Walrasian Equilibrium with production

All sections last edited 15 October 2018.

Convex sets and concave functions

Convex combination of two vectors

Consider any two vectors z^0 and z^1. A weighted average of these two vectors is

$$z^{\lambda} = (1-\lambda)z^0 + \lambda z^1 \text{ , } 0 < \lambda < 1$$

Such averages where the weights are both strictly positive and add to 1 are called the convex combinations of z^0 and z^1.

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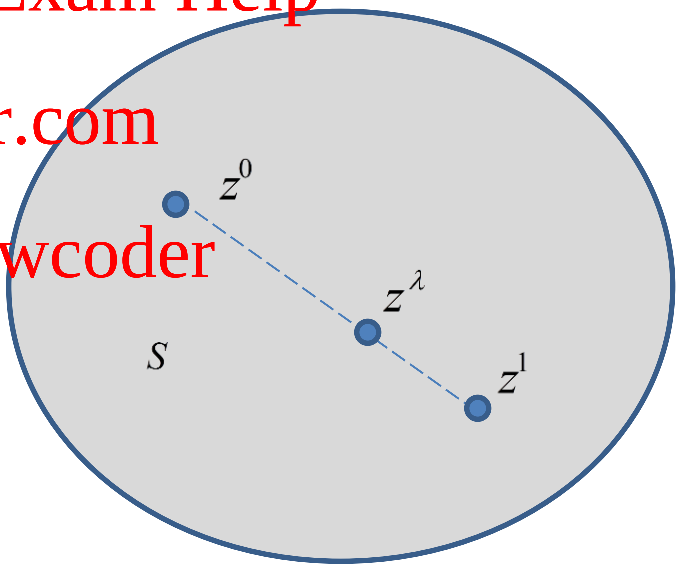
Such averages where the weighs are both strictly positive and add to 1 are called the convex combinations of z^0 and z^1.

Convex set

The set $S \subset \mathbb{R}^n$ is convex if for any z^0 and z^1 in *S*,

every convex combination is also in S



A convex set

Convex combination of two vectors

- - another view

Consider any two vectors z^0 and z^1.

The set of weighted average of these

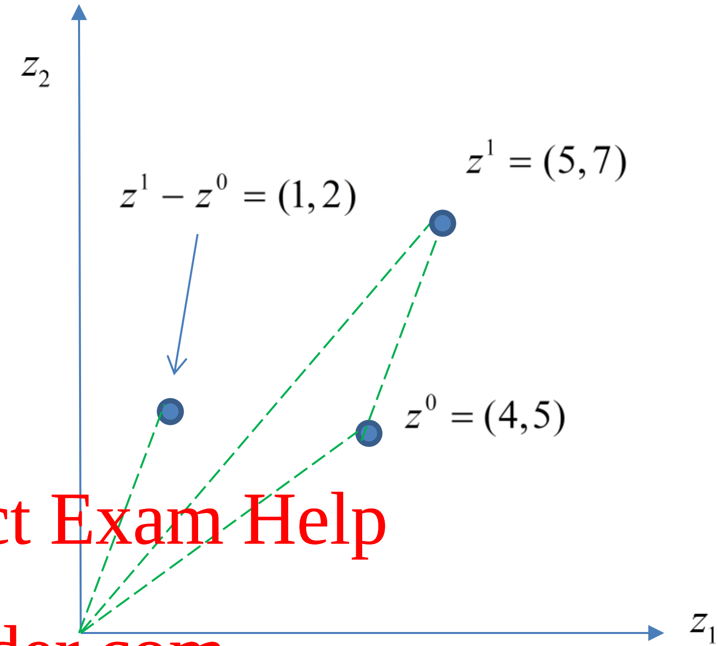
two vectors can be written as follows.

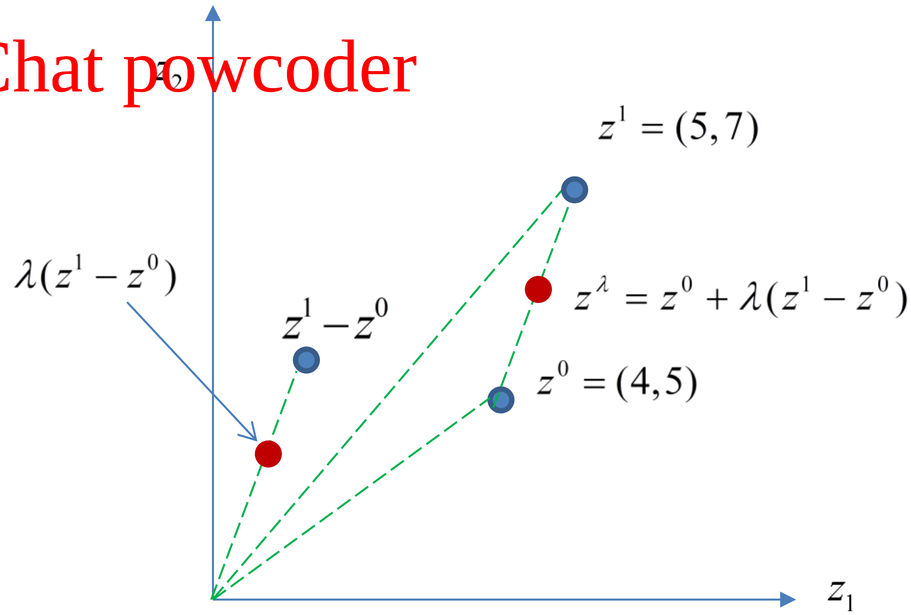
$$z^\lambda = (1-\lambda)z^0 + \lambda z^1 \text{ , } 0 < \lambda < 1$$

Rewrite the convex combination is follows:

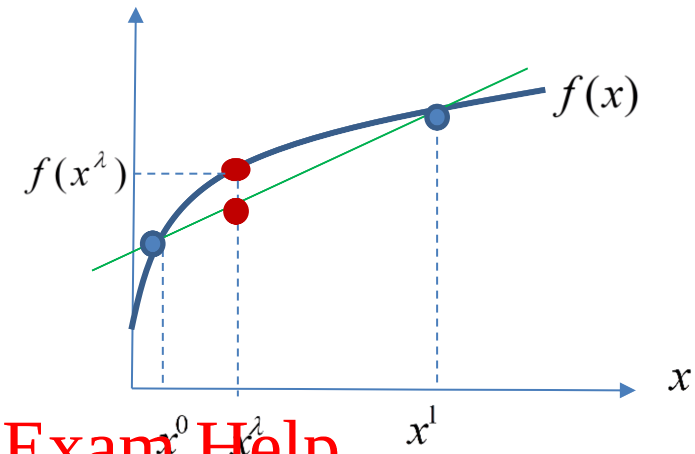
$$z^\lambda = z^0 + \lambda(z^1 - z^0)$$

