Advanced Structural Analysis and Earthquake Engineering
ENG5274

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2021 - Second Semester

Course information

Course information

The course is composed of two main parts:
Part I is on *structural elements*; Part II on *time-dependent* problems

The *aims* of the course are to develop:

- Structural elements:
  - an understanding of the mathematical models for *structural elements* including *beams*, plates and shells
  - an appreciation of boundary conditions and locking-related phenomena
  - an understanding of how to solve these models using the finite element method
- Time-dependent problems: dynamics and earthquake engineering
  - a knowledge of mathematical models for waves
  - an understanding of elastodynamics (stress wave propagation)
  - an understanding and appreciation of the solution of time-dependent problems using the finite difference and finite element methods

Intended learning outcomes

By the end of the course you should be able to:

- apply the finite element method to systematically derive the solution procedure for a range of structural elements with an emphasis on beams
- develop a basic finite element code to compute the deflection of a beam and the dynamic response of a rod
- appreciate key aspects of earthquake engineering

Contents

The course is broken into two main sections (3–4)

Course management

- We will use Moodle to manage the course
  - All announcements will be on Moodle
  - All submissions will be electronic and via Moodle
- The majority of the lectures are pre-recorded and the links available on Moodle
- Weekly problem sets (not for submission) will be uploaded to Moodle
- Some of the scheduled lecture slots will be used to answer questions and for worked examples
- The primary mechanism to ask questions is via the Moodle forum
  - Due to the size of the class, please ask questions via the forum and not email
- The complete set of course notes is available on Moodle
  - These will be updated regularly

Lectures and laboratory sessions

Preliminary material

- Lecture 1a — Course in pictures
- Lecture 2a — FEA - strong form
- Lecture 2b — FEA - weak form
- Lecture 2c — FEA - continuity
- Lecture 2d — FEA - discrete equations
- Lecture 2e — FEA - assembly and the global problem
- Lecture 2f — FEA - example problem
- Lecture 2g — FEA - Python code

Structural elements

Time-dependent problems

Course requirements and assessment

Refer to the course specification document on Moodle for more information

Minimum requirements for award of credits

- Must attend the degree examination
- Should attend the computational laboratory classes
- Must submit the two reports (additional to requirements in course specification)

The programming language *Python* will be used to illustrate key aspects of the theory and for the reports. A basic understanding of *Python* and of the *finite element method* is assumed.

Assessment

- Examination 70%
- Two reports (assignments) 30% of overall mark (equally weighted)

Reading material and resources

Comprehensive lecture notes will be provided on the Moodle site. These will be updated regularly.

In addition, the following reference books are recommended:

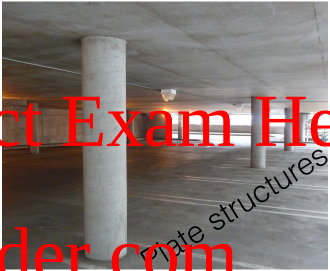
- J Fish and T Belytschko, "A First Course in Finite Elements" - this book is available online from the University of Glasgow library
- TJR Hughes, "The Finite Element Method: Linear Static and Dynamic Finite Element Analysis"

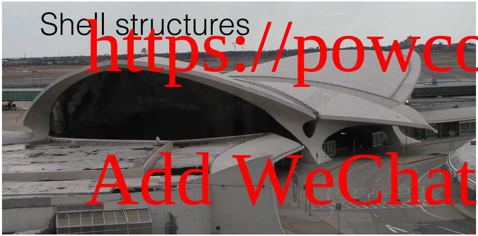
Software:

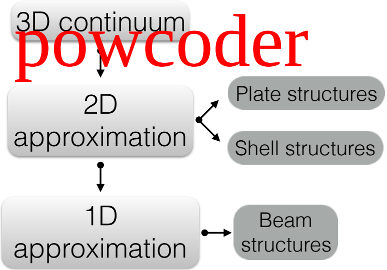
- We will be using Python version 3.8 or similar
- Recommend that you download Anaconda Individual Edition
- We will also make use of Google Colaboratory

The course in pictures

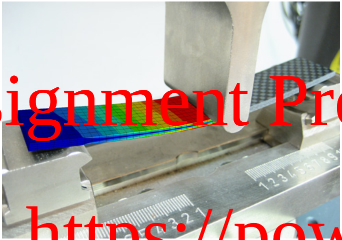
Course in pictures: structural elements

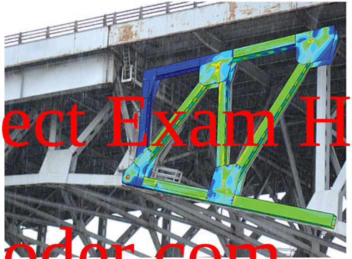






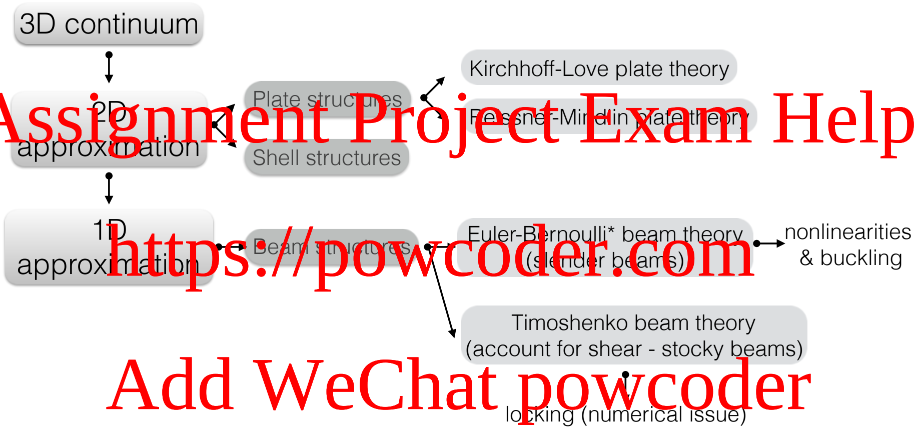
Approximate geometry, loading and boundary conditions to simplify analysis





Numerical modelling (FEA)

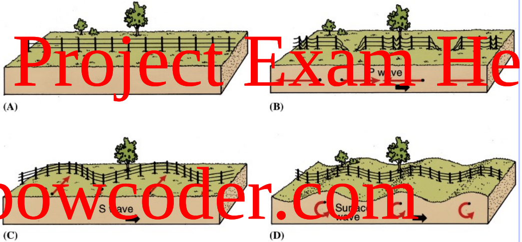
Course in pictures: Part I - structural elements



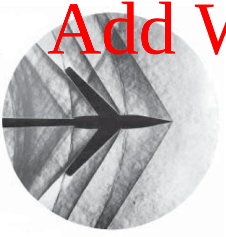
Course in pictures: Part II - dynamics

liquids

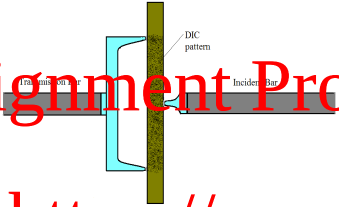
solids

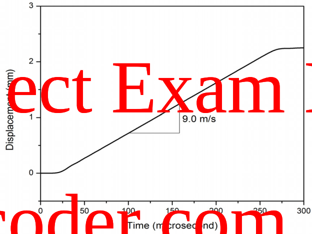


gases



3-point bending in a Kolsky compression bar





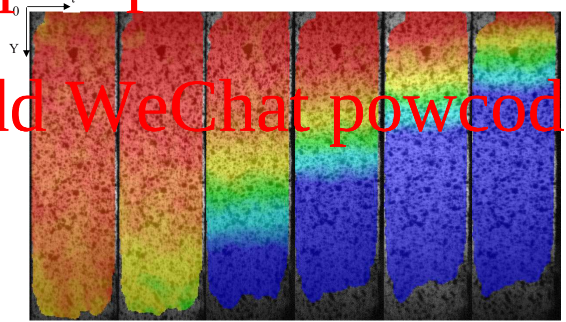
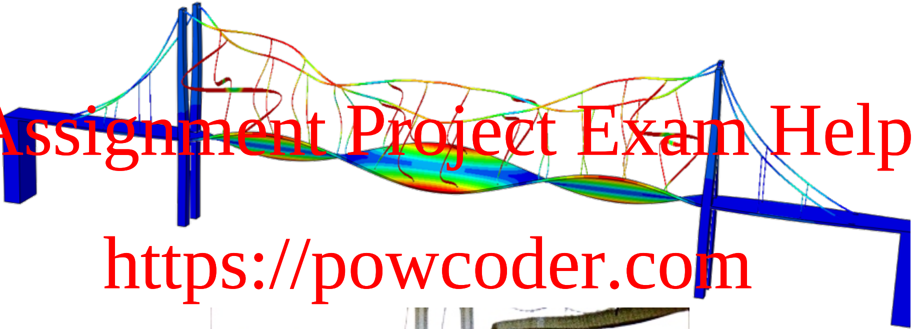
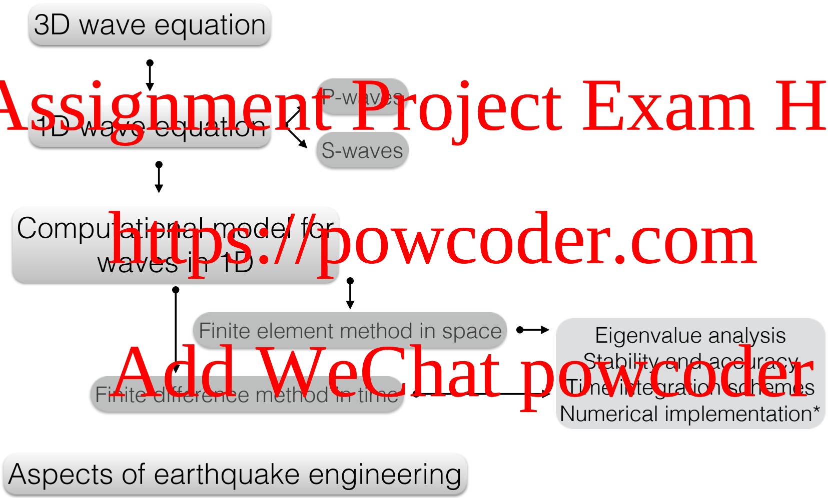


Fig. 3 Propagation of transverse wave (time interval: 5 microseconds)

Course in pictures: Part II - dynamics



Course in pictures: Part II - dynamics

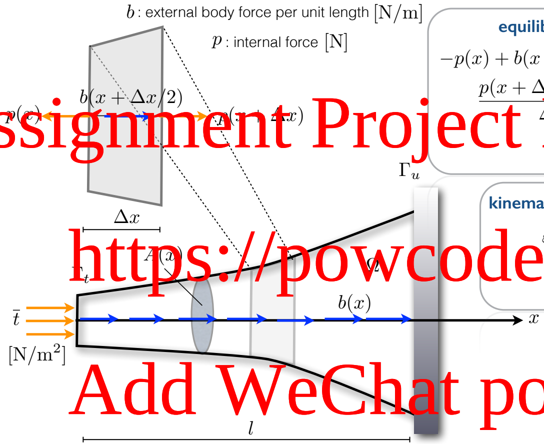


The finite element method: a refresher

Key steps in the finite element method

1. Strong form (governing equations and boundary conditions)
2. Weak form
3. Approximation of test function and trial solution
4. Discrete FE system of equations
5. Solution and postprocessing

Recap: FE formulation an axially-loaded elastic bar



equilibrium (balance of forces)

$$-p(x) + b(x + \Delta x/2)\Delta x + p(x + \Delta x) = 0$$

$$\frac{p(x + \Delta x) - p(x)}{\Delta x} + b(x + \Delta x/2) = 0$$

$$\implies \frac{\mathrm{d}p}{\mathrm{d}x} + b(x) = 0$$

kinematic (strain - displacement relation)

$$\varepsilon(x) = \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

$$\implies \varepsilon(x) = \frac{\mathrm{d}u(x)}{\mathrm{d}x} \qquad [-]$$

kinetics (stress)

$$\sigma(x) = \frac{p(x)}{A(x)} \qquad [\mathrm{N/m^2}]$$

constitutive relation

$$\sigma(x) = E(x)\varepsilon(x)$$

E : Young's modulus $[\mathrm{N/m^2}]$

governing equation and boundary conditions

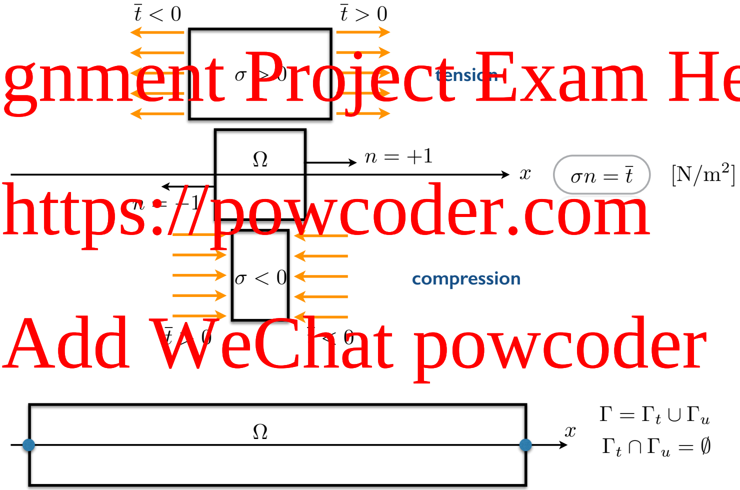
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + b = 0 \qquad \Omega := 0 < x < l$$

$$\sigma n = E\frac{\mathrm{d}u}{\mathrm{d}x}n = \bar{t} \qquad \text{on } \Gamma_t$$

$$u = \bar{u} \qquad \text{on } \Gamma_u$$

solve for $u(x)$

A note on sign conventions in linear elasticity



$$\Gamma = \Gamma_t \cup \Gamma_u$$
$$\Gamma_t \cap \Gamma_u = \emptyset$$

Weak form in 1D

Strong form for linear elasticity

$$\frac{\mathsf{d}}{\mathsf{d}x}\left(AE\frac{\mathsf{d}u}{\mathsf{d}x}\right) + b = 0 \qquad 0 < x < l \tag{1}$$

$$\sigma n = \left(E\frac{\mathsf{d}u}{\mathsf{d}x}\right) n = \bar{t} \qquad \text{on } \Gamma_t \tag{2}$$

$$u = \bar{u} \qquad \text{on } \Gamma_u \tag{3}$$

$u(x)$: *trial solution*
Introduce $\delta u(x)$: *arbitrary* weight (test) function

$$(1) \rightarrow \qquad \int_0^l \delta u\left(\frac{\mathsf{d}}{\mathsf{d}x}\left(AE\frac{\mathsf{d}u}{\mathsf{d}x}\right) + b\right)\,\mathsf{d}x = 0 \qquad \forall \delta u \tag{4}$$

$$(2) \rightarrow \qquad \delta u A\left(E\frac{\mathsf{d}u}{\mathsf{d}x} n - \bar{t}\right)\Big|_{\Gamma_t} \qquad \forall \delta u \tag{5}$$

Some properties of the test function δu

- The essential boundary condition is enforced directly (not weakly)
- $\delta u(\Gamma_u) = 0$ - the test function is zero on the essential boundary!

Weak form in 1D (linear elasticity)

Simplify the current integral expression:

- Decrease the degree of differentiability (continuity) required by the trial solution $u(x)$
- Make the problem symmetric (computational advantages)

Product rule

$$\int_0^l \frac{\mathrm{d}}{\mathrm{d}x}(\delta u f)\ \mathrm{d}x = \int_0^l \frac{\mathrm{d}\delta u}{\mathrm{d}x} f\ \mathrm{d}x + \int_0^l \delta u \frac{\mathrm{d}f}{\mathrm{d}x}\ \mathrm{d}x$$

Integration by parts

$$\int_0^l \delta u \frac{\mathrm{d}f}{\mathrm{d}x}\ \mathrm{d}x = \left[(\delta u f)|_{x=l} - (\delta u f)|_{x=0}\right] - \int_0^l f \frac{\mathrm{d}\delta u}{\mathrm{d}x}\ \mathrm{d}x$$

$$= (\delta u f n)|_{\Gamma} - \int_0^l f \frac{\mathrm{d}\delta u}{\mathrm{d}x}\ \mathrm{d}x$$

$$= (\delta u f n)|_{\Gamma_u} + (\delta u f n)|_{\Gamma_t} - \int_0^l f \frac{\mathrm{d}\delta u}{\mathrm{d}x}\ \mathrm{d}x$$

Weak form in 1D (linear elasticity)

$$Eq.\ (4)_1 \rightarrow \qquad \int_0^l \delta u \frac{\mathsf{d}}{\mathsf{d}x}\left(AE\frac{\mathsf{d}u}{\mathsf{d}x}\right)\,\mathsf{d}x = \left(\delta u A \underbrace{\left[E\frac{\mathsf{d}u}{\mathsf{d}x}\right]}_{\sigma} n\right)\Big|_{\Gamma} - \int_0^l \frac{\mathsf{d}\delta u}{\mathsf{d}x} AE \frac{\mathsf{d}u}{\mathsf{d}x}\,\mathsf{d}x$$

Make use of

- $\delta u = 0$ on Γ_u
- Natural boundary condition Eq. (5) : $\delta u A\left(E\frac{\mathsf{d}u}{\mathsf{d}x}n - \bar{t}\right)\Big|_{\Gamma_t}$

Weak form

Find $u(x)$ among the smooth functions that satisfy $u\big|_{\Gamma_u} = \bar{u}$ such that

$$\int_0^l \frac{\mathsf{d}\delta u}{\mathsf{d}x} AE \frac{\mathsf{d}u}{\mathsf{d}x}\,\mathsf{d}x = (\delta u A \bar{t})\big|_{\Gamma_t} + \int_0^l \delta u\, b\,\mathsf{d}x \quad \forall \delta u \text{ with } \delta u(\Gamma_u) = 0 \qquad (6)$$

- We can show that a solution $u(x)$ of the weak form Eq. (6) satisfies the strong form (equilibrium equation and natural boundary condition)
- We need to consider what properties the functions u and δu must possess for Eq. (6) to make sense (smoothness requirements)

Weak form in 1D (linear elasticity) - in detail

$$\int_\Omega \delta u \frac{\mathrm{d}}{\mathrm{d}x} \underbrace{\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)}_{g(x)} \mathrm{d}x + \int_\Omega \delta u b \, \mathrm{d}x = 0 \qquad (\star)$$

