1. Level sets, superlevel sets and consumer choice

Alex has utility function $U^A(x)=a_1v_A(x_1)+a_2v_A(x_2)$ where $v_A(\cdot)$ is a differentiable strictly concave function.

Bev has utility function $U^B(x)=a_1v_B(x_1)+a_2v_B(x_2)=a_1h(v_A(x_1))+a_2h(v_A(x_2))$ where $h(\cdot)$ is a differentiable strictly concave function.

(a) Show that if $x_1=x_2$ then $MRS_A(x)=MRS_B(x)=\dfrac{a_1}{a_2}$.

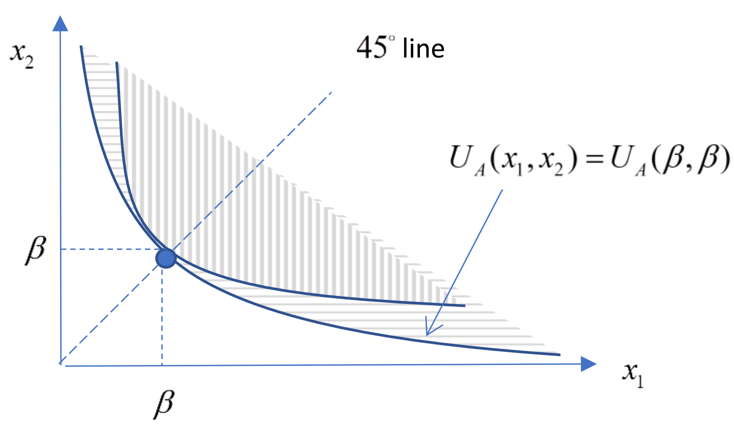
(b) Show that if $x_1>x_2$ then $\dfrac{a_1}{a_2}>MRS_A(x)>MRS_B(x)$

and that if $x_1<x_2$ then $\dfrac{a_1}{a_2}<MRS_A(x)<MRS_B(x)$.

(c) Show that if the two consumers have the same endowment and the prices satisfy $\dfrac{p_1}{p_2}<\dfrac{a_1}{a_2}$, then the choices of the two consumers $\overline{x}^A$ and $\overline{x}^B$ satisfy $\dfrac{\overline{x}_2^A}{\overline{x}_1^A}<\dfrac{\overline{x}_2^B}{\overline{x}_1^B}<1$.

(d) Is it also true that when $\dfrac{p_1}{p_2}>\dfrac{a_1}{a_2}$, then the choices of the two consumers $\overline{x}^A$ and $\overline{x}^B$ satisfy $\dfrac{\overline{x}_2^A}{\overline{x}_1^A}>\dfrac{\overline{x}_2^B}{\overline{x}_1^B}>1$.

(e) Explain why it must be the case that, as depicted, the vertically lined superlevel set for Bev, $S^B=\{x\,|\,U^B(x)\geq U^B(\beta,\beta)\}$ is a subset of the vertically and horizontally lined superlevel set for Alex, $S^A=\{x\,|\,U^A(x)\geq U^A(\beta,\beta)\}$.



2. Pareto efficient Allocations

Alex has utility function $U^A(x) = \pi_1 v_A(x_1) + \pi_2 v_A(x_2)$ where $v_A(\cdot)$ is a differentiable strictly increasing, strictly concave function. Bev has utility function $U^B(x) = \pi_1 v_B(x_1) + \pi_2 v_B(x_2)$ where $v_B(\cdot)$ is a differentiable strictly increasing, strictly concave function. Let $\omega^A > 0$ and $\omega^B > 0$ be the individual endowments and let $\omega = (\omega_1, \omega_2) >> 0$ be the aggregate endowment.

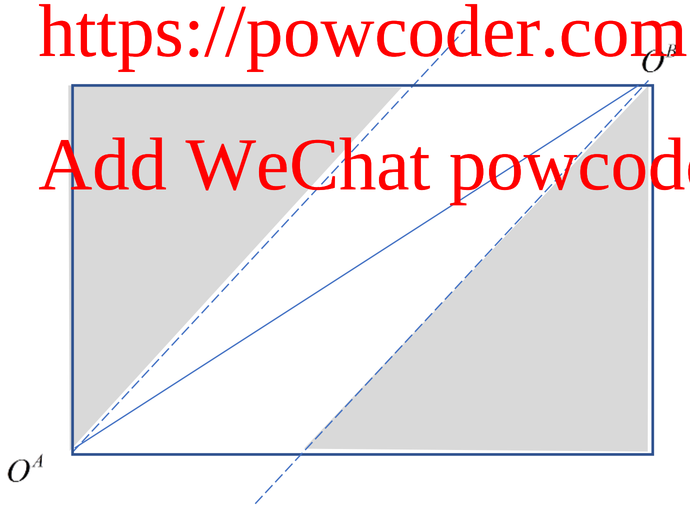
For parts (a) and (b) $\omega_1 = \omega_2$.

(a) Explain why the PE allocations are on the diagonal of the square Edgeworth Box.

(b) Explain why a price vector that supports a PE allocation must have a ratio $\dfrac{p_1}{p_2} = \dfrac{a_1}{a_2}$.

