# Econ/Math C103 – Fall 2020

# The Vickrey-Clark-Groves Mechanisms

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## 1 The Quasi-linear Model

In these lecture notes, we will build on our earlier mechanism design model. We will impose further assumptions on the structure of the social alternatives $X$ and individuals' type-dependent utility functions in order to obtain stronger results. More specifically, we will allow for monetary transfers between the social planner and the agents which may be interpreted as taxes/subsides, and assume that agents' utility functions are additively separable in their transfer. Under these assumptions, we will discuss an important class of dominant strategy mechanisms called the Vickrey-Clark-Groves (VCG) mechanisms.



Assume that $X = K \times \mathbb{R}^n$ for some arbitrary set $K$ called the set of "projects". A social outcome has two components $(k, t) \in K \times \mathbb{R}^n$ where $k \in K$ is the project selected, and $t = (t_1, \ldots, t_n)$ is the profile of monetary transfers from the social planner to the agents. The "project choice" term is coined because an important application of this model is when the set corresponds to $K$ different public projects. However, there are many other interesting applications of this model that are not related to the public project choice application. More generally, you should think of the set $K$ as any part of the social decision/allocation that is not about the monetary transfers.



We will assume that the utility of agent $i$ of type $\theta_i$, when the project choice is $k$ and he receives transfer $t_i \in \mathbb{R}$ is additively separable in the transfer, i.e.:

$$u_i((k, t_i); \theta_i) = v_i(k, \theta_i) + t_i \quad \text{for all } k \in K, \theta_i \in \Theta_i, t_i \in \mathbb{R}.$$

This is called the quasi-linear model, since the utility of the agent is linear in the monetary transfers $t_i$ that he receives. Quasi-linearity implies that for any pair of projects $k$ and $k'$, and any type $\theta_i$ of individual $i$, there exists some sum of money "\$$t_i$" in compensation of which $i$ would not mind changing the social outcome from $k$ to $k'$, namely $t_i > v_i(k, \theta_i) - v_i(k', \theta_i)$.

Because of the additional structure on $X = K \times \mathbb{R}^n$, a **social choice function (SCF)** $f : \Theta \to X$ can be broken down into two parts $f(\cdot) = (k(\cdot), t(\cdot))$ where $k(\cdot) : \Theta \to K$ is the **project-choice rule** which selects a project as a function of the type profile, and $t(\cdot) : \Theta \to \mathbb{R}^n$ is the **transfer rule** which determines the vector of transfers made to the agents as a function of the type profile.

For every $\theta = (\theta_1, \ldots, \theta_n) \in \Theta$, $k(\theta)$ denotes the project that is selected and $t(\theta) = (t_1(\theta), \ldots, t_n(\theta))$ denotes the vector of transfers to the agents when the type profile is $\theta$. Here $t_i(\theta)$ denotes the monetary transfer made *to* the agent $i$. A negative transfer $t_i(\theta)$ corresponds to a payment made by the agent.

The natural notion of efficiency in this model is called ex-post efficiency. A project choice rule is ex-post efficient if it is not possible to strictly improve the utility of all agents without providing them with positive net transfers.

**Definition 1** A project-choice rule $k(\cdot) : \Theta \to K$ is **ex-post efficient** if there does not exist a type profile $\theta \in \Theta$, a project $k' \in K$ and a transfer vector $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ such that:

(i) $\sum_{i=1}^n t_i = 0$, and

(ii) $v_i(k', \theta_i) + t_i > v_i(k(\theta), \theta_i)$ for any $i \in N$.



Using the fact that the utility functions are quasi-linear in the current setup, we can prove a very useful characterization of ex-post efficiency: A project choice rule is ex-post efficient if and only if it maximizes the sum of utilities at every type profile.



**Proposition 1** *A project-choice rule $k(\cdot) : \Theta \to K$ is ex-post efficient if and only if for all $\theta \in \Theta$ and $k' \in K$:*



$$\sum_{i=1}^n v_i(k(\theta), \theta_i) \geq \sum_{i=1}^n v_i(k', \theta_i).$$

**Proof:** Covered in class. □

## 2 Dominant Strategy Implementation

We will study a class of dominant strategy incentive compatible mechanisms. By the revelation principle, we can restrict attention to direct mechanisms that are truthfully implementable in dominant strategies, in which agents are directly asked to report their types and will have incentive to report their true types. Let's restate this definition in our current quasi-linear model.

**Definition 2** The direct mechanism $(\Theta; k(\cdot), t(\cdot))$ is **truthfully implementable in dominant strategies** (equivalently, dominant strategy incentive compatible/ strategy-proof) if for any $\theta \in \Theta$, $i \in N$ and $\hat{\theta}_i \in \Theta_i$:

$$v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(k(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i}).$$

Our central question will be the following: Given a project-choice rule $k(\cdot)$, can we design a supplementary transfer rule $t : \Theta \to \mathbb{R}^n$ such that the direct mechanism $(\Theta; k(\cdot), t(\cdot))$ is **truthfully implementable in dominant strategies**? The answer to this question turns out to be: Yes, if the project choice rule $k(\cdot)$ is ex-post efficient to start with.