Now consider the *first term* in $(\star)$ (using the product rule):

$$\int_\Omega \delta u \frac{\mathrm{d}g}{\mathrm{d}x} \, \mathrm{d}x = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}x}(\delta u g) \, \mathrm{d}x - \int_\Omega \frac{\mathrm{d}\delta u}{\mathrm{d}x} g \, \mathrm{d}x$$

Using the Fundamental Theorem of Calculus to expand the first term on the RHS above:

$$\int_\Omega \delta u \frac{\mathrm{d}g}{\mathrm{d}x} \, \mathrm{d}x = (\delta u g)\big|_{x=l} - (\delta u g)\big|_{x=0} - \int_\Omega \frac{\mathrm{d}\delta u}{\mathrm{d}x} g \, \mathrm{d}x$$

Recall that the boundary $\Gamma = \Gamma_u \cup \Gamma_t$ is the points $x = 0$ and $x = l$. From the definition of the normal n where $n = 1$ at $x = l$, and $n = -1$ at $x = 0$

$$\int_\Omega \delta u \frac{\mathrm{d}g}{\mathrm{d}x} \, \mathrm{d}x = (\delta u g n)\big|_{\Gamma_u} + (\delta u g n)\big|_{\Gamma_t} - \int_\Omega \frac{\mathrm{d}\delta u}{\mathrm{d}x} g \, \mathrm{d}x$$

Using the property of the test function that $\delta u = 0$ on the essential boundary Γ_u we get

$$\int_\Omega \delta u \frac{\mathrm{d}g}{\mathrm{d}x} \, \mathrm{d}x = (\delta u g n)\big|_{\Gamma_t} - \int_\Omega \frac{\mathrm{d}\delta u}{\mathrm{d}x} g \, \mathrm{d}x$$

Using natural boundary condition for the traction $AE(\mathrm{d}u/\mathrm{d}x)n = A\bar{t}$ on Γ_t and the definition of $g(x) = AE(\mathrm{d}u/\mathrm{d}x)$:

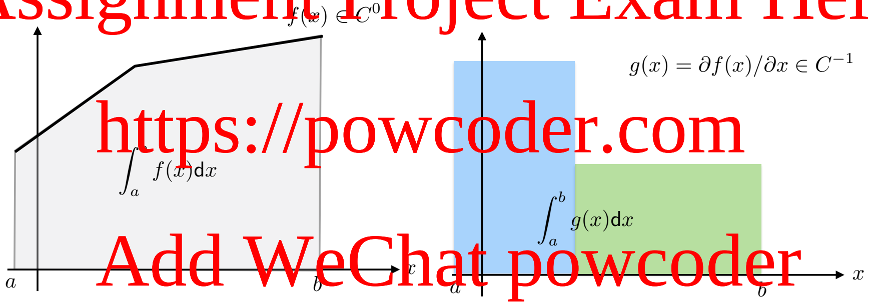
the first term on the LHS of $(\star)$ becomes $\longrightarrow \displaystyle\int_\Omega \delta u \frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) \mathrm{d}x = (\delta u A\bar{t})\big|_{\Gamma_t} - \int_\Omega \frac{\mathrm{d}\delta u}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) \mathrm{d}x$

finally we substitute the above into $(\star)$ to get $\longrightarrow \displaystyle\int_\Omega \frac{\mathrm{d}\delta u}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x = (\delta u A\bar{t})|_{\Gamma_t} + \int_\Omega \delta u b \, \mathrm{d}x$

Continuity

C^m continuous functions

$$C^m(a,b) = \{u : u, \mathrm{d}u/\mathrm{d}x, \ldots, \mathrm{d}^m u/\mathrm{d}x^m \text{ are all continuous functions}\}$$

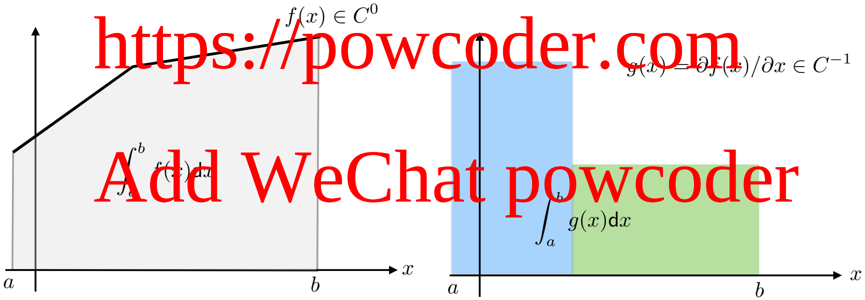


Integrability

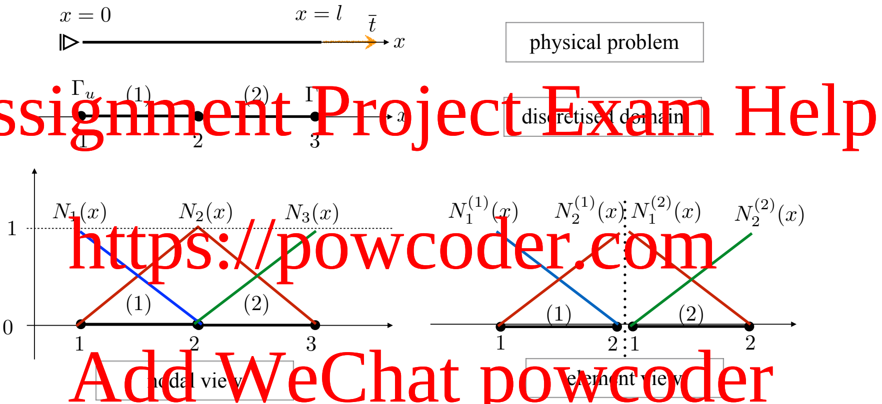
In order for the weak form to make sense we need to evaluate integrals of the general form

$$\int_0^l \frac{\mathsf{d}\delta u}{\mathsf{d}x} EA \frac{\mathsf{d}u}{\mathsf{d}x} \, \mathsf{d}x$$

If δu and u are C^0 functions then $\mathsf{d}\delta u/\mathsf{d}x \in C^{-1}$ and $\mathsf{d}u/\mathsf{d}x \in C^{-1}$ and their product is also in C^{-1}. All C^{-1} functions are integrable.



Discrete equations for a simple problem



$$N_2(x) = N_2^{(1)}(x) + N_1^{(2)}(x)$$

$$u(x) = \underbrace{\begin{bmatrix} N_1(x) & N_2(x) & N_3(x) \end{bmatrix}}_{\boldsymbol{N}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\boldsymbol{d}}$$

$$u^e(x) = \underbrace{\begin{bmatrix} N_1^e(x) & N_2^e(x) \end{bmatrix}}_{\boldsymbol{N}^e} \underbrace{\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}}_{\boldsymbol{d}^e}$$

Discrete equations for a simple problem

Weak form (W)

Find $u(x)$ that satisfies $u\big|_{\Gamma_u} = \overline{u}$ such that

$$\int_0^l \left[\frac{\mathrm{d}\delta u}{\mathrm{d}x}\right]^{\mathsf{T}} AE \left[\frac{\mathrm{d}u}{\mathrm{d}x}\right] \mathrm{d}x - (\delta u^{\mathsf{T}} A \overline{t})\big|_{\Gamma_t} - \int_0^l \delta u^{\mathsf{T}} b \, \mathrm{d}x = 0 \quad \forall \delta u \text{ with } \delta u\big|_{\Gamma_u} = 0$$

Take an element view

$$\int_0^l (\bullet)\mathrm{d}x = \sum_{e=1}^{n_{\text{el}}} \int_{x_1^e}^{x_2^e} (\bullet)\mathrm{d}x$$

Note: henceforth we label the boundaries as per the example problem

Approximate test functions and trial solutions

$$\delta u(x) \approx \delta u^h(x) = \boldsymbol{N}(x)\delta\boldsymbol{d} \qquad \text{and} \qquad u(x) \approx u^h(x) = \boldsymbol{N}(x)\boldsymbol{d}$$

$$u_1 = \overline{u}_1 \qquad \text{and} \qquad \delta u_1 = 0$$

Discrete equations for a simple problem

Substitute approximations into W:

$$\sum_{e=1}^{n_{\text{el}}} \left[\int_{x_1^e}^{x_2^e} \left[\frac{\mathrm{d}\delta u^e}{\mathrm{d}x}\right]^{\mathsf{T}} A^e E^e \left[\frac{\mathrm{d}u^e}{\mathrm{d}x}\right] \, \mathrm{d}x - (\delta u^{e\mathsf{T}} A^e \bar{t})|_{x=l} - \int_{x_1^e}^{x_2^e} \delta u^{e\mathsf{T}} b \, \mathrm{d}x \right] = 0$$

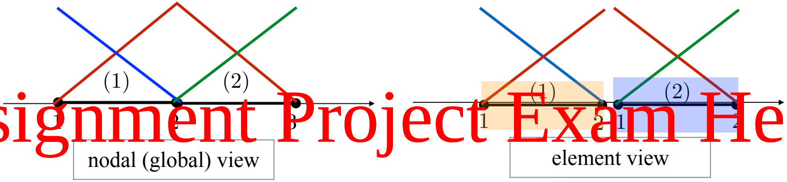
Use element-level approximations:

$$u^e(x) = \boldsymbol{N}^e \boldsymbol{d}^e \qquad \frac{\mathrm{d}u^e(x)}{\mathrm{d}x} = \boldsymbol{B}^e \boldsymbol{d}^e$$

$$\delta u^e(x)^{\mathsf{T}} = \delta \boldsymbol{d}^{e\mathsf{T}} \boldsymbol{N}^{e\mathsf{T}} \qquad \left[\frac{\mathrm{d}\delta u^e(x)}{\mathrm{d}x}\right]^{\mathsf{T}} = \delta \boldsymbol{d}^{e\mathsf{T}} \boldsymbol{B}^{e\mathsf{T}}$$

$$\sum_{e=1}^{n_{\text{el}}} \delta \boldsymbol{d}^{e\mathsf{T}} \left(\underbrace{\int_{x_1^e}^{x_2^e} \left[\boldsymbol{B}^{e\mathsf{T}} A^e E^e \boldsymbol{B}^e\right] \mathrm{d}x}_{\boldsymbol{K}^e} \, \boldsymbol{d}^e - \underbrace{(\boldsymbol{N}^{e\mathsf{T}} A^e \bar{t})|_{x=l}}_{\boldsymbol{f}_{\Gamma^e}} - \underbrace{\int_{x_1^e}^{x_2^e} \boldsymbol{N}^{e\mathsf{T}} b \, \mathrm{d}x}_{\boldsymbol{f}_{\Omega^e}} \right) = 0$$

Assembly



assembly

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \qquad \boldsymbol{K} = \mathop{\mathsf{A}}_{e} \boldsymbol{K}^e$$

$$\boldsymbol{K}^{(1)} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{bmatrix} \qquad \boldsymbol{K}^{(2)} = \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix}$$

$$\boldsymbol{f}_\Omega = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \qquad \boldsymbol{f}_\Omega = \mathop{\mathsf{A}}_{e} \boldsymbol{f}_\Omega^e$$

$$\boldsymbol{f}_\Omega^{(1)} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{bmatrix} \qquad \boldsymbol{f}_\Omega^{(2)} = \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{bmatrix}$$

$$\boldsymbol{d} = \mathop{\mathsf{A}}_{e} \boldsymbol{d}^e \qquad \delta\boldsymbol{d} = \mathop{\mathsf{A}}_{e} \delta\boldsymbol{d}^e \qquad \boldsymbol{f}_\Gamma = \mathop{\mathsf{A}}_{e} \boldsymbol{f}_\Gamma^e$$

Discrete equations for a simple problem

$$\sum_{e=1}^{n_{\text{el}}} \delta \boldsymbol{d}^{e\mathsf{T}} \left(\underbrace{\int_{x_1^e}^{x_2^e} \left[\boldsymbol{B}^{e\mathsf{T}} A^e E^e \boldsymbol{B}^e \right] \, \mathrm{d}x}_{\boldsymbol{K}^e} \, \boldsymbol{d}^e - \underbrace{(\boldsymbol{N}^{e\mathsf{T}} A^e \bar{t})|_{x=l}}_{\boldsymbol{f}_{\Gamma^e}} - \underbrace{\int_{x_1^e}^{x_2^e} \boldsymbol{N}^{e\mathsf{T}} b \, \mathrm{d}x}_{\boldsymbol{f}_{\Omega^e}} \right) = 0$$

Assemble the equations

$$\delta \boldsymbol{d}^{\mathsf{T}} \underbrace{\left[\boldsymbol{K} \boldsymbol{d} - \underbrace{(\boldsymbol{f}_\Gamma + \boldsymbol{f}_\Omega)}_{\boldsymbol{f}} \right]}_{\boldsymbol{r}} = 0 \qquad \forall \delta \boldsymbol{d} \text{ except } \delta u = 0 \text{ on } \Gamma_u$$

$$\implies \quad \boldsymbol{r} = \boldsymbol{K} \boldsymbol{d} - \boldsymbol{f}$$

- The residual vector $\boldsymbol{r}$ contains zeros except for the node on Γ_u!
- We don't have an equation to solve at Γ_u - we need to impose the essential boundary condition $u = \overline{u}$
- We will compute the non-zero entry in $\boldsymbol{r}$ after solving the problem
  - the non-zero entry is the reaction force to impose the constraint

Discrete equations for a simple problem

Write in terms of the global unknowns which for our problem

$$
\underbrace{\begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{r}} = \underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}}_{\boldsymbol{K}} \underbrace{\begin{bmatrix} \overline{u}_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\boldsymbol{d}} - \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{\boldsymbol{f}}
$$

$$
\implies \quad \underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}}_{\boldsymbol{K}} \underbrace{\begin{bmatrix} \overline{u}_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\boldsymbol{d}} = \underbrace{\begin{bmatrix} f_1 + r_1 \\ f_2 \\ f_3 \end{bmatrix}}_{\boldsymbol{f}+\boldsymbol{r}}
$$

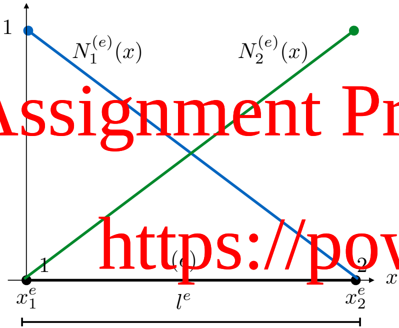
Need to impose boundary conditions using partition approach and solve:

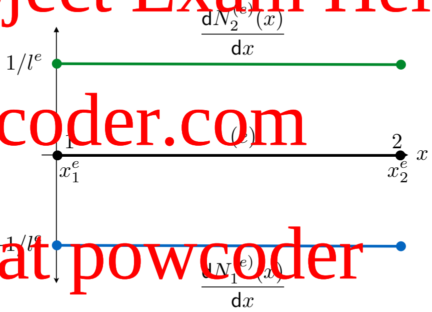
$$
\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_2 - K_{21}\overline{u}_1 \\ f_3 - K_{31}\overline{u}_1 \end{bmatrix}
$$

Then post-process for the reaction force:

$$
r_1 = \begin{bmatrix} K_{11} & K_{12} & K_{13} \end{bmatrix} \begin{bmatrix} \overline{u}_1 \\ u_2 \\ u_3 \end{bmatrix} - f_1
$$

Shape functions for two-noded elements





$$\boldsymbol{N}^e = \frac{1}{l^e}\begin{bmatrix}(x_2^e - x) & (x - x_1^e)\end{bmatrix}$$

$$\boldsymbol{B}^e = \frac{1}{l^e}\begin{bmatrix}-1 & 1\end{bmatrix}$$

Element matrices for two-noded elements

$$\boldsymbol{N}^e = \frac{1}{l^e}\begin{bmatrix}(x_2^e - x) & (x - x_1^e)\end{bmatrix} \qquad \text{and} \qquad \boldsymbol{B}^e = \frac{1}{l^e}\begin{bmatrix}-1 & 1\end{bmatrix}$$

$$\boldsymbol{K}^e = \int_{x_1^e}^{x_2^e} \boldsymbol{B}^{e\top} A^e E^e \boldsymbol{B}^e \,\mathrm{d}x = \int_{x_1^e}^{x_2^e} \frac{1}{l^e}\begin{bmatrix}-1\\ 1\end{bmatrix} A^e E^e \frac{1}{l^e}\begin{bmatrix}-1 & 1\end{bmatrix} \mathrm{d}x = \frac{A^e E^e}{l^{e2}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\int_{x_1^e}^{x_2^e} \mathrm{d}x$$

Element stiffness matrix $\boldsymbol{K}^e$

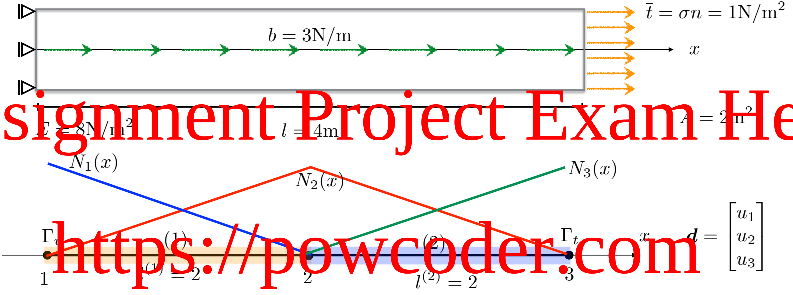
$$\boldsymbol{K}^e = \frac{A^e E^e}{l^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\boldsymbol{f}_\Omega^e = \int_{x_1^e}^{x_2^e} \boldsymbol{N}^{e\top} b(x)\,\mathrm{d}x = \int_{x_1^e}^{x_2^e} \boldsymbol{N}^{e\top}\boldsymbol{N}^e \,\mathrm{d}x\boldsymbol{b}^e = \frac{1}{l^{e2}}\int_{x_1^e}^{x_2^e}\begin{bmatrix}(x_2^e - x)^2 & (x_2^e - x)(x - x_1^e)\\ (x_2^e - x)(x - x_1^e) & (x - x_1)^{e2}\end{bmatrix}\mathrm{d}$$

Element body force vector $\boldsymbol{f}_\Omega^e$

$$\boldsymbol{f}_\Omega^e = \frac{l^e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}b_1^e\\ b_2^e\end{bmatrix}$$