Henceforth $\omega_1 > \omega_2$

(c) Explain why any PE allocation in which $x^A > 0$ and $x^B > 0$ must be between the two dotted 45° lines and not in the shaded regions depicted below.



Edgeworth Box with dotted 45° lines

(d) Suppose that the utility function for Alex is $U^A(x) = a_1 x_1^{1/2} + a_2 x_2^{1/2}$ and for Bev is $U^B(x) = a_1 h(x_1^{1/2}) + a_2 h(x_2^{1/2})$ where $h(\cdot)$ is a differentiable strictly concave function.

Show that that along the diagonal the marginal rates of substitution are different. Must the PE allocations must all lie below the diagonal hence closer to the dotted 45° line of Bev than that for Alex?

3. Walrasian Equilibrium Allocations

Alex and Bev have the utility functions of question 2(a)-(c). The aggregate endowment is $\omega = (\omega_1, \omega_2)$.

(a) If $\frac{p_1}{\pi_1} \geq \frac{p_2}{\pi_2}$ explain clearly why the sum of Alex and Bev's demand for commodity 2 exceeds the sum of demands for commodity 1.

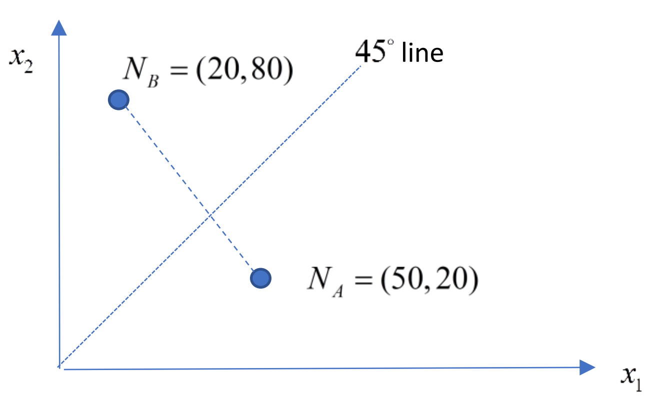
(b) Hence show that if $\omega_1 > \omega_2$, then the WE price ratio $\frac{p_1}{p_2} < \frac{\pi_1}{\pi_2}$. Equivalently, $\frac{p_1}{\pi_1} < \frac{p_2}{\pi_2}$.

(c) Suppose that there are three commodities and H consumers where

$U^h(x) = \sum_{j=1}^{3} a_j v_h(x_j^h),\ h = 1,...,H$ and $v_h(\cdot)$ is a strictly increasing, strictly concave differentiable function. Is it necessarily the case that $\omega_1 > \omega_2 > \omega_3$ implies that $\frac{p_1}{a_1} < \frac{p_2}{a_2} < \frac{p_3}{a_3}$? Explain carefully.

4. A two asset portfolio

An investor has a wealth of 120. There are two states. The dividends of asset A are uncertain. In state 1 the total dividend is 5000 and in state 2 it is 2000. The two states are equally likely. A 1% shareholding in asset A costs 120 so if the investor only purchases share in asset A his uncertain outcome is the lottery $(50, 20; \frac{1}{2}, \frac{1}{2})$. Asset B has a dividend of 2000 in state 1 and 8000 in state 2. A 1% shareholding costs 120 so if the investor purchases only shares in asset B his uncertain outcome is the lottery $(20, 80; \frac{1}{2}, \frac{1}{2})$. The two outcomes are depicted below.



(a) If instead the investor spends a fraction z of his wealth on asset A and $1-z$ on asset B show that his uncertain outcome is

$$(x_1, x_2; \tfrac{1}{2}, \tfrac{1}{2}) = (20 + 30z, 80 - 60z; \tfrac{1}{2}, \tfrac{1}{2}) \ . \qquad (1.1)$$

(b) Choose p_1 and p_2 so that $p_1 x_1 + p_2 x_2 = 120$.

Thus the outcomes from diversifying are the points on the line joining the non-diversified outcomes $N_A = (50, 20)$ and $N_B = (20, 80)$.

This line is thus an implicit budget constraint.

To solve for the optimal portfolio $(\overline{z}, 1-\overline{z})$ we can first solve for the optimal outcome. This is the best point satisfying the implicit budget constraint.

(c) Suppose $v(x) = \ln x$ so that expected utility $U(x) = \mathbb{E}[v(x)] = \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2$.

Solve for the optimal outcome satisfying the implicit budget constraint.

(d) Use (1.1) to solve for $\overline{z}$.

(e) Suppose $v(x) = \ln(8+x)$. Solve for the new optimal outcome $\overline{\overline{x}}$ and hence for the share of wealth spent on asset A.

(f) Suppose $v(x) = \ln(40+x)$. Solve for the new optimal outcome $\overline{\overline{x}}$ and hence for the share of wealth spent on asset A.

(g) Is it the case that for any differentiable strictly concave function $v(x)$, the optimal outcome will be above the 45° line?

1 point bonus

(h) Suppose $v(x) = \ln(50+x)$. What is the optimal portfolio?