The vector z^λ is a fraction λ of the way along the line segment connecting z^0 and z^1





Concave functions of 1 variable



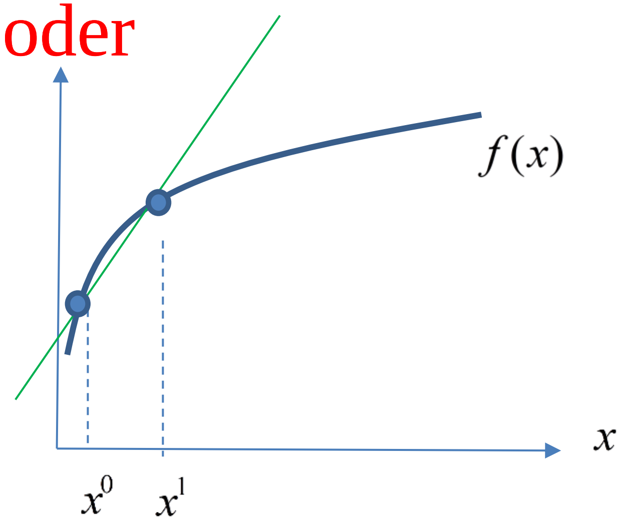
Definition 1: A function is concave if, for every x^0 and x^1, the graph of the function is above the line joining $(x^0, f(x^0))$ and $(x^1, f(x^1))$, i.e.

$$f(x^{\lambda}) \geq (1-\lambda) f(x^0) + \lambda f(x^1)$$

for every convex combination

$$x^{\lambda} = (1-\lambda)x^0 + \lambda x^1$$

Note that as the distance between x^1 and x^0 approaches zero, the line passing through two blue markers becomes the tangent line.

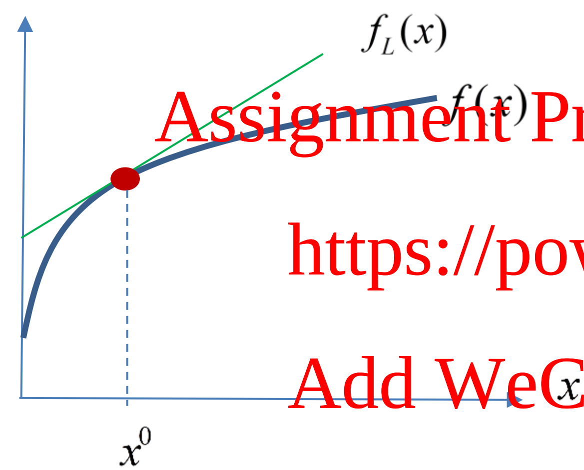


Tangent line is the linear approximation of the function f at x^0

$$f_L(x) \equiv f(x^0) + f'(x^0)(x - x^0) \ .$$

Note that the linear approximation has the same value at x^0 and the same first derivative (the slope.)

In the figure $f_L(x)$ is a line tangent to the graph of the function.



Definition 2: Differentiable concave function

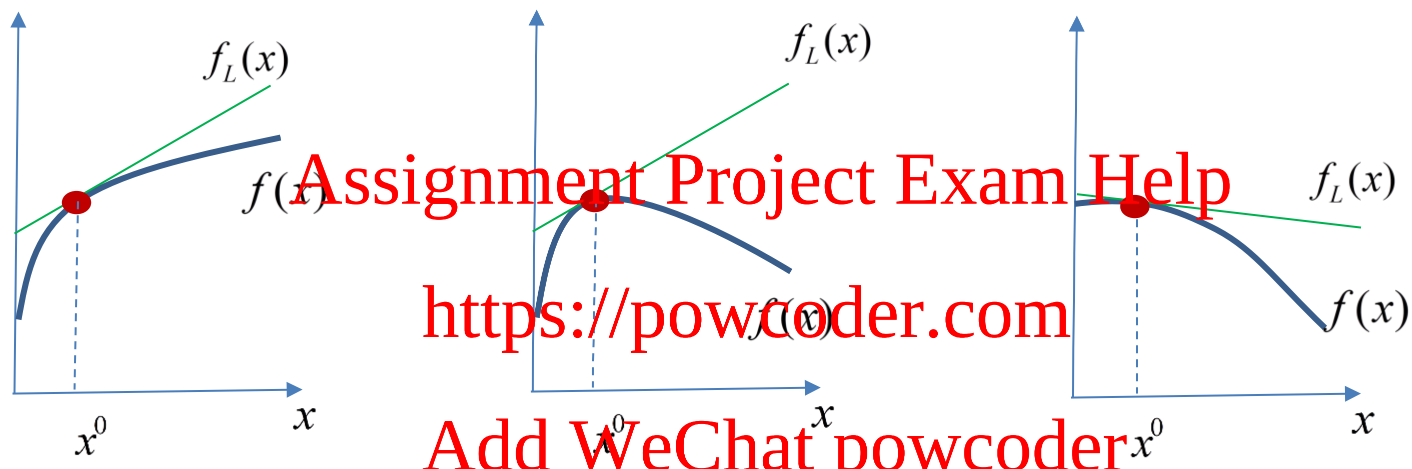
A differentiable function is concave if every tangent line is above the graph of the function. i.e.,

$$f(x) \le f(x^0) + f'(x^0)(x^1 - x^0)$$

Definition 3: Concave Function

A differentiable function f defined on an interval X is concave if $f'(x)$, the derivative of $f(x)$ is decreasing.

The three types of differentiable concave function are depicted below.



Note that in each case the linear approximations at any point x^0 lie above the graph of the function.

