MONASH University

MONASH BUSINESS SCHOOL

ETW3420: Principles of Forecasting and Applications

Topic 5: Exponential Smoothing

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Outline

Introduction

- Exponential smoothing was proposed in the late 1950s (Brown, 1959; Holt, 1957 and Winters, 1960 are key pioneering works) and has motivated some of the most successful forecasting methods.

- Was invented during WW2 by Robert G. Brown (1923-2013), who was involved in the design of a tracking system for fire-control information on the location of enemy submarines.

Introduction

- Later on, the principles of exponential smoothing were applied to business data, especially in the analysis of the demand for service parts in inventory systems by Brown in a book *Advanced Service Parts Inventory Control* (1982).

- Widely used in the areas of sales, inventory, logistics and production planning as well as in quality control, process control, financial planning and marketing planning.

Introduction

Data	Methods	Reference
Airline passengers	DA-A	Grubb and Mason (2001)
Ambulance demand calls	A-M	Baker and Fitzpatrick (1986)
Australian football margins of victory	N-N	Clarke (1993)
Auto parts	N-N	Gardner and Diaz-Saiz (2002)
Auto parts	N-N, Croston	Snyder (2002)
Auto parts	N-N, Croston	Syntetos and Boylan (2005)
Auto parts	N-N, Croston	Syntetos et al. (2005)
Call volumes to telemarketing centers	A-A, A-M	Bianchi et al. (1998)
Chemical products	N-N, Croston	García-Flores et al. (2003)
Computer network services	N-N	Masuda and Whang (1999)
Computer parts	DA-N	Gardner (1993)
Confectionery equipment repair parts	N-N, Croston	Strijbosch et al. (2000)
Consumer product sales (annual)	N-N, A-N, DA-N	Schnaars (1986)
Consumer food products	N-N	Koehler (1985)
Cookware sales	DA-N	Gardner and Anderson (1997)
Cookware sales	DA-N	Gardner, Anderson-Fletcher, and Wicks (2001)
Crime rates	N-N, A-N	Gorr, Olligschlaeger, and Thompson (2003)
Currency exchange rates	N-N, A-N, A-M	Dheeriya and Raj (2000)
Department store sales	N-N, A-N	Geurts and Kelly (1986)
Economic data (various)	N-N, A-N	Geriner and Ord (1991)
Economic, environmental data (various)	A-N	Wright (1986b)
Electric utility loads	A-N	Huss (1985a)
Electric utility sales	A-N	Huss (1985b)
Electricity demand	N-N, A-N	Price and Sharp (1986)
Electricity demand	A-M	Taylor (2003b)
Electricity demand forecast errors	N-N	Ramanathan, Engle, Granger, Vahid-Araghi, and Brace (1997)
Electricity supply	A-N	Sharp and Price (1990)

Introduction

Electrical service requests	A-M	Weintraub, Aboud, Fernandez, Laporte, and Ramirez (1999)
Electronics components	N-N, A-N	Flores et al. (1993)
Exports	N-N	Mahmoud, Motwani, and Rice (1990)
Financial futures prices	N-N	Sharda and Musser (1986)
Financial returns	N-N	Taylor (2004a)
Food product demand	N-N	Fairfield and Kingsman (1993)
Food product demand	N-N	Mercer and Tao (1996)
Hospital patient movements	A-M	Lin (1989)
Hotel revenue data	N-N, A-N	Weatherford and Kimes (2003)
IBM product sales	A-M	Wu et al. (1991)
Industrial data (various)	N-N, Croston	Willemain, Smart, Shockor, and DeSautels (1994) and Willemain et al. (2004)
Industrial fasteners	N-N, A-N	Adshead and Price (1987)
Industrial production differences	N-N	Öller (1986)
Industrial production index	A-A	Bodo and Signorini (1987)
Leading indicators	A-N	Holmes (1986)
Macroeconomic variables	A-M	Thury (1985)
Mail order sales	N-N	Chambers and Eglese (1988)
Mail volumes	A-M	Thomas (1993)
Manpower retention rates	A-N	Chu and Lin (1994)
Medicaid expenses	A-N	Williams and Miller (1999)
Medical supplies	A-N	Mathews and Diamantopoulos (1994)
Natural gas demand	N-N, A-N, A-M	Lee et al. (1993)
Stock index direction	N-N	Leung, Daouk, and Chen (2000)
Supermarket product sales	Many	Taylor (2004c)
Point-of-sale scanner data	N-N	Curry et al. (1995)

Introduction

- Forecasts are based on the extrapolation of past patterns with forecasting equations that are simple to update and maintain in a database.
- Capture **level** (a starting point for the forecasts), **trend** (a factor for growth or decline) and **seasonal factors** (for adjustment of seasonal variation) in data patterns.
- Forecasts produced using exponential smoothing methods are weighted average of past observations, with the weights decaying exponentially as the observations get older.
  - The more recent the observation, the higher the associated weight.

Outline

Components in Exponential Smoothing Method

An exponential smoothing method comprises one or more of the following components: the current level, the current trend, and the current seasonal index.

- The current *level* serves as the starting point of the forecast.
  - Calculated to represent an exponentially weighted average of the time series at the end of the fit period.
  - Can be regarded as the value the time series would now have if there were nothing at all unusual going on at present.

Components in Exponential Smoothing Method

- The current *trend* represents the amount by which we expect the time series to grow or decline per time period into the future.

  - Calculated as an exponentially weighted average of past period-to-period changes in the level of the series.
  - As such, recent growth or decline in the time series is (usually) given more weight than changes farther back in time.

- The current *seasonal* index is the degree by which the season’s values tends to exceed or fall short of the norm.

Outline

Simple methods

Time series $y_1, y_2, \ldots, y_T$.

Random walk forecasts

$$\hat{y}_{T+h|T} = y_T$$

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Average forecasts

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Random walk forecasts

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Average forecasts

$$\hat{y}_{T+h|T} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

- Want something in between that weights most recent data more highly.
- Simple exponential smoothing uses a weighted moving average with weights that decrease exponentially.

Simple Exponential Smoothing

Forecast equation

$$\begin{aligned}\hat{y}_{T+1|T} &= \alpha y_T + \alpha(1-\alpha)y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots + \alpha(1-\alpha)^{T-1} y_1 \\ &= \sum_{i=0}^{T-1} \alpha(1-\alpha)^i y_{T-i}\end{aligned}$$

where $0 \le \alpha \le 1$.

Forecast equation

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha)y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots + \alpha(1-\alpha)^{T-1} y_1$$
$$= \sum_{i=0}^{T-1} \alpha(1-\alpha)^i y_{T-i}$$

where $0 \le \alpha \le 1$.

- The one-step-ahead forecast at time $T+1$ is then a weighted average of all the observations in the series, $y_1, \cdots, y_T$.
- The rate at which the weights decrease is controlled by the parameter α. The rate that $(1-\alpha)^i$ decays to zero determines how much influence past values have on forecasts.
- There will be very little smoothing (or averaging) if $(1-\alpha)^i$ decays to zero very fast, i.e. when α is large (or close to 1).