Example problem: linear elasticity



stiffness matrix

$$\boldsymbol{K}^e = \frac{A^e E^e}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

$$\boldsymbol{K}^{(1)} = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} \qquad \boldsymbol{K}^{(2)} = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 8 & -8 & 0 \\ -8 & 16 & -8 \\ 0 & -8 & 8 \end{bmatrix}$$

body force vector

$$\boldsymbol{f}^e_\Omega = \frac{l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} b^e_1 \\ b^e_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\boldsymbol{f}^{(1)}_\Omega = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad \boldsymbol{f}^{(2)}_\Omega = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\boldsymbol{f}_\Omega = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

traction force vector

$$\boldsymbol{f}^{(1)}_\Gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{f}^{(2)}_\Gamma = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\boldsymbol{f}_\Gamma = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Example problem: linear elasticity. Partition and solution

$$\underbrace{\begin{bmatrix} 8 & -8 & 0 \\ -8 & 16 & -8 \\ 0 & -8 & 8 \end{bmatrix}}_{\boldsymbol{K}} \begin{bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}}_{\boldsymbol{f}_\Omega} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}_{\boldsymbol{f}_\Gamma} + \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 + 3 \\ 6 \\ 5 \end{bmatrix}$$

$$\Longrightarrow \quad \underbrace{\begin{bmatrix} 16 & -8 \\ -8 & 8 \end{bmatrix}}_{\tilde{\boldsymbol{K}}} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

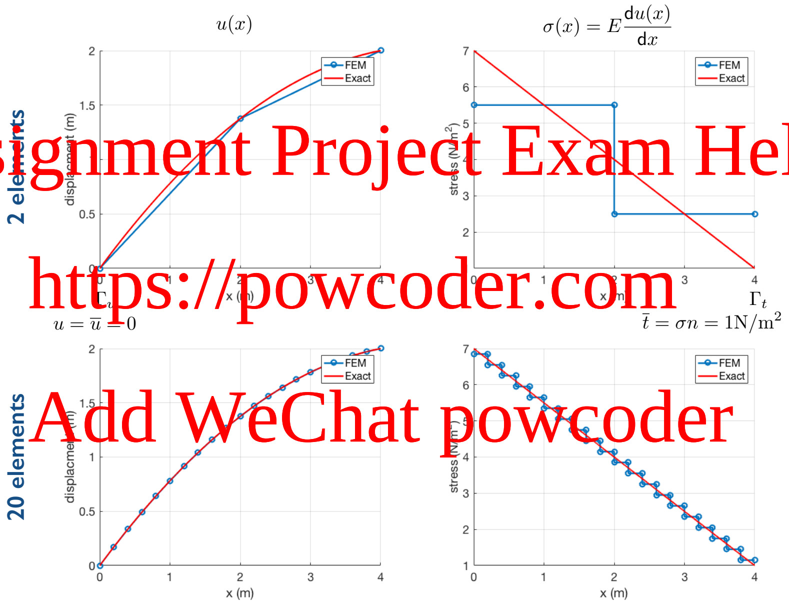
$$\Longrightarrow \quad \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1.375 \\ 2 \end{bmatrix}$$

$$\boldsymbol{d} = \begin{bmatrix} 0 \\ 1.375 \\ 2 \end{bmatrix} \qquad r_1 = \begin{bmatrix} K_{11} & K_{12} & K_{13} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \end{bmatrix} - f_1 = -14$$

Check (sum of forces):

$$bl + \bar{t}A + r_1 = 3 * 4 + 1 * 2 - 14 = 0 \checkmark$$

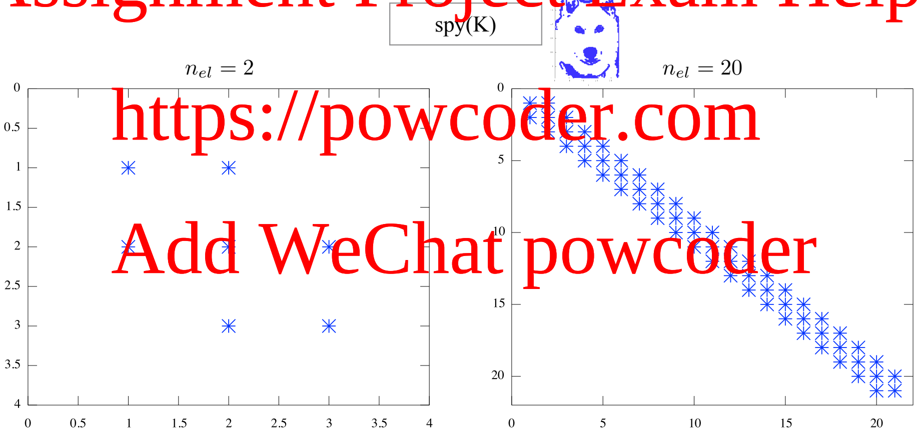
Example problem: linear elastic rod - results



see the Python code

Example problem: linear elastic rod - properties of $\boldsymbol{K}$

- Sparse (mainly contains zeros)
- Symmetric ($\boldsymbol{K} = \boldsymbol{K}^\mathsf{T}$)
- Banded (fixed band within which non-zero entries are clustered)



Key knowledge

- Make sure you understand all the 1D finite elements covered in the previous course (notes are available on Moodle)
- Ensure that your knowledge of Python is sufficient to understand the finite element example

Structural elements

Where are we?

1. Course information

2. The course in pictures

3. The finite element method: a refresher

4. Structural elements
   - Introduction and overview
   - Beam theory: notation and geometrically exact theory
   - Euler–Bernoulli beam theory
   - Timoshenko beams
   - Summary of plate and shell theory

5. Time-dependent problems: Dynamics
   - Overview
   - Introduction to wave propagation in elastic media
   - Finite element method for wave propagation

Overview

- Beams, plates and shells are some of the most important structural elements in engineering
- Analytical solutions for the response of structural elements require significant assumptions to be tractable:
  - infinitesimal (small) deformations
  - simple geometries
  - basic material (constitutive) models (e.g. linear elasticity)
  - therefore they can be of limited use for engineering applications
- Computational methods (e.g. the finite element method FEM) can solve a far greater range of problems
- Elements for beams, plates, and shells are available in the vast majority of finite element codes
  - intelligent use of such codes and interpretation of the results requires a basic understanding of the numerical models
  - "*essentially, all models are wrong, but some are useful*": George Box
    Very true of models for structural elements
- Advantages of the FEM:
  - handle geometrically complex geometries produced using CAD packages
  - handle complex material (constitutive) models: e.g. elasticity, plasticity, damage
  - account for dynamics and multi-physics
  - cost-effective and fast relative to experiment

Examples: civil and mechanical engineering

Civil engineering

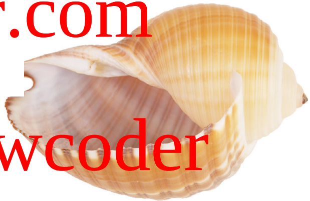
Great Court, British Museum

Mechanical engineering

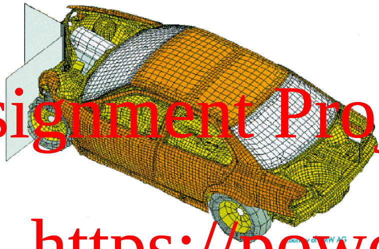
Examples: consumer products and nature

Consumer products

Nature

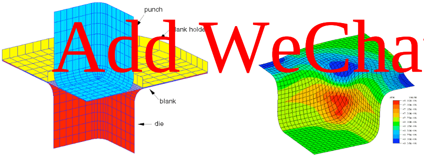


Examples: FE simulations

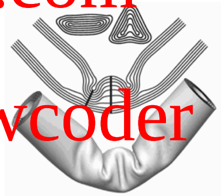


Virtual crash test (BMW)

Wrinkling of an inflated party balloon

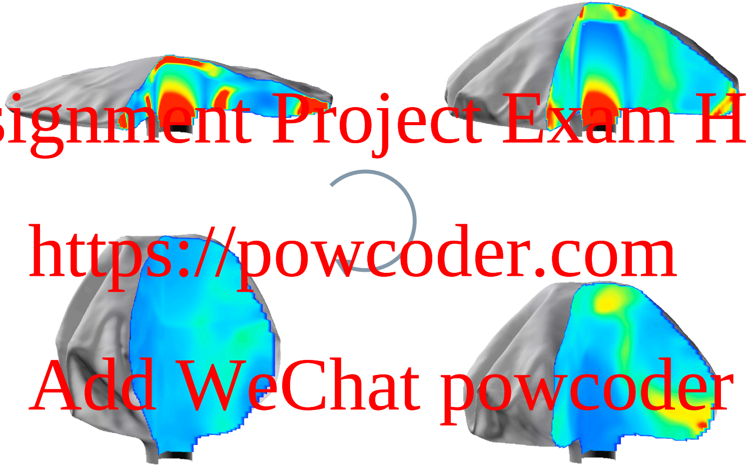


Sheet metal stamping (Abaqus)

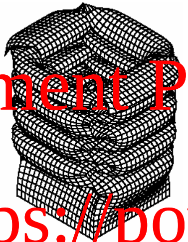


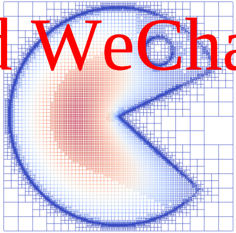
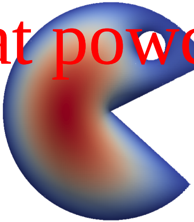
buckling of carbon nanotubes

Examples: FE simulations - inflation of an airbag



Examples: FE simulations

Why not use 3d solid elements?

- Potentially huge number of elements required to model thin structure
  - need to ensure that the elements are not overly distorted,
  - structural elements can be very efficient computationally

- All structural elements are constructed by restricting the 3d theory
- The type and nature of the restrictions define the applicability of the reduced model

Stuttgart 21: a case study (see Moodle resources for the full paper)

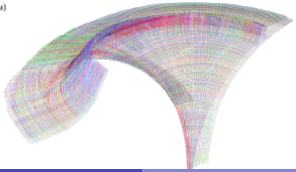
The Railway Station "Stuttgart 21"

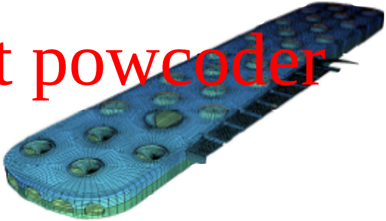
Structural Modelling and Fabrication of Double Curved Concrete Surfaces

Lucio Blandini, Albert Schuster, and Werner Sobek

It is often surprising how light and elegant concrete can appear once used for spatially curved surfaces. However the engineering of such complex structural elements is an extremely challenging task, which requires close cooperation between the different professionals involved. The structural engineer should not only master the understanding of the structural behaviour of shells and the material properties of concrete, but should also be capable to develop adequate computational tools to optimize the structural behaviour of the design.

a)

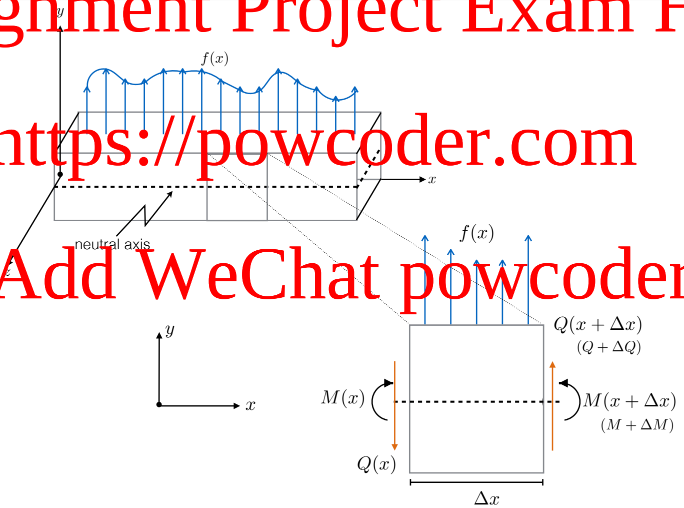




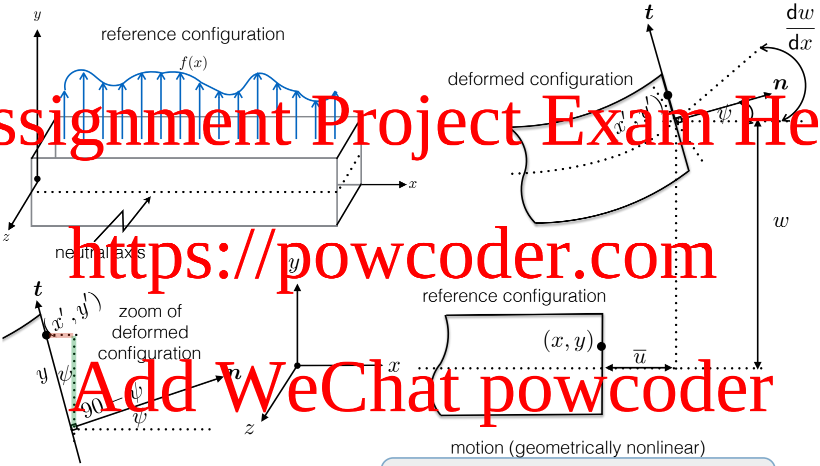
Where are we?

Notation

Nearly every reference book and finite element code adopts a different notation for structural elements!
Please compare the notation used here to Structural Analysis 4 as there are minor differences.



Beam theory: full (geometrically exact) theory



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + \overline{u}(x) \\ w(x) \end{bmatrix} + y \begin{bmatrix} -\sin\psi(x) \\ \cos\psi(x) \end{bmatrix}$$

Plane sections remain plane

Plane sections orthogonal to the neutral axis do not necessarily remain orthogonal to the neutral axis

$$\psi \neq \frac{\mathrm{d}w}{\mathrm{d}x}$$

Key questions

- Why is the previous theory called *geometrically nonlinear*?
- Why do we assume that *plane sections remain plane*?
- What deformation mode would be exclude if plane section were constrained to remain normal to the neutral axis?

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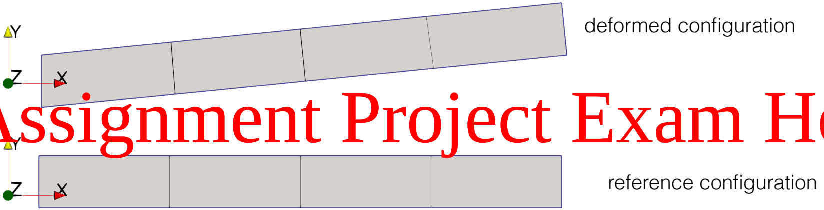
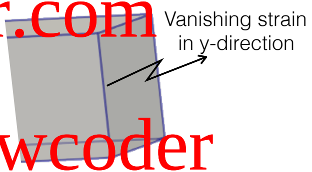
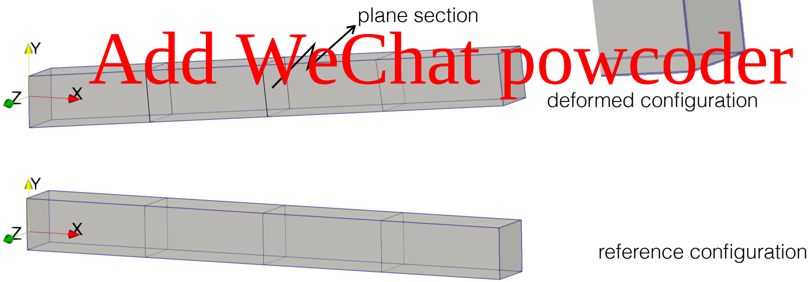
Beam theory: notation

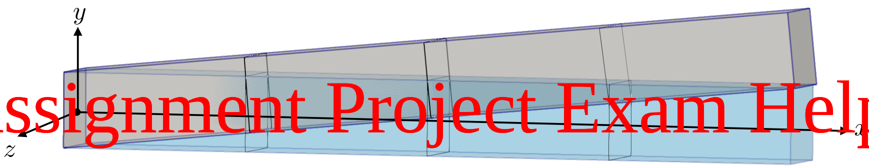


Plane sections remain plane
Assume small deformations.

