Econ/Math C103 - Fall 2020

The Envelope Theorem and Optimal Auctions

Haluk Ergin

The Envelope Theorem

Source:
Milgrom and Segal (2002)

The Envelope Theorem

Let A be an arbitrary set with generic element $\alpha \in A$. Let x, y denote generic elements of $[0,1]$.

Given a function $f : A \times [0,1] \to \mathbb{R}$, define:

$$\begin{aligned} V(x) &= \sup_{\alpha \in A} f(\alpha, x) \\ A^*(x) &= \{\alpha \in A : f(\alpha, x) = V(x)\}. \end{aligned}$$

If it exists, let $f_2(\alpha, x)$ denote the partial derivative of f with respect to its second argument.

Theorem *(M&S, 2002) Let $x \in (0,1)$ and $\alpha^*(x) \in A^*(x)$. If the derivatives $f_2(\alpha^*(x), x)$ and $V'(x)$ exist, then*

$$V'(x) = f_2(\alpha^*(x), x).$$

Absolute Continuity of V

If a function $g : [0,1] \to \mathbb{R}$ is *absolutely continuous*, then it is differentiable almost everywhere and it is equal to the integral of its derivative (i.e. the Fundamental Theorem of Calculus is applicable to g).

Examples of Absolutely continuous functions:
-Lipschitz continuous functions, e.g. differentiable functions with bounded derivative.
-Convex [or concave] functions which are continuous at the endpoints 0 and 1.

Corollary *Suppose that V is absolutely continuous and $f(\alpha, \cdot)$ is differentiable for every $\alpha \in A$. Suppose that $\alpha^*(y) \in A^*(y)$ for almost all $y \in [0,1]$. Then:*

$$V(x) = V(0) + \int_0^x f_2(\alpha^*(y), y)dy.$$

Optimal Auction Design
Myerson (1981)

The Model

A single indivisible object is to be allocated to n bidders: $N = \{1, 2, \ldots, n\}$.

Bidder i's valuation is denoted by X_i or x_i. It is distributed with density f_i and c.d.f. F_i, in a closed interval $\mathcal{X}_i = [\underline{x}_i, \bar{x}_i] \subset \mathbb{R}$. f_i is continuous and $f_i(x_i) > 0$ for any $x_i \in \mathcal{X}_i$. Valuations are independently distributed across bidders.

A profile of valuations: $x = (x_1, \ldots, x_n) \in \mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$. The joint density is: $f(x) = f_1(x_1) \times \ldots \times f_n(x_n)$.

Similarly, $x_{-i} \in \mathcal{X}_{-i}$ is a profile of valuations excluding x_i and f_{-i} is the joint density of valuations other than x_i.

Note: We still have independence, private values, and risk neutrality, but symmetry is dropped.

Direct Auction Mechanisms

A direct auction mechanism determines each agent's probability of receiving the object and her expected payment as a function of the reported types ($\equiv$ valuations).

Definition A **direct (auction) mechanism** is a pair (p, t) where $p : \mathcal{X} \to \mathbb{R}^n$ and $t : \mathcal{X} \to \mathbb{R}^n$ are functions, and p satisfies $p_j(x) \geq 0$ and $\sum_{i=1}^n p_i(x) \leq 1$ for all $j \in N$ and $x \in \mathcal{X}$.

$p_i(x)$ denotes the probability with which agent i receives the object and $t_i(x)$ denotes the expected payment of i, when the reported profile of valuations is x.

By the revelation principle, we can restrict attention to direct auction mechanisms that are truthfully implementable in BNE.

Feasible Mechanisms

Given a direct mechanism (p, t), the expected utility of agent i with valuation x_i from truthfully revealing her type when everybody else truthfully reveals their type is:

$$U_i(x_i) = \int_{\mathcal{X}_{-i}} \Big[p_i(x_i, x_{-i})x_i - t_i(x_i, x_{-i})\Big] f_{-i}(x_{-i})dx_{-i}.$$

Definition A direct mechanism (p, t) is **feasible** if it is

1. **Bayesian Incentive Compatible (BIC):** For all $i \in N$, $x_i, x_i' \in \mathcal{X}_i$

$$U_i(x_i) \geq \int_{\mathcal{X}_{-i}} \Big[p_i(x_i', x_{-i})x_i - t_i(x_i', x_{-i})\Big] f_{-i}(x_{-i})dx_{-i}.$$

2. **Individually Rational (IR):** For all $i \in N$, $x_i \in \mathcal{X}_i$

$$U_i(x_i) \geq 0.$$

Note: BIC $\equiv$ Truthfuly implementable in BNE.