Simple Exponential Smoothing

- The table below shows the weights attached to observations for 4 different values of α when forecasting using SES

	Weights assigned to observations for:			
Observation	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
y_T	0.2	0.4	0.6	0.8
y_{T-1}	0.16	0.24	0.24	0.16
y_{T-2}	0.128	0.144	0.096	0.032
y_{T-3}	0.1024	0.0864	0.0384	0.0064
y_{T-4}	$(0.2)(0.8)^4$	$(0.4)(0.6)^4$	$(0.6)(0.4)^4$	$(0.8)(0.2)^4$
y_{T-5}	$(0.2)(0.8)^5$	$(0.4)(0.6)^5$	$(0.6)(0.4)^5$	$(0.8)(0.2)^5$

Simple Exponential Smoothing

Component form

Forecast equation $\hat{y}_{t+h|t} = \ell_t$

Smoothing equation $\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$

- ℓ_t is the level (or the smoothed value) of the series at time t.
- $\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha)\hat{y}_{t|t-1}$
  Iterate to get exponentially weighted moving average form.

Weighted average form

$$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha(1 - \alpha)^j y_{T-j} + (1 - \alpha)^T \ell_0$$

Optimisation

- Need to choose value for α and ℓ_0
- Similarly to regression — we choose α and ℓ_0 by minimising SSE:

$$\text{SSE} = \sum_{t=1}^{T}(y_t - \hat{y}_{t|t-1})^2.$$

- Unlike regression there is no closed form solution — use numerical optimization.

Example: Oil production

```r
oildata <- window(oil, start=1996)
# Estimate parameters
fc <- ses(oildata, h=5)
summary(fc$model) #or summary(fc) for full print out of results
```

```
## Simple exponential smoothing
##
## Call:
##  ses(y = oildata, h = 5)
##
##   Smoothing parameters:
##     alpha = 0.8339
##
##   Initial states:
##     l = 446.5868
##
##   sigma:  29.83
##
```

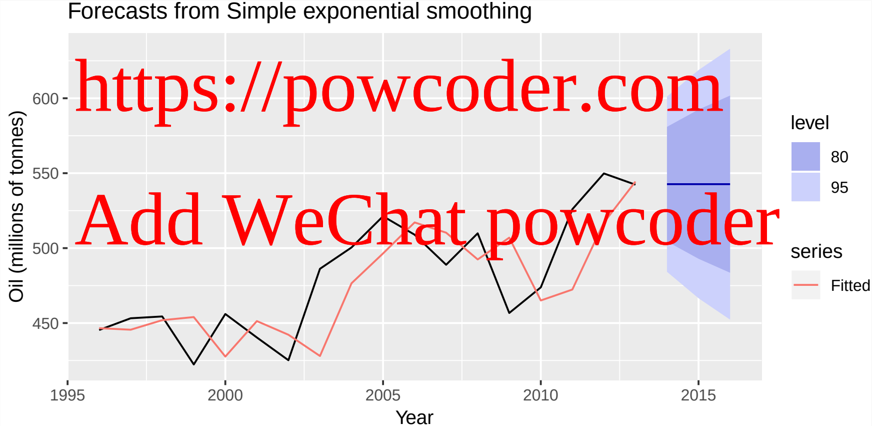
Example: Oil production

Year	Time	Observation	Level	Forecast
	t	y_t	ℓ_t	$\hat{y}_{t+1\|t}$
1995	0		446.59	
1996	1	445.36	445.57	446.59
1997	2	453.20	451.93	445.57
1998	3	454.41	454.00	451.93
1999	4	422.38	427.63	454.00
2000	5	456.04	451.32	427.63
2001	6	440.39	442.20	451.32
2002	7	425.19	428.02	442.20
2003	8	486.21	476.54	428.02
2004	9	500.43	496.46	476.54
2005	10	521.28	517.15	496.46
2006	11	508.95	510.31	517.15
2007	12	488.89	492.45	510.31
2008	13	509.87	506.98	492.45
2009	14	456.72	465.07	506.98
2010	15	473.82	472.36	465.07
2011	16	525.95	517.05	472.36
2012	17	549.83	544.39	517.05
2013	18	542.34	542.68	544.39
	h			$\hat{y}_{T+h\|T}$
2014	1			542.68
2015	2			542.68

Year Time Observation Level Forecast

Example: Oil production

```r
autoplot(fc) +
  autolayer(fitted(fc), series="Fitted") +
  ylab("Oil (millions of tonnes)") + xlab("Year")
```

Outline

Holt's linear trend

- Simple exponential smoothing is only applicable for the forecasting of data that displays no trend and seasonality.
- Holt(1957) extended SES to allow forecasting of data with a trend.
- This method involves a forecast equation and 2 smoothing equations - one for the *level* and one for the *trend*.

Holt's linear trend

Component form

Forecast	$\hat{y}_{t+h\|t} = \ell_t + hb_t$
Level	$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},$

Holt's linear trend

Component form

Forecast	$\hat{y}_{t+h\|t} = \ell_t + hb_t$
Level	$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},$

- Two smoothing parameters α and β^* ($0 \le \alpha, \beta^* \le 1$).
- ℓ_t level: weighted average between y_t and one-step ahead forecast for time t, ($\ell_{t-1} + b_{t-1} = \hat{y}_{t|t-1}$)
- b_t slope: weighted average of $(\ell_t - \ell_{t-1})$ and b_{t-1}, current and previous estimate of slope.
- Choose $\alpha, \beta^*, \ell_0, b_0$ to minimise SSE.

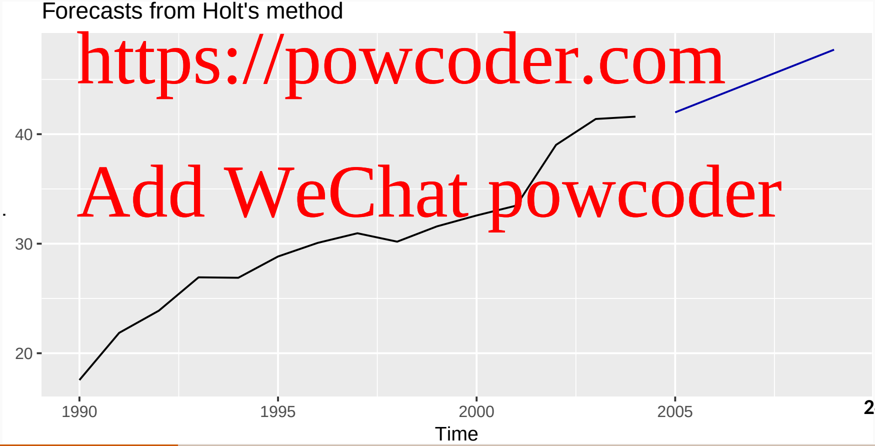
Holt's linear trend

- Note that the forecast function is no longer flat but trending.

- The h-step-ahead forecast is equal to the last estimated level plus h times the last estimated trend value. Hence, the forecasts are a linear function of h.