**Proposition 2** *Let $k(\cdot)$ be an ex-post efficient project-choice rule, and for each $i \in N$ let $h_i : \Theta_{-i} \to \mathbb{R}$ be an arbitrary real valued function defined on the types of all agents other than $i$. Then the following transfer rule:*

$$t_i(\theta) = \sum_{j \neq i} v_j(k(\theta), \theta_j) - h_i(\theta_{-i})$$

*coupled with the project-choice function $k(\cdot)$ is truthfully implementable in dominant strategies.*

**Proof:** Covered in class. $\square$

The transfer rule defined above along with the ex-post efficient project-choice rule $k(\cdot)$ is known as a **Vickrey-Clark-Groves (VCG) mechanism**. Note that there are many VCG mechanisms that can be associated with a given project-choice rule $k(\cdot)$, since we have freedom in selecting the functions $h_i$, as long as they do not depend on the type of agent $i$.

One interesting special case of VCG mechanism is known as the **pivotal mechanism** where the $h_i$ functions and the transfer rule is given by:

$$h_i(\theta_{-i}) = \max_{k' \in K} \left[ \sum_{j \neq i} v_j(k', \theta_j) \right]$$

$$t_i(\theta) = \sum_{j \neq i} v_j(k(\theta), \theta_j) - \max_{k' \in K} \left[ \sum_{j \neq i} v_j(k', \theta_j) \right]$$

Note that the pivotal transfers are never strictly positive, meaning that the social planner never has to contribute out of his pocket. Also note that under the pivotal transfer rule, if the agent $i$ makes a non zero payment (i.e. if $t_i(\theta) < 0$), then $k(\theta)$ is *not* a maximizer of the sum $\sum_{j \neq i} v_j(\cdot, \theta_j)$: that is, the ex-post efficient outcome for the smaller society $N \setminus \{i\}$ is different than the ex-post efficient outcome for the society $N$. In particular the agent $i$ does not make a payment unless he is pivotal in the sense that his presence tips over the social outcome from the maximizer of $\sum_{j \neq i} v_j(\cdot, \theta_j)$ to $k(\theta)$. We call this mechanism the pivotal mechanism since only pivotal agents make a non-zero payment.

# 3 Applications of VCG Mechanisms

## 3.1 Public Project Decision

The society has to reach a decision regarding whether to undertake a public project, e.g., build a bridge, or not. Then $K = \{0, 1\}$ where 0 corresponds to not building the bridge, and 1 corresponds to building it. Possible valuations of each agent $i$ is her net benefit from building the bridge: $\Theta_i = \mathbb{R}$. Note that individuals may have negative net values for the project. The utility functions are then given by $v_i(k, \theta_i) = k\theta_i$.

A project-choice rule $k(\cdot)$ is ex-post efficient if and only if it maximizes the sum of utilities, i.e., $k(\theta) = 0$ if $\sum_{i \in N} \theta_i < 0$ and $k(\theta) = 1$ if $\sum_{i \in N} \theta_i > 0$. For example, the project-choice rule $k^*(\cdot)$ defined by:

$$k^*(\theta) = 1 \iff \sum_{i \in N} \theta_i \geq 0$$

is ex-post efficient. The pivotal transfers for $k^*(\cdot)$ are given by:



$$t_i(\theta) = \begin{cases} \sum_{j \neq i} \theta_j & \text{if } \sum_{j \neq i} \theta_j < 0 \leq \sum_{j \in N} \theta_j \\ -\sum_{j \neq i} \theta_j & \text{if } \sum_{j \neq i} \theta_j \geq 0 > \sum_{j \in N} \theta_j \\ 0 & \text{otherwise.} \end{cases}$$



## 3.2 Allocation of a Single Indivisible Object: The Second-Price Auction



A single indivisible object is to be allocated to exactly one agent. The types $\Theta_i = [0, 1]$ denote the possible valuations of $i$ for the object. Utility of not receiving the object is zero.

In this problem, the social outcome determines who receives the object, so $K = N = \{1, \ldots, n\}$. The utilities are:

$$v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

The sum of utilities is therefore:

$$\sum_{i=1}^{n} v_i(k, \theta_i) = \theta_k, \quad k \in K.$$

Therefore a project-choice rule $k(\cdot)$ is ex-post efficient if and only if it allocates the object to an agent who has the highest valuation. Let $k^*(\cdot)$ denote such a project-choice rule.

As usual, given a type profile $\theta = (\theta_1, \ldots, \theta_n)$, let $\theta^l$ denote the $l$th highest valuation among $\theta_1, \ldots, \theta_n$ (the $l$th order statistic). Let's find the pivotal transfer rule associated with $k^*(\cdot)$:

$$t_i(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \max_{k' \in K} \left[ \sum_{j \neq i} v_j(k', \theta_j) \right] = \begin{cases} 0 - \theta^2 & = & -\theta^2 & \text{if } i = k^*(\theta) \\ \theta^1 - \theta^1 & = & 0 & \text{if } i \neq k^*(\theta) \end{cases}$$

Therefore under the pivotal mechanism, the highest valuation agent receives the object and pays the second highest valuation, just as in the dominant strategy equilibrium of the second-price auction.

