MONASH University

MONASH BUSINESS SCHOOL

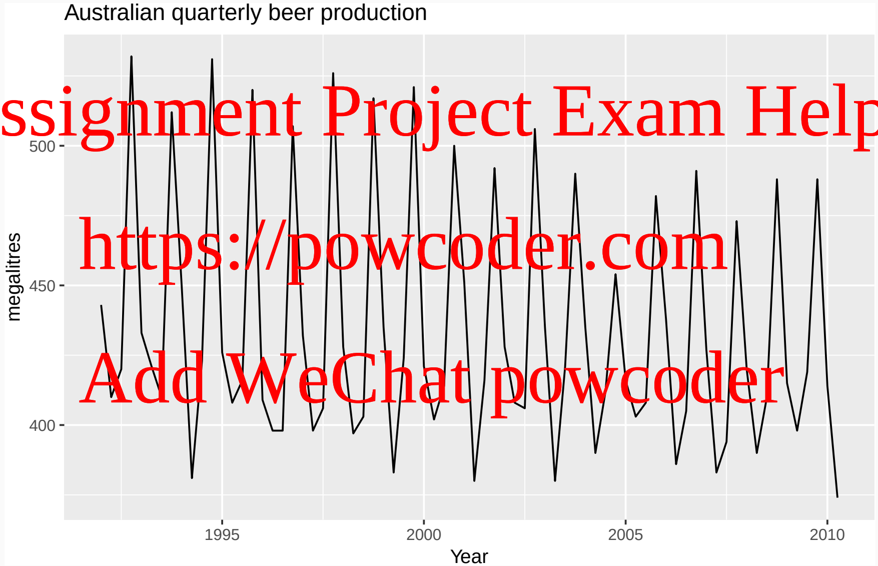
ETW3420: Principles of Forecasting and Applications

Topic 3: The Forecaster’s Toolbox

Dr. Jason Ng

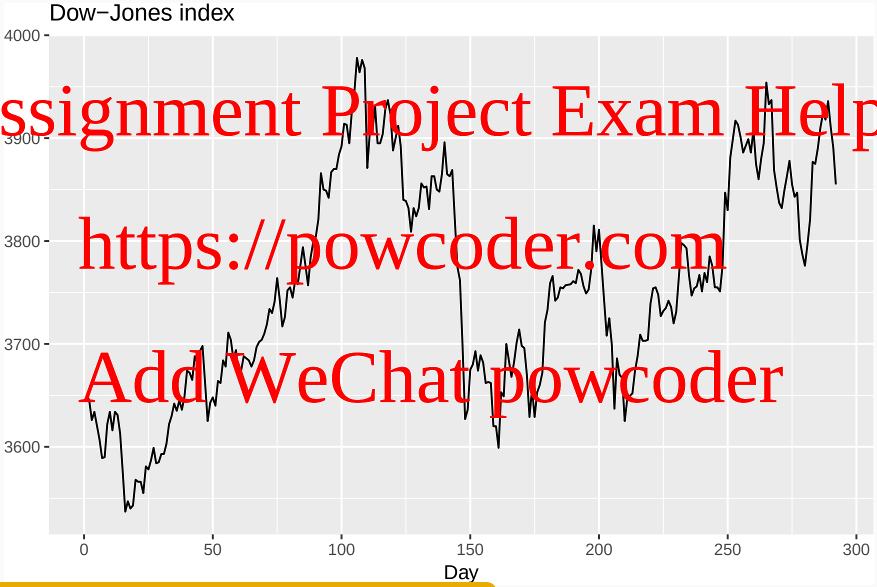
Outline

Some simple forecasting methods



How would you forecast these data?

Some simple forecasting methods



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1. Average method

- Forecast of all future values is equal to mean of historical data $\{y_1, \ldots, y_T\}$.
- Forecasts: $\hat{y}_{T+h|T} = \bar{y} = (y_1 + \cdots + y_T)/T$

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3. Seasonal naive method

- Forecasts equal to last value from same season.
- Forecasts: $\hat{y}_{T+h|T} = y_{T+h-m(k+1)}$, where m = seasonal period and k is the integer part of $(h-1)/m$.

Some simple forecasting methods

4. Drift method

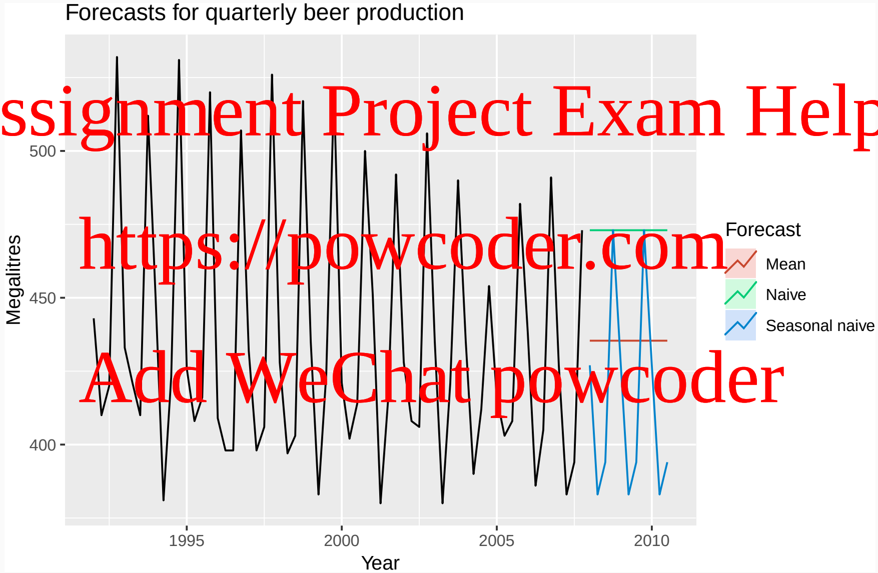
- Forecasts equal to last value plus average change.
- Forecasts:

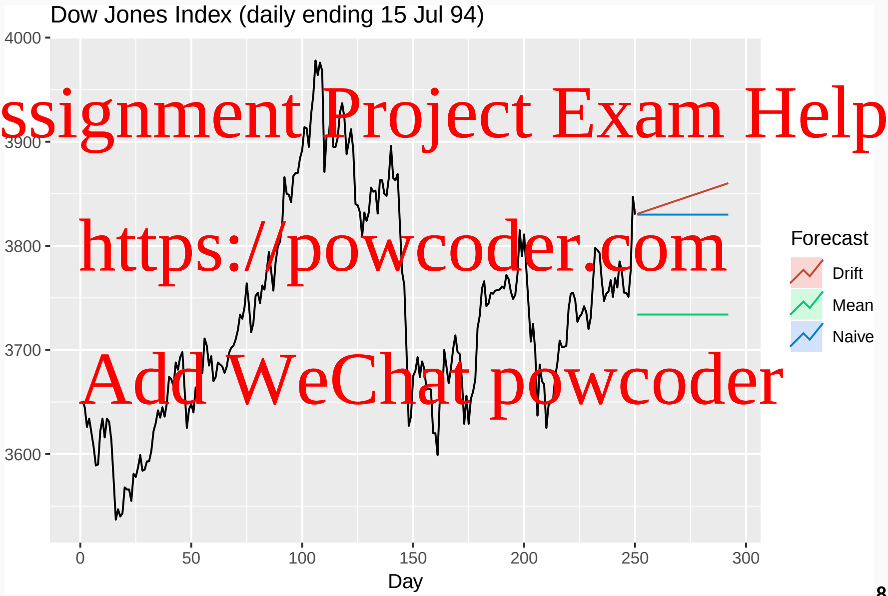
$$\hat{y}_{T+h|T} = y_T + \frac{h}{T-1}\sum_{t=2}^{T}(y_t - y_{t-1})$$

$$= y_T + \frac{h}{T-1}(y_T - y_1).$$

- Equivalent to extrapolating a line drawn between first and last observations.

Some simple forecasting methods




Dow Jones Index (daily ending 15 Jul 94)
4000
3900
3800
3700
3600
0
50
100
150
200
250
300
Day
Forecast
Drift
Mean
Naive

Outline

Variance stabilization

- If the data shows different variation at different levels of the series, then a transformation can be useful.

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Mathematical transformations for stabilizing variation

Square root	$w_t = \sqrt{y_t}$	$\downarrow$
Cube root	$w_t = \sqrt[3]{y_t}$	Increasing
Logarithm	$w_t = \log(y_t)$	strength

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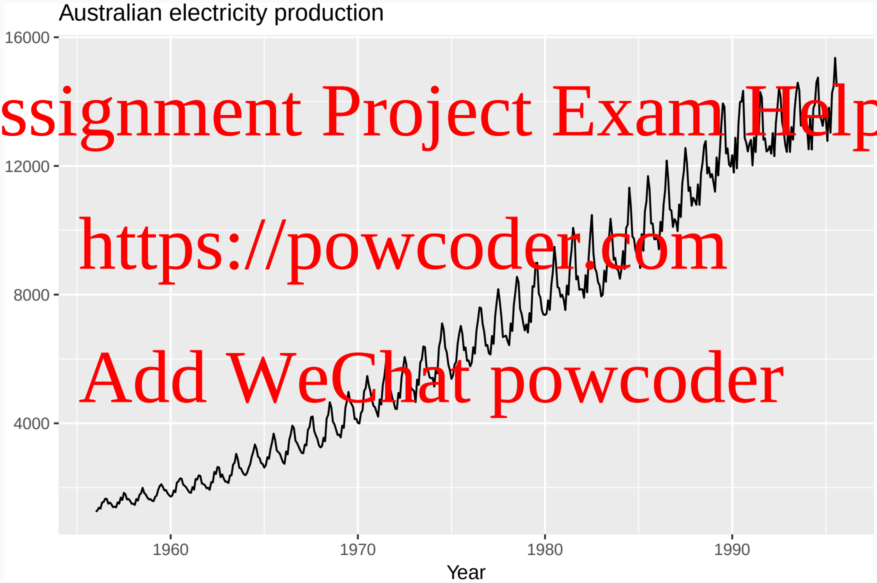
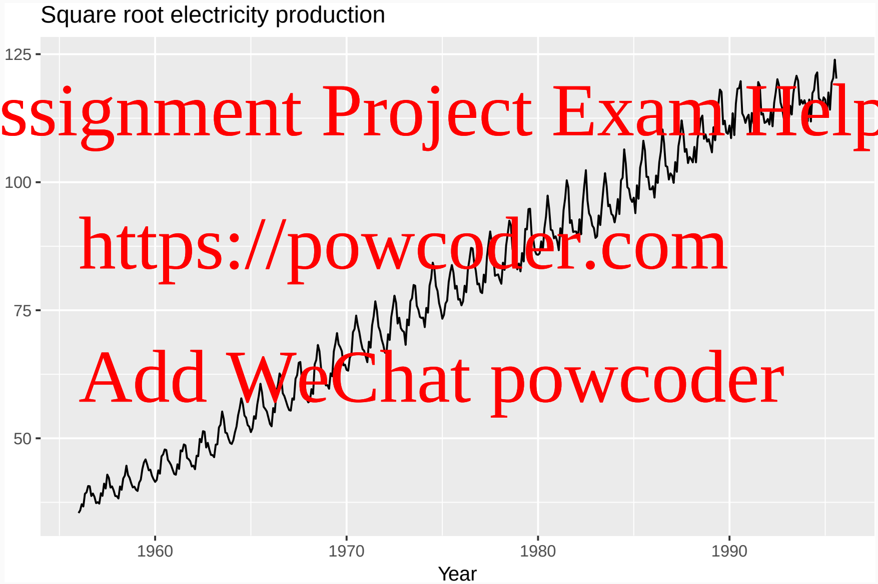
Logarithms, in particular, are useful because they are more interpretable: changes in a log value are **relative (percent) changes on the original scale.**

Variance stabilization



Variance stabilization



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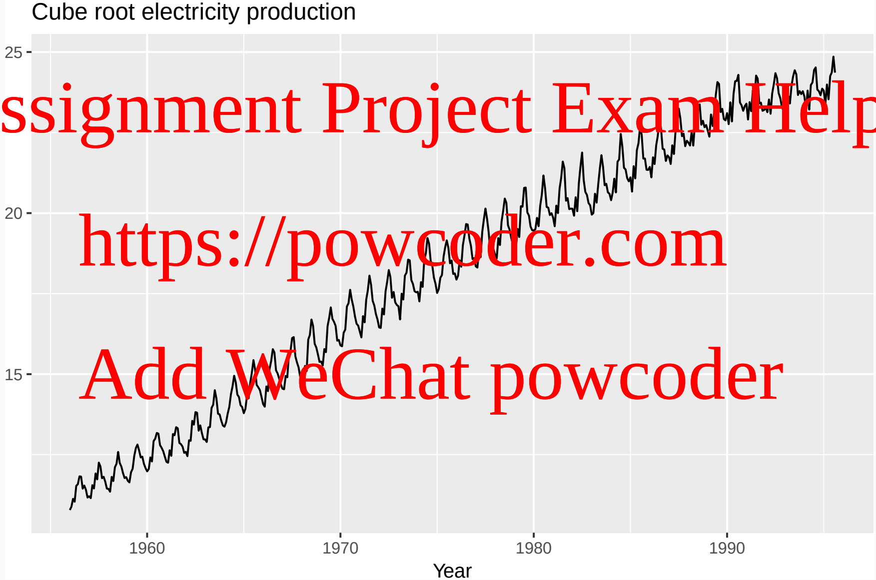

