\nu \leq \bar{u} \int_0^\infty |g(\nu)| d\nu. \end{aligned}$$

BIBO Stability

- Proof of necessity (\Rightarrow): By contradiction.

Suppose $\int_0^\infty |g(t)|dt = \lim_{T \rightarrow \infty} \int_0^T |g(t)|dt = \infty$.

For an arbitrary $T > 0$, apply an input

$$u(t) = \begin{cases} \bar{u} \cdot \text{sign}(g(T-t)), & 0 \leq t \leq T, \\ 0, & \text{else.} \end{cases}$$

$$\Rightarrow y(T) = \int_0^T g(t)u(T-t)dt$$

$$= \int_0^T g(t)\bar{u} \cdot \text{sign}(g(t))dt$$

$$= \bar{u} \cdot \int_0^T |g(t)|dt \rightarrow \infty. \text{ QED}$$

BIBO for LTI Systems with Rational Transfer Functions

- BIBO stability *iff*

- Transfer function is proper (why?), **and**

- All poles lie strictly in the LHP (why?).

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- If no *unstable pole-zero cancellations* in $B(s)/A(s)$,
BIBO stability \rightarrow Stability w.r.t. initial conditions

(but the reverse implication is not true)

- If no unstable pole-zero cancellations, **and** $B(s)/A(s)$ is proper

BIBO stability \Leftrightarrow Stability w.r.t. initial conditions

Why pay heed to a pole if cancelled by a zero anyway?

- Pole-zero cancellation said to occur if there is a zero β_k which equals a pole α_j .
- In this case, the factor $(s - \beta_k)$ in the numerator $B(s)$ formally cancels out the factor $(s - \alpha_j)$ in the denominator $A(s)$, and both factors disappear from the transfer function $G(s) = B(s)/A(s)$.
- So why do we care...?

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Why Cancelled Poles Still Count

- A cancelled pole α_j is no problem if initial conditions are zero. But in practice, initial conditions may not be exactly zero.
- Recall $y(t) =$ free response (due to nonzero initial conditions) + forced response (due to nonzero input).
- Cancelled pole α_j appears in the free response! Could affect transient performance if too slow.
- If α_j unstable \rightarrow Trouble as $t \rightarrow \infty$

Example of How Pole-Zero Cancellations Arise

Consider a DE system

$\ddot{y} - y = \dot{u} - u$, where $u(t) = 0, t < 0$.

$$\rightarrow (s^2 - 1)Y(s) - (s\dot{y}(0^-) + y(0^-)) = (s - 1)U(s)$$

$$\Leftrightarrow Y(s) = \frac{1}{s+1}U(s) + \frac{s\dot{y}(0^-) + y(0^-)}{(s+1)(s-1)}$$

System is BIBO stable, but not stable w.r.t. initial conditions

Example of System That's Stable w.r.t.
Initial Cond's, but isn't BIBO stable

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Determining stability for LTI systems

- Consider a transfer function

$$G(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0} = K \frac{(s - \beta_1) \dots (s - \beta_m)}{(s - \alpha_1) \dots (s - \alpha_n)} = K \frac{\prod_{k=1}^m (s - \beta_k)}{\prod_{k=1}^n (s - \alpha_k)}$$

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- Poles are the solutions of either equation:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$$

$$s^n + a'_{n-1} s^{n-1} + \dots + a'_1 s + a'_0 = (s - \alpha_1) \dots (s - \alpha_n) = 0$$

The second polynomial is “monic” and often used in tests.

Radical Facts



- A quadratic polynomial

$$as^2 + bs + c \text{ has zeros at } s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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- Roots of cubic and quartic polynomials can be found in terms of 3rd and 4th roots (*radicals*). But the formulas are impossible to remember.
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- Galois (1811-32): Polynomial equations of order 5 or higher cannot generally be expressed in terms of radicals!

So how to manually check stability for systems of order > 2 ?

A necessary condition for stability

- If a monic polynomial

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0$$

has all roots with negative real parts, then all its coefficients are positive; that is

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$$a_i > 0 \text{ for } i = 0, 1, \dots, n$$

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- If a coefficient is negative or zero, then some roots have positive or zero real part.
- This is also sufficient condition for first and second order polynomials.

Proof:

- Recall that an arbitrary monic polynomial in s with real coefficients can always be written as a product of monic linear and quadratic terms with real coefficients:

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- Both linear and quadratic terms have all roots with strictly negative real parts iff all their coefficients are positive.

Test:

- If some coefficient in the characteristic polynomial is negative or zero, then there are some poles with positive/zero real part.

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- If all coefficients in the characteristic polynomial are all positive, then use *Routh-Hurwitz criterion* to check whether there are any poles with positive/zero real part.

Examples

■ Consider

for $p(s) = s^2 + ks + (1-k)$
if $[p(s_0) = 0 \implies \Re(s_0) < 0]$ *i.e. all zeros have negative real part*
then $0 < k < 1$ *converse also true in this case since $n = 2$*

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This polynomial has all positive coefficients but two roots on the imaginary axis:

$$s^3 + s^2 + s + 1$$

has zeros at

$$s = -1, \pm j$$

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Routh-Hurwitz test

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- For polynomial with positive coefficients:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

we form the following (Routh) array:

$$\begin{array}{c|cccc}
 s^n & \gamma_{0,1} := a_n & \gamma_{0,2} := a_{n-2} & \gamma_{0,3} := a_{n-4} & \dots \\
 s^{n-1} & \gamma_{1,1} := a_{n-1} & \gamma_{1,2} := a_{n-3} & \gamma_{1,3} := a_{n-5} & \dots \\
 s^{n-2} & \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 s^1 & \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \dots \\
 s^0 & \gamma_{n,1} & \gamma_{n,2} & \gamma_{n,3} & \dots
 \end{array}$$

$$\gamma_{k,j} := -\frac{1}{\gamma_{k-1,1}} \begin{vmatrix} \gamma_{k-2,1} & \gamma_{k-2,j+1} \\ \gamma_{k-1,1} & \gamma_{k-1,j+1} \end{vmatrix}$$

