Econ/Math C103 - Fall 2020

Non-Cooperative Game Theory I:

Normal Form Games

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Non-cooperative Game Theory

How may a group of self-interested individuals behave if each of them is affected by the others' actions?

We need to specify

- Who are the players?
- What are the actions that each player can take?
- What is each player’s payoff resulting from everybody’s actions?

Prisoners' Dilemma

Player 2 / Player 1	Cooperate	Defect
Cooperate	3,3	0,5
Defect	5,0	1,1

A Three Player Game

Player 1 chooses the row: T, B; Player 2 chooses the column: L, M, R; & Player 3 chooses the matrix: **M1, M2**. Payoffs:

M1

	L	M	R
T	3,0,2	2,1,1	1,0,3
B	0,0,3	1,1,1	0,3,0

M2

	L	M	R
T	3,0,1	1,1,-1	0,2,3
B	0,1,3	1,1,2	3,0,0

Levels of interactive knowledge

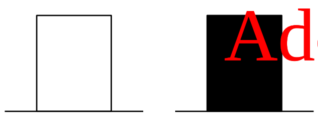
E: an event or statement (Examples: E=Jim is wearing a white hat; E=it rains outside; E=John is rational;…)

Levels of interactive knowledge:

1. Each player knows E
2. Each player knows that each player knows E
3. Each player knows that each player knows that each player knows E

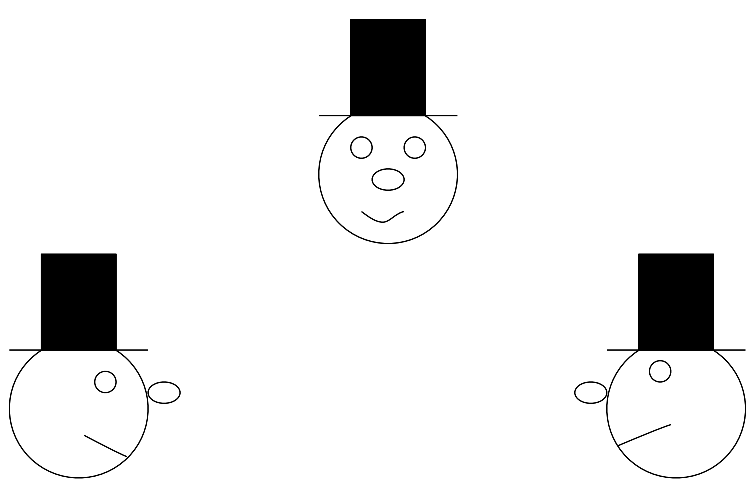
   ….

E is **common knowledge** among players if these hold ad infinitum.



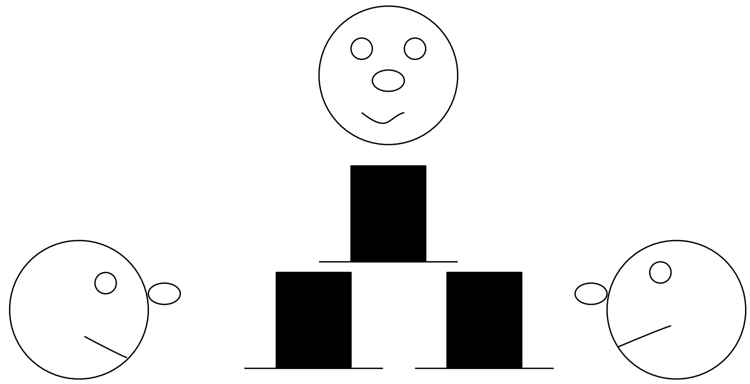
An example

E=There is at least one black hat



Note: 1. Each person knows E, 2. Each person knows that each person knows E. However E is not common knowledge.

Common knowledge



E is common knowledge

Normal form games

A **normal form game** consists of:

- a set of players N={1,2,…,n}
- a set of actions/strategies S_i for each i in N
- a vNM utility function:

$$u_i: S_1 \times S_2 \times \ldots \times S_n \rightarrow \mathbb{R} \qquad \text{for each } i \text{ in } N$$

S_i is also called the **pure strategies** of player i

Assumptions: -The game is common knowledge among the players.