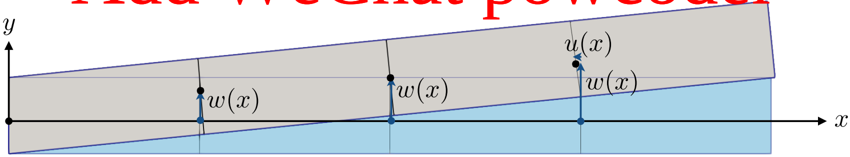




Beam theory: notation



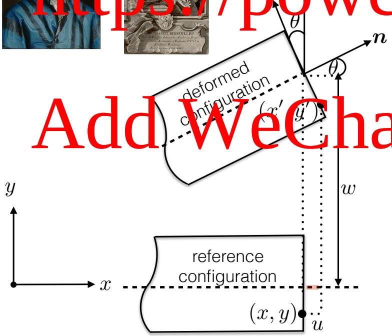
Want to develop a 1D theory for the defection of the beam $w(x)$



Euler–Bernoulli beam theory: motion

Key assumptions

- Plane sections remain plane
- Plane sections orthogonal to the neutral axis remain orthogonal to the neutral axis
- $\sigma_{yy} = 0$, i.e. no normal stresses orthogonal to the axis of the beam



assumed for simplicity $\overline{u} \equiv 0$

θ rotation of neutral axis (positive counterclockwise)

motion

$$u(x) = x' - x = -y \sin\theta(x)$$

if $\theta \ll 1$ then $\sin\theta \approx \theta$

$$u(x) \approx -y\theta(x)$$

$$\theta = \frac{\mathrm{d}w}{\mathrm{d}x}$$

$$\implies u(x) = -y\frac{\mathrm{d}w}{\mathrm{d}x}$$

Euler–Bernoulli beam theory: kinematics

Strain measures

$$\epsilon := \epsilon_{xx} = \frac{\mathrm{d}u}{\mathrm{d}x} = -y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \qquad \text{(axial strain = bending or flexural strain)}$$

$$\epsilon_{yy} = \frac{\mathrm{d}w}{\mathrm{d}y} = 0 \quad \text{(vanishing strain through the thickness)}$$

$$\gamma := 2\epsilon_{xy} = \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \quad \text{(vanishing transverse shear strain)}$$

- No change in angle between line segments in direction $\boldsymbol{n}$ and $\boldsymbol{t}$: hence $\gamma = 0$
- We have assumed $w = w(x)$, i.e. w does not vary with height (infinitesimal deformations): that is, no stretching of a line normal to the neutral axis

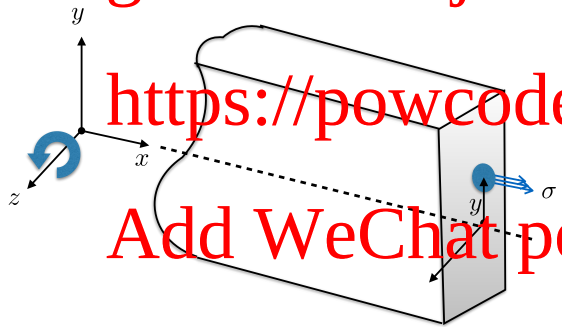
- We have now derived the *kinematics*, next we need to determine the *kinetics* - i.e. the stress-like quantities

Euler–Bernoulli beam theory: constitutive relations

Hooke's law

$$\sigma = \sigma_{xx} = E\epsilon_{xx} = -Ey\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -Ey\chi$$

Want to develop a 1D theory. Average over the y–z plane.



$$M_z = -\int y\sigma \, \mathrm{d}a$$

$$= \int y^2 E \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \, \mathrm{d}a$$

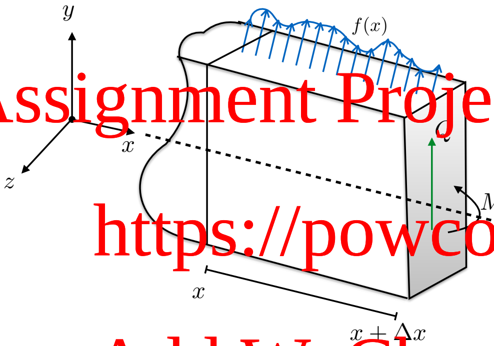
and if E is constant

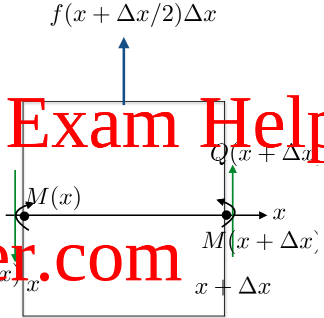
$$= E \underbrace{\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}}_{\chi} \underbrace{\int y^2 \, \mathrm{d}a}_{I}$$

constitutive relation: $\boxed{M_z = EI\chi}$

χ: curvature and I: moment of inertia

Euler–Bernoulli beam theory: balance of forces and moments





balance of forces:

$$Q(x + \Delta x) - Q(x) + f(x + \Delta x/2)\Delta x = 0$$
$$\implies \frac{\mathrm{d}Q}{\mathrm{d}x} = -f$$

balance of moments:

$$M(x + \Delta x) - M(x) + \Delta x Q(x + \Delta x) + \frac{1}{2}\Delta x^2 f(x + \Delta x/2)) = 0$$
$$\implies \frac{\mathrm{d}M}{\mathrm{d}x} = -Q$$

$$
\begin{aligned}
-f &= \frac{\mathrm{d}Q}{\mathrm{d}x} \\
&= -\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} \\
&= -EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}
\end{aligned}
$$

Governing differential equation (strong form)

$$
EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = f
$$

- Fourth-order differential equation
  - Implications for the finite element approximation and boundary conditions!

Euler–Bernoulli beam theory: governing differential equation and bc's

Governing differential equation (strong form)

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = f$$

Boundary conditions

$$w = \overline{w} \qquad \text{on } \Gamma_w$$

$$\frac{\partial w}{\partial x} = \overline{\theta} \qquad \text{on } \Gamma_\theta$$

$$Mn = EI\frac{\partial^2 w}{\partial x^2}n = \overline{M} \qquad \text{on } \Gamma_M$$

$$Qn = -EI\frac{\partial^3 w}{\partial x^3}n = \overline{Q} \qquad \text{on } \Gamma_Q$$

Note: $\overline{M}$ is *positive* when acting counter-clockwise. $\overline{Q}$ is *positive* when acting in the positive y-direction.

The n is to ensure consistency between the definition of positive internal moments and shear forces on faces and the sign convention

Boundary conditions

free end with applied load

$$Qn = \overline{Q} \quad \text{on } \Gamma_Q$$
$$Mn = \overline{M} \quad \text{on } \Gamma_M$$

clamped support

$$\overline{w} = 0 \quad \text{on } \Gamma_w$$
$$\overline{\theta} = 0 \quad \text{on } \Gamma_\theta$$

simple support

$$\overline{M} = 0 \quad \text{on } \Gamma_M$$
$$\overline{w} = 0 \quad \text{on } \Gamma_w$$

- Two boundary conditions are required at each boundary (4th order de)

$$\Gamma_Q \cap \Gamma_w = \emptyset \quad \text{and} \quad \Gamma_Q \cup \Gamma_w = \Gamma$$
$$\Gamma_M \cap \Gamma_\theta = \emptyset \quad \text{and} \quad \Gamma_M \cup \Gamma_\theta = \Gamma$$

Properties of the test functions

$$\delta w = 0 \quad \text{on } \Gamma_w$$
$$\partial \delta w / \partial x = 0 \quad \text{on } \Gamma_\theta$$

Weak form for Euler–Bernoulli beams

$$0 = \int_0^l \delta w EI \frac{\mathrm{d}^4 w}{\mathrm{d}x^4}\, \mathrm{d}x - \int_0^l \delta w f \,\mathrm{d}x = \underbrace{\int_0^l \delta w \frac{\mathrm{d}^2 M}{\mathrm{d}x^2}\, \mathrm{d}x}_{(A)} - \int_0^l \delta w f \,\mathrm{d}x$$

$$(A) = \int_0^l \frac{\mathrm{d}}{\mathrm{d}x}\left[\delta w \frac{\mathrm{d}M}{\mathrm{d}x}\right]\, \mathrm{d}x - \int_0^l \frac{\mathrm{d}\delta w}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x}\, \mathrm{d}x$$

$$= \left(\delta w \frac{\mathrm{d}M}{\mathrm{d}x} n\right)\Big|_\Gamma - \int_0^l \frac{\mathrm{d}\delta w}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x}\, \mathrm{d}x = -\left(\delta w \overline{Q}\right)\Big|_{\Gamma_Q} - \underbrace{\int_0^l \frac{\mathrm{d}\delta w}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x}\, \mathrm{d}x}_{(B)}$$

$$(B) = \int_0^l \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}\delta w}{\mathrm{d}x} M\right]\, \mathrm{d}x - \int_0^l \frac{\mathrm{d}^2 \delta w}{\mathrm{d}x^2} M\, \mathrm{d}x$$

$$= \left(\frac{\mathrm{d}\delta w}{\mathrm{d}x} M n\right)\Big|_\Gamma - \int_0^l \frac{\mathrm{d}^2 \delta w}{\mathrm{d}x^2} M\, \mathrm{d}x = \left(\frac{\mathrm{d}\delta w}{\mathrm{d}x}\overline{M}\right)\Big|_{\Gamma_M} - \int_0^l \frac{\mathrm{d}^2 \delta w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\, \mathrm{d}x$$

Weak form for Euler–Bernoulli beams

Find w such that

$$\int_0^l \frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x = \int_0^l \delta w f\,\mathrm{d}x + \left(\delta w \overline{Q}\right)\Big|_{\Gamma_Q} + \left(\frac{\mathrm{d}\delta w}{\mathrm{d}x}\overline{M}\right)\Big|_{\Gamma_M} \qquad \forall \delta w$$

where

$$\delta w = 0 \text{ on } \Gamma_w \qquad \text{and} \qquad \partial\delta w/\partial x = 0 \text{ on } \Gamma_\theta$$

$$Mn = EI\frac{\partial^2 w}{\partial x^2} n = \overline{M} \text{ on } \Gamma_M \qquad \text{and} \qquad Qn = -EI\frac{\partial^3 w}{\partial x^3} n = \overline{Q} \text{ on } \Gamma_Q$$

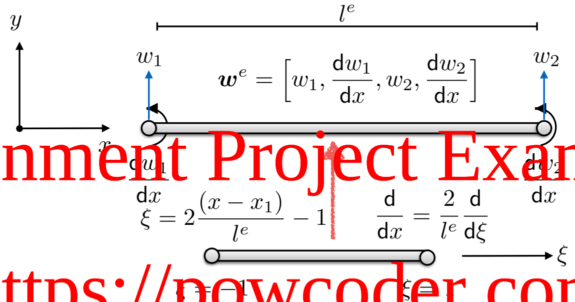
$$w = \overline{w} \text{ on } \Gamma_w \qquad \text{and} \qquad \frac{\partial w}{\partial x} = -\overline{\theta} \text{ on } \Gamma_\theta$$

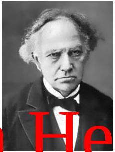
$$\Gamma_Q \cap \Gamma_w = \emptyset \qquad \text{and} \qquad \Gamma_Q \cup \Gamma_w = \Gamma$$

$$\Gamma_M \cap \Gamma_\theta = \emptyset \qquad \text{and} \qquad \Gamma_M \cup \Gamma_\theta = \Gamma$$

- It's now clear that Γ_Q and Γ_M correspond to the *natural* boundary conditions
- We require second-derivatives of the test and trial functions - C^1-continuity!
  - C^1-continuous finite elements are very complicated to construct in two and three dimensions
  - options include Natural Element Method, discontinuous Galerkin, isogeometric
- In one dimension we can use Hermite polynomials

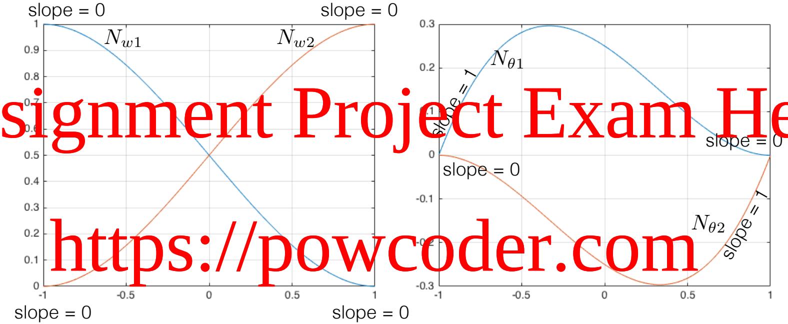
Hermite polynomials for Euler–Bernoulli beam elements





$$w^e = \boldsymbol{N}^e \boldsymbol{w}^e$$
$$\boldsymbol{N}^e = \{N_{w1}, N_{\theta 1}, N_{w2}, N_{\theta 2}\}$$
$$\frac{\mathrm{d}w^e}{\mathrm{d}x} = \boldsymbol{C}^e \boldsymbol{w}^e$$
$$\boldsymbol{C}^e = \left\{\frac{\mathrm{d}N_{w1}}{\mathrm{d}x}, \frac{\mathrm{d}N_{\theta 1}}{\mathrm{d}x}, \frac{\mathrm{d}N_{w2}}{\mathrm{d}x}, \frac{\mathrm{d}N_{\theta 2}}{\mathrm{d}x}\right\}$$
$$\frac{\mathrm{d}^2 w^e}{\mathrm{d}x^2} = \boldsymbol{B}^e \boldsymbol{w}^e$$
$$\boldsymbol{B}^e = \left\{\frac{\mathrm{d}^2 N_{w1}}{\mathrm{d}x^2}, \frac{\mathrm{d}^2 N_{\theta 1}}{\mathrm{d}x^2}, \frac{\mathrm{d}^2 N_{w2}}{\mathrm{d}x^2}, \frac{\mathrm{d}^2 N_{\theta 2}}{\mathrm{d}x^2}\right\}$$

$$\frac{\mathrm{d}(\bullet)}{\mathrm{d}x} = \frac{\mathrm{d}(\bullet)}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{\mathrm{d}(\bullet)}{\mathrm{d}\xi}\frac{2}{l^e}$$
$$\frac{\mathrm{d}^2(\bullet)}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}(\bullet)}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x}\right] = \left[\frac{\mathrm{d}^2(\bullet)}{\mathrm{d}\xi^2}\right]\left[\frac{\mathrm{d}\xi}{\mathrm{d}x}\right]^2$$
$$= \frac{\mathrm{d}^2(\bullet)}{\mathrm{d}\xi^2}\left[\frac{2}{l^e}\right]^2$$
$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}\xi}\mathrm{d}\xi = \frac{l^e}{2}\mathrm{d}\xi$$



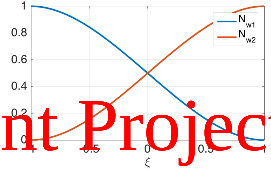
$$N_{w1} = \frac{1}{4}(1-\xi)^2(2+\xi)$$
$$N_{\theta 1} = \frac{l^e}{8}(1-\xi)^2(1+\xi)$$
$$N_{w2} = \frac{1}{4}(1+\xi)^2(2-\xi)$$
$$N_{\theta 2} = \frac{l^e}{8}(1+\xi)^2(\xi-1)$$

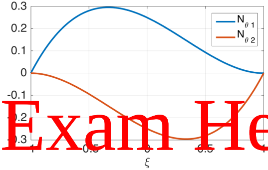
$$N_{wI}(x_J) = \delta_{IJ}$$
$$\frac{\mathrm{d}N_{\theta I}(x_J)}{\mathrm{d}x} = \delta_{IJ}$$

Note: N_θ depends on the length of the element.

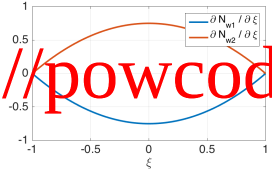
Hermite polynomials for Euler–Bernoulli beam elements

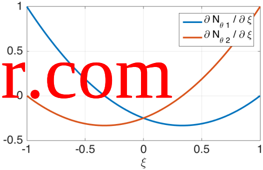
$$w^e = \boldsymbol{N}^e \boldsymbol{w}^e$$
$$\boldsymbol{N}^e = \{N_{w1}, N_{\theta 1}, N_{w2}, N_{\theta 2}\}$$



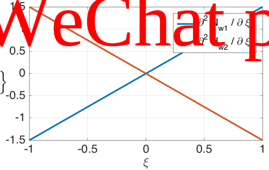


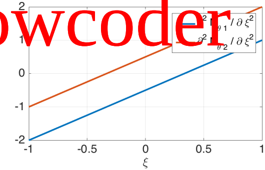
$$\frac{\mathrm{d}w^e}{\mathrm{d}x} = \boldsymbol{C}^e \boldsymbol{w}^e$$
$$\boldsymbol{C}^e = \left\{\frac{\mathrm{d}N_{w1}}{\mathrm{d}x}, \frac{\mathrm{d}N_{\theta 1}}{\mathrm{d}x}, \frac{\mathrm{d}N_{w2}}{\mathrm{d}x}, \frac{\mathrm{d}N_{\theta 2}}{\mathrm{d}x}\right\}$$





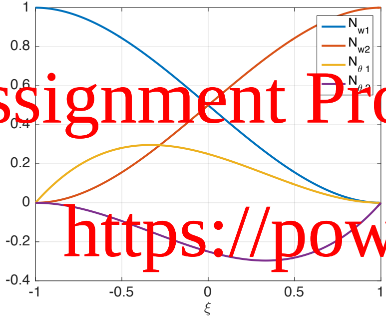
$$\frac{\mathrm{d}^2 w^e}{\mathrm{d}x^2} = \boldsymbol{B}^e \boldsymbol{w}^e$$
$$\boldsymbol{B}^e = \left\{\frac{\mathrm{d}^2 N_{w1}}{\mathrm{d}x^2}, \frac{\mathrm{d}^2 N_{\theta 1}}{\mathrm{d}x^2}, \frac{\mathrm{d}^2 N_{w2}}{\mathrm{d}x^2}, \frac{\mathrm{d}^2 N_{\theta 2}}{\mathrm{d}x^2}\right\}$$

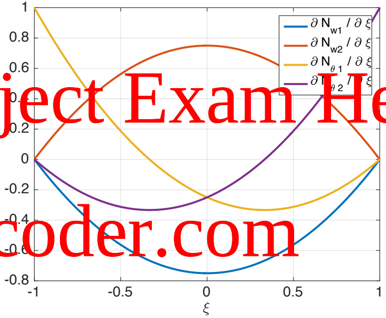




$$\int_0^l \frac{\mathrm{d}^2 \delta w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\, \mathrm{d}x = \int_0^l \delta w f \, \mathrm{d}x + \left(\delta w \overline{Q}\right)\Big|_{\Gamma_Q} + \left(\frac{\mathrm{d}\delta w}{\mathrm{d}x}\overline{M}\right)\Big|_{\Gamma_M} \qquad M = EI\frac{\partial^2 w}{\partial x^2} \quad \text{and} \quad Q = -EI\frac{\partial^3 w}{\partial x^3}$$

Hermite polynomials for Euler–Bernoulli beam elements





$$w^e = \boldsymbol{N}^e \boldsymbol{w}^e$$

$$\boldsymbol{N}^e = \{N_{w1}, N_{\theta 1}, N_{w2}, N_{\theta 2}\}$$

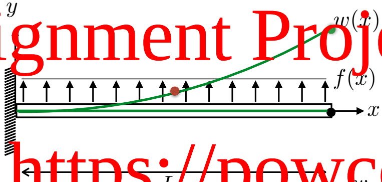
$$\frac{\mathrm{d}w^e}{\mathrm{d}x} = \boldsymbol{C}^e \boldsymbol{w}^e$$

$$\boldsymbol{C}^e = \left\{\frac{\mathrm{d}N_{w1}}{\mathrm{d}x}, \frac{\mathrm{d}N_{\theta 1}}{\mathrm{d}x}, \frac{\mathrm{d}N_{w2}}{\mathrm{d}x}, \frac{\mathrm{d}N_{\theta 2}}{\mathrm{d}x}\right\}$$

$$\boldsymbol{w}^e = \left\{w_1, \frac{\mathrm{d}w_1}{\mathrm{d}x}, w_2, \frac{\mathrm{d}w_2}{\mathrm{d}x}\right\}$$

Example: Hermite polynomials for Euler–Bernoulli beam elements

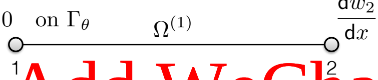
$$N_{w1} = \frac{1}{4}(1-\xi)^2(2+\xi)$$
$$N_{\theta 1} = \frac{l^e}{8}(1-\xi)^2(1+\xi)$$
$$N_{w2} = \frac{1}{4}(1+\xi)^2(2-\xi)$$
$$N_{\theta 2} = \frac{l^e}{8}(1+\xi)^2(\xi-1)$$



$$\boldsymbol{w}^e = \left\{ w_1, \frac{\mathrm{d}w_1}{\mathrm{d}x}, w_2, \frac{\mathrm{d}w_2}{\mathrm{d}x} \right\} = \left\{ 0, 0, w_2, \frac{\mathrm{d}w_2}{\mathrm{d}x} \right\}$$