Holt's method in R

```
window(ausair, start=1990, end=2004) %>%
  holt(h=5, PI=FALSE) %>%
  autoplot()
```

Damped trend method

- Forecasts generated by Holt’s method display a constant trend (increasing or decreasing) indefinitely into the future.
- More extreme are the forecasts generated by the exponential trend method which include exponential growth or decline.
- Empirical evidence indicates that these methods tend to over-forecast, especially for longer forecast horizons.
- Garnder and McKenzie (1985) introduced a **parameter** that “dampens” the trend to a flat line some time in the future.
- Methods that include a damped trend have proven to be successful; arguably the most popular individual methods when forecasts are required automatically for many series.

Damped trend method

Component form

$$\hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h) b_t$$

$$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.$$

Damped trend method

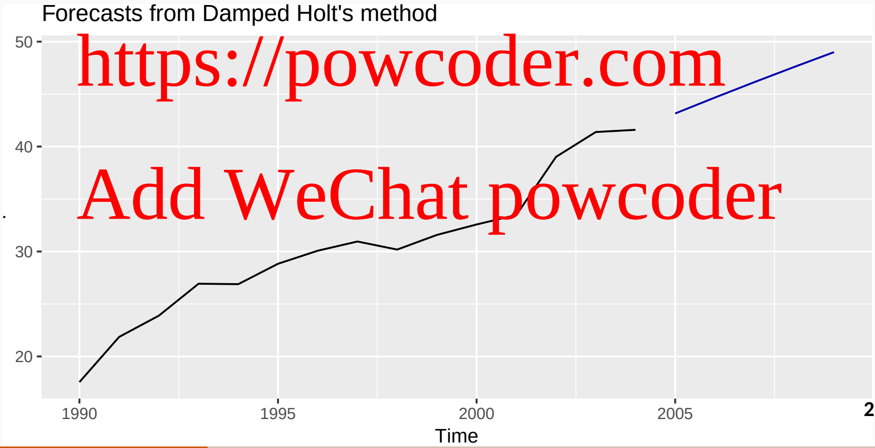
Component form

$$\hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \dots + \phi^h) b_t$$
$$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$$
$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.$$

- Damping parameter $0 < \phi < 1$.
- If $\phi = 1$, identical to Holt's linear trend.
- As $h \to \infty$, $\hat{y}_{T+h|T} \to \ell_T + \phi b_T/(1-\phi)$.(Recall geometric progression rule)
- Short-run forecasts trended, long-run forecasts constant.

Example: Air passengers

```
window(ausair, start=1990, end=2004) %>%
  holt(damped=TRUE, h=5, PI=FALSE) %>%
  autoplot()
```

Example: Sheep in Asia

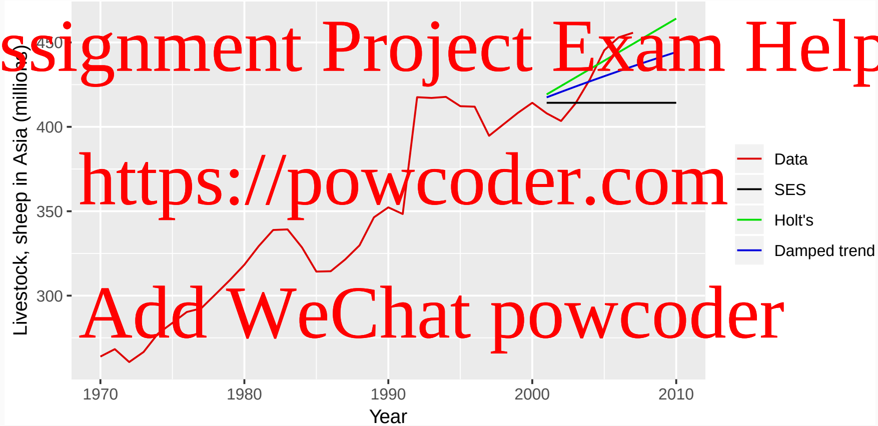
```r
livestock2 <- window(livestock, start=1970,
                     end=2000)
fit1 <- ses(livestock2)
fit2 <- holt(livestock2)
fit3 <- holt(livestock2, damped = TRUE)
```

```r
accuracy(fit1, livestock)
accuracy(fit2, livestock)
accuracy(fit3, livestock)
```

Example: Sheep in Asia

	SES	Linear trend	Damped trend
α	1.00	0.98	0.97
β^*		0.00	0.00
ϕ			0.98
ℓ_0	263.90	251.46	251.89
b_0		4.99	6.29
Training RMSE	14.77	13.98	14.00
Test RMSE	25.46	11.88	14.73
Test MAE	20.38	10.71	13.30
Test MAPE	4.60	2.54	3.07
Test MASE	2.26	1.19	1.48

Example: Sheep in Asia



Self-Practice

`eggs` contains the price of a dozen eggs in the United States from 1900-1993

1. Use SES and Holt's linear method (with and without damping) to forecast future data.
   [Hint: use h=100 so you can clearly see the differences between the options when plotting the forecasts.]

2. Which method gives the best training RMSE?

3. Are these RMSE values comparable?
4. Do the residuals from the best fitting method look like white noise?

Outline

1. Introduction
2. Components in Exponential Smoothing Method
3. Simple exponential smoothing
4. Trend methods
5. **Seasonal methods**
6. Taxonomy of exponential smoothing methods
7. Innovations state space models
8. ETS in R

Holt-Winters seasonal method

- Holt (1957) and Winters (1960) extended Holt's method to capture seasonality.
- The seasonal method comprises of the forecast equation and 3 smoothing equations - one for the level, trend and seasonal component.
- There are 2 variations to this method that differ in the nature of the seasonal method.
  - *Additive* method: seasonal variations are roughly constant through the series.
  - *Multiplicative* method: seasonal variations are changing proportional to the level of the series.

Holt-Winters additive method

Component form

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

$$\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$$

$$s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m},$$

- k = integer part of $(h-1)/m$. Ensures estimates from the final year are used for forecasting.
- Parameters: $0 \le \alpha \le 1$, $0 \le \beta^* \le 1$, $0 \le \gamma \le 1-\alpha$ and m = period of seasonality (e.g. m = 4 for quarterly data).

Holt-Winters additive method

Level equation weighted average between the *seasonally adjusted* observation $(y_t - s_{t-m})$ and the non-seasonal forecast $(\ell_{t-1} + b_{t-1})$ for time t.

Trend equation weighted average of current trend estimate $(\ell_t - \ell_{t-1})$ and previous trend estimate b_{t-1}.

Seasonal equation weighted average between current seasonal index $(y_t - \ell_{t-1} - b_{t-1})$ and the seasonal index of the same season last year (i.e. m time periods ago).

Holt-Winters additive method

- Seasonal component is usually expressed as
  $s_t = \gamma^*(y_t - \ell_t) + (1 - \gamma^*)s_{t-m}.$

- Substitute in for ℓ_t:
  $s_t = \gamma^*(1 - \alpha)(y_t - \ell_{t-1} - b_{t-1}) + [1 - \gamma^*(1 - \alpha)]s_{t-m}$
- We set $\gamma = \gamma^*(1 - \alpha)$.

- The usual parameter restriction is $0 \le \gamma^* \le 1$, which translates to $0 \le \gamma \le (1 - \alpha)$.

Holt-Winters multiplicative method

For when seasonal variations are changing proportional to the level of the series.

