Choice under uncertainty

Part 1

57 slides

1. Introduction to choice under uncertainty (two states)

Let X be a set of possible outcomes ("states of the world").

An element of X might be a consumption vector, health status, inches of rainfall etc.

Initially, simply think of each element of X as a consumption bundle. Let $\bar{x}$ be the most preferred element of X and let $\underline{x}$ be the least preferred element.

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Consumption prospects

Suppose that there are only two states of the world. $X = \{x_1, x_2\}$ Let π_1 be the probability that the state is x_1 so that $\pi_2 = 1 - \pi_1$ is the probability that the state is x_2.

We write this "consumption prospect" as follows:

$$(x;\pi) = (x_1, x_2; \pi_1, \pi_2)$$

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$$(x;\pi) = (x_1, x_2; \pi_1, \pi_2)$$

If we make the usual assumptions about preferences, but now on prospects, it follows that preferences over prospects can be represented by a continuous utility function

$$U(x_1, x_2, \pi_1, \pi_2)\ .$$

Prospect or "Lottery"

$$L = (x_1, x_2, ..., x_S; \pi_1, ..., \pi_S)$$

(outcomes; probabilities)

Consider two prospects or "lotteries", L_A and L_B

$$L_A = (x_1, x_2, ..., x_S; \pi_1^A, ..., \pi_S^A) \quad L_B = (c_1, c_2, ..., c_S; \pi_1^B, ..., \pi_S^B)$$

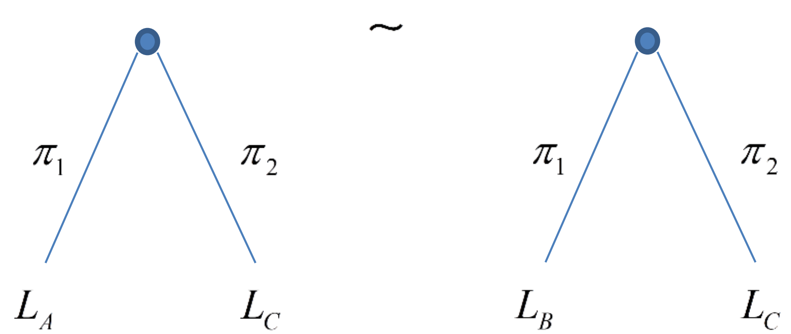
Independence Axiom (axiom of complex gambles).

Suppose that a consumer is indifferent between these two prospects (we write $L_A \sim L_B$).

Then for any probabilities π_1 and π_2 summing to 1 and any other lottery L_C

$$(L_A, L_C; \pi_1, \pi_2) \sim (L_B, L_C; \pi_1, \pi_2)$$

Tree representation



This axiom can be generalized as follows:

Suppose that a consumer is indifferent between the prospects L_A and L_B

and is also indifferent between the two prospects L_C and L_D ,

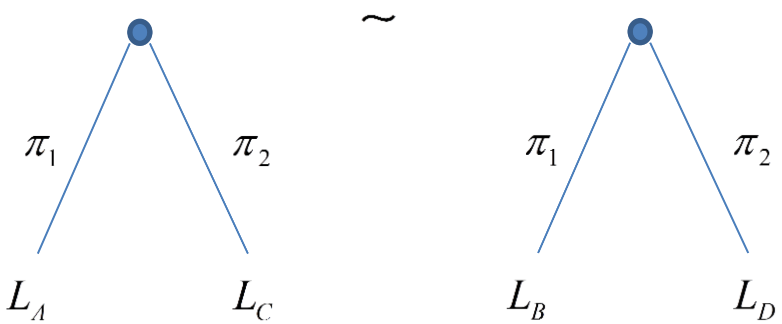
i.e. $L_A \sim L_B$ and $L_C \sim L_D$

Then for any probabilities π_1 and π_2 summing to 1,

$$(L_A, L_C; \pi_1, \pi_2) \sim (L_B, L_D; \pi_1, \pi_2)$$

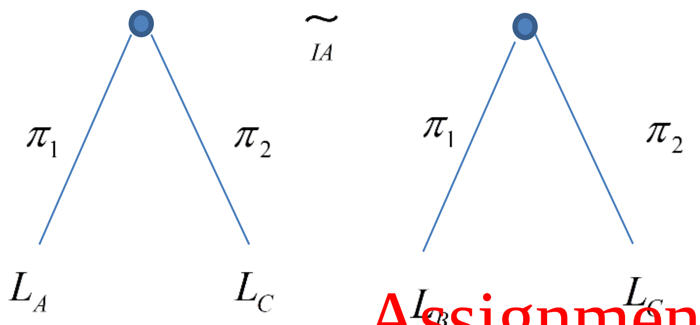
Tree representation

We wish to show that if $L_A \sim L_B$ and $L_C \sim L_D$ then



Proof: $L_A \sim L_B$ and $L_C \sim L_D$

Step 1: By the Independence Axiom, since $L_A \sim L_B$



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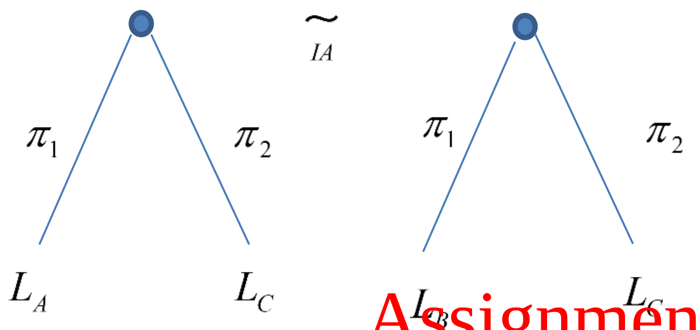
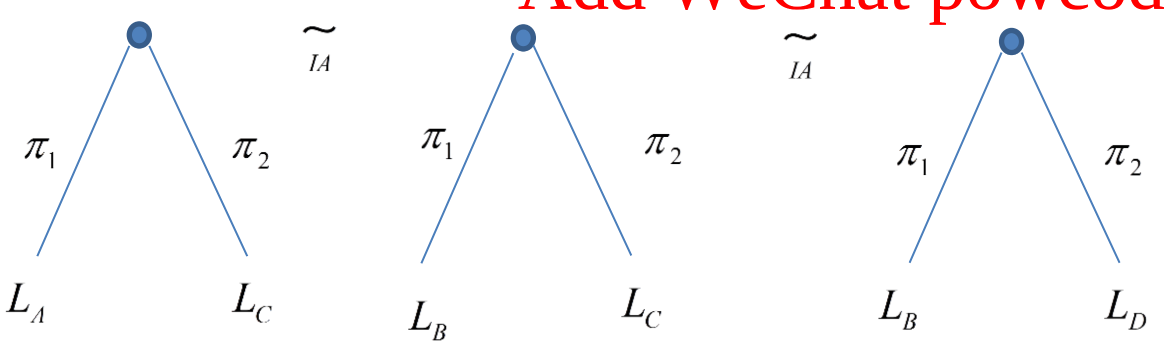


Step 2: By the Independence Axiom, since $L_C \sim L_D$



Expected utility

Consider some very good outcome $\bar{x}$ and very bad outcome $\underline{x}$ and outcomes x_1 and x_2 satisfying

$\underline{x} \prec x_1 \prec \bar{x}$ and $\underline{x} \prec x_2 \prec \bar{x}$

Reference lottery

$L_R(v) = (\bar{w}, \underline{w}, v, 1-v)$ so v is the probability of the very good outcome.

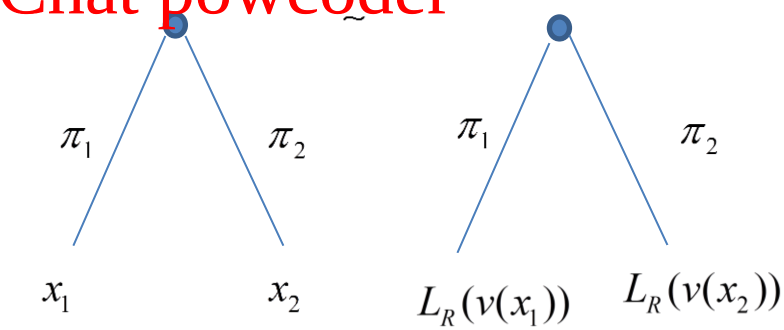
$L_R(0) \prec x_1 \prec L_R(1)$ and $L_R(0) \prec x_2 \prec L_R(1)$