Concave function of n variables

Definition 1: A function is concave if, for every x^0 and x^1,

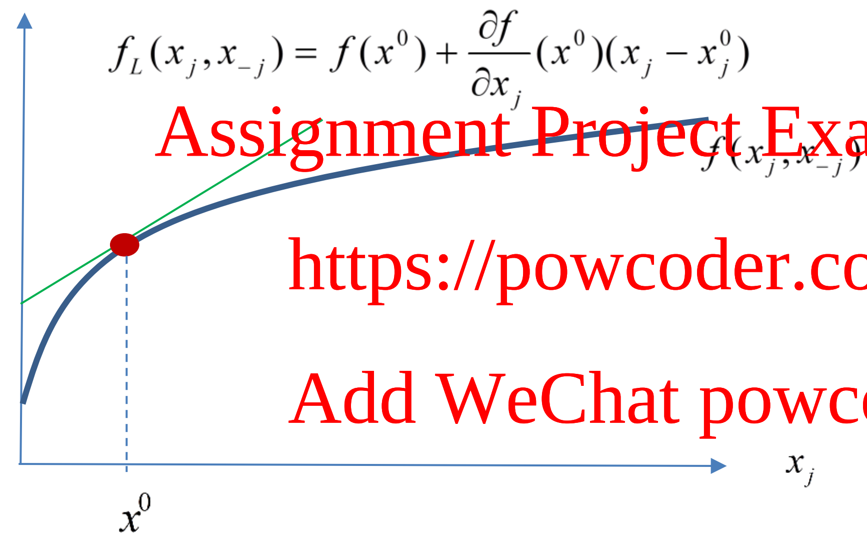
$$f(x^\lambda) \geq (1-\lambda) f(x^0) + \lambda f(x^1) \text{ for every convex combination } x^\lambda = (1-\lambda)x^0 + \lambda x^1,\ 0 < \lambda < 1$$

(Exactly the same as the definition when $n = 1$)

Linear approximation of the function f at x^0

$$f_L(x) \equiv f(x^0) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x^0)(x_j - x_j^0) \ .$$

Note that for each x_j the linear approximation has the same value at x^0 and the same first derivative (the slope.)



Definition 2: Differentiable Concave function

For any x^0 and x^1

$$f(x^1) \le f(x^0) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x^0)(x_j - x_j^0)$$

Group exercise: Appeal to one of these definitions to prove the first of the following important propositions.

Proposition

If $f(x)$ is concave, and $\bar{x}$ satisfies the necessary conditions for the maximization problem

$$\underset{x \geq 0}{Max}\{f(x)\}$$

then $\bar{x}$ solves the maximization problem.

Proposition

If $f(x)$ and $h(x)$ are concave, and $\bar{x}$ satisfies the necessary conditions for the maximization problem

$$\underset{x \geq 0}{Max}\{f(x) \mid h(x) \geq 0\}$$

then $\bar{x}$ is a solution of the maximization problem

Remark: This result continues to hold if there are multiple constraints $h_i(x) \geq 0$ and each function $h_i(x)$ is concave.

Concave functions of n variables

Proposition

1. The sum of concave functions is concave

2. If f is linear (i.e. $f(x) = a_0 + b \cdot x$) and g is concave then $h(x) = g(f(x))$ is concave.

3. An increasing concave function of a concave function is concave.

4. If $f(x)$ is homogeneous of degree 1 (i.e. $f(\theta x) = \theta f(x)$ for all $\theta > 0$) and the superlevel sets of $f(x)$ are convex, then $f(x)$ is concave.

Remark: The proof of 1-3 follows directly from the definition of a concave function. The proofs of 4 and 5 are very similar and more subtle. For the very few who may be interested, Proposition 4 is proved in a Technical Appendix.

Group exercise 1: Prove that the sum of concave functions is concave.

Group Exercise 2: If (i) $n = 1$ and (ii) both f and g are twice differentiable and concave and g is increasing, prove that $h(x) = g(f(x))$ is concave

Group Exercise 3: Prove the following result

Proposition: Concave functions have convex superlevel sets

If $f(x)$ is a concave function then the superlevel sets of $f(x)$ are convex sets. i.e.,

If x^0, x^1 are in the superlevel set $S = \{x \mid f(x) \geq k\}$ then every convex combination is in S .

Group Exercise 4: Output maximization with a fixed budget

A plant has the CES production function

$$F(z) = (z_1^{1/2} + z_2^{1/2})^2.$$

The CEO gives the plant manager a budget B and instructs her to maximize output. The input price vector is $r = (r_1, r_2)$. Solve for the maximum output $q(r, B)$. What is the firm's cost function?

Hint: Explain why $F(z)$ is concave.

2. Production sets and returns to scale (first 3 pages are a review)

Feasible plan

If an input-output vector (z,q) where $z=(z_1,....,z_m)$ and $q=(q_1,...,q_n)$ is a feasible plan if q can be produced using z.

Production set

The set of all feasible plans is called the firm's production set.

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Production sets

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Production function

If a firm produces one commodity the maximum output for some input vector z,

$$q=G(z)$$

is called the firm the firm's production function

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Production sets

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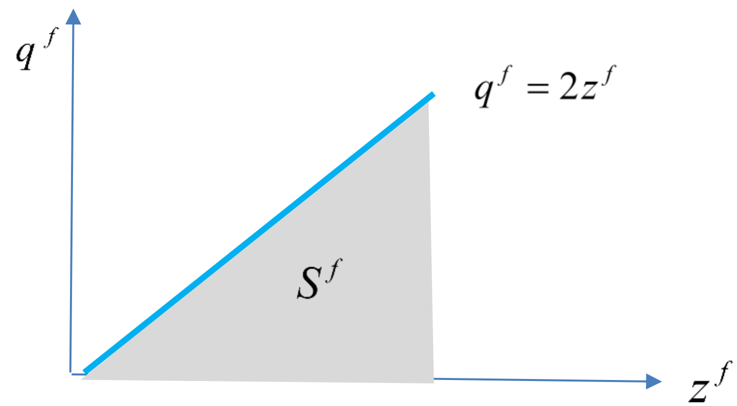
Example 1: One output and one input

$$S^f=\{(z^f,q^f)\}|0\le q_f\le 2z^f\}$$



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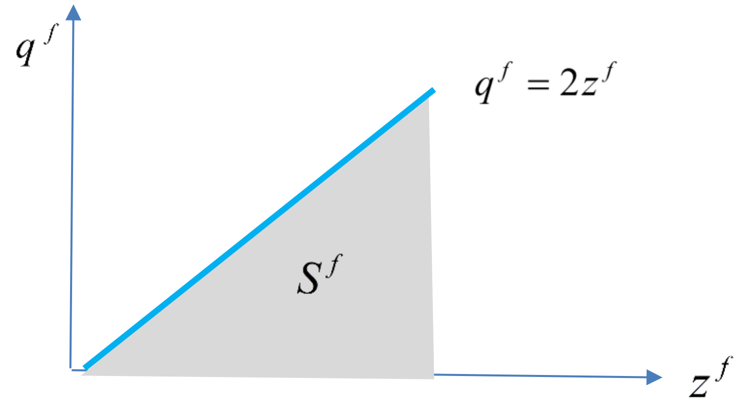