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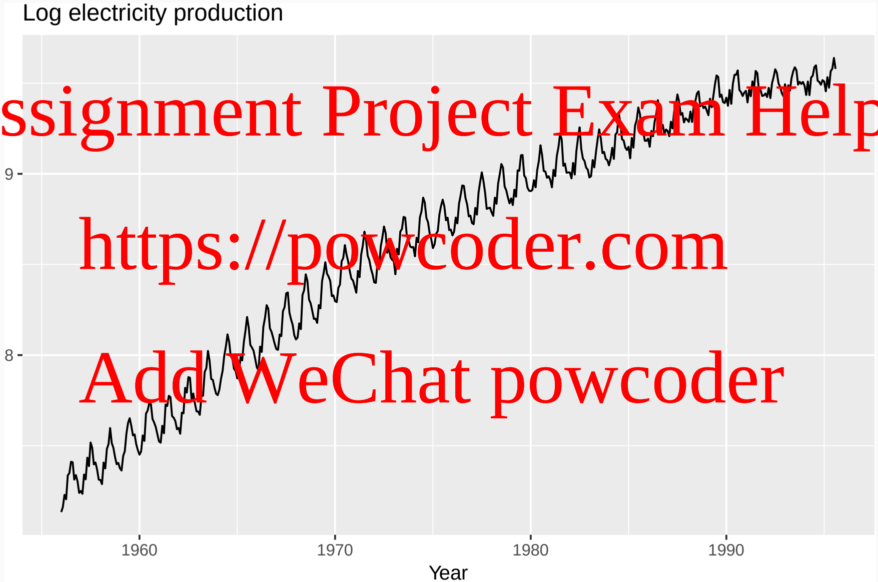
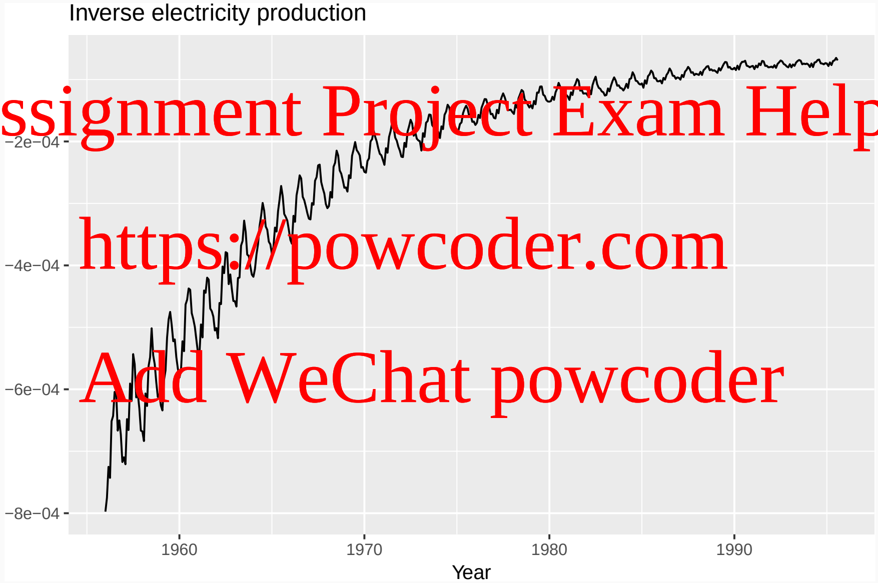


Variance stabilization



Box-Cox transformations

Each of these transformations is close to a member of the family of **Box-Cox transformations**:

$$w_t = \begin{cases} \log(y_t), & \lambda = 0; \\ (y_t^\lambda - 1)/\lambda, & \lambda \neq 0. \end{cases}$$

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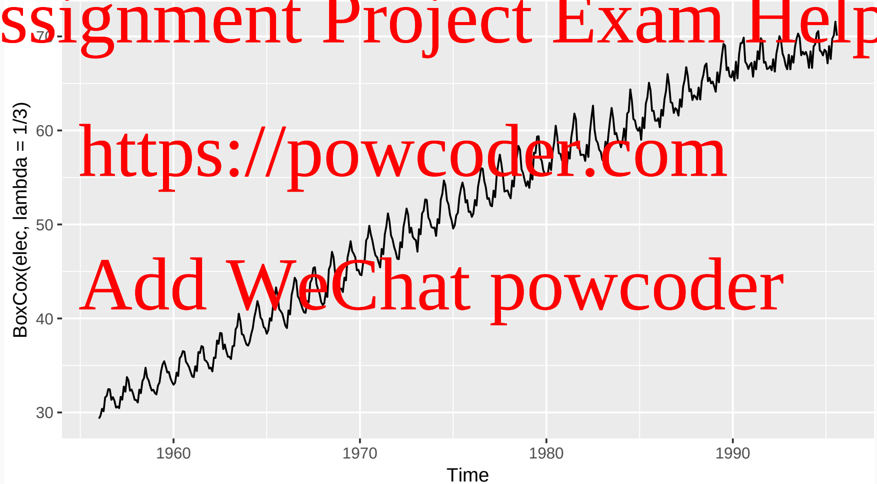
$$w_t = \begin{cases} \log(y_t), & \lambda = 0; \\ (y_t^\lambda - 1)/\lambda, & \lambda \neq 0. \end{cases}$$

- $\lambda = 1$: (No substantive transformation)
- $\lambda = \frac{1}{2}$: (Square root plus linear transformation)
- $\lambda = 0$: (Natural logarithm)
- $\lambda = -1$: (Inverse plus 1)

Box-Cox transformations

```
autoplot(BoxCox(elec,lambda=1/3))
```

Box-Cox transformations

- y_t^λ for λ close to zero behaves like logs.
- If some $y_t = 0$, then must have $\lambda > 0$
- if some $y_t < 0$, no power transformation is possible unless all y_t adjusted by **adding a constant to all values**.
- Simple values of λ are easier to explain.
- Results are relatively insensitive to λ.
- Often no transformation ($\lambda = 1$) needed.
- Transformation can have very large effect on PI.
- Choosing $\lambda = 0$ is a simple way to force forecasts to be positive

Automated Box-Cox transformations


```
(BoxCox.lambda(elec))
```

```
## [1] 0.2654076
```

Automated Box-Cox transformations


```
(BoxCox.lambda(elec))
```

```
## [1] 0.2654076
```


- This attempts to balance the seasonal fluctuations and random variation across the series.
- Always check the results.
- A low value of λ can give extremely large prediction intervals.