Then for some probabilities $v(x_1)$ and $v(x_2)$

$x_1 \sim L_R(v(x_1)) = (\bar{x}, \underline{x}; v(x_1), 1-v(x_1))$ and $x_2 \sim L_R(v(x_2)) = (\bar{x}, \underline{x}; v(x_2), 1-v(x_2))$

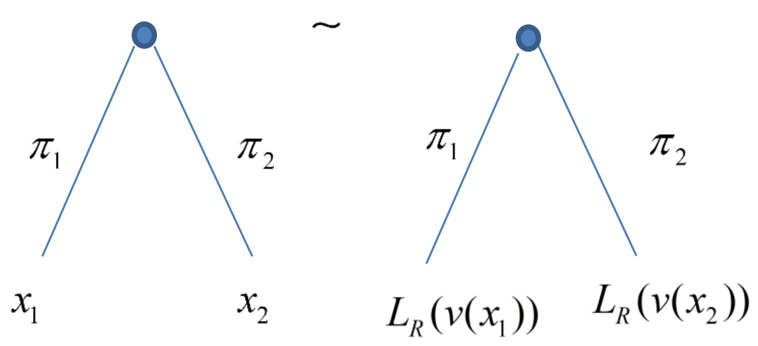
Then by the independence axiom

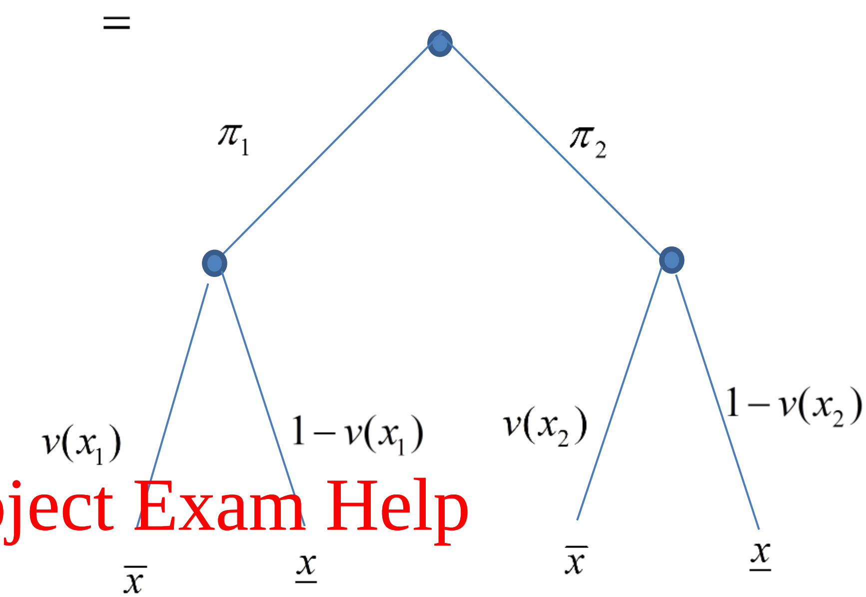
$(x_1, x_2; \pi_1, \pi_2) \sim (L_R(v(x_1)), L_R(v(x_2)); \pi_1, \pi_2)$



Definition: Expectation of $v(x)$

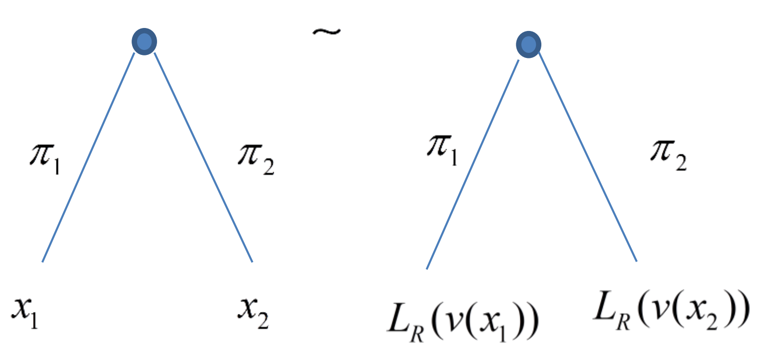
$$\mathbb{E}[v(x)] \equiv \pi_1 v(x_1) + \pi_2 v(x_2)$$