$\overline{w} = 0$ on Γ_w

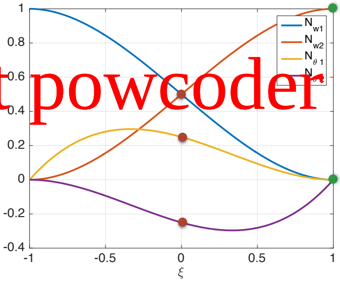
$\overline{\theta} = 0$ on Γ_θ



$$w^e = \boldsymbol{N}^e \boldsymbol{w}^e$$

$$= \begin{bmatrix} N_{w1} & N_{\theta 1} & N_{w2} & N_{\theta 2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ w_2 \\ \frac{\mathrm{d}w_2}{\mathrm{d}x} \end{bmatrix}$$

$$= N_{w2} w_2 + N_{\theta 2} \frac{\mathrm{d}w_2}{\mathrm{d}x}$$



Discrete problem for Euler–Bernoulli beams

Weak form

$$\int_0^l \frac{\mathrm{d}^2 \delta w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x = \int_0^l \delta w\, f\,\mathrm{d}x + \left(\delta w \overline{Q}\right)\Big|_{\Gamma_Q} + \left(\frac{\mathrm{d}\delta w}{\mathrm{d}x}\overline{M}\right)\Big|_{\Gamma_M}$$

Follow the same procedure as for elastic rods to obtain

Discrete problem

$$\boldsymbol{K}\boldsymbol{w} = \boldsymbol{f} + \boldsymbol{r}$$

$$\boldsymbol{K}^e = \int_{\Omega^e} E^e I^e \boldsymbol{B}^{e\mathsf{T}} \boldsymbol{B}^e\,\mathrm{d}x$$

$$\boldsymbol{f}^e = \underbrace{\int_{\Omega^e} \boldsymbol{N}^{e\mathsf{T}} f^e\,\mathrm{d}x}_{\boldsymbol{f}^e_\Omega} + \underbrace{\left(\boldsymbol{N}^{e\mathsf{T}}\overline{Q}\right)\Big|_{\Gamma_Q} + \left(\boldsymbol{C}^{e\mathsf{T}}\overline{M}\right)\Big|_{\Gamma_M}}_{\boldsymbol{f}^e_\Gamma}$$

Discrete problem for Euler–Bernoulli beams

Assuming (or approximating) $E^e I^e$ and f^e as constant on an element:

$$\boldsymbol{K}^e = \int_{\Omega^e} E^e I^e \boldsymbol{B}^{e\intercal} \boldsymbol{B}^e \, \mathrm{d}x = \frac{E^e I^e}{l^{e3}} \begin{bmatrix} 12 & 6l^e & -12 & 6l^e \\ & 4l^{e2} & -6l^e & 2l^{e2} \\ & & 12 & -6l^e \\ \text{sym} & & & 4l^{e2} \end{bmatrix}$$

$$\boldsymbol{f}^e_\Omega = \int_{\Omega^e} \boldsymbol{N}^{e\intercal} f^e \, \mathrm{d}x = \frac{f^e l^e}{2} \begin{bmatrix} 1 \\ l^e/6 \\ 1 \\ -l^e/6 \end{bmatrix}$$

As a special type of loading to include with $\boldsymbol{f}^e_\Omega$:

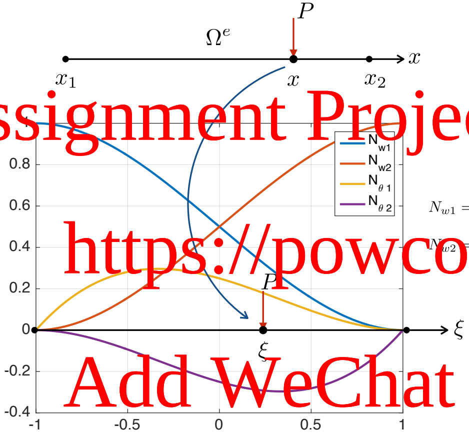
Point load acting P at $\xi = 2(x - x_1)/l^e - 1$

$$\boldsymbol{f}^e_P = P\boldsymbol{N}^T = P[N_{w1}(\xi) \ N_{\theta 1}(\xi) \ N_{w2}(\xi) \ N_{\theta 2}(\xi)]^T$$

$$N_{w1} = \frac{1}{4}(1-\xi)^2(2+\xi) \qquad N_{\theta 1} = \frac{l^e}{8}(1-\xi)^2(1+\xi)$$

$$N_{w2} = \frac{1}{4}(1+\xi)^2(2-\xi) \qquad N_{\theta 2} = \frac{l^e}{8}(1+\xi)^2(\xi-1)$$

Point load acting P at $\xi = 2(x - x_1)/l^e - 1$



$$x = x_1\left(1 - \frac{\xi + 1}{2}\right) + x_2\left(\frac{\xi + 1}{2}\right)$$

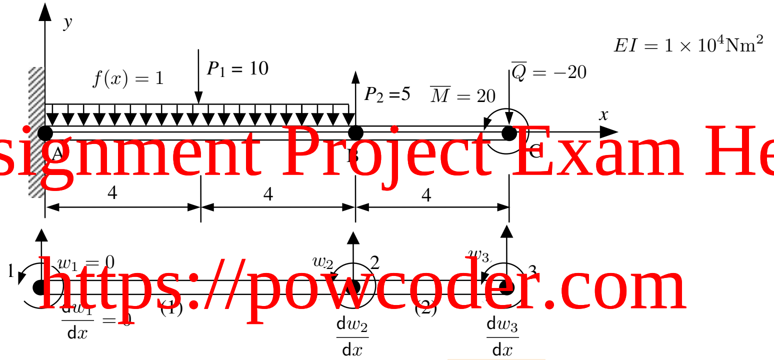
$$N_{w1} = \frac{1}{4}(1-\xi)^2(2+\xi) \qquad N_{\theta 1} = \frac{l^e}{8}(1-\xi)^2(1+\xi)$$

$$N_{w2} = \frac{1}{4}(1+\xi)^2(2-\xi) \qquad N_{\theta 2} = \frac{l^e}{8}(1+\xi)^2(\xi-1)$$

$$\xi = 2\frac{(x - x_1)}{l^e} - 1$$

$$\boldsymbol{f}_P^e = P\boldsymbol{N}^T = P[N_{w1}(\xi)\ N_{\theta 1}(\xi)\ N_{w2}(\xi)\ N_{\theta 2}(\xi)]^T$$

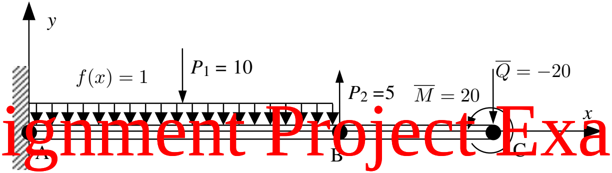
Euler–Bernoulli beam worked example



$$
\boldsymbol{K}^1 = 10^3 \begin{array}{cc} \left[\begin{array}{cccc} 0.23 & 0.94 & -0.23 & 0.94 \\ 0.94 & 5.00 & -0.94 & 2.50 \\ -0.23 & -0.94 & 0.23 & -0.94 \\ 0.94 & 2.50 & -0.94 & 5.00 \end{array}\right] & \begin{array}{c} [1] \\ \\ [2] \\ \\ \end{array} \\ \begin{array}{cc} \quad [1] & \qquad\qquad [2] \end{array} & \end{array}
\qquad
\boldsymbol{K}^2 = 10^3 \begin{array}{cc} \left[\begin{array}{cccc} 1.88 & 3.75 & -1.88 & 3.75 \\ 3.75 & 10.00 & -3.75 & 5.00 \\ -1.88 & -3.75 & 1.88 & -3.75 \\ 3.75 & 5.00 & -3.75 & 10.00 \end{array}\right] & \begin{array}{c} [2] \\ \\ [3] \\ \\ \end{array} \\ \begin{array}{cc} \quad [2] & \qquad\qquad [3] \end{array} & \end{array}
$$

$$
\mathbf{K} = 10^3 \begin{array}{cc} \left[\begin{array}{cccccc} 0.23 & 0.94 & -0.23 & 0.94 & 0 & 0 \\ 0.94 & 5.00 & -0.94 & 2.50 & 0 & 0 \\ -0.23 & -0.94 & 2.11 & 2.81 & -1.88 & 3.75 \\ 0.94 & 2.50 & 2.81 & 15.00 & -3.75 & 5.00 \\ 0 & 0 & -1.88 & -3.75 & 1.88 & -3.75 \\ 0 & 0 & 3.75 & 5.00 & -3.75 & 10.00 \end{array}\right] & \begin{array}{c} [1] \\ \\ [2] \\ \\ [3] \\ \\ \end{array} \\ \begin{array}{ccc} \quad [1] & \qquad\qquad [2] & \qquad\qquad [3] \end{array} & \end{array}
$$

Euler–Bernoulli beam worked example



element 1	element 2	global
$\boldsymbol{f}^1_\Omega = \begin{bmatrix} -9 \\ -15.3 \\ -9 \\ 15.3 \end{bmatrix} \begin{matrix} [1] \\ \\ [2] \\ \end{matrix}$	$\boldsymbol{f}^2_\Omega = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} [2] \\ \\ [3] \\ \end{matrix}$	$\boldsymbol{f}_\Omega = \begin{bmatrix} -9 \\ -15.3 \\ -4 \\ 15.3 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} [1] \\ \\ [2] \\ \\ [3] \\ \end{matrix}$

$$\boldsymbol{f}^1_P = (-10)[N_{w1}(0)\ N_{\theta 1}(0)\ N_{w2}(0)\ N_{\theta 2}(0)]^T$$

$$N_{w1} = \frac{1}{4}(1)^2(2) = \frac{1}{2} \qquad N_{\theta 1} = \frac{8}{8}(1)^2(1) = 1$$

$$N_{w2} = \frac{1}{4}(1)^2(1) = \frac{1}{2} \qquad N_{\theta 2} = \frac{8}{8}(1)^2(-1) = -1$$

$$\boldsymbol{f}^1_P = [-5\ -10\ -5\ 10]^T$$

$$\frac{fl^e}{2}\begin{bmatrix} 1 \\ l^e/6 \\ 1 \\ -l^e/6 \end{bmatrix} = \frac{-8}{2}\begin{bmatrix} 1 \\ 8/6 \\ 1 \\ -8/6 \end{bmatrix} = \begin{bmatrix} -4 \\ -16/3 \\ -4 \\ 16/3 \end{bmatrix}$$

$$\boldsymbol{f}_\Gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -20 \\ 20 \end{bmatrix} \begin{matrix} [1] \\ \\ [2] \\ \\ [3] \\ \end{matrix} \qquad \begin{matrix} \overline{Q} \rightarrow -20 \\ \overline{M} \rightarrow 20 \end{matrix}$$

$$\boldsymbol{f}^e_P = P[N_{w1}(\xi)\ N_{\theta 1}(\xi)\ N_{w2}(\xi)\ N_{\theta 2}(\xi)]^T$$

$$N_{w1} = \frac{1}{4}(1-\xi)^2(2+\xi) \qquad N_{\theta 1} = \frac{l^e}{8}(1-\xi)^2(1+\xi)$$

$$N_{w2} = \frac{1}{4}(1+\xi)^2(2-\xi) \qquad N_{\theta 2} = \frac{l^e}{8}(1+\xi)^2(\xi-1)$$

$$\boldsymbol{f}^e_\Omega = \int_{\Omega^e} \boldsymbol{N}^{e\mathsf{T}} f \, \mathrm{d}x = \frac{fl^e}{2}\begin{bmatrix} 1 \\ l^e/6 \\ 1 \\ -l^e/6 \end{bmatrix}$$

Euler–Bernoulli beam worked example

(I) matrix problem and solution

$$\left[\begin{array}{cc:cccc} 0.23 & 0.94 & -0.23 & 0.94 & 0 & 0 \\ 0.94 & 5.00 & -0.94 & 2.50 & 0 & 0 \\ \hdashline -0.23 & -0.94 & 2.11 & 2.81 & -1.88 & 3.75 \\ 0.94 & 2.50 & 2.81 & 15.00 & -3.75 & 5.00 \\ 0 & 0 & -1.88 & -3.75 & 1.88 & -3.75 \\ 0 & 0 & 3.75 & 5.00 & -3.75 & 10.00 \end{array}\right] \left[\begin{array}{c} w_1 = 0 \\ \frac{\mathrm{d}w_1}{\mathrm{d}x} = 0 \\ \hdashline w_2 \\ \frac{\mathrm{d}w_2}{\mathrm{d}x} \\ w_3 \\ \frac{\mathrm{d}w_3}{\mathrm{d}x} \end{array}\right] = \left[\begin{array}{c} -9 + r_{u1} \\ -15.3 + r_{\theta 1} \\ \hdashline -4 \\ 15.3 \\ -20 \\ 20 \end{array}\right]$$

(II) apply boundary conditions:

$$\begin{bmatrix} 2.11 & 2.81 & -1.88 & 3.75 \\ 2.81 & 15.00 & -3.75 & 5.00 \\ -1.88 & -3.75 & 1.88 & -3.75 \\ 3.75 & 5.00 & -3.75 & 10.00 \end{bmatrix} \begin{bmatrix} w_2 \\ \frac{\mathrm{d}w_2}{\mathrm{d}x} \\ w_3 \\ \frac{\mathrm{d}w_3}{\mathrm{d}x} \end{bmatrix} = \begin{bmatrix} -4 \\ 15.3 \\ -20 \\ 20 \end{bmatrix}$$

(III) solve:

$$\begin{bmatrix} w_2 \\ \frac{\mathrm{d}w_2}{\mathrm{d}x} \\ w_3 \\ \frac{\mathrm{d}w_3}{\mathrm{d}x} \end{bmatrix} = \begin{bmatrix} -0.55 \\ -0.\text{[illegible]}1 \\ -1.03 \\ -0.12 \end{bmatrix}, \quad \begin{bmatrix} r_{u1} \\ r_{\theta 1} \end{bmatrix} = \begin{bmatrix} 33 \\ 252 \end{bmatrix}$$

(IV) postprocess
(**note** computed moment and shear force discontinuous between elements):

$$M^{(1)} = EI\boldsymbol{B}^{(1)}\boldsymbol{w}^{(1)} = -240.64 + 25.785x$$

$$Q^{(1)} = -EI\left\{\frac{\mathrm{d}^3 N_{w1}}{\mathrm{d}x^3}, \frac{\mathrm{d}^3 N_{\theta 1}}{\mathrm{d}x^3}, \frac{\mathrm{d}^3 N_{w2}}{\mathrm{d}x^3}, \frac{\mathrm{d}^3 N_{\theta 3}}{\mathrm{d}x^3}\right\}\boldsymbol{w}^{(1)} = -25.785$$

$$M^{(2)} = EI\boldsymbol{B}^{(2)}\boldsymbol{w}^{(2)} = -104.5 + 39.75x$$

$$Q^{(2)} = -EI\left\{\frac{\mathrm{d}^3 N_{w1}}{\mathrm{d}x^3}, \frac{\mathrm{d}^3 N_{\theta 1}}{\mathrm{d}x^3}, \frac{\mathrm{d}^3 N_{w2}}{\mathrm{d}x^3}, \frac{\mathrm{d}^3 N_{\theta 3}}{\mathrm{d}x^3}\right\}\boldsymbol{w}^{(2)} = -39.7$$

$$w = \boldsymbol{N}\boldsymbol{w}$$

$$M = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = EI\boldsymbol{B}\boldsymbol{w}$$

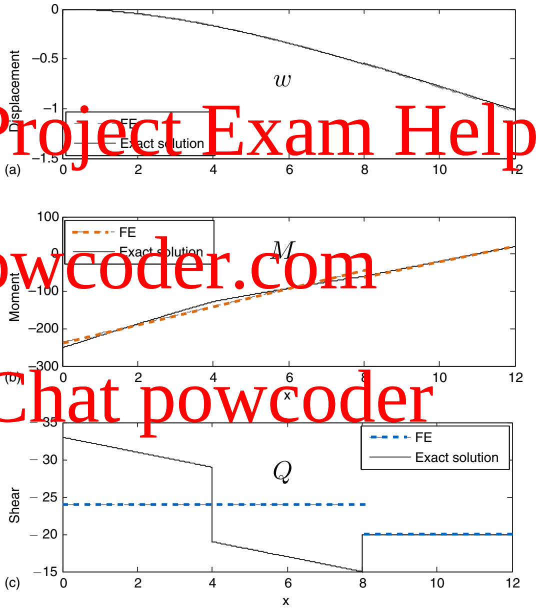
$$Q = -EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}$$

Euler–Bernoulli beam worked example

$$w = \boldsymbol{N}\boldsymbol{w}$$

$$M = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = EI\boldsymbol{B}\boldsymbol{w}$$

$$Q = -EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}$$



Inclusion of an axial load

Consider the case where the midline is now displaced by u^{M}:

Extended kinematics

$$u = u^{\mathsf{M}} - y\frac{\mathrm{d}w}{\mathrm{d}x}$$

Axial strain

$$\epsilon := \epsilon_{xx} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u^{\mathsf{M}}}{\mathrm{d}x} - y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \quad \text{(axial strain)}$$

- The axial strain is now the sum of the midline extensional strain and the strain due to bending (flexural strain)

Hooke's law

$$\sigma = E\epsilon = E\frac{\mathrm{d}u^{\mathsf{M}}}{\mathrm{d}x} - Ey\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

Treat the axial displacement u^{M} as before for an axially-loaded elastic bar - no need for Hermite polynomials for the new term!

Deficiencies in the Euler–Bernoulli beam theory

- The kinematic assumption that plane sections remain normal to the neutral axis results in a *zero transverse shear strain and stress*.
  - Implies the shear force is also zero
  - Implies a constant bending moment, since the shear force is the spatial derivative of the bending moment
  - Yet, we use Euler–Bernoulli beam elements for sections with a varying bending moment.

