ETW3420

Principles of Forecasting and Applications

Topic 6 Exercises - Part 2

Question 1

Consider the following AR(1) process

$$y_t = c + \phi_1 y_{t-1} + e_t \tag{1}$$

where

$$e_t \sim WN(0, \sigma^2)$$

(a) Prove that

$$E(y_t) = \mu = \frac{c}{1 - \phi_1} \quad \forall\, t \tag{2}$$

Hint: Take expectations on both sides of (1) and use the fact that y_t is stationary.

(b) Prove that

$$Var(y_t) = \gamma_0 = \frac{\sigma^2}{1 - \phi_1^2} \quad \forall\, t \tag{3}$$

Note: Since y_t is stationary,

$$Cov(y_{t-1}, e_t) = 0.$$

(c) Prove that

$$Cov(y_t, y_{t-j}) = \gamma_j = \frac{\sigma^2}{(1-\phi_1^2)}\phi_1^j, \quad \forall\, t \text{ and } \forall\, j \tag{4}$$

Hint: From (2) we have

$$c = \mu(1-\phi_1). \tag{5}$$

Substituting (5) into (1) and rearranging, we obtain:

$$y_t - \mu = \phi_1(y_{t-1} - \mu) + e_t \tag{6}$$

Multiplying both sides of (6) by $(y_{t-j} - \mu)$, we obtain:

$$(y_t - \mu)(y_{t-j} - \mu) = \phi_1(y_{t-1} - \mu)(y_{t-j} - \mu) + e_t(y_{t-j} - \mu). \tag{7}$$

Take expectations on both sides of (7) and use the following information:

i) $cov(y_t, y_{t-j}) = E[(y_t - \mu)(y_{t-j} - \mu)]$

ii) Fact that $E(e_t y_{t-j}) = 0$ for any $j \neq 0$.

iii) Consider pattern that emerges when we set $j = 1, j = 2, ...$

(d) Prove that

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \phi_1^j \quad \forall\, j \tag{8}$$

Question 2

Consider the following MA(1) process

$$y_t = e_t + \theta e_{t-1} \tag{9}$$

where

$$e_t \sim WN(0, \sigma^2).$$

(a) Prove that

$$E(y_t) = \mu = 0 \quad \forall\, t \tag{10}$$

Hint: Take expectations on both sides of (9).

(b) Prove that

$$Var(y_t) = \gamma_0 = (1 + \theta^2)\sigma^2 \quad \forall\, t \tag{11}$$

Hint: Take variance on both sides of (9).

(c) Prove that

$$Cov(y_t, y_{t-1}) = \theta\sigma^2$$
$$Cov(y_t, y_{t-j}) = 0 \quad \forall \quad j > 1$$

Hint: Use the following information:

i) $cov(y_t, y_{t-j}) = E[(y_t - \mu)(y_{t-j} - \mu)] = E(y_t y_{t-j})$

ii) Express $E(y_t y_{t-j})$ in terms of e_t

iii) $E(e_t e_{t-j}) = 0 \quad \forall \quad j > 0$

iv) Consider pattern that emerges when we set $j = 1, j = 2, ...$

Question 3

Consider `wmurders`, the number of women murdered each year (per 100,000 standard population) in the United States.

(a) By studying appropriate graphs of the series in R, find an appropriate ARIMA(p, d, q) model for these data.

(b) Should you include a constant in the model? Explain.

(c) Write this model in terms of the backshift operator.

(d) Fit the model using R and examine the residuals. Is the model satisfactory?

(e) Forecast three times ahead. Check your forecasts by hand to make sure that you know how they have been calculated.

(f) Create a plot of the series with forecasts and prediction intervals for the next three periods shown.

Question 4

Consider `austa`, the total international visitors to Australia (in millions) for the period 1980-2015. Produce a `ggtsdisplay` of the data and note its features.

```
ggtsdisplay(austa)
```

(a) Use `auto.arima()` to find an appropriate ARIMA model. What model was selected. Check that the residuals look like white noise. Plot forecasts for the next 10 periods.

```
#Auto fit an ARIMA model
(fit <- auto.arima(austa))

#Check residuals
checkresiduals(fit)
```

```
#Plot forecasts
fit %>% forecast(h=10) %>% autoplot()
```


(b) Execute the following codes. Can you explain the difference in results?

```
(Arima(austa, order = c(0,1,1)))


(Arima(austa, order = c(0,1,1), include.constant = T))
```

It boils down to whether an intercept, c, should be included/specified in the ARIMA model. In the context of R's parameterisation, it is whether the parameter μ is to be included. Recall that $c = \mu(1 - \phi_1 - ...\phi_p)$. In the R output, $\hat{\mu} = 0.173$. Since no AR coefficients were estimated, then $\hat{c} = \hat{\mu} = 0.173$.

- In the model specification, we have already decided that $d = 1$.

- If combined with $c = 0$, it implies that the long-term forecasts will go to a non-zero constant.

- If combined with $c \neq 0$, then the long term forecasts will follow a straight slope. (Refer to Slide 60 of lecture notes).

As `austa` displays an (upward) trend, we would expect the forecasts to also trend upwards, instead of being a non-zero constant. So we want to specify a constant in the ARIMA model, i.e. $c \neq 0$.

- Using `help(Arima)`, we read that there is an argument specification for `include.constant`.

- By default, the `Arima()` function sets $c = \mu = 0$ when $d > 0$ and provides an estimate of μ when $d = 0$.

- If `include.constant = T`, it will set `include.mean = T` if $d = 0$ and `include.drift = T` when $d > 0$. For the former, R labels the constant in the estimation output as `mean`. R labels the constant in the estimation output as `drift` for the latter.

- For $d > 1$, no constant is allowed by `Arima()` as a quadratic or higher order trend is particularly dangerous when forecasting.

- Referring back to our codes, `Arima(austa, order = c(0,1,1))` imposes a default setting of $c = \mu = 0$ when $d > 0$. This is not what we want because if $c = 0$ and $d = 1$, the long term forecasts is just a horizontal line.

- We want $c \neq 0$. Therefore, we need to specify `include.constant = TRUE` so that the parameter μ is estimated by R.

(c) Plot forecasts from an ARIMA(0,1,1) model with no drift and compare these to part a. Comment.

```r
austa %>% Arima(order=c(0,1,1), include.constant = FALSE) %>%
  forecast() %>% autoplot()
```

(d) Self-practice: Plot forecasts of `austa` from the following models and reconcile the plots with Slide 60 of your lecture notes. Be sure to take note of the values of d.

- ARIMA(0,0,1) with a constant
- ARIMA(0,0,0) with a constant
- ARIMA(0,2,1) with no constant.