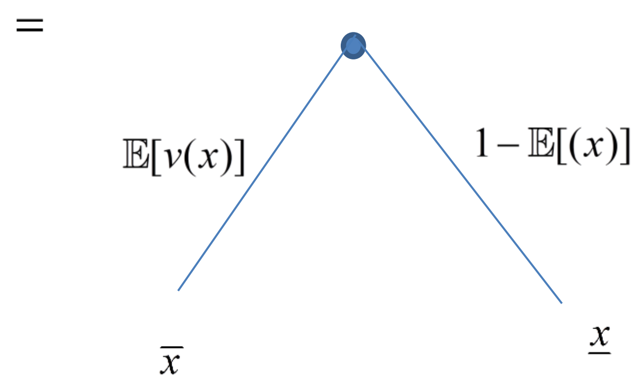




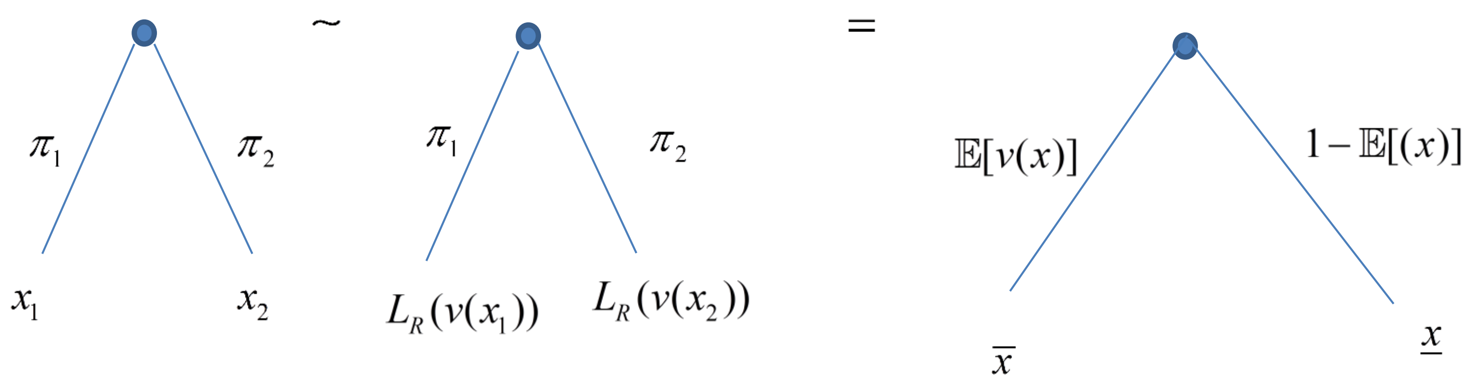
Note that in the big tree there are only two outcomes, $\bar{x}$ and $\underline{x}$. The probability of the very good outcome is $\pi_1 v(x_1) + \pi_2 v(x_2) = \mathbb{E}[v(x)]$

The probability of the very bad outcome is $1-\mathbb{E}[v(x)]$. Therefore





We showed that



i.e.

$$(x_1, x_2; \pi_1, \pi_2) \sim (\bar{x}, \underline{x}; \mathbb{E}[v], 1-\mathbb{E}[v])$$

Thus the expected win probability in the reference lottery is a representation of preferences over prospects.

An example:

A consumer with wealth $\hat{w}$ is offered a "fair gamble" . With probability $\frac{1}{2}$ his wealth will be $\hat{w}+x$ and with probability $\frac{1}{2}$ his wealth will be $\hat{w}-x$. If he rejects the gamble his wealth remains $\hat{w}$. Note that this is equivalent to a prospect with $x=0$

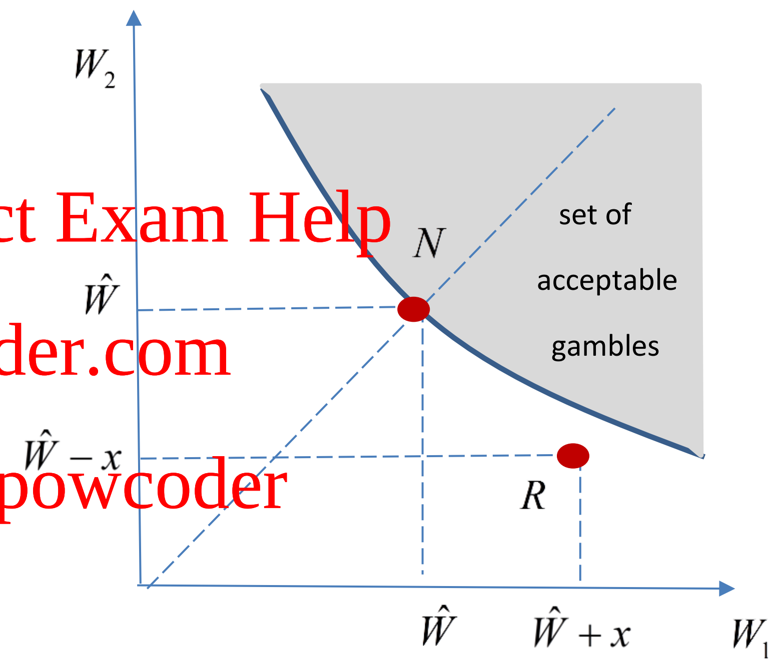
In prospect notation the two alternatives are