Characterization of Feasibility

Given a direct mechanism (p, t), the *expected* probability with which agent i receives the object if she announces x_i and everybody else truthfully reveals their type is:

$$Q_i(x_i) = \int_{\mathcal{X}_{-i}} p_i(x_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i}.$$

Lemma 1 *The direct mechanism (p, t) is feasible if and only if the following are satisfied for all $i \in N$:*

1. *For all $x_i, y_i \in \mathcal{X}_i$: $x_i \leq y_i \Longrightarrow Q_i(x_i) \leq Q_i(y_i)$.*
2. *For all $x_i \in \mathcal{X}_i$:*

$$U_i(x_i) = U_i(\underline{x}_i) + \int_{\underline{x}_i}^{x_i} Q_i(y_i) dy_i$$

3. *$U_i(\underline{x}_i) \geq 0$.*

The Seller's Expected Revenues from a Feasible Mechanism

Lemma 2 *If the direct mechanism (p, t) is feasible, then the seller's expected revenues is given by:*

$$\int_{\mathcal{X}} \left[\sum_{i=1}^{n} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) p_i(x) \right] f(x) dx - \sum_{i=1}^{n} U_i(\underline{x}_i).$$

The Revenue Equivalence Thm 1: Feasible Direct Mechanisms

The seller's expected revenues and the agents' expected payoffs in a feasible direct auction mechanism is completely determined by the probability function $p(\cdot)$ and the vector $(U_1(\underline{x}_1), \ldots, U_n(\underline{x}_n))$.

That is, if two feasible direct auction mechanisms allocate the object with the same probabilities and give the same expected payoffs to the lowest valuation bidders, then the expected revenues of the seller and the expected payoff of every agent of every valuation (not just the lowest valuation types) is also the same under the two mechanisms.

The Revenue Equivalence Thm 2: BNE of Arbitrary Auction Rules

Fix any two arbitrary auction rules A and B. Suppose that there is a BNE of A and a BNE of B such that: (i) the object is allocated with the same probabilities in the two BNE, and (ii) for every bidder $i \in N$ the expected payoff of the lowest-valuation type of i is the same and nonnegative in the two BNE.

Since every auction rule is an indirect mechanism, by the revelation principle, the BNE of an auction rule induces a direct mechanism that is truthfully implementable in BNE ($\equiv$ BIC).

By assumption, the direct mechanisms induced by the BNE of the auction rules A and B allocate the object with the same probabilities and give the same nonnegative expected payoffs to the lowest valuation bidders.

Therefore, the expected revenues of the seller and the expected payoff of every agent of every valuation must also be the same in the BNE of the auction rules A and B.

Example: The BNE of the First-Price, Second-Price, and All-Pay Auction Rules

In the BNE of the first-price and second-price auction we studied (note that the dominant strategy equilibrium of the second-price auction is in particular a BNE), and in the BNE of the all-pay auction in Problem Set 5, the object is always allocated to the highest valuation bidder and the payoffs of the lowest-valuation bidders is zero. Therefore, the seller's expected revenues and the bidders' expected payoffs are the same in all three auctions.

Note however that the same conclusion does not extend to the BNE of the second-price auction with reserve price $r > 0$ from Problem Set 5, because the object is not always allocated to the highest bidder (when the highest valuation is less than r).

Optimal Auctions

A direct auction mechanism (p, t) is **optimal** if it maximizes the seller's expected revenues subject to feasibility.

Corollary *Suppose that*

1. $p : \mathcal{X} \to \mathbb{R}^n$ *maximizes*

$$\int_{\mathcal{X}} \left[\sum_{i=1}^{n} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) p_i(x) \right] f(x) dx$$

subject to:

(a) *For all* $i \in N$, $x_i, y_i \in \mathcal{X}_i$: $x_i \leq y_i \Longrightarrow Q_i(x_i) \leq Q_i(y_i)$.

(b) $p_j(x) \geq 0$ *and* $\sum_{i=1}^{n} p_i(x) \leq 1$ *for all* $j \in N$ *and* $x \in \mathcal{X}$.

2. $t : \mathcal{X} \to \mathbb{R}^n$ *is given by:*

$$t_i(x) = p_i(x) x_i - \int_{\underline{x}_i}^{x_i} p_i(y_i, x_{-i}) dy_i.$$

Then, (p, t) *is an optimal direct auction mechanism.*

The Regular Case

Regularity assumption: The function

$$c_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}$$

is strictly increasing for each $i \in N$.

Corollary *Suppose that regularity is satisfied.*

Define $p : \mathcal{X} \to \mathbb{R}^n$ *as follows: If* $\max\{c_i(x_i) : i \in N\} < 0$, *then the seller keeps the object. Otherwise, the seller gives the object to the bidder with highest* $c_i(x_i)$ *value (breaking ties arbitrarily).*

Define $t : \mathcal{X} \to \mathbb{R}^n$ *by:*

$$t_i(x) = p_i(x) x_i - \int_{\underline{x}_i}^{x_i} p_i(y_i, x_{-i}) dy_i.$$

Then, (p, t) *is an optimal direct auction mechanism.*

An Example

Suppose the X_i is distributed uniformly on $\mathcal{X}_i = [0,1]$, for each $i \in N$. That is, $f_i(x_i) = 1$, $F_i(x_i) = x_i$, and

$$c_i(x_i) = 2x_i - 1.$$

Regularity is satisfied.

By symmetry of this example, the bidder with the highest $c_i(x_i)$ is the bidder with the highest valuation.

Therefore, the optimal auction allocates the object to the bidder with the highest valuation x_i iff $c_i(x_i) \geq 0$ iff $x_i \geq 1/2$. If the highest valuation is less than 1/2, then the seller keeps the object.

Notes:
-The optimal auction entails a reserve price of 1/2.
- The optimal auction is ex-post inefficient.

Optional Additional Reading

Chapters 3 and 5 of **Kr**, Chapter 23 of **MWG**, and

- Milgrom and Segal (2002), "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70, 583-601.

- Myerson (1981), "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58-73.