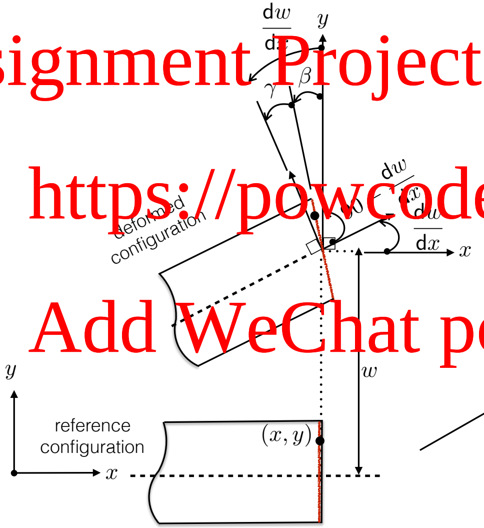
- The assumption of no normal stress perpendicular to the beam axis is inconsistent with the line loads applied on the top of the beam

$$M = EI \frac{\partial^2 w}{\partial x^2} \quad \text{and} \quad Q = -EI \frac{\partial^3 w}{\partial x^3}$$

Where are we?

Timoshenko beam theory: motion

Kinematic assumption: a plane section originally normal to the centroid remains plane, but in addition *shear deformations* occur. This is important for *stocky* beams!

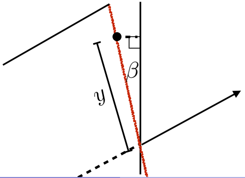


β : rotation angle of fibre

γ : shear angle

$$\frac{dw}{dx} = \gamma + \beta$$

$$u = -y \sin \beta$$
$$= -y\beta \quad \text{(for } \beta \text{ small)}$$



Timoshenko beam theory: kinematics

Strain measures

$$\epsilon := \epsilon_{xx} = \frac{\mathrm{d}u}{\mathrm{d}x} = -y \underbrace{\frac{\mathrm{d}\beta}{\mathrm{d}x}}_{\chi} \qquad \text{(axial strain varies linearly)}$$

$$\epsilon_{yy} = \frac{\mathrm{d}w}{\mathrm{d}y} = 0 \quad \text{(vanishing strain through the thickness)}$$

$$2\epsilon_{xy} = \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}x} = -\beta + \underbrace{\frac{\mathrm{d}w}{\mathrm{d}x}}_{(\gamma+\beta)} \quad \text{(constant transverse shear strain)}$$

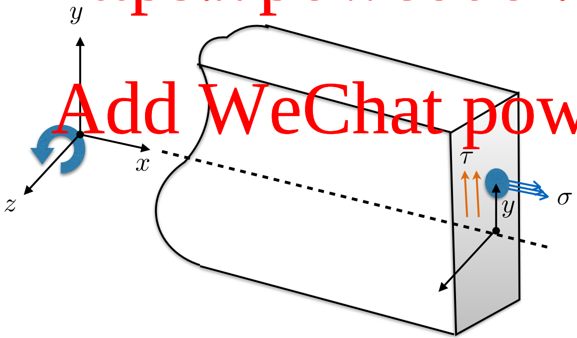
- We have now derived the *kinematics*, next we need to determine the *kinetics* - i.e. the stress-like quantities

Constitutive relations

$$\sigma = E\epsilon = -yE\frac{\mathrm{d}\beta}{\mathrm{d}x} \quad \text{and} \quad \tau = 2G\epsilon_{xy} = -G\beta + G\frac{\mathrm{d}w}{\mathrm{d}x} \qquad (G: \text{ shear modulus})$$

$$M = -\int y\sigma \,\mathrm{d}a = \int y^2 E\frac{\mathrm{d}\beta}{\mathrm{d}x}\,\mathrm{d}a = EI\chi$$

$$Q = \int \tau \,\mathrm{d}a = G\int \left[-\beta + \frac{\mathrm{d}w}{\mathrm{d}x}\right]\mathrm{d}a = GA\underbrace{\left[-\beta + \frac{\mathrm{d}w}{\mathrm{d}x}\right]}_{\gamma}$$



$$I = \int y^2 \,\mathrm{d}a$$

$$\chi = \frac{\mathrm{d}\beta}{\mathrm{d}x}$$

Timoshenko beam theory: balance of forces and moments

Balance of forces (as in EB theory)

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -f$$

Balance of moments (as in EB theory)

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -Q$$

NB: But now we have independent definitions for M and Q:

Strong form

$$-GA\frac{\mathrm{d}\beta}{\mathrm{d}x} + GA\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -f \qquad \text{(force balance)}$$

$$EI\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} - GA\left[\beta - \frac{\mathrm{d}w}{\mathrm{d}x}\right] = 0 \qquad \text{(moment balance)}$$

Coupled system of *second-order* differential equations for w and β

Timoshenko beam theory: weak formulation

Test the force balance with an arbitrary displacement δw. Test the moment balance with an arbitrary rotation $\delta\beta$, to obtain:

$$\int \delta w GA \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x - \int \delta w GA \frac{\mathrm{d}\beta}{\mathrm{d}x}\,\mathrm{d}x = -\int \delta w f \qquad (\star)$$

$$\int \delta\beta EI \frac{\mathrm{d}^2 \beta}{\mathrm{d}x^2}\,\mathrm{d}x - \int \delta\beta GA\beta\,\mathrm{d}x + \int \delta\beta GA \frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x = 0 \qquad (\star\star)$$

After an integration by parts and a rearranging of terms:

Weak formulation

$$(\star) \quad \rightarrow \quad \int \frac{\mathrm{d}\delta w}{\mathrm{d}x} GA \underbrace{\left[\frac{\mathrm{d}w}{\mathrm{d}x} - \beta\right]}_{\gamma} \mathrm{d}x = \int \delta w f\,\mathrm{d}x + \int_{\Gamma_Q} \delta w GA \underbrace{\left[\frac{\mathrm{d}w}{\mathrm{d}x} - \beta\right]}_{\gamma} n\,\mathrm{d}a$$

$$(\star\star) \quad \rightarrow \quad \int \frac{\mathrm{d}\delta\beta}{\mathrm{d}x} EI \underbrace{\frac{\mathrm{d}\beta}{\mathrm{d}x}}_{\chi} \mathrm{d}x - \int \delta\beta GA \underbrace{\left[\frac{\mathrm{d}w}{\mathrm{d}x} - \beta\right]}_{\gamma} \mathrm{d}x = \int_{\Gamma_M} \delta\beta EI \frac{\mathrm{d}\beta}{\mathrm{d}x} n\,\mathrm{d}a$$

- We only require C^0 interpolations for w and β!
- We have twice as many degrees of freedom

Timoshenko beam theory: weak formulation

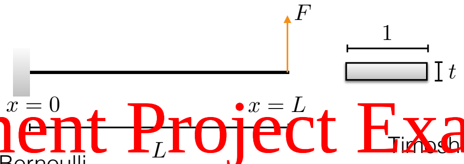
State the weak form as one equation: $(\star) + (\star\star)$:

$$\underbrace{\int \frac{\mathrm{d}\delta\beta}{\mathrm{d}x} EI \underbrace{\frac{\mathrm{d}\beta}{\mathrm{d}x}}_{\kappa} \, \mathrm{d}x}_{\text{bending}} + \underbrace{\int \underbrace{\left[\frac{\mathrm{d}\delta w}{\mathrm{d}x} - \delta\beta\right]}_{\delta\gamma} GA \underbrace{\left[\frac{\mathrm{d}w}{\mathrm{d}x} - \beta\right]}_{\gamma} \, \mathrm{d}x}_{\text{shear}}$$

$$= \int \delta w f \, \mathrm{d}x + \int_{\Gamma_Q} \delta w GA \underbrace{\left[\frac{\mathrm{d}w}{\mathrm{d}x} - \beta\right]}_{\gamma} n \, \mathrm{d}a + \int_{\Gamma_M} \delta\beta EI \frac{\mathrm{d}\beta}{\mathrm{d}x} n \, \mathrm{d}a$$

- Again we can see the contributions to the stiffness (on the LHS) from the *bending* and the *shear* terms

Comparison of Euler–Bernoulli and Timoshenko beam theory



Euler-Bernoulli

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0$$

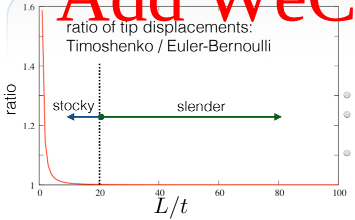
x=0	**boundary conditions**	x=L
$\overline{w} = 0$		$\overline{Q} = F$
$\overline{\theta} = 0$		$\overline{M} = 0$

Timoshenko

$$-GA\frac{\mathrm{d}\beta}{\mathrm{d}x} + GA\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0$$

$$EI\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} - GA\left[\beta - \frac{\mathrm{d}w}{\mathrm{d}x}\right] = 0$$

x=0	**boundary conditions**	x=L
$\overline{w} = 0$		$GA\left[\frac{\mathrm{d}w}{\mathrm{d}x} - \beta\right] = F$
$\overline{\beta} = 0$		$EI\frac{\mathrm{d}\beta}{\mathrm{d}x} = 0$



- for slender beam (L/t > 20): both theories give the same results
- for stocky beams (L/t < 10): Timoshenko theory is more realistic as it accounts for shear deformation
- taking into account shear deformation effectively *lowers the stiffness of the beam*, resulting in a larger deflection under a static load

Discrete problem for Timoshenko beams

Let's use *standard linear* C^0 interpolations for both w and β on an element:

$$w^e = \underbrace{\begin{bmatrix} N^1_{e,w} & 0 & N^2_{e,w} & 0 \end{bmatrix}}_{\boldsymbol{N}_{e,w}} \underbrace{\begin{bmatrix} w^1 \\ \beta^1 \\ w^2 \\ \beta^2 \end{bmatrix}}_{\boldsymbol{u}_e} \quad \text{and} \quad \beta^e = \underbrace{\begin{bmatrix} 0 & N^1_{e,\beta} & 0 & N^2_{e,\beta} \end{bmatrix}}_{\boldsymbol{N}_{e,\beta}} \begin{bmatrix} w^1 \\ \beta^1 \\ w^2 \\ \beta^2 \end{bmatrix}$$

Shear angle and curvature on an element

$$\gamma^e = \frac{\mathrm{d}w^e}{\mathrm{d}x} - \beta^e = \underbrace{\begin{bmatrix} \dfrac{\mathrm{d}N^1_{e,w}}{\mathrm{d}x} & -N^1_{e,\beta} & \dfrac{\mathrm{d}N^2_{e,w}}{\mathrm{d}x} & -N^2_{e,\beta} \end{bmatrix}}_{\boldsymbol{B}_{e,S}} \underbrace{\begin{bmatrix} w^1 \\ \beta^1 \\ w^2 \\ \beta^2 \end{bmatrix}}_{\boldsymbol{u}_e}$$

$$\chi^e = \frac{\mathrm{d}\beta^e}{\mathrm{d}x} = \underbrace{\begin{bmatrix} 0 & \dfrac{\mathrm{d}N^1_{e,\beta}}{\mathrm{d}x} & 0 & \dfrac{\mathrm{d}N^2_{e,\beta}}{\mathrm{d}x} \end{bmatrix}}_{\boldsymbol{B}_{e,B}} \begin{bmatrix} w^1 \\ \beta^1 \\ w^2 \\ \beta^2 \end{bmatrix}$$

- γ (shear) interpolation contains *both* constant and linear contributions
- χ (curvature) interpolation contains *only* constant contributions

Timoshenko beam theory: discrete problem

$$\boldsymbol{K}\boldsymbol{w} = \boldsymbol{f} + \boldsymbol{r}$$

$$\boldsymbol{K}^e = \boldsymbol{K}^e_B + \boldsymbol{K}^e_S$$

$$\boldsymbol{K}^e_B = \int_{\Omega^e} \boldsymbol{B}^{e\top}_B E^e I^e \boldsymbol{B}^e_B \, \mathrm{d}x, \quad \boldsymbol{K}^e_S = \int_{\Omega^e} \boldsymbol{B}^{e\top}_S \kappa G^e A^e \boldsymbol{B}^e_S \, \mathrm{d}x, \quad \boldsymbol{f}^e = \int \boldsymbol{N}^{e\top}_w f^e \, \mathrm{d}x + \text{b.c.}$$

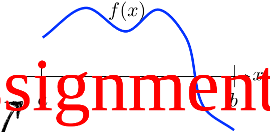
- Again we see the contributions to the stiffness from the bending and the shear terms
- $\kappa < 1$ is a *shear correction factor*. From elasticity theory γ is parabolic across the thickness but constant in Timoshenko beam theory
  - For a beam with rectangular cross-section: $\kappa = 5/6$

Quadrature

- Integration is done numerically in finite element programs - known as *quadrature*
- We need to briefly discuss quadrature before proceeding

Numerical integration - Gauss quadrature

physical domain



evaluate numerically

$$I = \int_a^b f(x)\,\mathrm{d}x$$

$$x = \frac{1}{2}(a+b) + \frac{1}{2}\xi(b-a)$$

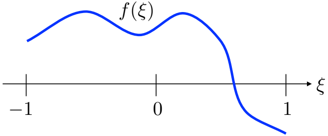
$$x = x_1 N_1(\xi) + x_2 N_2(\xi) = a\frac{1-\xi}{2} + b\frac{\xi+1}{2}$$

$$I = J \underbrace{\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi}_{\widehat{I}}$$

$$\mathrm{d}x = \frac{1}{2}(b-a)\mathrm{d}\xi = \frac{l}{2}\mathrm{d}\xi = J\mathrm{d}\xi$$

$$\widehat{I} \approx \sum_{i=1}^{n_{\mathrm{gp}}} W_i f(\xi_i) = \underbrace{\begin{bmatrix} W_1 & W_2 & \cdots & W_{n_{\mathrm{gp}}} \end{bmatrix}}_{\boldsymbol{W}} \underbrace{\begin{bmatrix} f(\xi_1) \\ f(\xi_2) \\ \vdots \\ f(\xi_{n_{\mathrm{gp}}}) \end{bmatrix}}_{\boldsymbol{f}}$$

parent domain

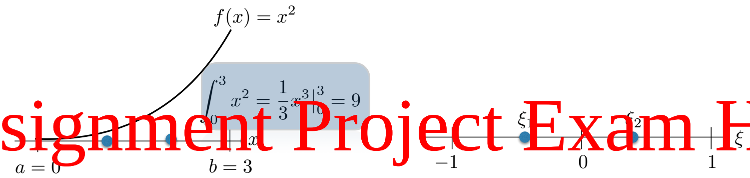


The number of Gauss points (n_{gp}) required to exactly integrate a polynomial of degree p is given by

$$n_{\text{gp}} \geq \frac{p+1}{2}$$

n_{gp}	Location, ξ_i	Weights, W_i
1	0.0	2.0
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0
3	± 0.7745966692	0.555 555 5556
	0.0	0.888 888 8889
4	± 0.8611363116	0.347 854 8451
	± 0.3399810436	0.652 145 1549
5	± 0.9061798459	0.236 926 8851
	± 0.5384693101	0.478 628 6705
	0.0	0.568 888 8889
6	± 0.9324695142	0.171 324 4924
	± 0.6612093865	0.360 761 5730
	± 0.2386191861	0.467 913 9346

Numerical integration - Gauss quadrature - example



$$\boxed{\int_0^3 x^2 = \frac{1}{3}x^3\Big|_0^3 = 9}$$

$$n_{\text{gp}} \geq \frac{2+1}{2} \implies n_{\text{gp}} = 2 \qquad W_1 = W_2 = 1 \qquad \xi_1 = -\xi_2 = -\frac{1}{\sqrt{3}} \qquad J = \frac{3}{2}$$

$$x(\xi_1) = 3\frac{\xi_1 + 1}{2} = (3/2)(-1+\sqrt{3})/\sqrt{3} = (1/2)(-\sqrt{3}+3)$$

$$x(\xi_2) = 3\frac{\xi_2 + 1}{2} = (3/2)(1+\sqrt{3})/\sqrt{3} = (1/2)(\sqrt{3}+3)$$

$$f(x(\xi_1)) = (1/4)(3 - 6\sqrt{3} + 9)$$

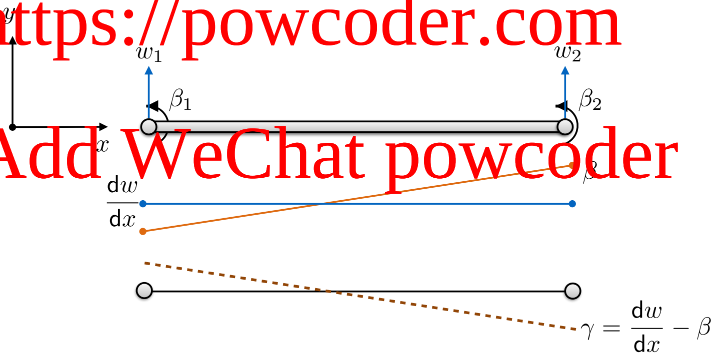
$$f(x(\xi_2)) = (1/4)(3 + 6\sqrt{3} + 9)$$

$$\implies \widehat{I} = 24/4 = 6$$

$$\boxed{I = J\widehat{I} = \frac{3}{2}6 = 9}$$

Shear locking in Timoshenko beams

- We expect that as $t \to 0$ (the Euler–Bernoulli limit) then $\gamma \to 0$
- This is a problem as we choose linear shape functions for both w and β, but $\gamma = \mathrm{d}w/\mathrm{d}x - \beta$ and adding a constant and a linear will never give zero!
- Thus shear strains can not be arbitrarily small everywhere and *an erroneous shear strain stiffness* appears in the shear stiffness contribution
  - computed FE solution much smaller than exact solution
  - causes the FE solution to converge very slowly upon refinement to the exact solution
- Know as shear locking!