Note that the production function

$$q^f = G^f(z^f) = 2z_f$$

Is homogeneous of degree one

$$G^f(\theta z^f) = \theta^f G(z^f).$$

Such a firm is said to exhibit constant returns to scale

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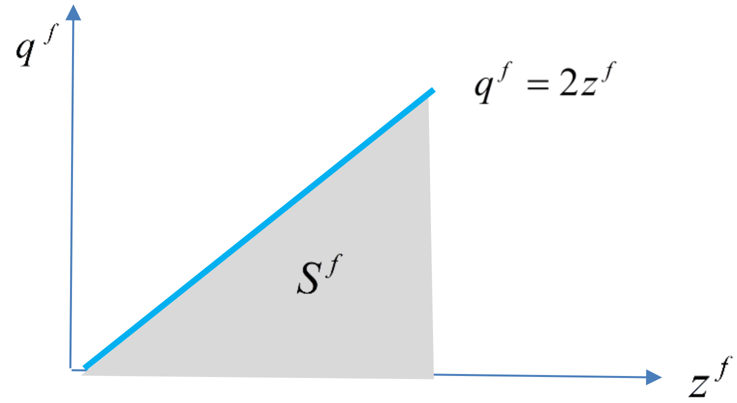

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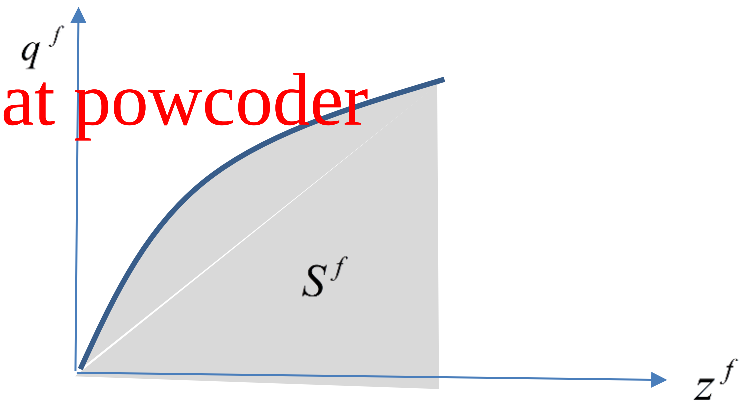
Example 2: One output and one input

$$S^f = \{(z^f, q^f) \ge 0 | q_f^2 \le z^f\}$$



Equivalently

$$S^f = \{(z^f, q^f) \ge 0 | q_f \le (z^f)^{1/2}\}$$

Group Exercise: Show that S^f is convex

Example 3: two inputs and one output

$$S^f = \{(z,q) \geq 0 \mid h^f(z,q) = A(z_1)^{1/3}(z_2)^{2/3} - q \geq 0\}$$

Note that the production function is concave (why?)

Hence $h(z,q)$ is concave. (why)

Remark: We have proved that the superlevel set of a concave function are convex so S^f is a convex set

Example 4: one input and two outputs

$$S^f = \{(z,q) \geq 0 \mid h^f(z,q) = z^2 - a_1 q_1^2 - a_2 q_2^2 \geq 0\}$$

Equivalently

$$S^f = \{(z,q) \geq 0 \mid h^f(z,q) = z - (a_1 q_1^2 + a_2 q_2^2)^{1/2} \geq 0\}$$

Class Exercise: Explain why S^f is a convex set

Aggregate production set

Let $\{S^f\}_{f=1}^F$ be the production sets of the F firms in the economy.

The aggregate production set is

$$S = S^1 + ... + S^F$$

That is

$(z,q) \in S$ if there exist feasible plans $\{(z^f, q^f)\}_{f=1}^F$ such that $(z,q) = \sum_{f=1}^{F} (z^f, q^f)$.

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Example 1: $S^f = \{(z^f, q^f) \geq 0 \,|\, 2z^f - q^f \geq 0\}$

In this simple case each unit of output requires 2 units of input so it does not matter whether one firm produces all the output or both produce some of the output. The aggregate production set is therefore $S = \{(z,q) \geq 0 \,|\, 2z - q \geq 0\}$.

Example 2: $S^f = \{(z^f, q^f) \mid (q^f)^2 \le z^f\}$

(a) Show that with two firms the aggregate production set is $S = \{(z,q) \mid q^2 \le 2z\}$

(b) What is the industry production set if there are 4 firms?

Group Exercise

HINT: The maximum output of the two firms is

$$q = Max\{q^1 + q^2 \mid (q^1)^2 \le z^1,\ (q^2)^2 \le z^2,\ z^1 + z^2 \le z\}.$$

Rather than use the Lagrange method with 3 constraints, note that for any z^1 and z^2 output is maximized by choosing q^1 and q^2 so that the first two inequalities are binding.

Method 1: Then $z^1 = (q^1)^2$ and $z^2 = (q^2)^2$. So the problem is reduced to a one constraint problem.

$$q = \underset{q^1,q^2}{Max}\{q^1 + q^2 \mid (q^1)^2 + (q^2)^2 \le z\}$$

Method 2: Since $q^f = (z^f)^{1/2}$ it follows that maximized output is

$$q = \underset{q^1,q^2}{Max}\{q^1 + q^2 = (z^1)^{1/2} + (z^2)^{1/2} \mid z^1 + z^2 \le z\}$$

HINT: In either approach the Necessary Conditions imply that $(z^1, q^1) = (z^2, q^2)$. Use this result to find a relationship between z and q.

Aggregation Theorem for price taking firms

Proposition: If there are 2 firms in an industry, prices are fixed and $(\bar{z}^f, \bar{q}^f)$ is profit maximizing for firm f, $f = 1,2$ then $(z,q) = (\bar{z}_1 + \bar{z}_2, \bar{q}_1 + \bar{q}_2)$ is industry profit-maximizing.

