C103 (Spring 2020) - Final Solutions

1. Consider the TTC algorithm for the given housing market:

   Step 1: Agents 1, 2 and 6 point to 5; agents 4 and 5 point to 1; and 3 points to 4. The only cycle formed consists of 1 and 5. Therefore, 1 receives e and 5 receives a. The remaining agents and objects are $N^1 = \{2, 3, 4, 6\}$ and $X^1 = \{b, c, d, f\}$.

   Step 2: All agents (including 4) point to 4. The only cycle is the self-cycle consisting of 4, so 4 receives d. The remaining agents and objects are $N^2 = \{2, 3, 6\}$ and $X^2 = \{b, c, f\}$.

   Step 3: Agent 2 points to 3; 3 points to 6; and 6 points to 2. We have a three-cycle where 2 receives c, 3 receives f, and 6 receives b. There are no remaining agents and objects, i.e., $N^3 = X^3 = \emptyset$, so the algorithm terminates.

   The resulting assignment is $\mu^{TTC} = (1e, 2c, 3f, 4d, 5a, 6b)$. By the Roth-Postlewaite Theorem μ^{TTC} gives the unique assignment in the core.

   Take any price vector $p = (p_a, p_b, p_c, p_d, p_e, p_f)$ such that the price of an object is decreasing in the step in which that object is allocated in the TTC algorithm (for example $p = (3, 1, 1, 2, 3, 1)$). Then, (μ^{TTC}, p) is a Walrasian equilibrium.

2. (a) *Step 1:* U is strictly dominated by $\frac{1}{2}\delta_M + \frac{1}{2}\delta_D$.

   Step 2: m is strictly dominated by $\frac{1}{2}\delta_L + \frac{1}{2}\delta_R$.

   Step 3: D is strictly dominated by M.

   Step 4: L is strictly dominated by R.

   So (M, R) is the only pure strategy profile that survives IESDS.

   (b) *Step 1:* Player 1 is rational.

   Step 2: Player 2 knows that Player 1 is rational. Player 2 is rational.

   Step 3: Player 1 knows that Player 2 knows that Player 1 is rational. Player 1 knows that Player 2 is rational. Player 1 is rational.

   Step 4: Player 2 knows that Player 1 knows that Player 2 knows that Player 1 is rational. Player 2 knows that Player 1 knows that Player 2 is rational. Player 2 knows that Player 1 is rational. Player 2 is rational.

   (c) By a result from class, the set of strategies played with positive probability in a NE survive IESDS. Therefore, there can not be any NE (mixed or pure strategy) other than (M, R). Furthermore, since by Nash's result every finite normal-form game has a NE, (M, R) must be a NE. So (M, R) is the unique NE of this game.

3. (a) The ex-post efficient allocations of the objects and the pivotal mechanism transfers for the two type profiles are:

$$
\begin{aligned}
k^*(\theta_1, \theta_2, \theta_3) &= (2, 0, 0) \\
t_1(\theta_1, \theta_2, \theta_3) &= (0 + 0) - (4 + 4) = -8 \\
t_2(\theta_1, \theta_2, \theta_3) &= (10 + 0) - (10 + 0) = 0 \\
t_3(\theta_1, \theta_2, \theta_3) &= (10 + 0) - (10 + 0) = 0 \\
k^*(\theta_1, \hat{\theta}_2, \hat{\theta}_3) &= (0, 1, 1) \\
t_1(\theta_1, \hat{\theta}_2, \hat{\theta}_3) &= (9 + 9) - (9 + 9) = 0 \\
t_2(\theta_1, \hat{\theta}_2, \hat{\theta}_3) &= (0 + 9) - (10 + 0) = -1 \\
t_3(\theta_1, \hat{\theta}_2, \hat{\theta}_3) &= (0 + 9) - (10 + 0) = -1
\end{aligned}
$$

(b) Two things about the above example are noteworthy. First, at the true-type profile θ, agents 2 and 3 can *jointly* misrepresent their types as $\hat{\theta}_2$ and $\hat{\theta}_3$ and obtain a payoff of 3=4-1 each, instead of zero. So even though VCG mechanisms are strategy-proof/dsic, they may be prone to collusive manipulation. Secondly, when the pivotal mechanism interpreted as a package auction, the revenues of the seller are not monotone: Even though all valuations are higher at the second profile, the seller's total revenue is less (2 instead of 8).