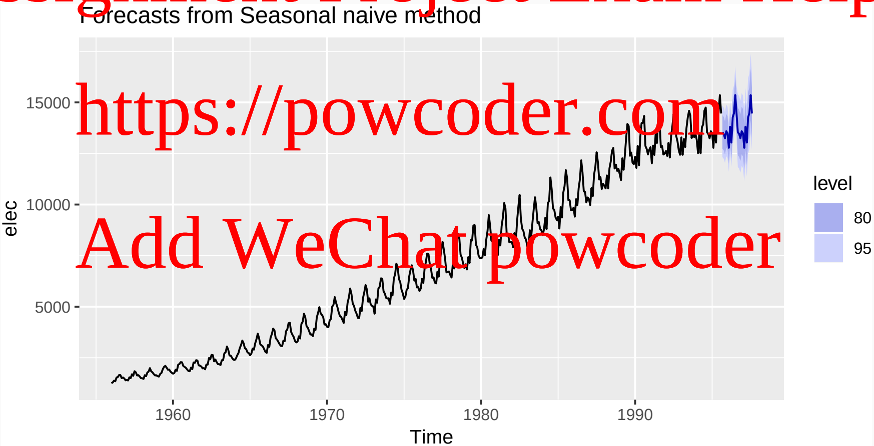
Back-transformation

We must reverse the transformation (or *back-transform*) to obtain forecasts on the original scale. The reverse Box-Cox transformations are given by

$$y_t = \begin{cases} \exp(w_t), & \lambda = 0; \\ (\lambda w_t + 1)^{1/\lambda}, & \lambda \neq 0 \end{cases}$$

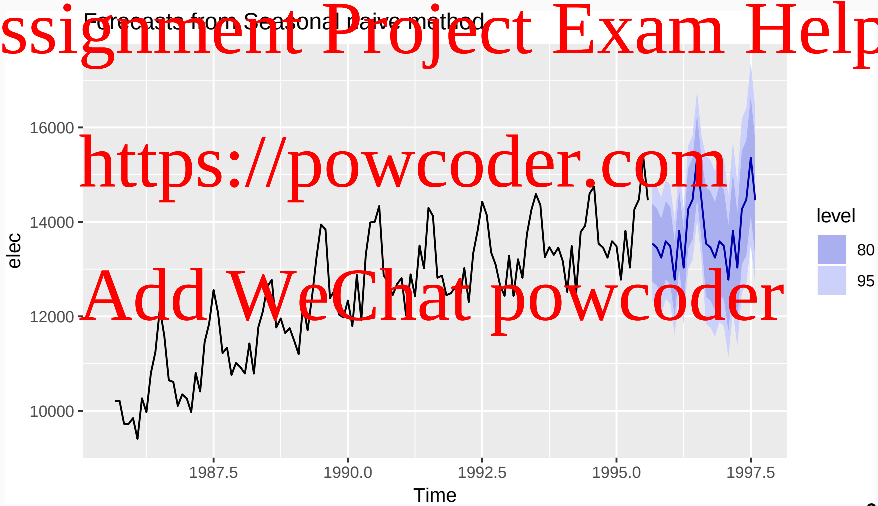
Back-transformation

```
fit <- snaive(elec, lambda=1/3)
autoplot(fit)
```

Back-transformation

```r
autoplot(fit, include=120)
```

Bias adjustment

- Back-transformed point forecasts are medians.
- Back-transformed PI have the correct coverage.

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Back-transformed means

Let X have mean μ and variance σ^2.

Let $f(x)$ be back-transformation function, and $Y = f(X)$.

Taylor series expansion about the point $X = \mu$:

$$f(X) = f(\mu) + (X - \mu)f'(\mu) + \frac{1}{2}(X - \mu)^2 f''(\mu)$$

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$$f(X) = f(\mu) + (X - \mu)f'(\mu) + \frac{1}{2}(X - \mu)^2 f''(\mu)$$

$$\text{E}[Y] = \text{E}[f(X)] = f(\mu) + \tfrac{1}{2}\sigma^2 f''(\mu)$$

Bias adjustment

Box-Cox back-transformation:

$$y_t = \begin{cases} \exp(w_t) & \lambda = 0; \\ (\lambda W_t + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f(x) = \begin{cases} exp(x) & \lambda = 0; \\ (\lambda x + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f''(x) = \begin{cases} exp(x) & \lambda = 0; \\ (1 - \lambda)(\lambda x + 1)^{1/\lambda - 2} & \lambda \neq 0. \end{cases}$$

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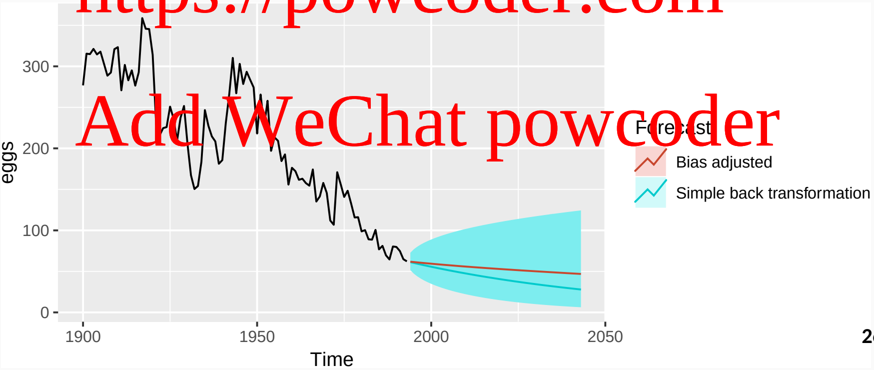
$$\mathrm{E}[Y] = \begin{cases} e^{\mu}\left[1 + \frac{\sigma^2}{2}\right] & \lambda = 0; \\ (\lambda\mu + 1)^{1/\lambda}\left[1 + \frac{\sigma^2(1-\lambda)}{2(\lambda\mu+1)^2}\right] & \lambda \neq 0. \end{cases}$$

Bias adjustment

```
fc <- rwf(eggs, drift=TRUE, lambda=0, h=50, level=80)
fc2 <- rwf(eggs, drift=TRUE, lambda=0, h=50, level=80,
  biasadj=TRUE)
autoplot(eggs) +
  autolayer(fc, series="Simple back transformation") +
  autolayer(fc2, series="Bias adjusted", PI=FALSE) +
  guides(colour=guide_legend(title="Forecast"))
```

Outline

Fitted values

- $\hat{y}_{t|t-1}$ is the forecast of y_t based on observations $y_1, \dots, y_{t-1}$.
- We call these "fitted values".
- Sometimes drop the subscript: $\hat{y}_t \equiv \hat{y}_{t|t-1}$.