-Players choose their actions simultaneously.

Frequently used notations

- A strategy of player i typically denoted by s_i in S_i
- Strategy profiles of all players:

$$S=S_1\times S_2\times\ldots\times S_n$$

with a typical member $s=(s_1,s_2,\ldots,s_n)$ in S

- The strategy profile of all players except player i:

$$S_{-i}=S_1\times\ldots\times S_{i-1}\times S_{i+1}\times\ldots\times S_n$$

with a typical member $s_{-i}=(s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$ in S_{-i}

How to play?

Player i is ***rational*** if she maximizes the expected value of u_i, given her knowledge and beliefs about how the others will play.

Prisoners' Dilemma

Player 2 / Player 1	Cooperate	Defect
Cooperate	3,3	0,5
Defect	5,0	1,1

Dominant-strategy equilibrium

- A pure strategy s_i^* **weakly dominates** s_i if

  $$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for any } s_{-i} \text{ in } S_{-i}$$

  and the inequality is strict for some s_{-i}.
- A strategy s_i^* is a **dominant strategy** if s_i^* weakly dominates every other strategy s_i.
- A strategy profile s^* is a **dominant-strategy equilibrium** if s_i^* is a dominant strategy for each player i.

Second-price auction

- An object is auctioned to two bidders 1 and 2.
- The value of the object to bidder i is v_i. Utility of player i from not buying the object is 0, and from buying the object at price p is $v_i – p$.
- Each bidder i bids b_i without seeing the other's bid.
- The highest bidder gets the object and pays the other's bid (=second highest bid).
- If bids are the same, then one of the bidders is chosen with equal probability, he gets the object and pays $p=b_1=b_2$.

Proposition: It is a dominant strategy equilibrium for each player to bid her valuation.

Proof: Covered in class.

A 2x3 Game

Player 1 \ Player 2	Left	Middle	Right
Top	2,0	0,1	1,2
Bottom	0,0	1,2	2,1

Randomization & mixed strategies

A **mixed strategy** σ_i of player i is a probability distribution over S_i.

Let $\sigma=(\sigma_1,\sigma_2,\ldots, \sigma_n)$, when each player j in N plays the mixed strategy σ_j the expected utility of player i is:

$U_i(\sigma)= \sum_{s=(s_1, s_2,\ldots,s_n) \text{ in } S} [\sigma_1(s_1)\sigma_2(s_2)\ldots\sigma_n(s_n)]\, u_i(s)$

where $\sigma_j(s_j)$ is the probability that player j plays the pure strategy s_j. (when S_j is finite)

Example: Prisoners' Dilemma

If $\sigma_1(C)=2/3$, $\sigma_1(D)=1/3$, $\sigma_2(C)=\sigma_2(D)=1/2$, then:

$$
\begin{aligned}
U_1(\sigma) &= \sigma_1(C)\sigma_2(C)u_1(C,C) + \sigma_1(C)\sigma_2(D)u_1(C,D) \\
&\quad + \sigma_1(D)\sigma_2(C)u_1(D,C) + \sigma_1(D)\sigma_2(D)u_1(D,D) \\
&=2
\end{aligned}
$$

Pl 1 \ Pl 2	C	D
C	3,3	0,5
D	5,0	1,1

Strict domination

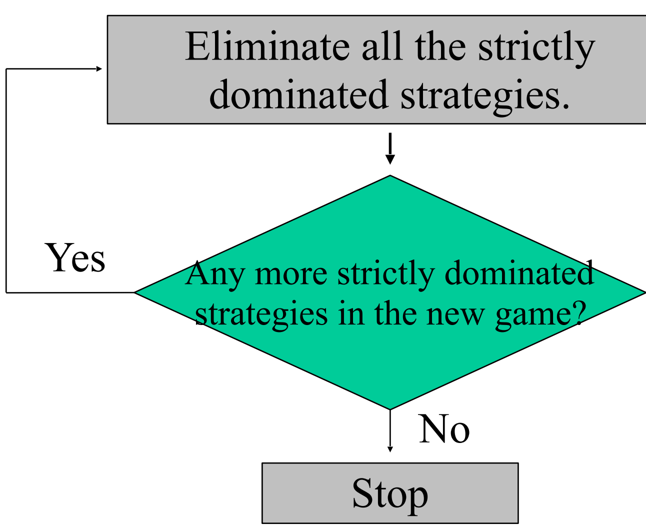
A mixed strategy σ_i^* **strictly dominates** a pure strategy s_i if