$$(w_1, w_2; \pi_1, \pi_2) = (\hat{w}, \hat{w}; \tfrac{1}{2}, \tfrac{1}{2})$$

and

$$(w_1, w_2; \pi_1, \pi_2) = (\hat{w}+x, \hat{w}-x; \tfrac{1}{2}, \tfrac{1}{2}).$$

These are depicted in the figure assuming $x > 0$.



Expected utility

$$U(w_1, w_2, \pi_1, \pi_2) = \mathbb{E}[v] = \pi_1 v(w_1) + \pi_2 v(w_2)$$

Class discussion

MRS if $v(w)$ is a concave function

Convex preferences

The two prospects are depicted opposite.

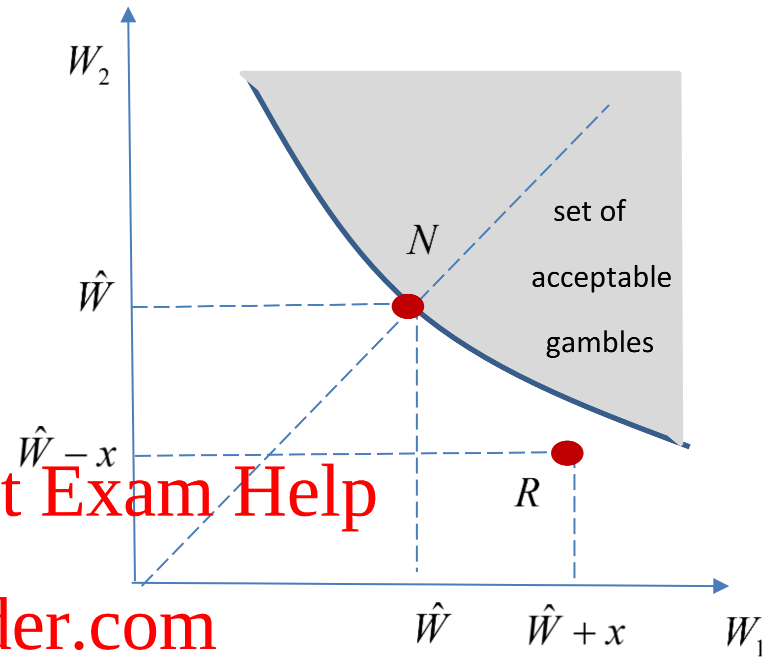
The level set for $U(w_1, w_2; \frac{1}{2}, \frac{1}{2})$ through the riskless

prospect N is depicted.

Note that the superlevel set

$$U(w_1, w_2; \tfrac{1}{2}, \tfrac{1}{2}) \geq U(\hat{w}, \hat{w}; \tfrac{1}{2}, \tfrac{1}{2})$$

is a convex set.