Timoshenko beam theory: discrete problem

$$\boldsymbol{K}\boldsymbol{w} = \boldsymbol{f} + \boldsymbol{r}$$

$$\boldsymbol{K}^e = \boldsymbol{K}^e_B + \boldsymbol{K}^e_S$$

$$\boldsymbol{K}^e_B = \int_{\Omega^e} \boldsymbol{B}^{e\top}_B E^e I^e \boldsymbol{B}^e_B \, \mathrm{d}x, \quad \boldsymbol{K}^e_S = \int_{\Omega^e} \boldsymbol{B}^{e\top}_S \kappa G^e A^e \boldsymbol{B}^e_S \, \mathrm{d}x, \quad \boldsymbol{f}^e = \int \boldsymbol{N}^{e\top}_w f^e \, \mathrm{d}x + \text{b.c.}$$

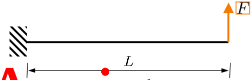
- The bending stiffness contribution $\boldsymbol{K}^e_B$ contains products of constant functions and can be integrated exactly using a *one-point* quadrature rule
- The shear stiffness contribution $\boldsymbol{K}^e_S$ contains products of linear functions and can be integrated exactly using a *two-point* quadrature rule

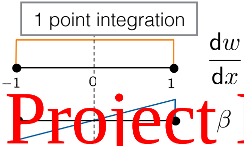
$$\boldsymbol{K}^e_B = \frac{E^e I^e}{l^e}\begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ \text{sym} & & & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{K}^e_S = \frac{\kappa G^e A^e}{l^e}\begin{bmatrix} 1 & l^e/2 & -1 & l^e/2 \\ & {l^e}^2/3 & -l^e/2 & {l^e}^2/6 \\ & & 1 & -l^e/2 \\ \text{sym} & & & {l^e}^2/3 \end{bmatrix}$$

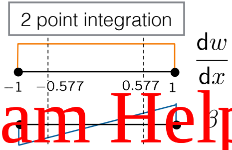
Recall: the number of Gauss points (n_{gp}) required to exactly integrate a polynomial of degree p is given by

$$n_{\text{gp}} \geq \frac{p+1}{2}$$

Shear locking in Timoshenko beams: an example







Normalized Tip Displacement

Number of element	One-point	Two-point	
	Thick beam		
1	0.762	0.416×10^{-1}	Thick beam
2	0.940	0.445	
4	0.985	0.762	
8	0.996	0.927	
16	0.999	0.981	
	Thin beam		
1	0.750	0.200×10^{-4}	Slender / thin beam
2	0.938	0.800×10^{-4}	
4	0.984	0.320×10^{-4}	
8	0.996	0.128×10^{-3}	
16	0.999	0.512×10^{-3}	

From Hughes, The Finite Element Method

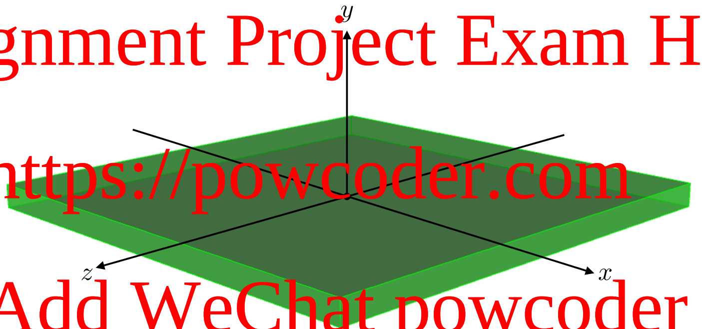
Shear locking in Timoshenko beams: a remedy

- If the deflections w and the rotations β are interpolated with the same shape functions (i.e. $N_{e,w} \equiv N_{e,\beta}$) there is a tendency to lock: i.e. to exhibit a too stiff numerical response - poor convergence
- Under-integrating the terms in the stiffness matrix resolves the problem
  - the bending contribution $\boldsymbol{K}^e_B$ is exactly integrated using a one point rule!
  - the shear contribution $\boldsymbol{K}^e_S$ is under-integrated by one order
- There are far more mathematically rigorous approaches such as mixed variational principles but we shall not discuss them here (see Hughes for an introduction)

Shape functions	Linear	Quadratic	Cubic
Reduced quadrature rule	One-point	Two-point	Three-point

Where are we?

thickness in *y*-direction small relative to other dimensions



$w(x,z)$ deflection in *y*-direction

Kirchhoff–Love plate theory

- Classical theory suitable for *thin* plates.
- Direct extension of Euler–Bernoulli beam theory to 2D:
  - Fibres originally orthogonal to the mid-surface remain straight after deformation
  - Fibres originally orthogonal to the mid-surface remain orthogonal after deformation
  - The normal stress orthogonal to the mid-surface is zero
  - Cannot account for shear deformation

- The requirement of C^1 continuous interpolations for the transverse displacement $w(x, z)$ (in y-direction) becomes more restrictive as we are now in two dimensions

Degrees of freedom

- $w(x, z)$: transverse displacement
- the slopes in the two directions, $\mathrm{d}w/\mathrm{d}x$, $\mathrm{d}w/\mathrm{d}z$

Boundary conditions

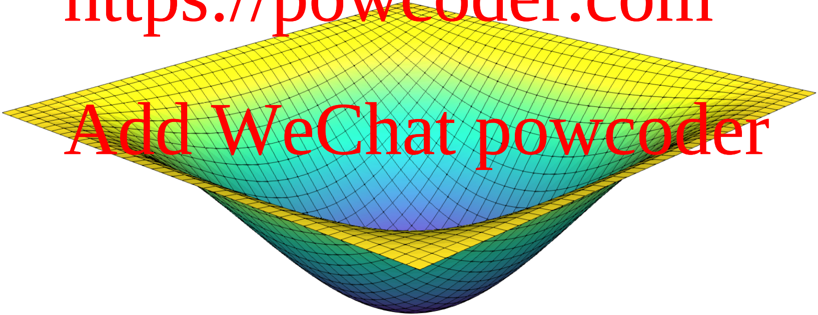
$$\text{clamped: } w = 0, \ \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \text{ and } \frac{\mathrm{d}w}{\mathrm{d}z} = 0$$

$$\text{column support: } w = 0$$

$$\text{hinged along an axis: } w = 0, \ \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \text{ or } \frac{\mathrm{d}w}{\mathrm{d}z} = 0$$

Reissner–Mindlin plate theory

- The most widely implemented plate theory in commercial FE codes
- Extension of the Timoshenko beam theory to 2D:
    - Fibres originally orthogonal to the mid-surface remain straight but not necessarily normal after deformation;
    - The normal stress orthogonal to the mid-surface is zero
- The kinematic and static assumptions are the same as Timoshenko beam theory, and so are the degrees of freedom: translational and rotational.
- Reissner–Mindlin theory also exhibits *shear locking* for thin plates



Time-dependent problems: Dynamics

Where are we?

Overview

- The balance relation for an elastic rod was obtained from a force balance (*internal* stresses developing due to the application of *external* tractions and body forces)
- In order to account for inertial forces we need to extend Newton's second law ($F^{\text{ext}} - F^{\text{int}} = ma$) to continuous bodies
- Why do we need to account for inertial forces?

Northridge Earthquake (Los Angeles)

Tacoma Narrows Bridge (Washington)

impact

Where are we?

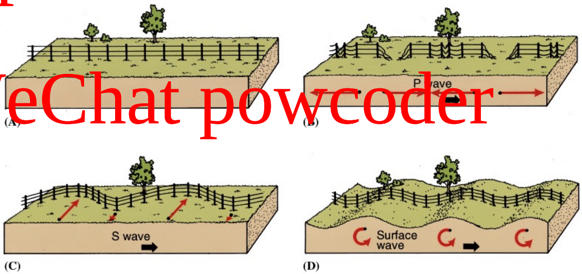
P, S and surface waves generated by an earthquake

Seismic waves are produced by earthquakes:

- volcanic eruptions,
- magma movement,
- large landslides,
- large man-made explosions

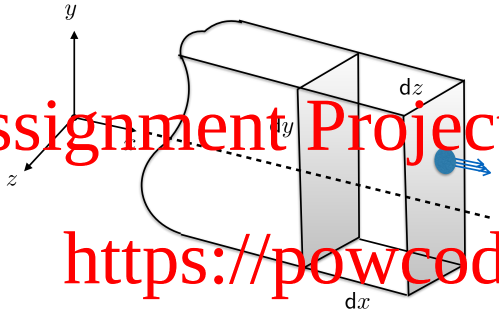
They are a form of acoustic wave. Vast majority are associated with natural earthquakes.

- P wave - primary (longitudinal): fastest moving
- S wave - secondary (shear)
- surface wave



See www.zmescience.com/other/feature-post/the-types-of-seismic-waves/

Longitudinal (P) waves in one dimension



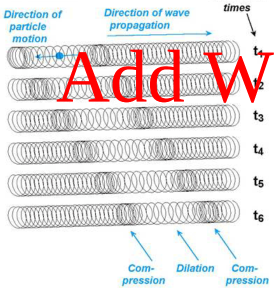
balance of momentum:

$$\mathrm{d}\sigma(\mathrm{d}y\mathrm{d}z) = \widetilde{\rho}(\mathrm{d}x\mathrm{d}y\mathrm{d}z)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$$

$$\Longrightarrow \quad \frac{\mathrm{d}\sigma}{\mathrm{d}x} = \widetilde{\rho}\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$$

$$\Longrightarrow \quad \boxed{E\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \widetilde{\rho}\frac{\mathrm{d}^2u}{\mathrm{d}t^2}}$$

longitudinal wave propagation

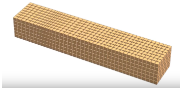


$$\rho = \widetilde{\rho}A$$

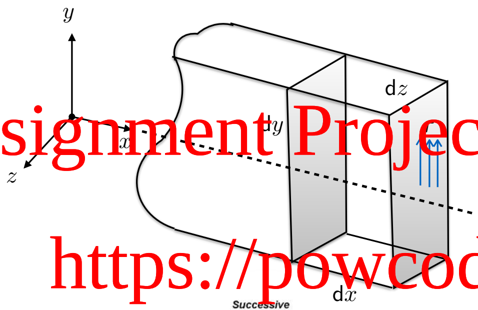
$$[\mathrm{kg/m}] = [\mathrm{kg/m^3}][\mathrm{m^2}]$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \underbrace{\left(\frac{E}{\widetilde{\rho}}\right)}_{c^2}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$$

c : speed of propagation of a longitudinal wave



Transverse (S) waves - one dimension



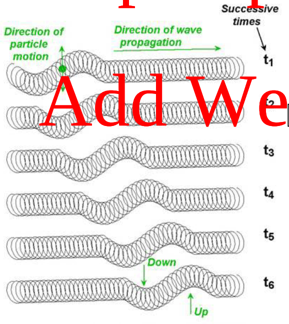
balance of momentum:

$$\mathrm{d}\tau(\mathrm{d}y\mathrm{d}z) = \widetilde{\rho}(\mathrm{d}x\mathrm{d}y\mathrm{d}z)\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}$$

$$\implies \quad \frac{\mathrm{d}\tau}{\mathrm{d}x} = \widetilde{\rho}\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}$$

$$\implies \quad \boxed{G\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \widetilde{\rho}\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}}$$

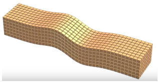
shear wave propagation



$$\mu = \widetilde{\rho}A$$

$$[\mathrm{kg/m}] = [\mathrm{kg/m^3}][\mathrm{m^2}]$$

$$\boxed{\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = \underbrace{\left(\frac{G}{\widetilde{\rho}}\right)}_{c_S^2}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}}$$



c_S : speed of propagation of a shear wave

Wave propagation in three dimensions

In three-dimensional elasticity theory

$$\lambda = \frac{\nu E}{(1-2\nu)(1+2\nu)}$$ first Lamé parameter

$$G = \frac{E}{2(1+\nu)}$$ shear modulus (second Lamé parameter)

Longitudinal or P (primary) waves

$$c_p = \sqrt{\frac{\lambda + 2G}{\rho}}$$

Shear or S (secondary) waves

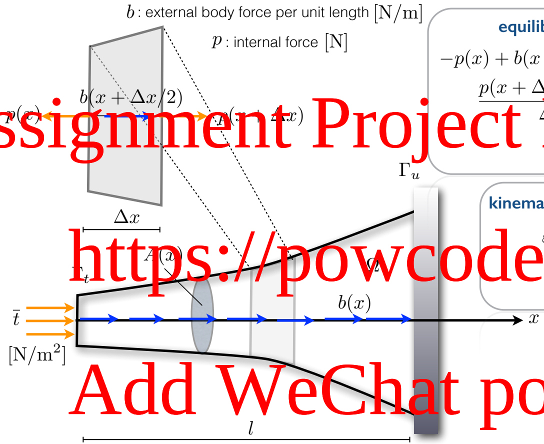
$$c_s = \sqrt{\frac{G}{\rho}}$$

$$c_p^2 = \frac{\lambda + 2G}{\rho} > \frac{G}{\rho} = c_s^2$$

P waves travel faster than S waves!

Where are we?

Recap: FE formulation an axially-loaded elastic bar



equilibrium (balance of forces)

$$-p(x) + b(x + \Delta x/2)\Delta x + p(x + \Delta x) = 0$$

$$\frac{p(x + \Delta x) - p(x)}{\Delta x} + b(x + \Delta x/2) = 0$$

$$\Longrightarrow \frac{\mathrm{d}p}{\mathrm{d}x} + b(x) = 0$$

kinematic (strain - displacement relation)

$$\varepsilon(x) = \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

$$\Longrightarrow \varepsilon(x) = \frac{\mathrm{d}u(x)}{\mathrm{d}x} \qquad [-]$$

kinetics (stress)

$$\sigma(x) = \frac{p(x)}{A(x)} \qquad [\mathrm{N/m^2}]$$

constitutive relation

$$\sigma(x) = E(x)\varepsilon(x)$$

E : Young's modulus $[\mathrm{N/m^2}]$

governing equation and boundary conditions

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + b = 0 \qquad \Omega := 0 < x < l$$

$$\sigma n = E\frac{\mathrm{d}u}{\mathrm{d}x}n = \bar{t} \qquad \text{on } \Gamma_t$$

$$u = \bar{u} \qquad \text{on } \Gamma_u$$

solve for $u(x)$

FEM for wave propagation

Governing equations

$$\underbrace{(A\widetilde{\rho})}_{\rho}\,\ddot{u} = \frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) \qquad \text{in } \Omega\times]0,T[\qquad \text{(balance of linear momentum)}$$

$$[\mathrm{m}^2][\mathrm{kg/m}^3][\mathrm{m/s}^2] = [\mathrm{N/m}] \qquad [/\mathrm{m}][\mathrm{m}^2][\mathrm{N/m}^2] = [\mathrm{N/m}]\checkmark$$

$$u = \overline{u} \qquad \text{on } \Gamma_u\times]0,T[\qquad \text{(essential b.c.)}$$

$$t = \sigma n = \bar{t} \qquad \text{on } \Gamma_t\times]0,T[\qquad \text{(natural b.c.)}$$

$$u(x,0) = u_0(x) \qquad x\in\Omega \qquad \text{(initial condition for displacement)}$$

$$\dot{u}(x,0) = \dot{u}_0(x) \qquad x\in\Omega \qquad \text{(initial condition for velocity)}$$

Inertial term $\rho\ddot{u}$ and the *initial conditions* are all that's new!

We could divide the balance of linear momentum by A *but* that would change the definition of $\boldsymbol{K}^e$ from earlier! We introduce $\tilde{\rho}$ [kg/m^3] for consistency.
(Warning: different textbooks do this differently)

1 N $= 1$ kg m/s^2

FEM for wave propagation

Weak form of inertial term

$$\underbrace{(A\widetilde{\rho})}_{\rho}\,\ddot{u} \longrightarrow \int_0^l \delta u \rho \ddot{u}\,\mathrm{d}x$$

FE approximation of displacement, velocity and acceleration

$$u = \boldsymbol{N}\boldsymbol{d}, \qquad \dot{u} = \boldsymbol{N}\dot{\boldsymbol{d}} \qquad \ddot{u} = \boldsymbol{N}\ddot{\boldsymbol{d}}$$

$$\int_\Omega \delta u \rho \ddot{u}\,\mathrm{d}x \quad \rightarrow \text{ a contribution to the FEM problem of: } \quad \underbrace{\int_\Omega \rho \boldsymbol{N}^T \boldsymbol{N}\,\mathrm{d}x}_{\boldsymbol{M}}\ \ddot{\boldsymbol{d}}$$

Mass matrix

The mass matrix $\boldsymbol{M}$ represents how mass is distributed in the finite element mesh that approximates the continuum body

Recap: Element matrices for two-noded elements

$$\boldsymbol{N}^e = \frac{1}{l^e}\Big[(x_2^e - x) \;\; (x - x_1^e)\Big] \qquad \text{and} \qquad \boldsymbol{B}^e = \frac{1}{l^e}\Big[-1 \;\; 1\Big]$$

$$\boldsymbol{K}^e = \int_{x_1^e}^{x_2^e} \boldsymbol{B}^{e\top} A^e E^e \boldsymbol{B}^e \, \mathrm{d}x = \int_{x_1^e}^{x_2^e} \frac{1}{l^e}\begin{bmatrix} -1 \\ 1 \end{bmatrix} A^e E^e \frac{1}{l^e}\begin{bmatrix} -1 & 1 \end{bmatrix} \mathrm{d}x = \frac{A^e E^e}{l^{e2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_1^e}^{x_2^e} \mathrm{d}x$$

Element stiffness matrix $\boldsymbol{K}^e$

$$\boldsymbol{K}^e = \frac{A^e E^e}{l^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{f}_\Omega^e = \int_{x_1^e}^{x_2^e} \boldsymbol{N}^{e\top} b(x)\, \mathrm{d}x = \int_{x_1^e}^{x_2^e} \boldsymbol{N}^{e\top} \boldsymbol{N}^e \, \mathrm{d}x \boldsymbol{b}^e = \frac{1}{l^{e2}} \int_{x_1^e}^{x_2^e} \begin{bmatrix} (x_2^e - x)^2 & (x_2^e - x)(x - x_1^e) \\ (x_2^e - x)(x - x_1^e) & (x - x_1)^{e2} \end{bmatrix} \mathrm{d}x$$

Element body force vector $\boldsymbol{f}_\Omega^e$

$$\boldsymbol{f}_\Omega^e = \frac{l^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b_1^e \\ b_2^e \end{bmatrix}$$

The discrete problem

Discrete problem

$$
\begin{aligned}
\boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} &= \boldsymbol{F} \\
\boldsymbol{d}(0) &= \boldsymbol{d}_0 \\
\dot{\boldsymbol{d}}(0) &= \dot{\boldsymbol{d}}_0
\end{aligned}
$$

or alternatively as:

$$
\begin{aligned}
\boldsymbol{M}\boldsymbol{a} + \boldsymbol{K}\boldsymbol{d} &= \boldsymbol{F} \\
\boldsymbol{d}(0) &= \boldsymbol{d}_0 \\
\boldsymbol{v}(0) &= \boldsymbol{v}_0
\end{aligned}
$$

We can compute $\boldsymbol{a}_0$ from

$$\boldsymbol{M}\boldsymbol{a}_0 = \boldsymbol{F}(t=0) - \boldsymbol{K}\boldsymbol{d}_0$$

where

mass matrix

$$\boldsymbol{M} = \mathop{\mathsf{A}}_{e} \boldsymbol{M}^e \qquad \text{where} \qquad \boldsymbol{M}^e = \int_{\Omega^e} \rho \boldsymbol{N}_e^T \boldsymbol{N}_e \, \mathrm{d}x$$

Mass matrix

Linear elements - see body forces for 1D elasticity

$$\boldsymbol{M}^e = \frac{\rho l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{(\widetilde{\rho}A)l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Quadratic elements

$$\boldsymbol{M}^e = \frac{\rho l^e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} = \frac{(\widetilde{\rho}A)l^e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

- $\sum_{I,J} \boldsymbol{M}^e_{IJ} = \rho l^e =$ mass of an element
- negative entries in mass matrix for quadratic elements - why?