Component form

$$\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}.$$

$$\ell_t = \alpha \frac{y_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

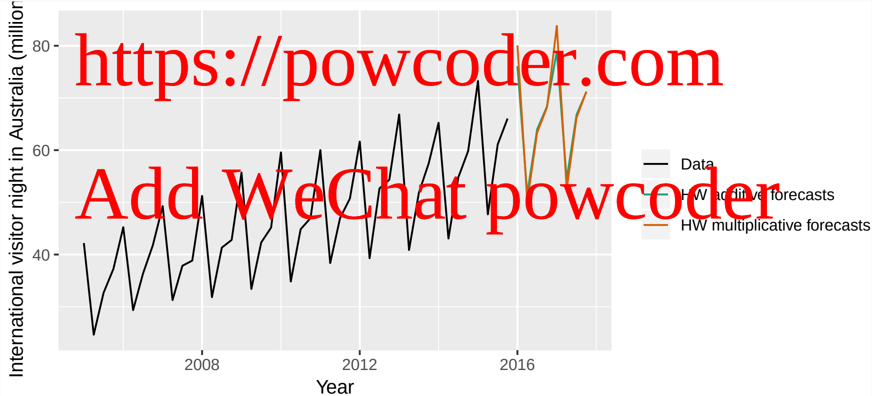
$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$$

$$s_t = \gamma \frac{y_t}{(\ell_{t-1} + b_{t-1})} + (1-\gamma)s_{t-m}$$

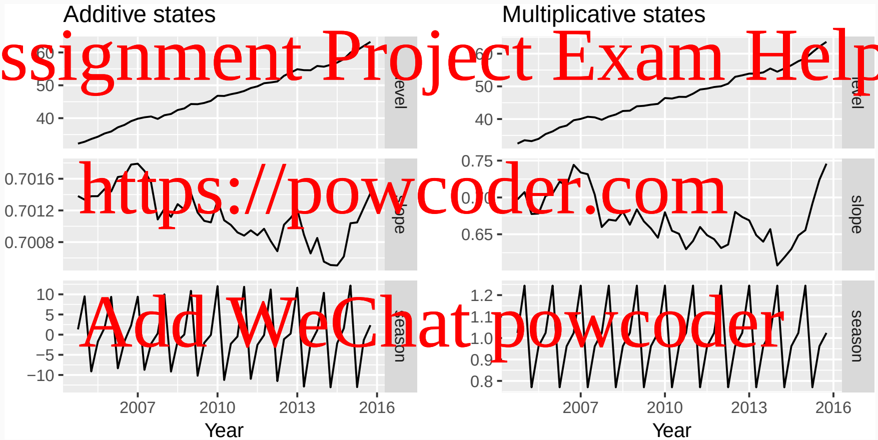
- k is integer part of $(h-1)/m$.
- With additive method s_t is in absolute terms: within each year $\sum_i s_i \approx 0$.
- With multiplicative method s_t is in relative terms: within each year $\sum_i s_i \approx m$.

Example: Visitor Nights

```
aust <- window(austourists,start=2005)
fit1 <- hw(aust,seasonal="additive")
fit2 <- hw(aust,seasonal="multiplicative")
```

Estimated components



Holt-Winters damped method

Often the single most accurate forecasting method for seasonal data:

$$
\begin{aligned}
\hat{y}_{t+h|t} &= [\ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t]s_{t+h-m(k+1)} \\
\ell_t &= \alpha(y_t/s_{t-m}) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1} \\
s_t &= \gamma \frac{y_t}{(\ell_{t-1} + \phi b_{t-1})} + (1-\gamma)s_{t-m}
\end{aligned}
$$

Self-Practice

Apply Holt-Winters multiplicative method to the `gas` data.

1. Why is multiplicative seasonality necessary here?
2. Experiment with making the trend damped.
3. Check that the residuals from the best method look like white noise.

Outline

Exponential smoothing methods

- Exponential smoothing methods are not restricted to those we have presented so far.
- By considering variations in the combination of the trend and seasonal components, 15 exponential smoothing methods are possible (see Table on next slide).
- Each method is labelled by a pair of letters (T,S) defining the type of 'Trend' and 'Seasonal' components.
  - E.g. (**A**,**M**): **A**dditive trend and **M**ultiplicative seasonality
  - E.g. (**M**,**N**): **M**ultiplicative (or exponential) trend and **No** seasonality

Exponential smoothing methods

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A_d	(Additive damped)	(A_d,N)	(A_d,A)	(A_d,M)

(N,N): Simple exponential smoothing

(A,N): Holt's linear method

(A_d,N): Additive damped trend method

(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A_d,M): Damped multiplicative Holt-Winters' method

There are also multiplicative trend methods (not recommended).

Recursive formulae

Trend	Seasonal N	Seasonal A	Seasonal M
N	$\hat{y}_{t+h\|t} = \ell_t$ $\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$	$\hat{y}_{t+h\|t} = \ell_t + s_{t+h-m(k+1)}$ $\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)\ell_{t-1}$ $s_t = \gamma(y_t - \ell_{t-1}) + (1-\gamma)s_{t-m}$	$\hat{y}_{t+h\|t} = \ell_t s_{t+h-m(k+1)}$ $\ell_t = \alpha(y_t/s_{t-m}) + (1-\alpha)\ell_{t-1}$ $s_t = \gamma(y_t/\ell_{t-1}) + (1-\gamma)s_{t-m}$
A	$\hat{y}_{t+h\|t} = \ell_t + hb_t$ $\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$ $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$	$\hat{y}_{t+h\|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$ $\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$ $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$ $s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$	$\hat{y}_{t+h\|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$ $\ell_t = \alpha(y_t/s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$ $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$ $s_t = \gamma(y_t/(\ell_{t-1} + b_{t-1})) + (1-\gamma)s_{t-m}$
A$_\mathbf{d}$	$\hat{y}_{t+h\|t} = \ell_t + \phi_h b_t$ $\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + \phi b_{t-1})$ $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}$	$\hat{y}_{t+h\|t} = \ell_t + \phi_h b_t + s_{t+h-m(k+1)}$ $\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1})$ $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}$ $s_t = \gamma(y_t - \ell_{t-1} - \phi b_{t-1}) + (1-\gamma)s_{t-m}$	$\hat{y}_{t+h\|t} = (\ell_t + \phi_h b_t)s_{t+h-m(k+1)}$ $\ell_t = \alpha(y_t/s_{t-m}) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1})$ $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}$ $s_t = \gamma(y_t/(\ell_{t-1} + \phi b_{t-1})) + (1-\gamma)s_{t-m}$

R functions

- Simple exponential smoothing: no trend.
  `ses(y)`

- Holt's method: linear trend.
  `holt(y)`
- Damped trend method.
  `holt(y, damped=TRUE)`

- Holt-Winters methods

```
hw(y, damped=TRUE, seasonal="additive")
hw(y, damped=FALSE, seasonal="additive")
hw(y, damped=TRUE, seasonal="multiplicative")
hw(y, damped=FALSE, seasonal="multiplicative")
```


- Combination of no trend with seasonality not possible using these functions.