$$U_i(\sigma_i^*, s_{-i}) > U_i(s_i, s_{-i}) \quad \text{for any } s_{-i}$$

Player i is *rational* if she maximizes her expected utility, given her knowledge and beliefs about how the others will play.

RATIONALITY implies Never play a strictly dominated strategy.

Iterated Elimination of Strictly Dominated Strategies (IESDS)



Prediction relies on the **common knowledge of rationality**

A 3x3 game

Player 1 \ Player 2	L	m	R
T	3,0	1,1	0,3
M	1,0	0,10	1,0
B	0,3	1,1	3,0

Assume: Players are rational and 2 knows that 1 is rational.

Player 1 \ Player 2	L	M	R
T	3,0	1,1	0,3
M	1,0	0,10	1,0
B	0,3	1,1	3,0

1 is rational:

	L	M	R
T	3,0	1,1	0,3
B	0,3	1,1	3,0

2 knows 1 is rational and 2 is rational:

	L	R
T	3,0	0,3
B	0,3	3,0

Simplified price-competition

Firm 1 \ Firm 2	High	Medium	Low
High	6,6	0,10	0,8
Medium	10,0	5,5	0,8
Low	8,0	8,0	4,4

Summary

- If players are rational (and cautious), then they play the dominant-strategy equilibrium whenever it exists
  - But typically, it does not exist
- If it is common knowledge that players are rational, then they will play a strategy-profile that survives IESDS:
  - Typically, there are too many strategies that survive IESDS
- Next, a stronger assumption: The players are rational and they hold correct beliefs about the other players' strategies.

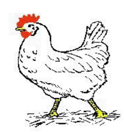
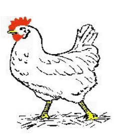
Nash Equilibrium in pure strategies

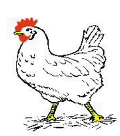
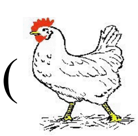
A pure strategy-profile $s^*=(s_1^*,\ldots,s_n^*)$ is a **Nash equilibrium (NE)** if no player has an incentive to deviate when the others play according to s^*, i.e. if for any i and s_i in S_i :

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

Assumption: Players are rational and have correct conjectures about others' strategies.

Chicken

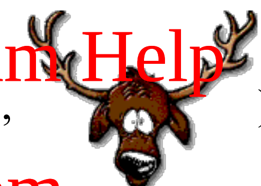
		
	-1,-1	1,0
	0,1	1/2,1/2

Two pure strategy NE: ( , ) and ( , )

Stag Hunt

		
	2,2	4,0
	0,4	6,6

Two pure strategy NE:

( , ) and ( , )

Matching Pennies

Player 2 / Player 1	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

No pure strategy NE.

Nash Equilibrium in mixed strategies

A mixed strategy-profile $\sigma^*=(\sigma_1{}^*,\ldots,\sigma_n{}^*)$ is a **Nash equilibrium** if no player has an incentive to deviate when the others randomize according to σ^*, i.e. if for any i and σ_i :

$$U_i(\sigma_i{}^*, \sigma_{-i}{}^*) \geq U_i(\sigma_i, \sigma_{-i}{}^*)$$

Assumption: Players are rational and have correct conjectures about others' strategies.

Best reply

A mixed strategy σ_i of player i is a ***best reply*** to σ_{-i} if it maximizes $U_i(\cdot, \sigma_{-i})$.

Notation: Let $B_i(\sigma_{-i})$ denote the set of best replies of player i to σ_{-i}.