4. We will give a counterexample. Let $X = \{x, y, z\}$, $e(x) = 2$, $e(y) = 1.5$, and $e(z) = 0$, and let $g(z) > g(y) > g(x)$. Note that $z = c(\{x, y, z\})$, but $x = c(\{x, z\})$. So c fails Sen's condition α. Therefore by the revealed preference theorem, there is no rational preference R on X such that $c = c^R$.

5. The virtual valuations of the bidders are:

$$c_1(x_1) = x_1 - \frac{1 - x_1}{1} = 2x_1 - 1 \quad \text{and} \quad c_2(x_2) = x_2 - \frac{1 - \frac{1}{2}x_2}{\frac{1}{2}} = 2x_2 - 2$$

which are both strictly increasing, so the regularity condition is satisfied. By Myerson's result, the allocation probabilities in a revenue-maximizing IR and BIC direct auction mechanism are as follows: The highest virtual valuation bidder receives the object if her virtual valuation is nonnegative; and the seller keeps the object if both virtual valuations are negative.

Bidder 1 gets the object if and only if $c_1(x_1) \geq 0$ and $c_1(x_1) \geq c_2(x_2)$ if and only if

$$x_1 \geq \frac{1}{2} \quad \text{and} \quad x_1 + \frac{1}{2} \geq x_2$$

Bidder 2 gets the object if and only if $c_2(x_2) \geq 0$ and $c_2(x_2) \geq c_1(x_1)$ if and only if

$$x_2 \geq 1 \quad \text{and} \quad x_2 \geq x_1 + \frac{1}{2}$$

(Note: For valuation pairs satisfying $x_1 \geq \frac{1}{2}$, $x_2 \geq 1$ and $x_2 = x_1 + \frac{1}{2}$, either bidder can get the object.)

The object is not always allocated to one of the two bidders: If $x_1 < \frac{1}{2}$ and $x_2 < 1$, then the seller keeps the object.

The object is not always allocated to the bidder with the highest valuation: If $x_2 - \frac{1}{2} \leq x_1 < x_2$ and $x_1 \geq \frac{1}{2}$ (e.g. $x_1 = 0.75$ and $x_2 = 1$), then bidder 2 has the highest valuation but bidder 1 receives the object.

6. Since all agents find those on the other side acceptable, in all stable matchings all agents must be matched (otherwise any unmatched man-woman would form a blocking pair). Therefore, there are $3! = 6$ matchings that are candidate stable matchings. We will argue that exactly three out of those six matchings are stable.

   We can use the DA algorithm to find the woman-optimal and man-optimal stable matchings (both algorithms terminate in the first step): $\mu_W = (1c, 2a, 3b)$ and $\mu_M = (1a, 2b, 3c)$. We know from the Gale-Shapley Theorem that both μ_M and μ_W are stable.

   The matching $\mu = (1b, 2c, 3a)$ is also stable. To see this, we will use the following property of the given preference profile: For any given pair of agents i, j from different sides, i top ranks j if and only if j bottom ranks i. Note that at μ, all agents are matched to their second-ranked partner. Suppose a pair of agents i, j forms a blocking pair. Then i must top rank j, implying that j bottom ranks i, so i, j can't form a blocking pair. Therefore, μ is pairwise stable. It is also individually rational since all agents find those on the other side acceptable, showing that μ is a stable matching.

We next show that none of the three other matchings where all agents are matched is stable. In any such matching ν, at least one woman w is matched to her top choice and at least one woman w' is matched to her bottom choice. Let $m = \nu(w)$. Since w top ranks m, m must bottom rank w. So (m, w') form a blocking pair, showing that ν is not stable.[1]

[1] More specifically, $\nu = (1c, 2b, 3a)$ is blocked by $(1, b)$; $\nu = (1b, 2a, 3c)$ is blocked by $(2, c)$; and $\nu = (1a, 2c, 3b)$ is blocked by $(3, a)$.