Outline

Methods v Models

Exponential smoothing methods

- Algorithms that return point forecasts.
- Do not cater for prediction intervals.

Methods v Models

Exponential smoothing methods

- Algorithms that return point forecasts.
- Do not cater for prediction intervals.

Innovations state space models

- Generate same point forecasts but can also generate forecast intervals.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for “proper” model selection, based on model selection criterion.

ETS models

- Each model has an *observation* equation and *transition* equations, one for each state (level, trend, seasonal), i.e., state space models.
- Two models for each method: one with additive and one with multiplicative errors, i.e., in total 18 models.
- ETS(Error,Trend,Seasonal):
  - Error = $\{A, M\}$
  - Trend = $\{N, A, A_d\}$
  - Seasonal = $\{N, A, M\}$.

Exponential smoothing methods

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
N	(None)	N,N	N,A	N,M
A	(Additive)	A,N	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M

Exponential smoothing methods

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
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General notation E T S : ExponenTial Smoothing

Error Trend Seasonal

Exponential smoothing methods

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General notation E T S : ExponenTial Smoothing

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Examples:

A,N,N: Simple exponential smoothing with additive errors

A,A,N: Holt's linear method with additive errors

M,A,M: Multiplicative Holt-Winters' method with multiplicative errors

Exponential smoothing methods

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N	(None)	N,N	N,A	N,M
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General notation E T S : ExponenTial Smoothing

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Examples:

A,N,N: Simple exponential smoothing with additive errors

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There are 18 separate models in the ETS framework

A model for SES

Component form

Forecast equation $\hat{y}_{t+h|t} = \ell_t$

Smoothing equation $\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$

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Forecast error: $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$.

Error correction form

$$y_t = \ell_{t-1} + e_t$$

$$\ell_t = \ell_{t-1} + \alpha(y_t - \ell_{t-1})$$

$$= \ell_{t-1} + \alpha e_t$$

A model for SES

Component form

$$\begin{aligned} \text{Forecast equation} \quad & \hat{y}_{t+h|t} = \ell_t \\ \text{Smoothing equation} \quad & \ell_t = \alpha y_t + (1-\alpha)\ell_{t-1} \end{aligned}$$

Forecast error: $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$.

Error correction form

$$\begin{aligned} y_t &= \ell_{t-1} + e_t \\ \ell_t &= \ell_{t-1} + \alpha(y_t - \ell_{t-1}) \\ &= \ell_{t-1} + \alpha e_t \end{aligned}$$

Specify probability distribution for e_t, we assume

$e_t = \varepsilon_t \sim \text{NID}(0, \sigma^2)$.

ETS(A,N,N)

Measurement equation	$y_t = \ell_{t-1} + \varepsilon_t$
State equation	$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- "innovations" or "single source of error" because same error process, ε_t.
- Measurement equation: relationship between observations and states.
- Transition equation(s): evolution of the state(s) through time.

ETS(A,A,N)

Recap Holt Linear's method:

Component form

Forecast	$\hat{y}_{t+h\|t} = \ell_t + hb_t$
Level	$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},$

ETS(A,A,N)

Recap Holt Linear’s method:

Component form

Forecast	$\hat{y}_{t+h\|t} = \ell_t + hb_t$
Level	$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},$

ETS(A,A,N)

Holt's linear method with additive errors.

- Assume $\varepsilon_t = y_t - \ell_{t-1} - b_{t-1} \sim \text{NID}(0, \sigma^2)$.
- Substituting into the error correction equations for Holt's linear method

$$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$$

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \alpha\beta^*\varepsilon_t$$

- For simplicity, set $\beta = \alpha\beta^*$.

Self-Practice

- Write down the model for ETS(A,Ad,N)

ETS(A,A,A)

Holt-Winters additive method with additive errors.

$$
\begin{aligned}
\text{Forecast equation} \quad & \hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)} \\
\text{Observation equation} \quad & y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t \\
\text{State equations} \quad & \ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
& b_t = b_{t-1} + \beta\varepsilon_t \\
& s_t = s_{t-m} + \gamma\varepsilon_t
\end{aligned}
$$

- Forecast errors: $\varepsilon_t = y_t - \hat{y}_{t|t-1}$
- k is integer part of $(h-1)/m$.

- Write down the model for ETS(A,N,A)

ETS(M,N,N)

SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
  - $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

ETS(M,N,N)

SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
  - $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

Measurement equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
State equation	$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$

ETS(M,N,N)

SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
  - $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

Measurement equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
State equation	$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$

- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.

ETS(M,A,N)

Holt's linear method with multiplicative errors.

- Assume $\varepsilon_t = \frac{y_t - (\ell_{t-1}+b_{t-1})}{(\ell_{t-1}+b_{t-1})}$
- Following a similar approach as above, the innovations state space model underlying Holt's linear method with multiplicative errors is specified as

$$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$$

$$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$$

$$b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$$

where again $\beta = \alpha\beta^*$ and $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

Additive error models

Trend	Seasonal N	Seasonal A	Seasonal M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/\ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + b_{t-1})$
A$_d$	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = \phi b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + \phi b_{t-1})$

Multiplicative error models

Trend	Seasonal N	Seasonal A	Seasonal M
N	$y_t = \ell_{t-1}(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$	$y_t = (\ell_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+\alpha(\ell_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1}+b_{t-1})(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1}+b_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+b_{t-1}+\alpha(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1}+b_{t-1})s_{t-m}(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$
A$_\mathbf{d}$	$y_t = (\ell_{t-1}+\phi b_{t-1})(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+\phi b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1}+\phi b_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+\phi b_{t-1}+\alpha(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1}+\phi b_{t-1})s_{t-m}(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+\phi b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$

Estimating ETS models

- Smoothing parameters α, β, γ and ϕ, and the initial states ℓ_0, $b_0, s_0, s_{-1}, \dots, s_{-m+1}$ are estimated by maximising the "likelihood" = the probability of the data arising from the specified model.

- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

- We will estimate models with the `ets()` function in the forecast package.

Let $\boldsymbol{x}_t = (\ell_t, b_t, s_t, s_{t-1}, \dots, s_{t-m+1})$ and $\varepsilon_t \overset{\text{iid}}{\sim} \mathrm{N}(0, \sigma^2)$.

$$y_t = \underbrace{h(\boldsymbol{x}_{t-1})}_{\mu_t} + \underbrace{k(\boldsymbol{x}_{t-1})\varepsilon_t}_{e_t}$$

$$\boldsymbol{x}_t = f(\boldsymbol{x}_{t-1}) + g(\boldsymbol{x}_{t-1})\varepsilon_t$$

Additive errors $k(x) = 1$. $\quad y_t = \mu_t + \varepsilon_t$.

Multiplicative errors

$k(\boldsymbol{x}_{t-1}) = \mu_t$. $\quad y_t = \mu_t(1 + \varepsilon_t)$.

$\varepsilon_t = (y_t - \mu_t)/\mu_t$ is relative error.