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<u>Proof</u>: Let Π^f be maximized profit of firm f Since the industry can mimic the two firms, industry profit cannot be lower. Suppose it is higher. Then for some feasible $(\hat{z}^f, \hat{q}^f)$, $f = 1,2$,

$$p \cdot (\hat{q}^1 + \hat{q}^2) - r \cdot (\hat{z}^1 + \hat{z}^2) > \bar{\Pi}^1 + \bar{\Pi}^2 .$$

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$$p \cdot (\hat{q}^1 + \hat{q}^2) - r \cdot (\hat{z}^1 + \hat{z}^2) > \bar{\Pi}^1 + \bar{\Pi}^2 .$$

Rearranging the terms,

$$(p \cdot \hat{q}^1 - r \cdot \hat{z}^1 - \bar{\Pi}^1) + (p \cdot \hat{q}^2 - r \cdot \hat{z}^2 - \bar{\Pi}^2) > 0$$

Then either

$$p \cdot \hat{q}^1 - r \cdot \hat{z}^1 > \bar{\Pi}^1 \text{ or } p \cdot \hat{q}^2 - r \cdot \hat{z}^2 > \bar{\Pi}^2$$

But then $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^1, \bar{q}^1)$ cannot both be profit-maximizing.

QED

Remark: Arguing in this way we can aggregate to the entire economy.

3. Walrasian equilibrium (WE) with Identical homothetic preferences & constant returns to scale

Consumer h has utility function $U(x_1^h, x_2^h) = x_1^h x_2^h$. The aggregate endowment is $\omega = (a,1)$. All firms have the same linear technology. Firm f can produce 2 units of commodity 2 for every unit of commodity 1. That is the production function of firm f is $q^f = 2z^f$

Then the aggregate production function is $q = 2z$.

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Then the aggregate production function is $q = 2z$.

Aggregate feasible set

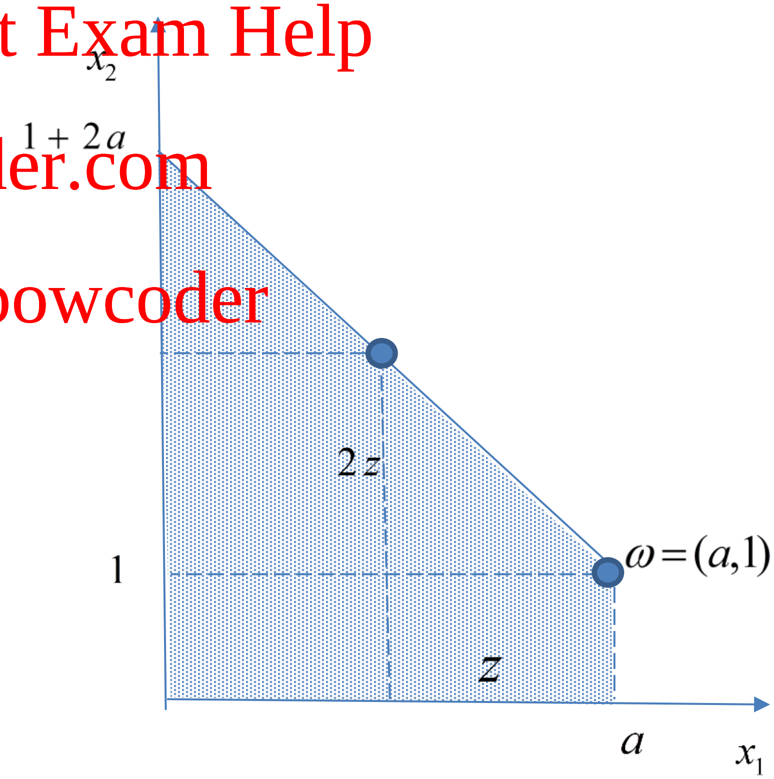
If the industry purchases z units of commodity 1

it can produce $q = 2z$ units of commodity 2

Then total supply of each commodity is

$$x = (a - z, 1 + 2z) \ .$$

This is depicted opposite.



Maximizing the utility of the representative consumer

Simply solve for the utility maximizing point

In the aggregate production set.

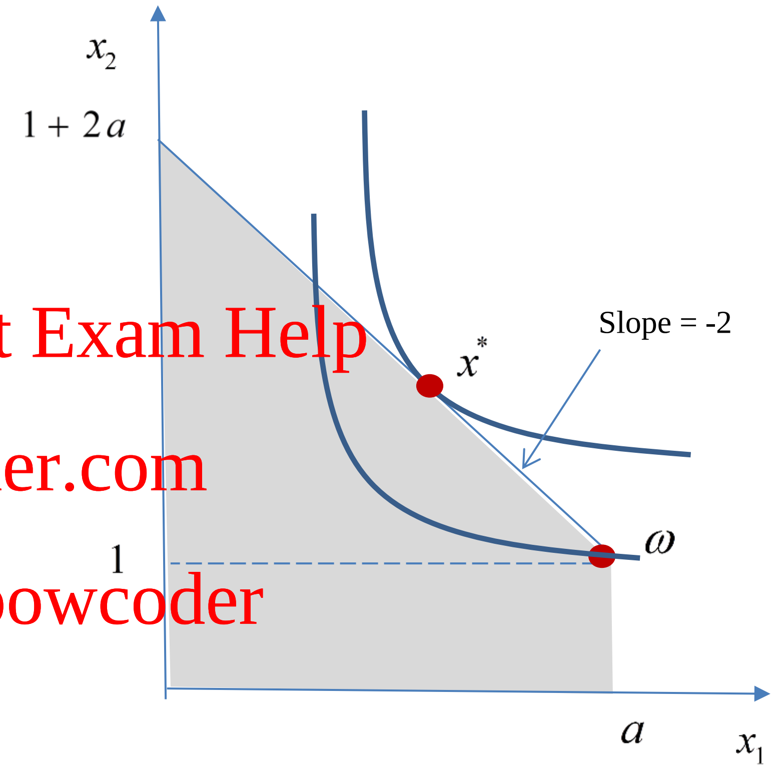
$$U(x_1^r, x_2^r) = x_1^r x_2^r = (a-z)(1+2z)$$

$$= a + (2a-1)z - 2z^2$$

$U'(z) = (2a-1) - 4z$.