Recall: Element stiffness matrix $\boldsymbol{K}^e$

$$\boldsymbol{K}^e = \frac{A^e E^e}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- $\sum_{I,J} \boldsymbol{K}^e_{IJ} = 0$

Viscous damping

- Include velocity proportional damping (viscous dashpot) to account for energy loss to mechanisms we don't account for directly: e.g. friction, heat
- Important for structural dynamics!

$$\boldsymbol{Ma} + \boldsymbol{Cv} + \boldsymbol{Kd} = \boldsymbol{F}$$

where $\boldsymbol{C}$ is the *viscous damping matrix*.

Rayleigh damping

Convenient to make damping proportional to mass and stiffness:

$$\boldsymbol{C} = a\boldsymbol{M} + b\boldsymbol{K}$$

Choose parameters a and b to obtain desired damping characteristics.

Eigenvalue and frequency analysis

Recall:

$$\rho \ddot{u} = \frac{\partial}{\partial x}\left(AE \frac{\partial u}{\partial x} \right) \qquad (\star)$$

Assume that $\ddot{u} \propto u$ (simple harmonic motion):

$$u = \sin \omega(t - t_0), \qquad \dot{u} = \omega \cos \omega(t - t_0), \qquad \ddot{u} = -\omega^2 \sin \omega(t - t_0)$$

The *frequencies* and *mode shapes* for the *free vibration of an elastic rod* are given by:

$$(\star) \longrightarrow -\rho\omega^2 \sin \omega(t - t_0) = \frac{\partial}{\partial x}\left(AE \frac{\partial(\sin \omega(t - t_0))}{\partial x} \right)$$

$$\longrightarrow \lambda \rho u + \frac{\mathrm{d}}{\mathrm{d}x}\left(AE \frac{\mathrm{d}u}{\mathrm{d}x} \right) = 0 \qquad (\star\star)$$

where

λ_k : an eigenvalue, $\qquad \lambda_k = \omega_k^2, \qquad u_k$: an eigenfunction (mode shape)

Natural frequency ω [rad/s] is the frequency at which a system tends to oscillate in the absence of any driving or damping force.
If we know the *maximum natural frequency* we can compute the period $T = 2\pi/\omega$ [s] and use this to inform the choice of time-step size in the time-integration algorithm.

Finite element approximation of eigenvalue problem

$(\star\star)$ has countably infinite solutions.

Properties of the eigenvalues and the eigenfunctions

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

$$\int_\Omega u_k \rho u_l \, \mathrm{d}x = \delta_{kl} \qquad \text{(orthogonality)}$$

Let's assume "standard" *homogeneous* boundary conditions:

$$u = 0 \text{ at } x = 0 \qquad \text{and} \qquad E \frac{\mathrm{d}u}{\mathrm{d}x} n \Big|_{x=l} = \sigma n \Big|_{x=l} = 0$$

Use FEM to approximate $(\star\star)$ (multiply by a test function δu, integrate over the domain, and perform an integration by parts) gives the generalised eigenvalue problem:

$$[\boldsymbol{K} - \lambda_k \boldsymbol{M}]\, \boldsymbol{\Psi}_k = \boldsymbol{0} \qquad (\star\star\star)$$

The vector $\boldsymbol{\Psi}_k$ is the k-th eigenvector - a global vector of nodal displacements.
The problem $(\star\star\star)$ has a finite number n_{eq} of solutions. One for each degree of freedom in the finite element domain.

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_{n_{\text{eq}}}$$

Mass lumping

Recall: for problems in 1D we required $n_{\text{gp}} > (p+1)/2$ Gauss points to fully integrate a polynomial of degree p using Gaussian quadrature.

Consistent mass matrix

If the mass matrix is integrated using a sufficient quadrature rule and computed from an element contribution:

$$\boldsymbol{M}^e = \int_{\Omega_e} \rho \boldsymbol{N}_e^T \boldsymbol{N}_e \, \mathrm{d}x$$

then it is *consistent*. For a linear element in 1D, $p = 2$ in the mass matrix and $n_{\text{gp}} \equiv 2$.

Sometimes it's good to break the rules: *mass lumping*

- a diagonal mass matrix is attractive as it can lead to very efficient algorithms when used in conjunction with explicit time-integration schemes

Mass lumping by nodal quadrature

Recall: consistent mass matrices

$$\boldsymbol{M}^e_{\text{con}} = \frac{\rho l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{M}^e_{\text{con}} = \frac{\rho l^e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

To construct a *lumped* element mass matrix, the general idea is to choose a quadrature scheme where the quadrature points are at the nodes of an element.
For example, using the trapezoidal rule with linear shape functions:

$$\boldsymbol{M}^e_{\text{lumped}} = \frac{\rho l^e}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using Simpson's rule with quadratic shape functions:

$$\boldsymbol{M}^e_{\text{lumped}} = \frac{\rho l^e}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

More generally we can use a Lobatto quadrature rule.
In both cases were are concentrating the nodal contributions into a single diagonal term!

Estimating the critical time-step size

- The stability of *conditionally-stable* time-integration schemes is dependent on the time step size Δt. Such schemes become unstable if $\Delta t > \Delta t_{\text{crit}}$.
- We can estimate Δt_{crit} using the maximum eigenvalue $\lambda_{n_{\text{eq}}}$ from the global eigenvalue problem
  - this is however expensive!

- It can be shown that:

$$\lambda_{n_{\text{eq}}} \leq \max_e(\lambda^e_{\text{max}}) =: \lambda^h_{\text{max}}$$

where λ^e_{max} is the *maximum eigenvalue* of element e, and λ^h_{max} is the maximum eigenvalue of the *whole system*.

Element eigenvalue problem

$$[\boldsymbol{K}^e - \lambda^e \boldsymbol{M}^e]\,\boldsymbol{\Psi}^e = \boldsymbol{0}$$

To solve: convert to a standard eigenvalue problem:

$$\left[\underbrace{[\boldsymbol{M}^e]^{-1}\boldsymbol{K}^e}_{\boldsymbol{A}^e} - \lambda^e \boldsymbol{I}\right]\boldsymbol{\Psi}^e = \boldsymbol{0} \qquad \text{where } \boldsymbol{M}^{e\,-1}\boldsymbol{M} = \boldsymbol{I}$$

$$\implies \det\left[\boldsymbol{A}^e - \lambda^e \boldsymbol{I}\right] = 0 \quad \text{(characteristic polynomial)}$$

Example: Eigenvalue problem for linear element and consistent mass matrix

$$\boldsymbol{K}^e = \frac{A^e E^e}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{M}^e = \frac{\rho l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{(\widetilde{\rho} A^e) l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$\left[\underbrace{\left[\frac{6E}{\widetilde{\rho} l_e^2}\right]}_{\tilde{d} := 6c^2/l_e^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right] \boldsymbol{\Psi}^e = 0$$

convert to a standard eigenvalue problem

$$\implies \left[\tilde{d}\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \boldsymbol{\Psi}^e = 0$$

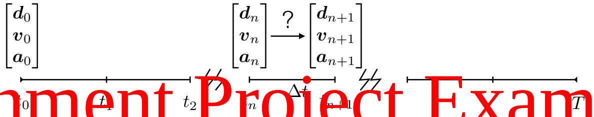
$$\implies \left[\tilde{d}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \boldsymbol{\Psi}^e = 0$$

$$\implies \det \begin{bmatrix} \tilde{d} - \lambda & -\tilde{d} \\ -\tilde{d} & \tilde{d} - \lambda \end{bmatrix} = 0 \implies (\tilde{d} - \lambda)(\tilde{d} - \lambda) - \tilde{d}^2 = 0$$

$$-2\tilde{d}\lambda + \lambda^2 = \lambda(\lambda - 2\tilde{d}) = 0$$

$$\implies \omega_{\text{max}}^2 = \lambda_{\text{max}}^e = 2\tilde{d} = \frac{12c^2}{l_e^2} = 12\left[\frac{c}{l_e}\right]^2$$

Note: the critical frequency depends on the ratio of the wave speed to the element length c/l^e. Its inverse relates to how long it takes a stress wave to pass through an element.



$[t_0 = 0, \ldots, t_n, t_{n+1}, \ldots, T]$ and $\Delta t := t_{n+1} - t_n$

Initial conditions

$$\boldsymbol{d}(0) = \boldsymbol{d}_0 \qquad \text{and} \qquad \boldsymbol{v}(0) = \boldsymbol{v}_0$$

We can compute $\boldsymbol{a}_0$ from

$$\boldsymbol{M}\boldsymbol{a}_0 = \boldsymbol{F}(t=0) - \boldsymbol{C}\boldsymbol{v}_0 - \boldsymbol{K}\boldsymbol{d}_0$$

Discrete problem

Given the state $[\boldsymbol{d}_n,\, \boldsymbol{v}_n,\, \boldsymbol{a}_n]$ and $\boldsymbol{F}_{n+1}$ solve:

$$\boldsymbol{M}\boldsymbol{a}_{n+1} + \boldsymbol{C}\boldsymbol{v}_{n+1} + \boldsymbol{K}\boldsymbol{d}_{n+1} = \boldsymbol{F}_{n+1}$$

for $[\boldsymbol{d}_{n+1},\, \boldsymbol{v}_{n+1},\, \boldsymbol{a}_{n+1}]$

One-step algorithm ($\boldsymbol{a}$-form) with a Newmark time-integration scheme

Discrete problem

Given the state $[\boldsymbol{d}_n, \boldsymbol{v}_n, \boldsymbol{a}_n]$ and $\boldsymbol{F}_{n+1}$ solve: $\boldsymbol{M}\boldsymbol{a}_{n+1} + \boldsymbol{C}\boldsymbol{v}_{n+1} + \boldsymbol{K}\boldsymbol{d}_{n+1} = \boldsymbol{F}_{n+1}$ $(\star)$

Define predictors (β and γ determine the *stability* and *accuracy*):

$$\widetilde{\boldsymbol{d}} = \boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{\Delta t^2}{2}(1-2\beta)\boldsymbol{a}_n \quad \text{and} \quad \widetilde{\boldsymbol{v}} = \boldsymbol{v}_n + (1-\gamma)\Delta t \boldsymbol{a}_n \, .$$

Correctors for Newmark scheme

$$\boldsymbol{d}_{n+1} = \widetilde{\boldsymbol{d}} + \beta \Delta t^2 \boldsymbol{a}_{n+1} \quad \text{and} \quad \boldsymbol{v}_{n+1} = \widetilde{\boldsymbol{v}} + \gamma \Delta t \boldsymbol{a}_{n+1} \, .$$

Finite-difference formulae for evolution of approximate solution:

$$\boldsymbol{d}_{n+1} = \boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{\Delta t^2}{2}(1-2\beta)\boldsymbol{a}_n + \beta \Delta t^2 \boldsymbol{a}_{n+1}$$

$$\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + (1-\gamma)\Delta t \boldsymbol{a}_n + \gamma \Delta t \boldsymbol{a}_{n+1}$$

Substitute into $(\star)$ - solve for $\boldsymbol{a}_{n+1}$ and update state from correctors

$$\underbrace{\left[\boldsymbol{M} + \gamma \Delta t \boldsymbol{C} + \beta \Delta t^2 \boldsymbol{K}\right]}_{\boldsymbol{A}} \boldsymbol{a}_{n+1} = \underbrace{\boldsymbol{F}_{n+1} - \boldsymbol{C}\widetilde{\boldsymbol{v}} - \boldsymbol{K}\widetilde{\boldsymbol{d}}}_{\boldsymbol{R}} \, .$$

The Newmark family of schemes

Method	Type	β	γ	Stability*	Order accuracy
Average acceleration	Implicit	$\frac{1}{4}$	$\frac{1}{2}$	Unconditional	2
Linear acceleration	Implicit	$\frac{1}{6}$	$\frac{1}{2}$	$\Omega_{\text{crit}} = 2\sqrt{3}$	2
Fox Goodwin	Implicit	$\frac{1}{12}$	$\frac{1}{2}$	$\Omega_{\text{crit}} = \sqrt{6}$	2
Central difference	Explicit	0	$\frac{1}{2}$	$\Omega_{\text{crit}} = 2$	2

*Stability is based on analysis of a single degree of freedom undamped problem. Damping increases the Δt_{crit}. $\Omega_{\text{crit}} = (\gamma/2 - \beta)^{-1/2}$ (critical sampling frequency).

Stability

$$\text{Unconditional stability: } 2\beta \geq \gamma \geq \frac{1}{2}$$

$$\text{Conditional stability: } \gamma \geq \frac{1}{2}, \qquad \beta < \frac{\gamma}{2}$$

$$\omega^h_{\text{max}} \Delta t \leq \Omega_{\text{crit}} \quad \leftrightarrow \quad \frac{\Delta t}{T} \leq \frac{\Omega_{\text{crit}}}{2\pi}$$

Accuracy

A time-integration scheme is n-th order accurate if the error e is proportional to the time-step size Δt to the power of n

$$e(\Delta t) = C\Delta t^n$$

where C is a constant.

All the schemes we considered were second order accurate: the errors decreases by a factor of 4 if we halve the time step size.

Explicit versus implicit schemes

Consider an undamped problem $\boldsymbol{C} = \boldsymbol{0}$

Explicit scheme: e.g. Central difference ($\beta = 0$)

$$\boldsymbol{d}_{n+1} = \widetilde{\boldsymbol{d}}$$

$$\boldsymbol{M}\boldsymbol{a}_{n+1} = \boldsymbol{F}_{n+1} - \boldsymbol{K}\widetilde{\boldsymbol{d}}$$

- The displacement $\boldsymbol{d}_{n+1}$ is computed using only information at t_n
- If $\boldsymbol{M}$ is lumped (diagonal) then no matrix inversion required
  - leads to a very efficient scheme but requires a small time step
  - well suited for high strain rate problems (e.g. impact) that require a small time step

Stability: some examples

Solved the element eigenvalue problem:

$$\left[\boldsymbol{K}^e - \lambda^e \boldsymbol{M}^e\right] \boldsymbol{\Psi}^e = \boldsymbol{0}$$

and have $\lambda^h_{\max}$

$$\lambda_{n_{eq}} \leq \max_e(\lambda^e_{\max}) =: \lambda^h_{\max}$$

Assume we are using a *central difference scheme*: $\Omega_{\text{crit}} = 2$

Two-node linear element with a lumped mass matrix:

$$\omega^h_{\max} = \frac{2c}{l^e} \quad \text{thus} \quad \Delta t \leq \frac{2}{\omega^h_{\max}} = \frac{l^e}{c}$$

Two-node linear element with a consistent mass matrix:

$$\omega^h_{\max} = \frac{2\sqrt{3}c}{l^e} \quad \text{thus} \quad \Delta t \leq \frac{2}{\omega^h_{\max}} = \frac{l^e}{\sqrt{3}c}$$

Consistent mass matrices tend to yield smaller critical time steps than lumped mass matrices. Another advantage of mass lumping.

Exercise: Compute the critical time step for a three-node quadratic element.

Algorithm for elastodynamics using Newmark scheme

Data: Material and load, Domain, Mesh, Time
Assemble $\boldsymbol{M}$ *and* $\boldsymbol{K}$;
for e **to** n_e **do**
  $\boldsymbol{M} \leftarrow \boldsymbol{M}^e$;
  $\boldsymbol{K} \leftarrow \boldsymbol{K}^e$;

end
Result: $\boldsymbol{A} := \boldsymbol{M} + \beta \Delta t^2 \boldsymbol{K}$
Data: Initial conditions $\boldsymbol{d}_0$ and $\boldsymbol{v}_0$ and loading $\boldsymbol{F}_0$
Solve for initial acceleration $\boldsymbol{a}_0$: $\boldsymbol{M}\boldsymbol{a}_0 = \boldsymbol{F}_0 - \boldsymbol{K}\boldsymbol{d}_0$;

Assemble $\boldsymbol{R}$ *and solve* $\boldsymbol{A}\boldsymbol{a}_{n+1} = \boldsymbol{R}$ *for accelerations for all* n;
for $n = 1$ **to** n_t **do**
  Data: $\boldsymbol{a}_n \leftarrow \boldsymbol{a}$, $\boldsymbol{v}_n \leftarrow \boldsymbol{v}$, $\boldsymbol{d}_n \leftarrow \boldsymbol{d}$
  $\widetilde{\boldsymbol{d}} = \boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{\Delta t^2}{2}(1 - 2\beta)\boldsymbol{a}_n$;
  $\widetilde{\boldsymbol{v}} = \boldsymbol{v}_n + (1 - \gamma)\Delta t \boldsymbol{a}_n$;

  $\boldsymbol{R} = \boldsymbol{F}_{n+1} - \boldsymbol{K}\widetilde{\boldsymbol{d}}$;
  Solve $\boldsymbol{A}\boldsymbol{a}_{n+1} = \boldsymbol{R}$ *subject to boundary conditions*;
  $\boldsymbol{d}_{n+1} \leftarrow \widetilde{\boldsymbol{d}} + \beta \Delta t^2 \boldsymbol{a}_{n+1}$;
  $\boldsymbol{v}_{n+1} \leftarrow \widetilde{\boldsymbol{v}} + \gamma \Delta t \boldsymbol{a}_{n+1}$;
end
Result: Complete time history of problem at all nodal points

Some comments on damping

- The higher modes of the discrete equations are artefacts of the spatial discretisation (the mesh)
- It is generally desirable and often necessary to remove the high-frequency modes
- The Newmark algorithm itself can introduce *algorithmic damping*:
  - $\gamma > 1/2$ introduces high-frequency dissipation
  - for fixed $\gamma > 1/2$, select β to maximise high-frequency dissipation
  - an ideal value of $\beta = (\gamma + 1/2)^2/4$
  - one drawback: only $\gamma = 1/2$ is second-order accurate!
- Viscous damping (i.e. $\boldsymbol{C} \neq \boldsymbol{0}$):
  - Viscous damping damps an intermediate band of frequencies without having a significant effect on the all-important high modes

Stress waves and boundary conditions

- A longitudinal stress wave can be compressive (- sign) or tensile (+ sign)
- A wave will reflect without inverting off a fixed (essential) boundary
- A wave will reflect and invert of a free boundary