Estimation

$$L^*(\theta, \mathbf{x}_0) = n \log\left(\sum_{t=1}^{n} \varepsilon_t^2 / k^2(\mathbf{x}_{t-1})\right) + 2\sum_{t=1}^{n} \log|k(\mathbf{x}_{t-1})|$$

$$= -2\log(\text{Likelihood}) + \text{constant}$$

- Estimate parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \phi)$ and initial states $\mathbf{x}_0 = (\ell_0, b_0, s_0, s_{-1}, \dots, s_{-m+1})$ by minimizing L^*.

Parameter restrictions

Usual region

- Traditional restrictions in the methods $0 < \alpha, \beta^*, \gamma^*, \phi < 1$ (equations interpreted as weighted averages).
- In models we set $\beta = \alpha\beta^*$ and $\gamma = (1 - \alpha)\gamma^*$.
- Therefore $0 < \alpha < 1, \quad 0 < \beta < \alpha \quad$ and $0 < \gamma < 1 - \alpha$.
- $0.8 < \phi < 0.98$ — to prevent numerical difficulties.

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- Therefore $0 < \alpha < 1, \quad 0 < \beta < \alpha \quad$ and $0 < \gamma < 1 - \alpha$.
- $0.8 < \phi < 0.98$ — to prevent numerical difficulties.

Admissible region

- To prevent observations in the distant past having a continuing effect on current forecasts.
- Usually (but not always) less restrictive than the *traditional* region.
- For example for ETS(A,N,N):
  traditional $0 < \alpha < 1$ — *admissible* is $0 < \alpha < 2$.

Model selection

Akaike's Information Criterion

$$\text{AIC} = -2\log(L) + 2k$$

where L is the likelihood and k is the number of parameters initial states estimated in the model.

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Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2(k+1)(k+2)}{T-k}$$

which is the AIC corrected (for small sample bias).

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which is the AIC corrected (for small sample bias).

Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k(\log(T) - 2).$$

Some unstable models

- Some of the combinations of (Error, Trend, Seasonal) can lead to numerical difficulties; see equations with division by a state.

- These are: ETS(A,N,M), ETS(A,A,M), ETS(A,A_d,M).
- Models with multiplicative errors are useful for strictly positive data, but are not numerically stable with data containing zeros or negative values. In that case only the six fully additive models will be applied.

Exponential smoothing models

Additive Error

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	~~A,N,M~~
A	(Additive)	A,A,N	A,A,A	~~A,A,M~~
A_d	(Additive damped)	A,A_d,N	A,A_d,A	~~A,A_d,M~~

Multiplicative Error

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A_d	(Additive damped)	M,A_d,N	M,A_d,A	M,A_d,M

Example: International tourists

```r
aust <- window(austourists, start=2005)
fit <- ets(aust)
summary(fit)
```

```
## ETS(M,A,M)
##
## Call:
##  ets(y = aust)
##
##   Smoothing parameters:
##     alpha = 0.1908
##     beta  = 0.0392
##     gamma = 2e-04
##
##   Initial states:
##     l = 32.3679
##     b = 0.9281
##     s = 1.022 0.9628 0.7683 1.247
##
##   sigma:  0.0383
##
##   AIC  AICc   BIC
## 224.9 230.2 240.9
```

Example: International tourists

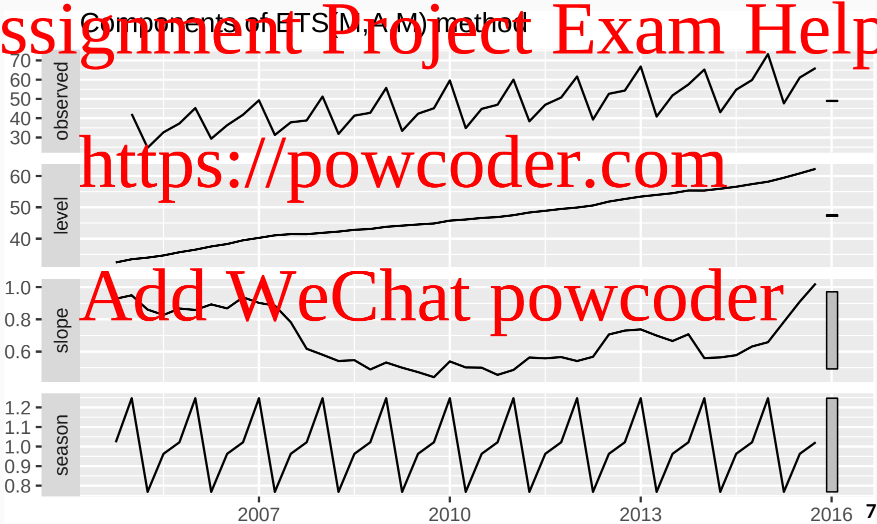
Model selected: ETS(M,A,M)

$$
\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t_1})\varepsilon_t \\
s_t &= s_{t-m}(1 + \gamma\varepsilon_t).
\end{aligned}
$$

$\hat{\alpha} = 0.1908$, $\hat{\beta} = 0.0392$, and $\hat{\gamma} = 0.00019$.

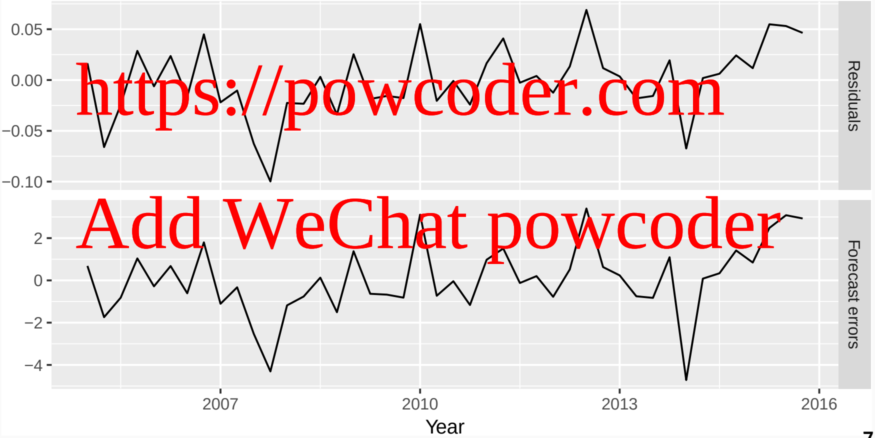
Example: International tourists

```r
autoplot(fit)
```

Example: International tourists

```
cbind('Residuals' = residuals(fit),
      'Forecast errors' = residuals(fit, type='response')) %>%
  autoplot(facet=TRUE) + xlab("Year") + ylab("")
```

Residuals

Response residuals

$$\hat{e}_t = y_t - \hat{y}_{t|t-1}$$

Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$

Forecasting with ETS models

Point forecasts: iterate the equations for $t = T + 1, T + 2, \ldots, T + h$ and set all $\varepsilon_t = 0$ for $t > T$.

Forecasting with ETS models

Point forecasts: iterate the equations for $t = T+1, T+2, \dots, T+h$ and set all $\varepsilon_t = 0$ for $t > T$.

- Not the same as $E(y_{t+h}|\mathbf{x}_t)$ unless trend and seasonality are both additive.
- Point forecasts for ETS(A,x,y) are identical to ETS(M,x,y) if the parameters are the same.