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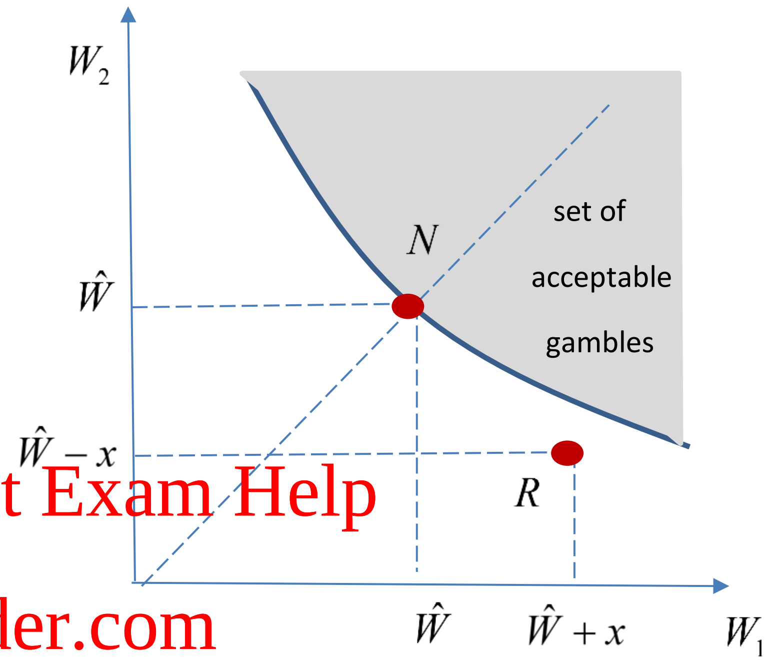

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is a convex set.

This is the set of acceptable gambles for the consumer.

As depicted the consumer strictly prefers the riskless prospect N to the risky prospect R .

Most individuals, when offered such a gamble (say over $\$5$) will not take this gamble.

2. Risk aversion

Class Discussion: Which alternative would you choose?

$N: \ (w_1, w_2; \pi_1, \pi_2) = (\hat{w}, \hat{w}; \pi_1, \pi_2)$ $\qquad$ $R: \ (w_1, w_2; \pi_1, \pi_2) = (\hat{w}+x, \hat{w}-x; \pi_1, \pi_2)$ where $\pi_1 = \dfrac{50}{100}$

What if the gamble were "favorable" rather than "fair"

$R: \quad (w_1, w_2; \pi_1, \pi_2) = (\hat{w}+x, \hat{w}-x; \pi_1, \pi_2)$ where (i) $\pi_1 = \dfrac{55}{100}$ (ii) $\pi_1 = \dfrac{60}{100}$ (iii) $\pi_1 = \dfrac{75}{100}$

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What is the smallest integer n such that you would gamble if $\pi_1 = \frac{n}{100}$?

Preference elicitation

In an attempt to elicit your preferences write down your number n (and your first name) on a piece of paper. The two participants with the lowest number n will be given the riskless opportunity.

Let the three lowest integers be n_1, n_2, n_3 . The win probability will not be $\frac{n_1}{100}$ or $\frac{n_2}{100}$. Both will get the higher win probability $\frac{n_3}{100}$.

2. Risk preferences

$$U(x,\pi)=\pi_1 v(x_1)+\pi_2 v(x_2) \quad \text{or} \quad U(x,\pi)=\mathbb{E}[v]$$

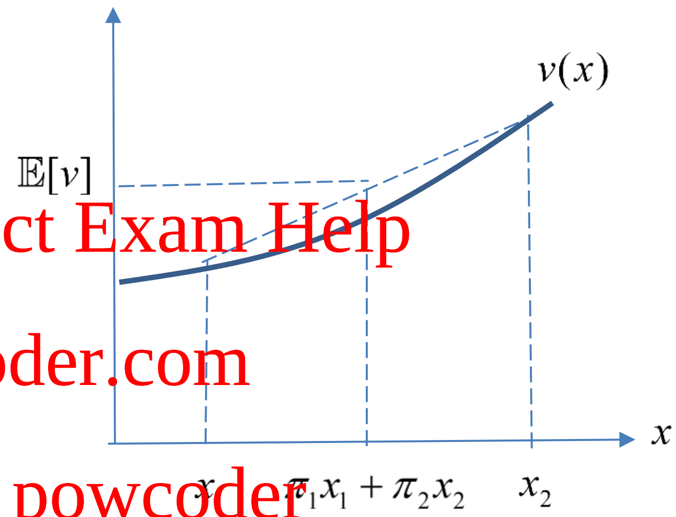
Risk preferring consumer

Consider the two wealth levels x_1 and $x_2 > x_1$.

$$v(\pi_1 x_1+\pi_2 x_2) < \pi_1 v(x_1)+\pi_2 v(x_2)$$

If $v(x)$ is convex, then the slope of $v(x)$

is strictly increasing as shown in the top figure.