Important Note: A mixed strategy profile $\sigma^*=(\sigma_1{}^*,\ldots, \sigma_n{}^*)$ is a Nash equilibrium if and only if for every player i, $\sigma_i{}^*$ is a best reply to $\sigma_{-i}{}^*$.

Proposition: σ_i is a best reply to σ_{-i} iff for any pure strategy s_i played with positive probability (i.e. $\sigma_i(s_i)>0$), s_i gives i more expected payoff than any other pure strategy:

$$U_i(s_i,\sigma_{-i}) \geq U_i(s_i', \sigma_{-i}) \text{ for any } s_i' \text{ in } S_i$$

Proof: $U_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \text{ in } S_i} \sigma_i(s_i)\, U_i(s_i,\sigma_{-i})$, therefore σ_i is an optimal response to σ_{-i} if and only if each s_i with $\sigma_i(s_i)>0$ is an optimal response.

In particular when σ_i is a best reply to σ_{-i}, i is *indifferent* between any two strategies that he plays with positive probability.

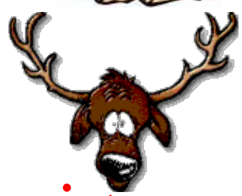
Nash equilibrium existence

Theorem: (John F. Nash, 1950) Every normal form game with finitely many pure strategies has a mixed strategy Nash equilibrium.

Example: (A game without a mixed strategy Nash equilibrium) Consider two players where each player i chooses an integer s_i. Player i receives 1 if $s_i > s_j$, and 0 otherwise ($j \neq i$).

Stag Hunt



2,2	4,0
0,4	6,6

Mixed strategy equilibrium in the stag hunt game

Suppose that player 1 believes that player 2 plays Rabbit with probability β.

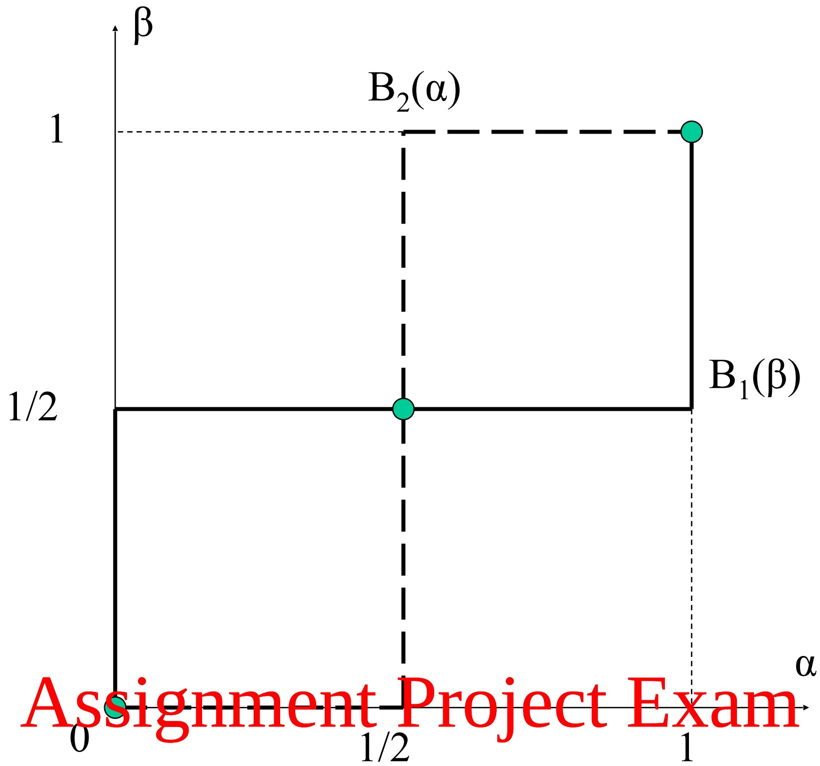
Player 1's payoff from playing Rabbit or Stag:

$U_1(R,\beta)=2\beta+4(1-\beta)$; $U_1(S,\beta)=0\beta+6(1-\beta)$

$$U_1(R,\beta) > U_1(S,\beta) \text{ , } \beta > 1/2$$

α: the probability with which 1 plays Rabbit.