### 3.3 Allocation of Many Identical Indivisible Objects: The Uniform Price Auction

In this example $m$ indivisible object are to be allocated to exactly $m$ agents. We assume that $m < n$, i.e. there are more agents than objects. Each agent can consume at most one object. The types $\Theta_i = [0, 1]$ denote the possible valuations of $i$ for the object. Utility of not receiving the object is again zero.



A possible social outcome corresponds to a subset of people who are allocated these objects, i.e. $K = \{M \subset N : |M| = m\}$. The utilities are as in before:



$$v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } i \in k \\ 0 & \text{if } i \notin k. \end{cases}$$



The sum of utilities is therefore:

$$\sum_{i=1}^{n} v_i(k, \theta_i) = \sum_{i \in k} \theta_i, \quad k \in K.$$

Therefore a project-choice rule $k(\cdot)$ is ex-post efficient if and only if it allocates the object to agents with $m$ highest valuations: $\theta^1, \ldots, \theta^m$. (Remember that $\theta^1 \geq \theta^2 \geq \ldots \geq \theta^m \geq \theta^{m+1} \geq \ldots \theta^n$).

Let $k^*(\cdot)$ be such an ex-post efficient project-choice rule. Then, the pivotal transfers for $k^*(\cdot)$ are given by:

$$t_i(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \max_{k' \in K} \left[ \sum_{j \neq i} v_j(k', \theta_j) \right]$$

$$= \begin{cases} (\theta^1 + \ldots + \theta^m - \theta_i) - (\theta^1 + \ldots + \theta^m - \theta_i + \theta^{m+1}) & = & -\theta^{m+1} & \text{if } i \in k^*(\theta) \\ (\theta^1 + \ldots + \theta^m) - (\theta^1 + \ldots + \theta^m) & = & 0 & \text{if } i \notin k^*(\theta). \end{cases}$$

So under the pivotal mechanism, the $m$ highest valuation agents receive the objects and each of them pays the $(m+1)th$ highest valuation, which is the natural generalization of the dominant strategy equilibrium of the second price auction. The $(m+1)th$-price auction that we have just derived is also known as the **uniform price auction** since all agents who win an object make the same payment, even if they have different valuations.

### 3.4 Package Auctions, Example 1

Two identical objects are to be allocated among three agents. Each agent can consume zero, one, or two objects. The types of each agent is given by $\Theta_i = \{(v_1, v_2) \in \mathbb{R}^2_+ | v_1 \leq v_2\}$ where $v_1$ denotes utility of consuming just one object and $v_2$ is the utility of consuming two objects. A possible allocation must specify how many objects each agent receives, that is $k = (k_1, k_2, k_3)$ where $k_i$ is the number of objects $i$ is allocated. Hence

$$K = \{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}.$$



Let us compute the ex-post efficient project choice and the pivotal transfers for the following type profile:



| | $\theta_1$ | $\theta_2$ | $\theta_3$ |
|---|---|---|---|
| $v_1$ | 3 | 4 | 1 |
| $v_2$ | 4 | 5 | 6 |



It can be checked that the ex-post efficient allocation is given by $k^*(\theta) = (1,1,0)$. The corresponding pivotal transfers are:

$$t_1(\theta) = 4 - 6 = -2,$$

$$t_2(\theta) = 3 - 6 = -3,$$

$$t_3(\theta) = 7 - 7 = 0.$$

Note in particular that agents 1 and 2 both receive a single identical object yet they pay different amounts under the pivotal mechanism.

### 3.5 Package Auctions, Example 2

Two distinct objects $A$ and $B$ are to be allocated among two agents. Each agent can consume no object, only $A$, only $B$, or both $A$ and $B$. The types of each agent is given by $\Theta_i = \{(v_A, v_B, v_{AB}) \in \mathbb{R}^3_+ | v_A, v_B \leq v_{AB}\}$ where $v_o$ denotes utility of consuming just the object $o \in \{A, B\}$ and $v_{AB}$ is the utility of consuming both objects. A possible allocation must specify the distribution of the objects to the agents, that is $k = (k_1, k_2)$ where $k_i \in \{\emptyset, A, B, AB\}$ is the allocation of agent $i$. Hence

$$K = \{(AB, \emptyset), (A, B), (B, A), (\emptyset, AB)\}.$$

For the following type profile:

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $v_A$ | 0 | 9 |
| $v_B$ | 0 | 10 |
| $v_{AB}$ | 12 | 10 |

,

it can be checked that the ex-post efficient project choice is given by $k^*(\theta) = (AB, \emptyset)$. The corresponding pivotal transfers are:

$$t_1(\theta) = 0 - 10 = -10,$$



$$t_2(\theta) = 12 - 12 = 0.$$

The seller's revenue is 10.



# 4 Optional Additional Reading

Chapter 23 of **MWG**.