Case (i) $a \geq \frac{1}{2}$. Then $\bar{z} = \frac{1}{4}(2a-1)$

Hence $\bar{x} = (a-\bar{z}, 1+2\bar{z}) = (\frac{1}{2}a + \frac{1}{4}, a + \frac{1}{2})$



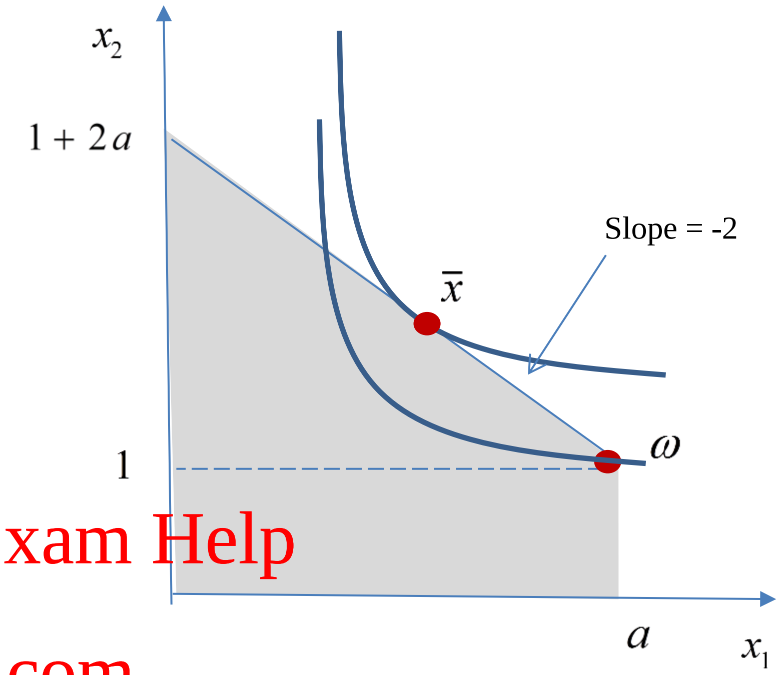
Case (ii) $a < \frac{1}{2}$. Then $\bar{z} = 0$

Hence $\bar{x} = (a, 1)$

Supporting prices

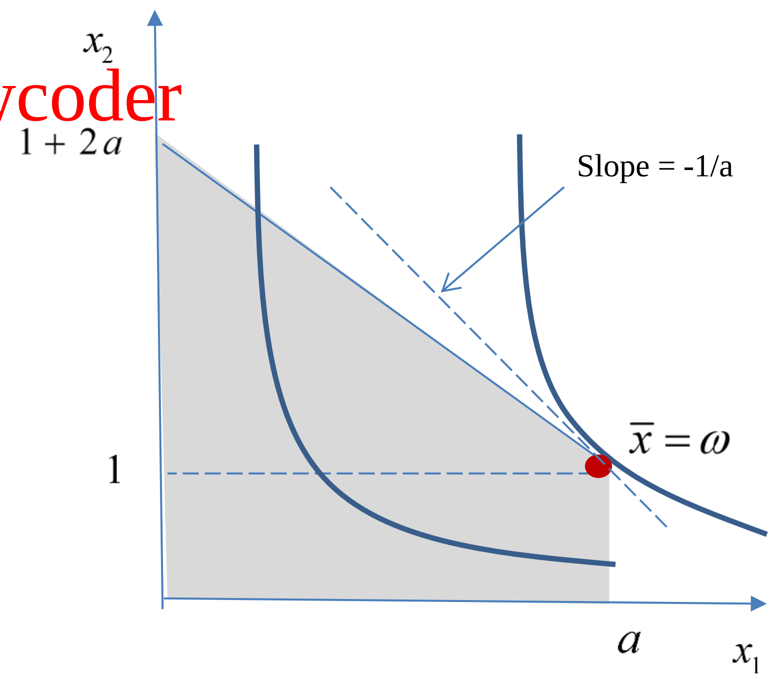
At what prices will the representative consumer not wish to trade?

Case 1: $\dfrac{p_1}{p_2} = MRS(\bar{x}) = \dfrac{\partial U}{\partial x_1}(\bar{x}) / \dfrac{\partial U}{\partial x_2}(\bar{x}) = \dfrac{\bar{x}_2}{\bar{x}_1} = 2$.



Case 2:

$$\frac{p_1}{p_2} = MRS(\bar{x}) = \frac{\partial U}{\partial x_1}(\bar{x}) / \frac{\partial U}{\partial x_2}(\bar{x}) = \frac{\bar{x}_2}{\bar{x}_1} = \frac{1}{a}$$



Profit maximization

The profit of firm f is

$$\Pi^f = p_2 q_2^f - p_1 z_1^f = p_2 2 z_1^f - p_1 z_1^f = z_1^f (2p_2 - p_1) .$$

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Profit maximization

The profit of firm f is

$$\Pi^f = p_2 q_2^f - p_1 z_1^f = p_2 2 z_1^f - p_1 z_1^f = z_1^f (2p_2 - p_1) \ .$$

If $\frac{p_1}{p_2} > 2$: the profit maximizing firm will purchase no inputs and so produce no output.

If $\frac{p_1}{p_2} < 2$: No profit maximizing plan

If $\frac{p_1}{p_2} = 2$: any input-output vector $(z_1, q_2) = (z_1, 2z_1)$ is profit maximizing.

Note that equilibrium profit must be zero.

Group Exercise: Why must Walrasian Equilibrium profit be zero if the production functions exhibits constant returns to scale?

Second example:

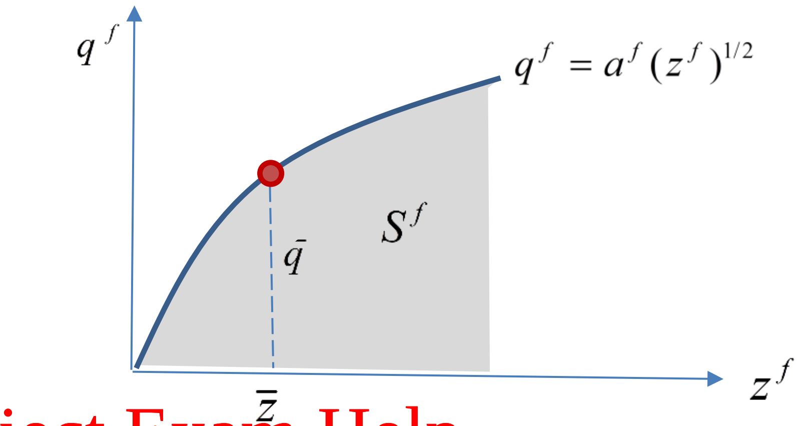
One output and one input

$$S^f = \{(z^f, q^f) \geq 0 \mid q^f \leq a^f (z^f)^{1/2}\}$$

There are two firms $(a^1, a^2) = (3,4)$

The aggregate endowment is $\omega = (12,0)$

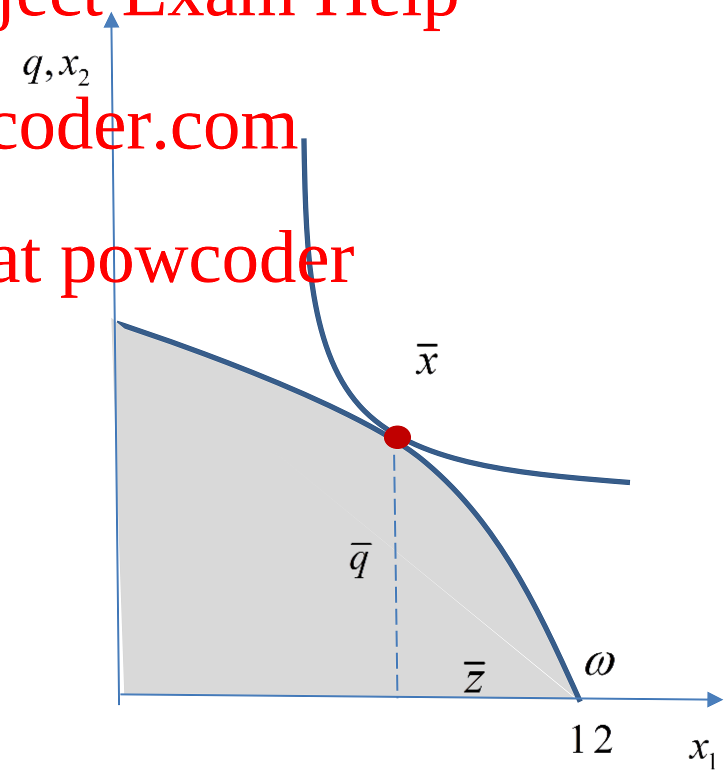
Consumer preferences are as in the previous example. $u(x) = \ln U(x) = \ln x_1 + \ln x_2$



Exercise

Show that the aggregate production set can be written as follows:

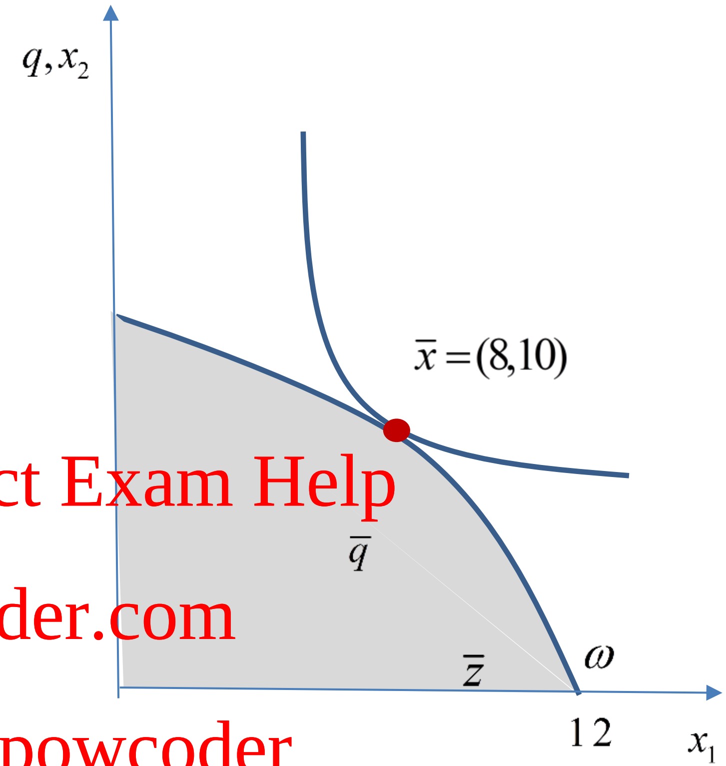
$$S = \{(z,q) \geq 0 \mid q \leq 5z^{1/2}\}$$



Step 1: Solve for the utility maximizing consumption

Step 2: Find prices that support the optimum

Step 3: Check to see if firms are profit maximizers



Step 1:

$$(x_1, x_2) = (\omega - z_1, q_2) = (12 - z_1, 5z_1^{1/2})$$

Define $u(x) = \ln U(x) = \ln x_1 + \ln x_2$

$$u = \ln(12 - z_1) + \ln(z_1^{1/2})$$

$$= \ln(12 - z_1) + \tfrac{1}{2}\ln z_1$$

Exercise: Why is $u(z_1)$ concave?

$$u'(z_1) = -\frac{1}{12 - z_1} + \frac{\frac{1}{2}}{z_1}$$

This has a unique critical point $\bar{z}_1 = 4$.

Then

$$(\bar{x}_1, \bar{x}_2) = (\omega - z_1, q_2) = (12 - z_1, 5z_1^{1/2}) = (8,10)$$

Step 2: Supporting the optimum

$$\frac{\partial u}{\partial x}(\bar{x})=(\frac{\partial u}{\partial x_1}(\bar{x}),\frac{\partial u}{\partial x_2}(\bar{x}))=(\frac{1}{\bar{x}_1},\frac{1}{\bar{x}_2})=(\frac{1}{8},\frac{1}{10})=\frac{1}{80}(10,8)\ .$$

Necessary conditions

$$\frac{\partial u}{\partial x}(\bar{x})=\lambda p\ .$$

Then $\frac{\partial u}{\partial x}(\bar{x})$ or any scalar multiple is a supporting price vector.