Best replies in the Stag-Hunt game



A 3x3 game

Player 1 \ Player 2	L	m	R
T	3,0	1,1	0,3
M	1,0	0,10	1,0
B	0,3	1,1	3,0

Nash equilibria and IESDS, Fact I

Proposition 1: Let $\sigma^*=(\sigma_1^*,\ldots,\sigma_n^*)$ be a mixed strategy Nash equilibrium. Then any pure strategy profile s that comes about with positive probability under σ^* (i.e. $\sigma_i^*(s_i)>0$ for any i) survives IESDS.

Proof: Covered in class.

Simplified price-competition

Firm 1 \ Firm 2	High	Medium	Low
High	6,6	0,10	0,8
Medium	10,0	5,5	0,8
Low	8,0	8,0	4,4

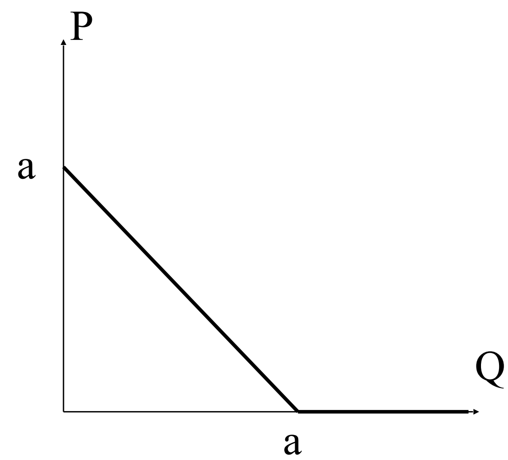
Nash equilibria and IESDS, Fact II

Proposition 2: If only one pure strategy profile s^* survives IESDS, then s^* is a Nash equilibrium of the game. Moreover, the game has no other (pure or mixed strategy) Nash equilibrium.

Proof: Covered in class.

Cournot duopoly

- $N = \{1,2\}$ two firms;
- Simultaneously, each firm i produces $q_i \geq 0$ units of a good at marginal cost $c > 0$,
- and sells the good at price
  $P(Q) = \max\{0, a-Q\}$ $a>c$
  where $Q = q_1+q_2$.



Profits of firm i:

$$\pi_i(q_1,q_2) = q_i[P(q_1+q_2)-c] = \begin{cases} q_i[a-q_1-q_2-c] & \text{if } q_1+q_2<a \\ -q_i c & \text{otherwise} \end{cases}$$

Cournot duopoly best replies

$$B_i(q_j) = \begin{cases} 0 & \text{if } q_j \geq a-c \\ (a-q_j-c)/2 & \text{otherwise} \end{cases}$$

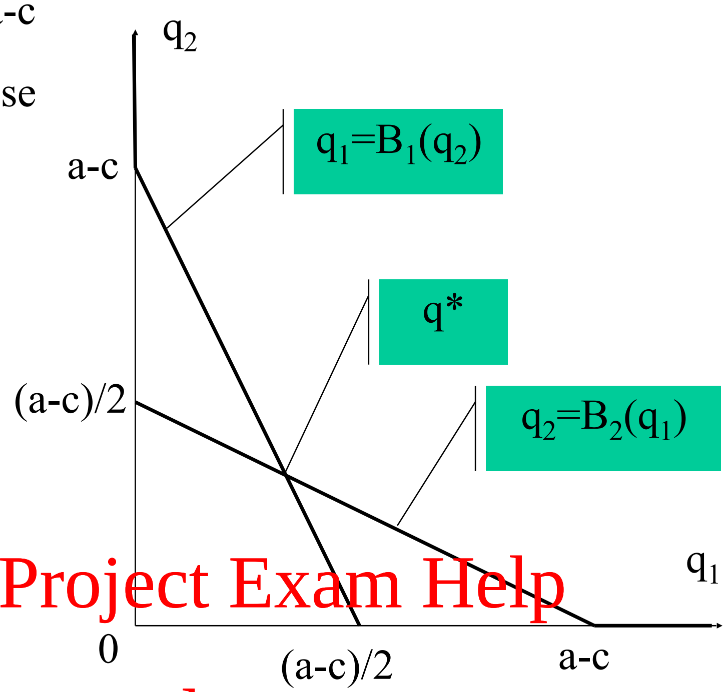
Nash equilibrium:

$$q_1^* = (a - q_2^* - c)/2$$

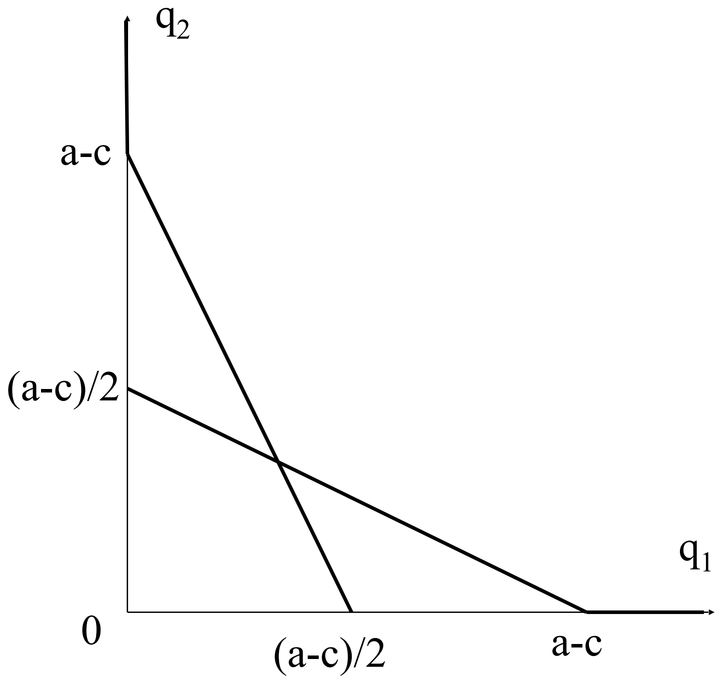
$$q_2^* = (a - q_1^* - c)/2$$

imply:

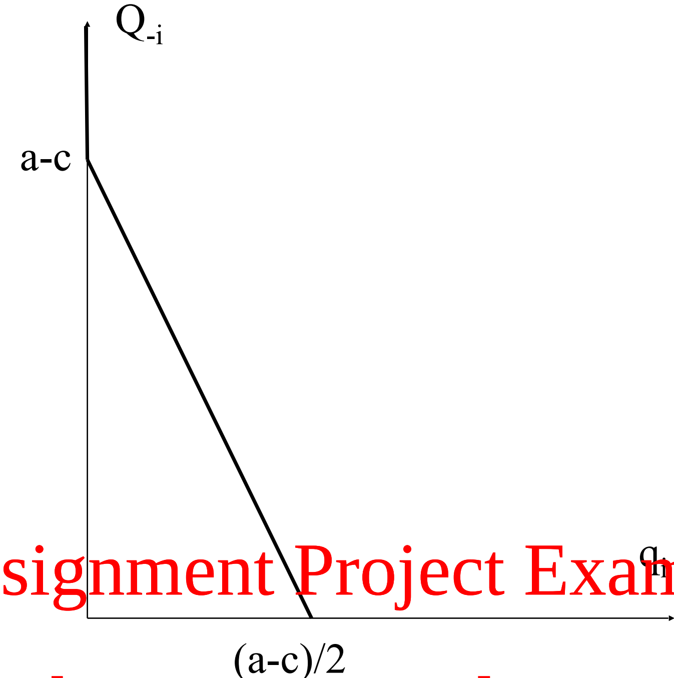
$$q_1^* = q_2^* = (a-c)/3$$



IESDS in the Cournot duopoly



IESDS in the Cournot Oligopoly with n>2



Commons problem

- N = {1,2,…,n} players, each with unlimited money;
- Simultaneously, each player i contributes $x_i \geq 0$ to produce $y = x_1 + \ldots + x_n$ unit of some public good, yielding payoff

$$u_i(x_i, y) = y^{1/2} - x_i.$$

Optional Additional Reading

- Chapters 1-4 of Osborne and Rubinstein.
- Chapter 7-8 of MWG