Consumer prefers risk

$$U(x,\pi)=\pi_1 v(x_1)+\pi_2 v(x_2)$$

Risk averse consumer

$$v(\pi_1 x_1+\pi_2 x_2)>\pi_1 v(x_1)+\pi_2 v(x_2)\ .$$

In the lower figure $u(x)$ is strictly concave so that

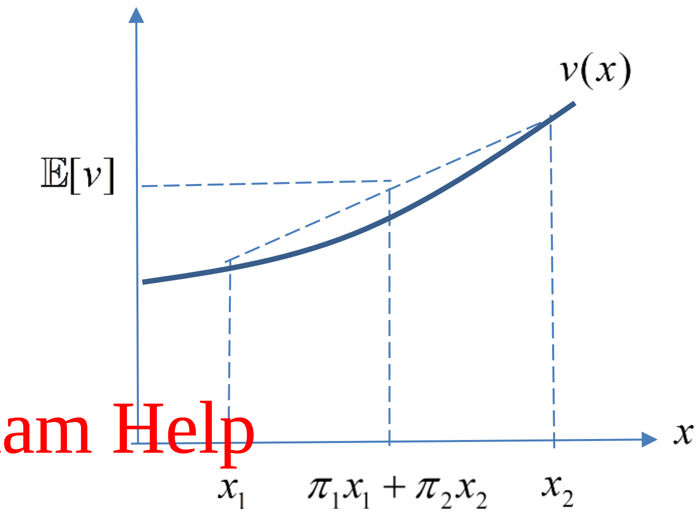
$$v(\pi_1 x_1+\pi_2 x_2)>\pi_1 v(x_1)+\pi_2 v(x_2)=\mathbb{E}[v].$$

In practice consumers exhibit aversion to such a risk.

Thus we will (almost) always assume that the

expected utility function $v(x)$ is a strictly increasing

strictly concave function.



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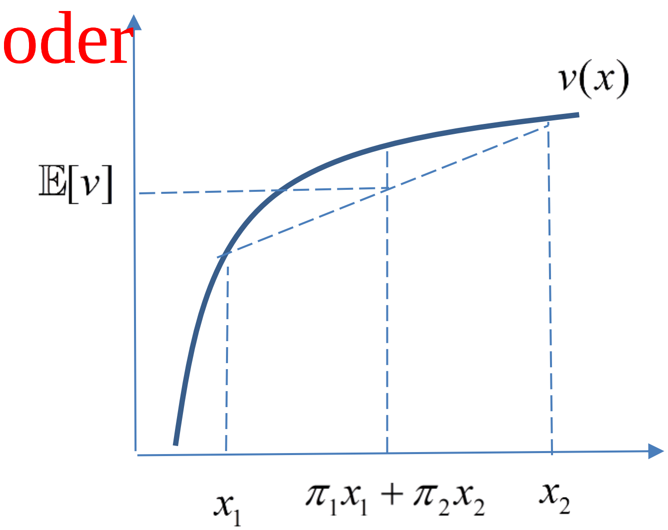

Risk averse consumer

Class Discussion:

If consumers are risk averse why do they go to Las Vegas?

3. Acceptable gambles: Improving the odds to make the gamble just acceptable.

New risky alternative: $(w_1, w_2; \pi_1, \pi_2) = (\hat{w}+x, \hat{w}-x; \tfrac{1}{2}+\alpha, \tfrac{1}{2}-\alpha)$.

Choose α so that the consumer is indifferent between gambling and not gambling.

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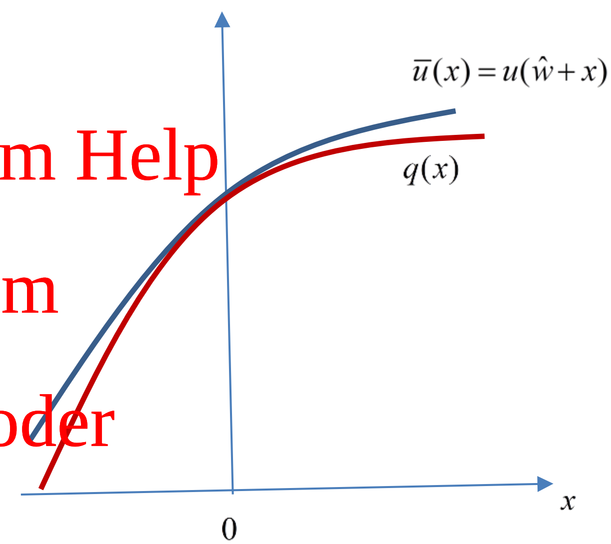
For small x we can use the quadratic approximation of the utility function

Quadratic approximation of his utility



As long as x is small we can approximate his utility

as a quadratic. Suppose $u(w+x) = \ln(w+x)$.