Hence $p=(10,8)$ is a supporting price vector

Step 3: Profit maximization

$$\pi = p_2 q_2 - p_1 z_1 = 8(5z^{1/2}) - 10 z_1$$

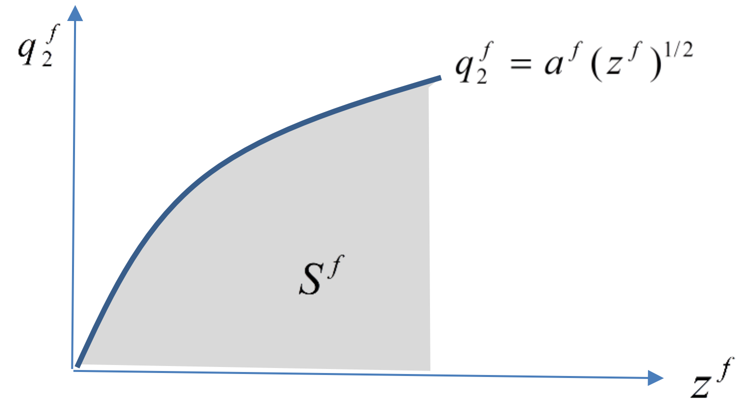
$$\pi'(z_1)=20z_1^{-1/2}-10=\frac{20}{z_1^{1/2}}-10\ .$$

So profit is maximized at $\bar{z}_1=4$ and maximized profit is $\pi(\bar{z}_1)=40$

Answer to exercise:

One output and one input

$$S^f = \{(z^f, q^f) \geq 0 \,|\, q^f \leq a^f (z^f)^{1/2}\}$$



There are two firms $(a^1, a^2) = (3, 4)$

(a) Show that the aggregate production set

can be written as follows:

$$S = \{(z, q) \geq 0 \,|\, q \leq 5z^{1/2}\}$$

If the allocation of the input to firm 1 is z^1, then maximized output is $q^1 = 3(z^1)^{1/2}$. Similarly $q^2 = 4(z^2)^{1/2}$ and so

$$q^1 + q^2 = 3(z^1)^{1/2} + 4(z^2)^{1/2}$$

Maximized industry output is therefore

$$q = Max\{q^1 + q^2 = 3(z^1)^{1/2} + 4(z^2)^{1/2} \,|\, z^1 + z^2 \leq z\}$$

The problem is concave so the necessary condition are sufficient. We look for a solution with $(z^1, z^2) >> 0$. The Lagrangian is

$$L = 3(z^1)^{1/2} + 4(z^2)^{1/2} + \lambda(z - z^1 - z^2)$$

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$$\frac{\partial L}{\partial q^1} = \frac{3}{2}(z^1)^{-1/2} - \lambda = 0\ , \quad \frac{\partial L}{\partial q^1} = \frac{4}{2}(z^1)^{-1/2} - \lambda = 0$$

Therefore

$$\frac{3}{(z^1)^{1/2}} = \frac{4}{(z^2)^{1/2}}$$

Squaring and appealing to the Ratio Rule,

$$\frac{9}{z^1} = \frac{16}{z^2} = \frac{25}{z^1 + z^2} = \frac{25}{z}$$

Therefore

$z^1 = \frac{9}{25}z$ and $z^2 = \frac{16}{25}z$ and so $q^1 = 3(z^1)^{1/2} = \frac{9}{5}z^{1/2}$ and $q^2 = 4(z^2)^{1/2} = \frac{16}{5}z^{1/2}$.

So $q = q^1 + q^2 = 5z^{1/2}$.

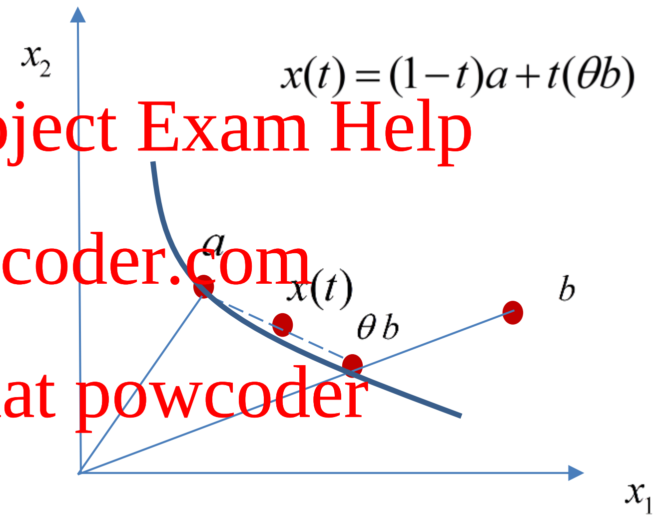
Technical Appendix: (Definitely ***not*** required material!)

Proposition: If $f(x)$ exhibits constant returns to scale and the superlevel sets of f are convex, then, for any non-negative vectors a and b, the function is super-additive, i.e.

$$f(a+b) \geq f(a)+f(b)$$

For some $\theta > 0$,

$$f(a) = f(\theta b) \underset{CRS}{=} \theta f(b)$$



Therefore

$$f(b) = \frac{1}{a} f(a)$$

Choose $t = \dfrac{1}{1+\theta}$. Then $1-t = 1-\dfrac{1}{1+\theta} = \dfrac{\theta}{1+\theta}$

Since a and θb are in the superlevel set, $S = \{x \mid f(x) \geq f(a)\}$

It follows that

$$f(x(t)) = f(\frac{\theta}{1+\theta}a + \frac{1}{1+\theta}\theta b) \geq f(a)$$

We have shown that

$$f(x(t)) = f(\frac{\theta}{1+\theta}a + \frac{1}{1+\theta}\theta b) \geq f(a), \quad \text{where } f(b) = \frac{1}{a}f(a) \qquad \text{(0-1)}$$

i.e.

$$f(\frac{\theta}{1+\theta}a + \frac{\theta b}{1+\theta}) = f(\frac{\theta}{1+\theta}(a+b)) \underset{CRS}{=} \frac{\theta}{1+\theta}f(a+b) \geq f(a)$$

Therefore

$$f(a+b) \geq \frac{1+\theta}{\theta}f(a) = \frac{1}{\theta}f(a) + f(a) = f(a) + \frac{1}{\theta}f(a)$$

Appealing to (0-1)

$$f(a+b) \geq f(a) + f(b). \qquad \text{QED}$$

Choose $a = (1-\lambda)x^0$ and $b = \lambda x^1$, Then

$$f((1-\lambda)x^0 + \lambda x^1) \geq f((1-\lambda)x^0) + f(\lambda x^1)$$

Appealing to constant returns to scale $f(\theta z) = \theta f(z)$. Therefore

$$f((1-\lambda)x^0+\lambda x^1)\geq(1-\lambda)f(x^0)+\lambda f(x^1)$$