For example:

- $\hat{y}_t = \bar{y}$ for average method.
- $\hat{y}_t = y_{t-1} + (y_T - y_1)/(T - 1)$ for drift method.

Forecasting residuals

Residuals in forecasting: difference between observed value and its fitted value: $\hat{e}_t = y_t - \hat{y}_{t|t-1}$.

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1. $\{e_t\}$ uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
2. $\{e_t\}$ have mean zero. If they don't, then forecasts are biased.

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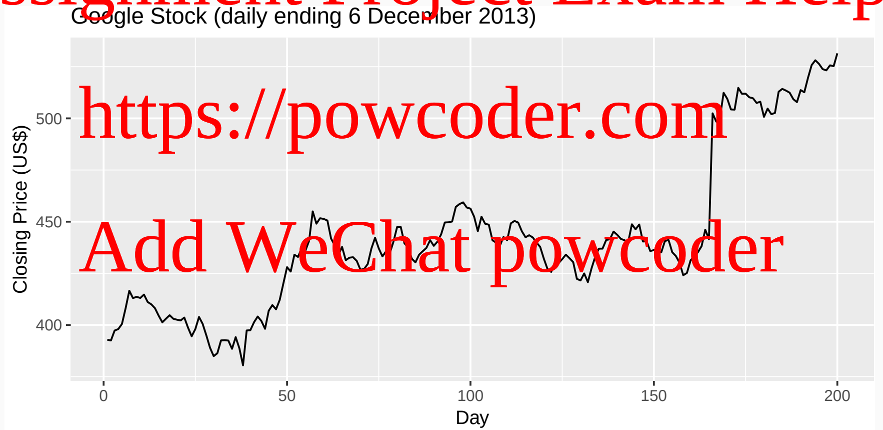
Useful properties (for prediction intervals)

3. $\{e_t\}$ have constant variance.
4. $\{e_t\}$ are normally distributed.

Example: Google stock price

```r
autoplot(goog200) +
  xlab("Day") + ylab("Closing Price (US$)") +
  ggtitle("Google Stock (daily ending 6 December 2013)")
```



Example: Google stock price

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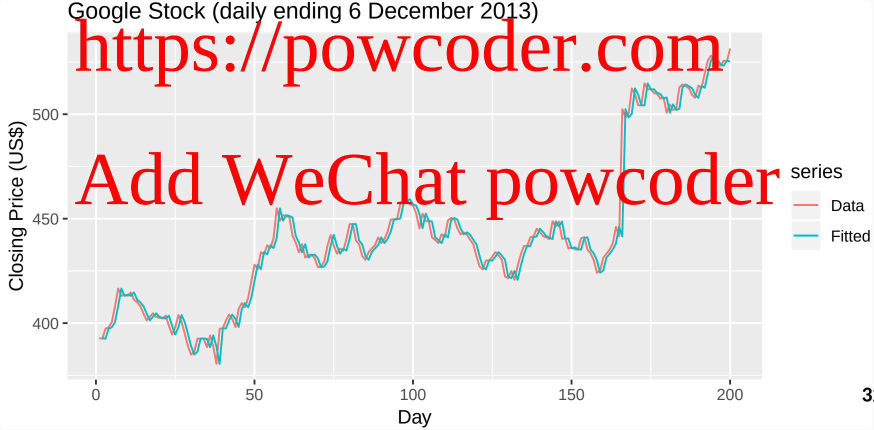
$$\hat{e}_t = y_t - y_{t-1}$$

Note: $\hat{e}_t$ are one-step-ahead forecast residuals

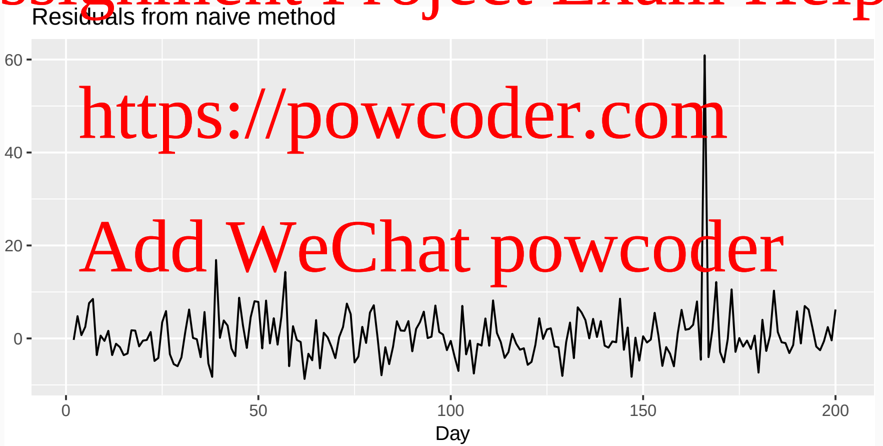
Example: Google stock price

```
fits <- fitted(naive(goog200))
autoplot(goog200, series="Data") +
  autolayer(fits, series="Fitted") +
  xlab("Day") + ylab("Closing Price (US$)") +
  ggtitle("Google Stock (daily ending 6 December 2013)")
```

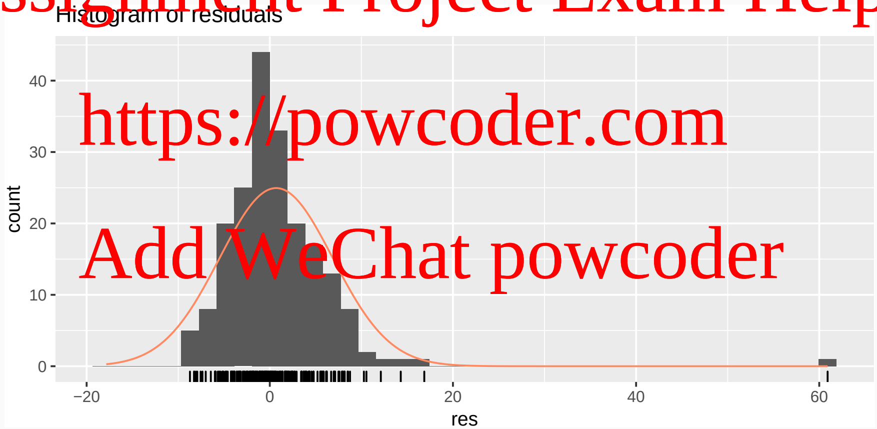
Example: Google stock price

```r
res <- residuals(naive(goog200))
autoplot(res) + xlab("Day") + ylab("") +
  ggtitle("Residuals from naive method")
```