Example: ETS(A,A,N)

$$
\begin{aligned}
y_{T+1} &= \ell_T + b_T + \varepsilon_{T+1} \\
\hat{y}_{T+1|T} &= \ell_T + b_T \\
y_{T+2} &= \ell_{T+1} + b_{T+1} + \varepsilon_{T+2} \\
&= (\ell_T + b_T + \alpha\varepsilon_{T+1}) + (b_T + \beta\varepsilon_{T+1}) + \varepsilon_{T+2} \\
\hat{y}_{T+2|T} &= \ell_T + 2b_T
\end{aligned}
$$

etc.

Example: ETS(M,A,N)

$$
\begin{aligned}
y_{T+1} &= (\ell_T + b_T)(1 + \varepsilon_{T+1}) \\
\hat{y}_{T+1|T} &= \ell_T + b_T. \\
y_{T+2} &= (\ell_{T+1} + b_{T+1})(1 + \varepsilon_{T+2}) \\
&= \left\{(\ell_T + b_T)(1 + \alpha\varepsilon_{T+1}) + [b_T + \beta(\ell_T + b_T)\varepsilon_{T+1}]\right\}(1 + \varepsilon_{T+2}) \\
\hat{y}_{T+2|T} &= \ell_T + 2b_T.
\end{aligned}
$$

etc.

Forecasting with ETS models

Prediction intervals: cannot be generated using the methods, only the models.

- The prediction intervals will differ between models with additive and multiplicative errors.
- Exact formulae for some models.
- More general to simulate future sample paths, conditional on the last estimate of the states, and to obtain prediction intervals from the percentiles of these simulated future paths. (Next slide)
- Options are available in R using the `forecast` function in the forecast package.

Prediction intervals (Simulation)

Prediction Interval via Simulation

Simulate Q number of future sample paths,

$$y^{(1)}_{T+1}|x_T, y^{(1)}_{T+2}|x_T, \ldots, y^{(1)}_{T+h}|x_T$$

$$y^{(2)}_{T+1}|x_T, y^{(2)}_{T+2}|x_T, \ldots, y^{(2)}_{T+h}|x_T$$

$$\vdots$$

$$y^{(Q)}_{T+1}|x_T, y^{(Q)}_{T+2}|x_T, \ldots, y^{(Q)}_{T+h}|x_T$$

Therefore, the point forecast of $\widehat{y}_{T+i|T} = \frac{1}{Q}\sum_{j=1}^{Q} y^{(j)}_{T+i}|x_T$ and the prediction interval will be the associated percentiles of the simulated $y^{(j)}_{T+i}|x_T$ values for $j = 1, 2, \ldots Q$.

Prediction intervals (Analytical)

PI for most ETS models: $\hat{y}_{T+h|T} \pm c\sigma_h$, where c depends on coverage probability and σ_h is forecast standard deviation.

(A,N,N) $\quad \sigma_h = \sigma^2\left[1 + \alpha^2(h-1)\right]$

(A,A,N) $\quad \sigma_h = \sigma^2\left[1 + (h-1)\{\alpha^2 + \alpha\beta h + \tfrac{1}{6}\beta^2 h(2h-1)\}\right]$

(A,A$_d$,N) $\quad \sigma_h = \sigma^2\Big[1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2}\left\{2\alpha(1-\phi) + \beta\phi\right\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)}\left\{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\right\}\Big]$

(A,N,A) $\quad \sigma_h = \sigma^2\left[1 + \alpha^2(h-1) + \gamma k(2\alpha+\gamma)\right]$

(A,A,A) $\quad \sigma_h = \sigma^2\Big[1 + (h-1)\{\alpha^2 + \alpha\beta h + \tfrac{1}{6}\beta^2 h(2h-1)\} + \gamma k\{2\alpha + \gamma + \beta m(k+1)$

(A,A$_d$,A) $\quad \sigma_h = \sigma^2\Big[1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2}\left\{2\alpha(1-\phi) + \beta\phi\right\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)}\left\{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\right\} + \gamma k(2\alpha+\gamma) + \frac{2\beta\gamma\phi}{(1-\phi)(1-\phi^m)}\left\{k(1-\phi^m) - \phi^m(1-\phi^{mk})\right\}\Big]$

Outline

Example: drug sales

```r
ets(h02)
```

```
## ETS(M,Ad,M)
##
## Call:
##  ets(y = h02)
##
##   Smoothing parameters:
##     alpha = 0.1953
##     beta  = 1e-04
##     gamma = 1e-04
##     phi   = 0.9798
##
##   Initial states:
##     l = 0.3945
##     b = 0.0085
##     s = 0.874 0.8197 0.7644 0.7693 0.6941 1.284
##            1.326 1.177 1.162 1.095 1.042 0.9924
##
##   sigma:  0.0676
##
##      AIC     AICc      BIC
## -122.91 -119.21  -63.18
```

Example: drug sales

```r
ets(h02, model="AAA", damped=FALSE)
```

```
## ETS(A,A,A)
##
## Call:
##  ets(y = h02, model = "AAA", damped = FALSE)
##
##   Smoothing parameters:
##     alpha = 0.1672
##     beta  = 0.0084
##     gamma = 1e-04
##
##   Initial states:
##     l = 0.3895
##     b = 0.0116
##     s = -0.1058 -0.1359 -0.1875 -0.1803 -0.2414 0.2097
##            0.2493 0.1426 0.1411 0.0823 0.0293 -0.0033
##
##   sigma:  0.0642
##
##     AIC    AICc     BIC
## -18.26 -14.97   38.14
```

The `ets()` function

- Automatically chooses a model by default using the AIC, AICc or BIC.
- Can handle any combination of trend, seasonality and damping

- Ensures the parameters are admissible (equivalent to invertible)

- Produces an object of class "ets".

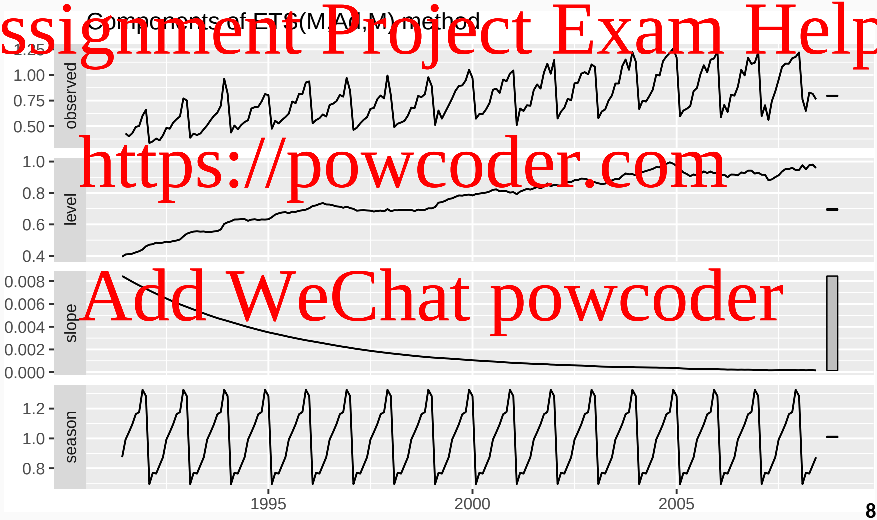
ets objects

- **Methods:** `coef()`, `autoplot()`, `plot()`, `summary()`, `residuals()`, `fitted()`, `simulate()` and `forecast()`

- `autoplot()` shows time plots of the original time series along with the extracted components (level, growth and seasonal).