Define $\bar{u}(x) = \ln(w+x)$.

Then (i) $\bar{u}(0) = \ln w$ (ii) $\bar{u}'(0) = \dfrac{1}{w}$ and (iii) $\bar{u}''(0) = -\dfrac{1}{w^2}$

Consider the quadratic function

$$q(x) = \ln w + (\frac{1}{w})x - \tfrac{1}{2}(\frac{1}{w^2})x^2 \ . \qquad (3.1)$$

If you check you will find that $\bar{u}(x)$ and $q(x)$ have the same, value, first derivative and second derivative at $x = 0$. We then use this quadratic approximation to compute the gambler's (approximated) expected gain.

With probability $\frac{1}{2}+\alpha$ his payoff is $q(x)$ and with probability $\frac{1}{2}-\alpha$ his payoff is $q(-x)$. Therefore his expected payoff is

$$\mathbb{E}[q(x)]=(\tfrac{1}{2}+\alpha)q(x)+(\tfrac{1}{2}-\alpha)q(-x)$$

Substituting from (3.1)

$$\mathbb{E}[q(x)]=(\tfrac{1}{2}+\alpha)[\ln w+(\frac{1}{w})x-\tfrac{1}{2}(\frac{1}{w^2})x^2$$

$$+(\tfrac{1}{2}-\alpha)[\ln w+(\frac{1}{w})(-x)-\tfrac{1}{2}(\frac{1}{w^2})(-x)^2 .$$

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Collecting terms,

$$\mathbb{E}[q(x)]=\ln w+2\alpha(\frac{1}{w})x-\tfrac{1}{2}(\frac{1}{w^2})x^2.$$

If the gambler rejects the opportunity his utility is $\ln w$. Thus his expected gain is

$$\mathbb{E}[q(x)]-\ln w=2\alpha(\frac{1}{w})x-\tfrac{1}{2}(\frac{1}{w^2})x^2=\frac{2x}{w}[\alpha-\tfrac{1}{4}(\frac{1}{w})x].$$

Thus the gambler should take the small gamble if and only if $\alpha>\frac{1}{4}(\frac{1}{w})x$.

The general case: quadratic approximation of his utility

$$q(x) = v(\hat{w}) + v'(\hat{w})x + \tfrac{1}{2}v''(\hat{w})x^2$$

Class Exercise: Confirm that the value and the first two derivatives of $\bar{u}(x) \equiv v(\hat{w}+x)$ and $q(x)$ are equal at $x = 0$.

The expected value utility of the risky alternative is

$$\mathbb{E}[u(\hat{w}+x)] \approx \mathbb{E}[q(x)] = (\tfrac{1}{2}+\alpha)q(x) + (\tfrac{1}{2}-\alpha)q(-x)$$

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$$+(\tfrac{1}{2}-\alpha)[v(\hat{w}) + v'(\hat{w})(-x) - \tfrac{1}{2}v''(\hat{w})(-x)^2$$

Collecting terms,

$$\mathbb{E}[q(x)] = v(\hat{w}) + 2\alpha v'(\hat{w})x - \tfrac{1}{2}v''(\hat{w})x^2.$$

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$$+(\tfrac{1}{2}-\alpha)[v(\hat{w}) + v'(\hat{w})(-x) + \tfrac{1}{2}v''(\hat{w})(-x)^2]$$

Collecting terms,

$$\mathbb{E}[q(x)] = v(\hat{w}) + 2\alpha v'(\hat{w})x + \tfrac{1}{2}v''(\hat{w})x^2.$$

The gain in expected utility is therefore

$$\mathbb{E}[q(x)] - v(\hat{w}) = 2\alpha v'(\hat{w})x + \tfrac{1}{2}v''(\hat{w})x^2$$

$$= 2v'(\hat{w})x[\alpha - \tfrac{1}{4}(-\frac{v''(\hat{w})}{v'(\hat{w})})x]$$

Thus the probability of the good outcome must be increased by $\alpha = \tfrac{1}{4}(-\frac{v''(\hat{w})}{v'(\hat{w})})x$.