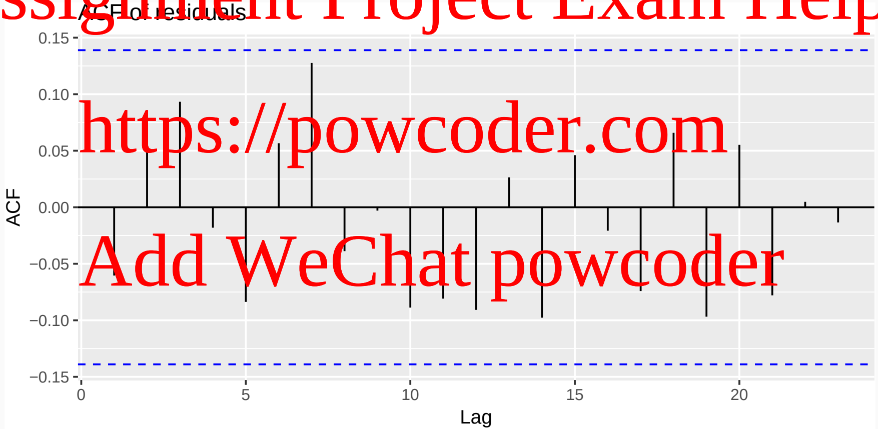
Example: Google stock price

```
gghistogram(res, add.normal=TRUE) +
  ggtitle("Histogram of residuals")
```

Example: Google stock price

```r
ggAcf(res) + ggtitle("ACF of residuals")
```

ACF of residuals

- We assume that the residuals are white noise (uncorrelated, mean zero, constant variance). If they aren't, then there is information left in the residuals that should be used in computing forecasts.

- So a standard residual diagnostic is to check the ACF of the residuals of a forecasting method.

- We *expect* these to look like white noise.

Portmanteau tests

Consider a *whole set* of r_k values, and develop a test to see whether the set is significantly different from a zero set.

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Box-Pierce test

$$Q = T\sum_{k=1}^{h} r_k^2$$

where h is max lag being considered and T is number of observations.

- If each r_k close to zero, Q will be **small**.
- If some r_k values large (positive or negative), Q will be **large**.

Portmanteau tests

Consider a *whole set* of r_k values, and develop a test to see whether the set is significantly different from a zero set.

Ljung-Box test

$$Q^* = T(T+2)\sum_{k=1}^{h}(T-k)^{-1}r_k^2$$

where h is max lag being considered and T is number of observations.

- My preferences: $h = 10$ for non-seasonal data, $h = 2m$ for seasonal data.
- Better performance, especially in small samples.

Portmanteau tests

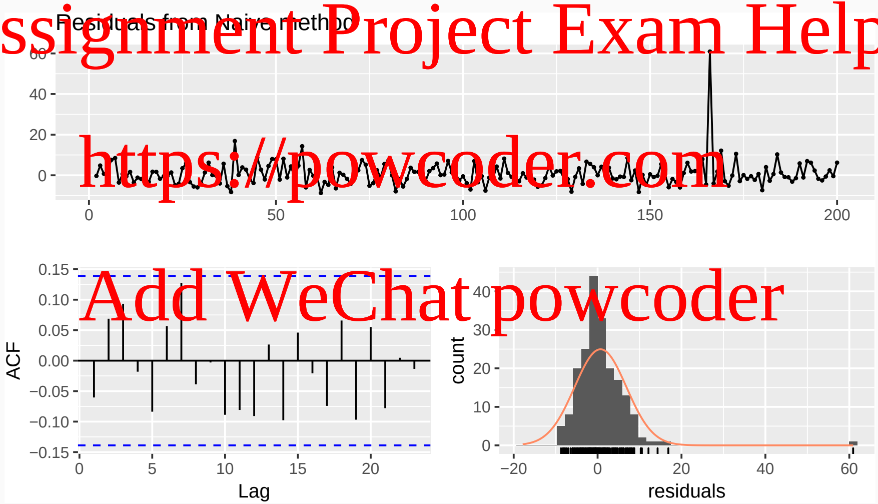
- If data are WN, $Q^* \sim \chi^2(h - K)$, where K = no. parameters in model.
- When applied to raw data, set $K = 0$.
- For the Google example:


```
# lag=h and fitdf=K
Box.test(res, lag=10, fitdf=0, type="Ljung")
```

```
##
##  Box-Ljung test
##
## data:  res
## X-squared = 11.031, df = 10, p-value =
## 0.3551
```

checkresiduals function

```
checkresiduals(naive(goog200))
```

##

`checkresiduals` function


```
##
##  Ljung-Box test
##
## data:  Residuals from Naive method
## Q* = 11.031, df = 10, p-value = 0.3551
##
## Model df: 0.   Total lags used: 10
```

Self-Practice

Compute seasonal naive forecasts for quarterly Australian beer production from 1992.


```
beer <- window(ausbeer, start=1992)
fc <- snaive(beer)
autoplot(fc)
```


Test if the residuals are white noise.


```
checkresiduals(fc)
```

What do you conclude?

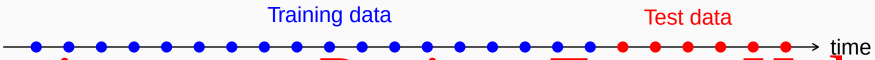
Outline

1. Some simple forecasting methods
2. Box-Cox transformations
3. Residual diagnostics
4. **Evaluating forecast accuracy: The traditional approach**
5. Evaluating forecast accuracy: The modern approach
6. Prediction intervals

Partitioning

- Partitioning is the 5th step in the forecasting process.
- It refers to the **splitting** of the data set into two parts: **training** set (70% - 80%) and **test** (validation) set.
- We develop our forecasting model/method using the training set (e.g. model identification and estimation).
- The developed model is then used to produce forecasts for the test set period. Actual values in the test set are then compared with the forecasted values for the test set to compute forecast errors.
- These forecast errors are then summarized to produce measures of forecasting accuracy.

Training and test sets



- A model which fits the training data well will not necessarily forecast well.
- A perfect fit can always be obtained by using a model with enough parameters.
- Over-fitting a model to data is just as bad as failing to identify a systematic pattern in the data.
- The test set must not be used for *any* aspect of model development or calculation of forecasts.
- Forecast accuracy is based only on the test set.

Forecast errors

Forecast "error": the difference between an observed value and its forecast.