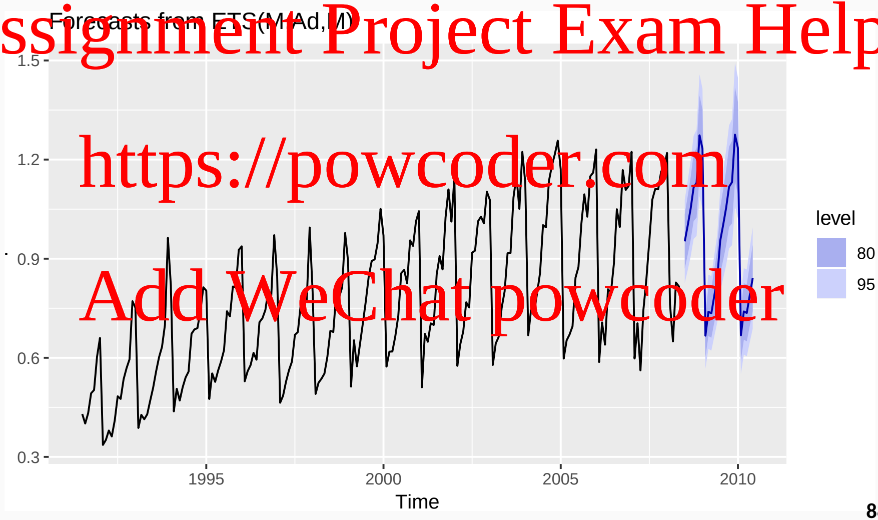
Example: drug sales

```r
h02 %>% ets() %>% autoplot()
```



Example: drug sales

```
h02 %>% ets() %>% forecast() %>% autoplot()
```

Example: drug sales


```r
h02 %>% ets() %>% accuracy()
```

```
##                    ME    RMSE     MAE    MPE  MAPE  MASE
## Training set 0.003873 0.05097 0.03904 0.1125 5.046 0.644 0
```



```r
h02 %>% ets(model="AAA", damped=FALSE) %>% accuracy()
```



```
##                     ME   RMSE     MAE    MPE  MAPE   MASE
## Training set -0.006447 0.0616 0.04949 -1.258 7.142 0.8164
```

The `ets()` function

`ets()` function also allows refitting model to new data set.

```r
train <- window(h02, end=c(2004,12))
test <- window(h02, start=2005)
fit1 <- ets(train)
fit2 <- ets(test, model = fit1)
accuracy(fit2)
```

```
##                   ME    RMSE     MAE     MPE  MAPE   MASE    ACF1
## Training set 0.00144 0.05406 0.04314 -0.4332 5.218 0.6785 -0.4121
```

```r
accuracy(forecast(fit1,10), test)
```

```
##                     ME    RMSE     MAE      MPE   MAPE  MASE     ACF1
## Training set  0.003427 0.04453 0.03290   0.1589  4.364 0.558  0.02236
## Test set     -0.077245 0.09158 0.07955 -10.0413 10.252 1.349 -0.04361
##              Theil's U
## Training set        NA
## Test set        0.6333
```

The `ets()` function in R

```r
ets(y, model = "ZZZ", damped = NULL,
  additive.only = FALSE,
  lambda = NULL, biasadj = FALSE,
  lower = c(rep(1e-04, 3), 0.8),
  upper = c(rep(0.9999, 3), 0.98),
  opt.crit = c("lik","amse","mse","sigma","mae"),
  nmse = 3,
  bounds = c("both", "usual", "admissible"),
  ic = c("aicc", "aic", "bic"),
  restrict = TRUE,
  allow.multiplicative.trend = FALSE, ...)
```

The `ets()` function in R

- `y`
  The time series to be forecast.
- `model`
  use the ETS classification and notation: “N” for none, “A” for additive, “M” for multiplicative, or “Z” for automatic selection. Default `ZZZ` all components are selected using the information criterion.
- `damped`
- If `damped=TRUE`, then a damped trend will be used (either A_d or M_d).
- `damped=FALSE`, then a non-damped trend will used.
- If `damped=NULL` (default), then either a damped or a non-damped trend will be selected according to the information criterion chosen.

The `ets()` function in R

- `additive.only`

  Only models with additive components will be considered if `additive.only=TRUE`. Otherwise all models will be considered.

- `lambda`

  Box-Cox transformation parameter. It will be ignored if `lambda=NULL` (default). Otherwise, the time series will be transformed before the model is estimated. When `lambda` is not `NULL`, `additive.only` is set to `TRUE`.

- `biadadj`

  Uses bias-adjustment when undoing Box-Cox transformation for fitted values.

The `ets()` function in R

- `lower,upper` bounds for the parameter estimates of α, β^*, γ^* and ϕ.
- `opt.crit=lik` (default) optimisation criterion used for estimation.
- `bounds` Constraints on the parameters.
  - *usual* region – `"bounds=usual"`;
  - *admissible* region – `"bounds=admissible"`;
  - `"bounds=both"` (default) requires the parameters to satisfy both sets of constraints.
- `ic=aicc` (default) information criterion to be used in selecting models.
- `restrict=TRUE` (default) models that cause numerical problems not considered in model selection.
- `allow.multiplicative.trend` allows models with a multiplicative trend.

The `forecast()` function in R

```
forecast(object,
  h=ifelse(object$m>1, 2*object$m, 10),
  level=c(80,95), fan=FALSE,
  simulate=FALSE, bootstrap=FALSE,
  npaths=5000, PI=TRUE,
  lambda=object$lambda, biasadj=FALSE,...)
```

- `object`: the object returned by the `ets()` function.
- `h`: the number of periods to be forecast.
- `level`: the confidence level for the prediction intervals.
- `fan`: if `fan=TRUE`, suitable for fan plots.

The `forecast()` function in R

- `simulate`: If `TRUE`, prediction intervals generated via simulation rather than analytic formulae. Even if `FALSE` simulation will be used if no algebraic formulae exist.
- `bootstrap`: If `bootstrap=TRUE` and `simulate=TRUE`, then simulated prediction intervals use re-sampled errors rather than normally distributed errors.
- `npaths`: The number of sample paths used in computing simulated prediction intervals.
- `PI`: If `PI=TRUE`, then prediction intervals are produced; otherwise only point forecasts are calculated. If `PI=FALSE`, then `level`, `fan`, `simulate`, `bootstrap` and `npaths` are all ignored.

The `forecast()` function in R

- `lambda`: The Box-Cox transformation parameter. Ignored if `lambda=NULL`. Otherwise, forecasts are back-transformed via inverse Box-Cox transformation.
- `biasadj`: Apply bias adjustment after Box-Cox?

Self-Practice

- Use `ets()` on some of these series:

  bicoal, *chicken*, *dole*, *usdeaths*, *bricksq*, *lynx*, *ibmclose*, *eggs*, *bricksq*, *ausbeer*

- Does it always give good forecasts?