- Example of computing entries in the third row

$$\begin{array}{c|cccc}
 s^n & \gamma_{0,1} := a_n & \gamma_{0,2} := a_{n-2} & \gamma_{0,3} := a_{n-4} & \dots \\
 s^{n-1} & \gamma_{1,1} := a_{n-1} & \gamma_{1,2} := a_{n-3} & \gamma_{1,3} := a_{n-5} & \dots \\
 s^{n-2} & \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 s^1 & \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \dots \\
 s^0 & \gamma_{n,1} & \gamma_{n,2} & \gamma_{n,3} & \dots
 \end{array}$$

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$$\gamma_{2,1} := -\frac{1}{\gamma_{1,1}} \begin{vmatrix} \gamma_{0,1} & \gamma_{0,2} \\ \gamma_{1,1} & \gamma_{1,2} \end{vmatrix}$$

- Example of computing entries in the third row

$$\begin{array}{c|cccc}
 s^n & \gamma_{0,1} := a_n & \gamma_{0,2} := a_{n-2} & \gamma_{0,3} := a_{n-4} & \dots \\
 s^{n-1} & \gamma_{1,1} := a_{n-1} & \gamma_{1,2} := a_{n-3} & \gamma_{1,3} := a_{n-5} & \dots \\
 s^{n-2} & \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 s^1 & \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \dots \\
 s^0 & \gamma_{n,1} & \gamma_{n,2} & \gamma_{n,3} & \dots
 \end{array}$$

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$$\gamma_{2,2} := -\frac{1}{\gamma_{1,1}} \begin{vmatrix} \gamma_{0,1} & \gamma_{0,3} \\ \gamma_{1,1} & \gamma_{1,3} \end{vmatrix}$$

Routh-Hurwitz test

- The polynomial with positive coefficients

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

has all roots with strictly negative real parts if
and only if all elements in the first column of
the Routh array do not change their sign.

- No need to solve for the roots!

Example

- For a first order polynomial

$a_1s + a_0$
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$$\begin{array}{c|c} s^1 & a_1 \\ s^0 & a_0 \end{array}$$

Example

- For a quadratic polynomial

$a_2s^2 + a_1s + a_0$
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$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & a_0 = -\frac{a_2 \cdot 0 - a_0 \cdot a_1}{a_1} & 0 \end{array}$$

Example

- For third order polynomials

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Routh array is

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$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & -\frac{a_3 \cdot a_0 - a_1 \cdot a_2}{a_2} & 0 \\ s^0 & a_0 & \end{array}$$

Example

- For fourth order polynomials we have:

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For $p(s) = \sum_{k=0}^4 a_k s^k$, $a_4 > 0$,

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$$\begin{array}{l|l} s^4 & a_4 \qquad a_2 \qquad a_0 \\ s^3 & a_3 \qquad a_1 \qquad 0 \\ s^2 & b_0 \doteq -\frac{a_4 a_1 - a_2 a_3}{a_3} \quad b_1 \doteq -\frac{a_4 \cdot 0 - a_0 a_3}{a_3} = a_0 \quad 0 \\ s^1 & c \doteq -\frac{a_3 b_1 - a_1 b_0}{b_0} \qquad 0 \\ s^0 & -\frac{b_0 \cdot 0 - b_1 c}{c} = b_1 \end{array}$$

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Example

- Consider a feedback system with

$$G_0(s) = \frac{1}{s(s+1.5)} \quad C(s) = K + \frac{s}{0.1s+1}$$

- The characteristic polynomial is

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$$0.1s^3 + 1.15s^2 + (0.1K + 2.5)s + K$$

- Find all values of K for which the system is stable.

Example

- We use formulas for 3rd order polynomials:

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$$\begin{array}{c|c} s^3 & 0.1 \\ s^2 & 1.15 \\ s^1 & \frac{3K}{230} + 2.5 \\ s^0 & K \end{array} \quad (0.1K + 2.5)$$

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- The system is stable if

$$K > 0$$

Example

- Consider the same system.
- For which values of K would all roots of the characteristic polynomial have real parts strictly smaller than -1?

- We modify the polynomial by introducing

$$w = s + 1 \qquad \Re(w) < 0 \quad \Leftrightarrow \quad \Re(s) < -1$$

$$\begin{aligned} &0.1(w - 1)^3 + 1.15(w - 1)^2 + (0.1K + 2.5)(w - 1) + K \\ &= 0.1w^3 + 0.85w^2 + (0.1K + 0.5)w + (0.9K - 1.45) \end{aligned}$$

Example

- Routh array is

$$\begin{array}{c|cc} w^3 & 0.1 & 0.1K + 0.5 \\ w^2 & 0.85 & 0.9K - 1.45 \\ w^1 & \frac{1}{170}(-K + 114) & \\ w^0 & 0.9K - 1.45 & \end{array}$$

- Hence, we have the conditions:

$$0.9K - 1.45 > 0 \text{ and } \frac{1}{170}(114 - K) > 0$$

$$\text{or } \frac{29}{18} < K < 114$$

Exercise

- Form the Routh array for the following polynomial:

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$$s^3 + s^2 + s + 1$$

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