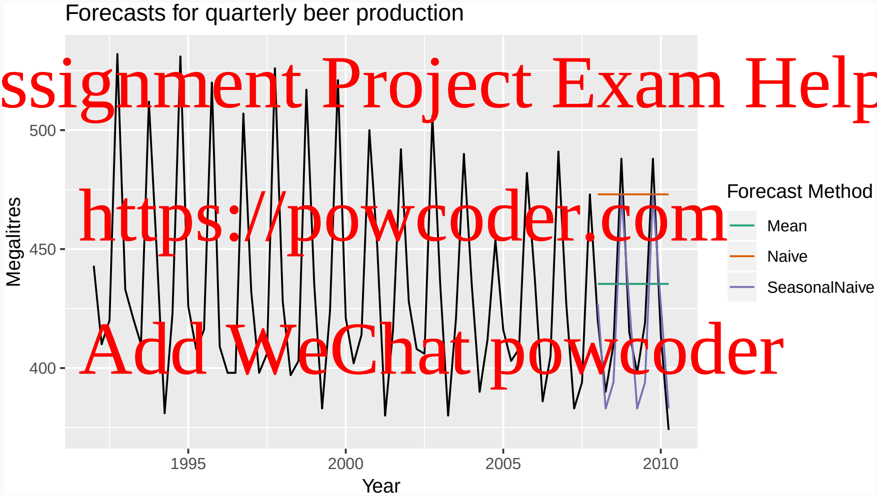
$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T},$$

where the training data is given by $\{y_1, \dots, y_T\}$

- Unlike residuals, forecast errors on the test set involve multi-step forecasts.

- These are *true* forecast errors as the test data is not used in computing $\hat{y}_{T+h|T}$.

Measures of forecast accuracy



Measures of forecast accuracy

$$
\begin{aligned}
y_{T+h} &= (T+h)\text{th observation, } h = 1, \dots, H \\
\hat{y}_{T+h|T} &= \text{its forecast based on data up to time } T. \\
e_{T+h} &= y_{T+h} - \hat{y}_{T+h|T}
\end{aligned}
$$

1. Mean Absolute Error

$$MAE = \frac{1}{H}\sum_{h=1}^{H} |e_{T+h}|$$

2. Mean Squared Error

$$MSE = \frac{1}{H}\sum_{h=1}^{H} e_{T+h}^2$$

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$$
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e_{T+h} &= y_{T+h} - \hat{y}_{T+h|T}
\end{aligned}
$$

3. Root Mean Squared Error

$$RMSE = \sqrt{\frac{1}{H}\sum_{h=1}^{H} e_{T+h}^2}$$

4. Mean Absolute Percentage Error

$$MAPE = \frac{1}{H}\sum_{h=1}^{H}\left|\frac{e_{T+h}}{y_{T+h}}\right| \times 100$$

- MAE, MSE, RMSE are all **scale dependent**.
- MAPE is **scale independent** but is only sensible if $y_t \gg 0$ for all t.
- The Mean Absolute Scaled Error (MASE) was subsequently developed by Hyndman and Koehler (IJF, 2006) to be able to handle zero counts, and also be a scale independent measure.

Measures of forecast accuracy

5. Mean Absolute Scaled Error

$$\text{MASE} = \frac{1}{H}\sum_{h=1}^{H}\frac{|e_{T+h}|}{Q}$$

where Q is a stable measure of the scale of the time series $\{y_t\}$.

For **non-seasonal time series**,

$$Q = (T-1)^{-1}\sum_{t=2}^{T}|y_t - y_{t-1}|$$

works well. Then MASE is equivalent to MAE relative to a naive method.

Measures of forecast accuracy

5. Mean Absolute Scaled Error

$$\text{MASE} = \frac{1}{H}\sum_{h=1}^{H}\frac{|e_{T+h}|}{Q}$$

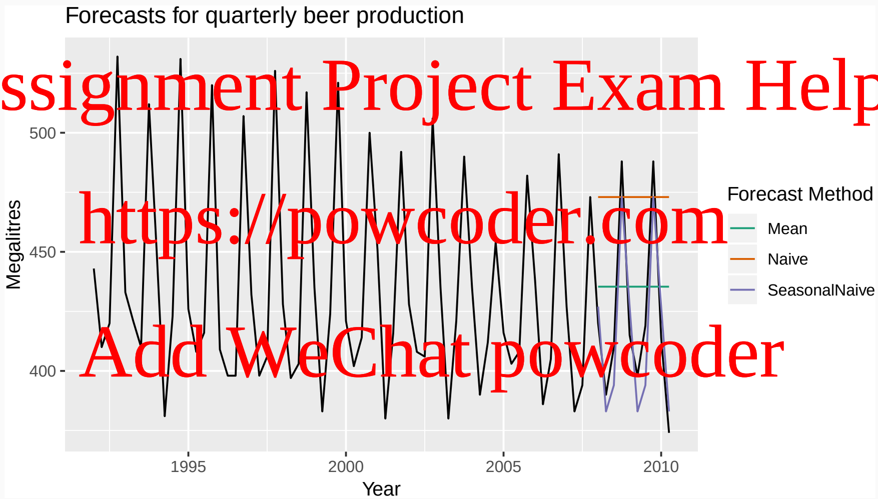
where Q is a stable measure of the scale of the time series $\{y_t\}$.

For **seasonal time series**,

$$Q = (T-m)^{-1}\sum_{t=m+1}^{T}|y_t - y_{t-m}|$$

works well. Then MASE is equivalent to MAE relative to a seasonal naive method.

Measures of forecast accuracy



Measures of forecast accuracy - Computing in R


```
beer2 <- window(ausbeer, start=1992, end=c(2007,4)) #training
beer3 <- window(ausbeer, start=2008) #test
beerfit1 <- meanf(beer2, h=10)
beerfit2 <- rwf(beer2, h=10)
beerfit3 <- snaive(beer2, h=10)
accuracy(beerfit1, beer3)
accuracy(beerfit2, beer3)
accuracy(beerfit3, beer3)
```

Measures of forecast accuracy - Computing in R


```
##                            RMSE    MAE
## Mean method            38.44724 34.825
## Naive method           62.69290 57.400
## Seasonal naive method  14.31084 13.400
##                             MAPE      MASE
## Mean method             8.283390 2.4353147
## Naive method           14.184424 4.0139860
## Seasonal naive method   3.168503 0.9370629
```

Quiz: true or false?

1. Good forecast methods should have normally distributed residuals.
2. A model with small residuals will give good forecasts.

3. The best measure of forecast accuracy is MAPE.
4. If your model doesn't forecast well, you should make it more complicated.

5. Always choose the model with the best forecast accuracy as measured on the test set.

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6. Prediction intervals

Time series cross-validation

- Time series cross-validation is a more sophisticated version of training/test sets.

- There are a series of test sets, each consisting of a **single observation**.
- The corresponding training set consists only of observations that occurred prior to the observation that forms the test set.

- Since it is not possible to obtain a reliable forecast based on a small training set, the earliest observations are not considered as test sets.

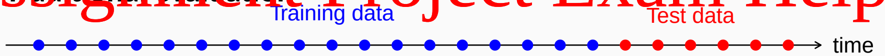
- The following diagram compares the traditional and cross-validation approaches, with the latter illustraing the series of training and test sets, where the blue observations form the training sets, and the red observations form the test sets.

Time series cross-validation

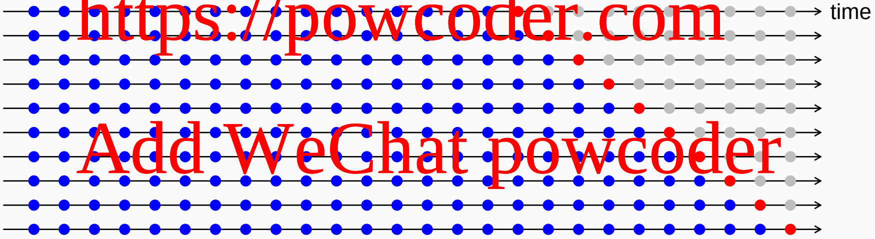
Traditional evaluation

Time series cross-validation

Traditional evaluation



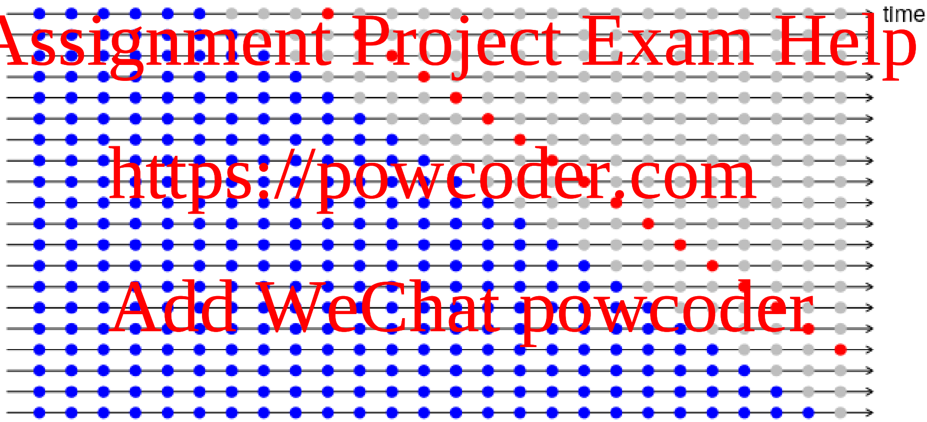
Time series cross-validation



Time series cross-validation

- The forecast accuracy is computed by averaging over the test sets.
- Also known as "evaluation on a rolling forecasting origin" because the "origin" at which the forecast is based rolls forward in time.
- Cross-validation procedure based on a rolling forecasting origin can be modified to allow multi-step errors to be used.
- Next diagram shows the example if we are interested in models that produce good 4-step-ahead forecasts.

Time series cross-validation



tsCV function:


```
#Obtain Residuals
e <- tsCV(goog200, rwf, drift=TRUE, h=1)

#Compute RMSE
sqrt(mean(e^2, na.rm=TRUE))
```



```
## [1] 6.233245
```


A good way to choose the best forecasting model is to find the model with the smallest RMSE computed using time series cross-validation.

Pipe operator

Ugly code:

```
sqrt(mean(tsCV(goog200, rwf, drift=TRUE, h = 1)^2,
na.rm=TRUE))
```


- In the above (ugly) code, we are nesting functions within functions within functions, so you have to read the code from the inside out, making it difficult to understand what is being computed.

Better with a pipe:


```
#Obtain residuals
goog200 %>%
  tsCV(forecastfunction=rwf, drift=TRUE, h=1) -> e
#Compute RMSE
e^2 %>% mean(na.rm=TRUE) %>% sqrt
```

Pipe operator

- When using the pipe operator `%>%`, the left hand side of each pipe is passed as the first argument to the function on the right hand side. This is consistent with the way we read from left to right in English.
- When using pipes, all other arguments must be named, which also helps readability.
- When using pipes, it is natural to use the right arrow assignment `->` rather than the left arrow. For example, the first line above can be read as "Take the `goog200` series, pass it to `rwf()` with `drift=TRUE`, compute the resulting residuals, and store them as `res`".

Outline

Prediction intervals

- A forecast $\hat{y}_{T+h|T}$ is (usually) the mean of the conditional distribution $y_{T+h} \mid y_1, \dots, y_T$.
- A prediction interval gives a region within which we expect y_{T+h} to lie with a specified probability.
- Assuming forecast errors are normally distributed, then a 95% PI is

$$\hat{y}_{T+h|T} \pm 1.96\hat{\sigma}_h$$

where $\hat{\sigma}_h$ is the st dev of the h-step distribution.

- When $h = 1$, $\hat{\sigma}_h$ can be estimated from the residuals.

Prediction intervals

Naive forecast with prediction interval:

```
res_sd <- sqrt(mean(res^2, na.rm=TRUE))
c(tail(goog200,1)) + 1.96 * res_sd * c(-1,1)
```

```
## [1] 519.3103 543.6462
```

```
naive(goog200, level=95)
```

```
##     Point Forecast    Lo 95    Hi 95
## 201       531.4783 519.3104 543.6461
## 202       531.4783 514.2703 548.6862
## 203       531.4783 510.4029 552.5536
## 204       531.4783 507.1425 555.8140
## 205       531.4783 504.2701 558.6865
## 206       531.4783 501.6732 561.2833
```

Prediction intervals

- Point forecasts are often useless without prediction intervals.
- Prediction intervals require a stochastic model (with random errors, etc).
- Multi-step forecasts for time series require a more sophisticated approach (with PI getting wider as the forecast horizon increases).

Prediction intervals

Assume residuals are normal, uncorrelated, sd = $\hat{\sigma}$:

Mean forecasts: $\hat{\sigma}_h = \hat{\sigma}\sqrt{1 + 1/T}$

Naive forecasts: $\hat{\sigma}_h = \hat{\sigma}\sqrt{h}$

Seasonal naive forecasts $\hat{\sigma}_h = \hat{\sigma}\sqrt{k + 1}$

Drift forecasts: $\hat{\sigma}_h = \hat{\sigma}\sqrt{h(1 + h/T)}$.

where k is the integer part of $(h - 1)/m$.

Note that when $h = 1$ and T is large, these all give the same approximate value $\hat{\sigma}$.

Prediction intervals

- Computed automatically using: `naive()`, `snaive()`, `rwf()`, `meanf()`, etc.

- Use `level` argument to control coverage.
- Check residual assumptions before believing them.
- Usually too narrow due to unaccounted uncertainty.