Walrasian Equilibrium with production

First two sections recently edited.

Convex sets and concave functions

Convex combination of two vectors

Consider any two vectors z^0 and z^1. A weighted average of these two vectors is

$$z^\lambda = (1-\lambda)z^0 + \lambda z^1 \text{ , } 0<\lambda<1$$

Such averages where the weights are both strictly positive and add to 1 are called the convex combinations of z^0 and z^1.

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Convex sets and concave functions

Convex combination of two vectors

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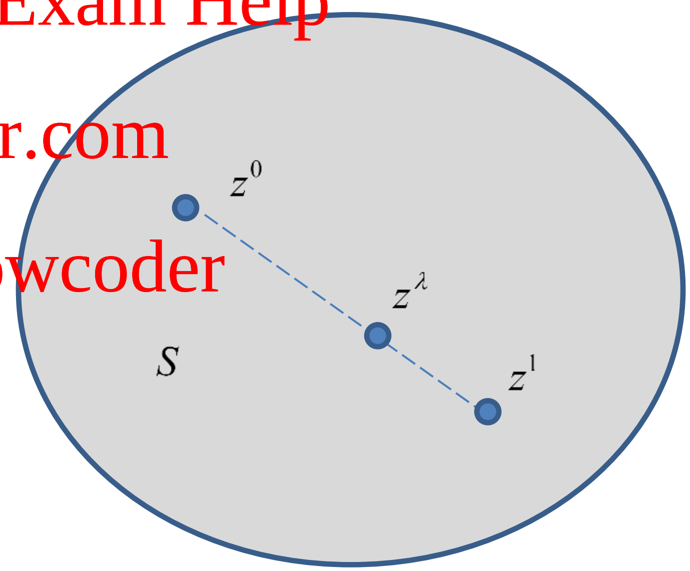
Such averages where the weighs are both strictly positive and add to 1 are called the convex combinations of z^0 and z^1.

Convex set

The set $S \subset \mathbb{R}^n$ is convex if for any z^0 and z^1 in *S*,

every convex combination is also in S



A convex set

Convex combination of two vectors

- - another view

Consider any two vectors z^0 and z^1.

The set of weighted average of these

two vectors can be written as follows.

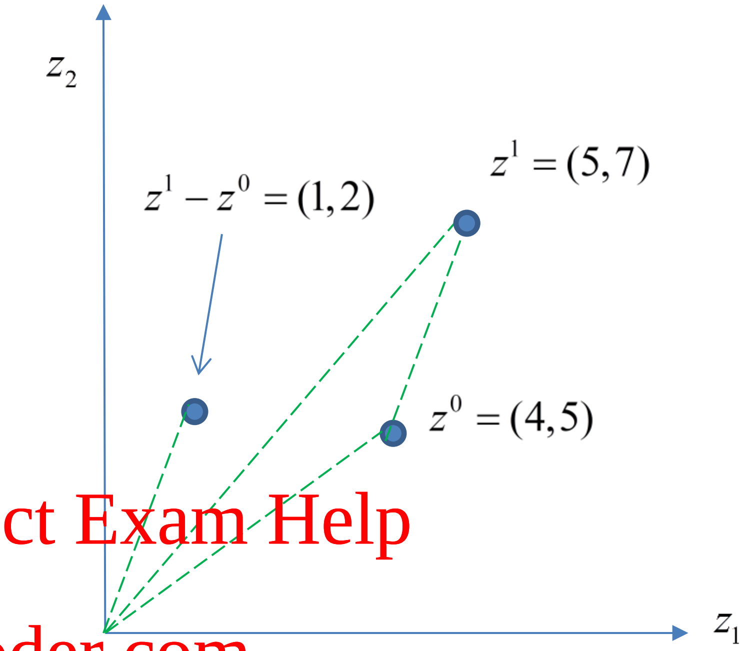
$$z^\lambda = (1-\lambda)z^0 + \lambda z^1 \text{ , } 0<\lambda<1$$

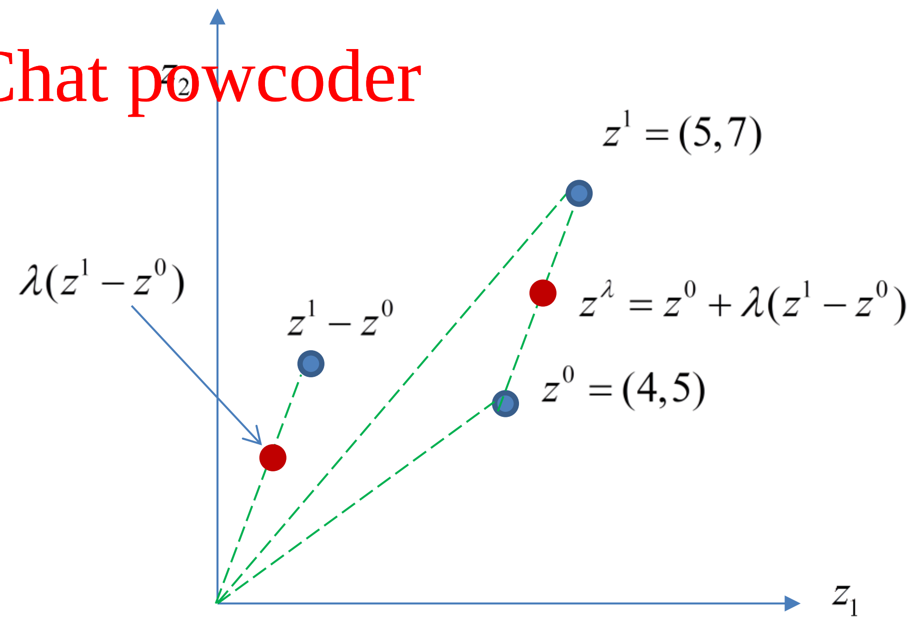
Rewrite the convex combination is follows:

$$z^\lambda = z^0 + \lambda(z^1 - z^0)$$

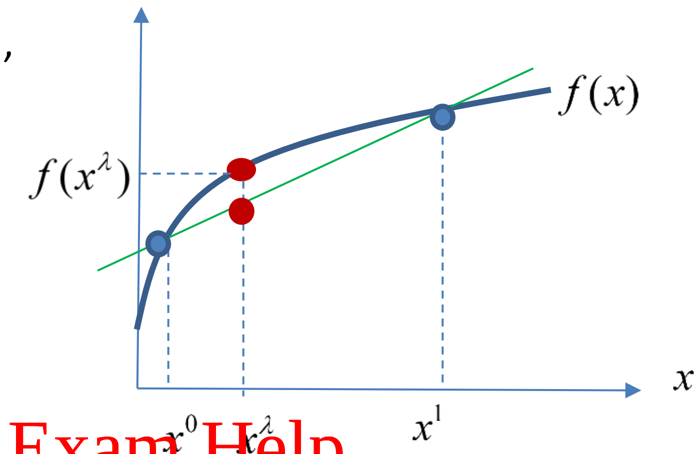
The vector z^λ is a fraction λ

of the way along the line joining z^0 and z^1





Concave functions of 1 variable



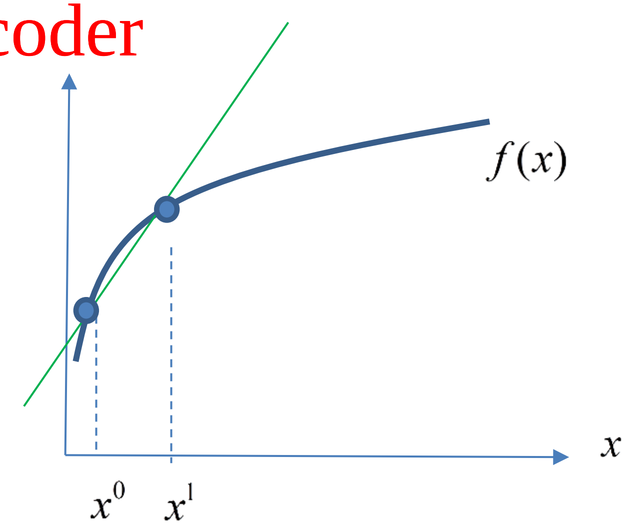
Definition 1: A function is concave if, for every x^0 and x^1, the graph of the function is above the line joining $(x^0, f(x^0))$ and $(x^1, f(x^1))$, i.e.

$$f(x^\lambda) \geq (1-\lambda) f(x^0) + \lambda f(x^1)$$

for every convex combination

$$x^\lambda = (1-\lambda)x^0 + \lambda x^1$$

Note that as the distance between x^1 and x^0 approaches zero, the line passing through two blue markers becomes the tangent line.

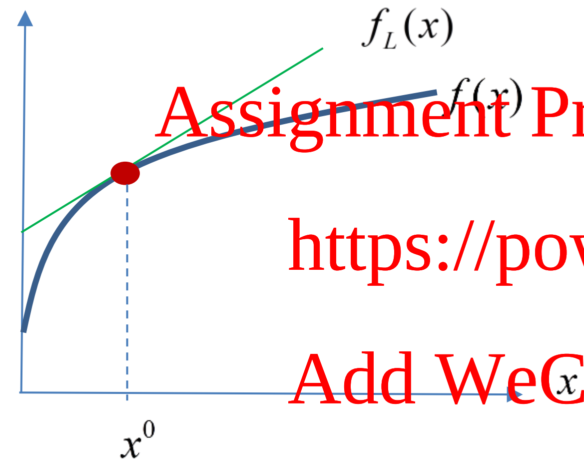


Tangent line is the linear approximation of the function f at x^0

$$f_L(x) \equiv f(x^0) + f'(x^0)(x - x^0) \text{ .}$$

Note that the linear approximation has the same value at x^0 and the same first derivative (the slope.)

In the figure $f_L(x)$ is a line tangent to the graph of the function.



Definition 2: Differentiable concave function

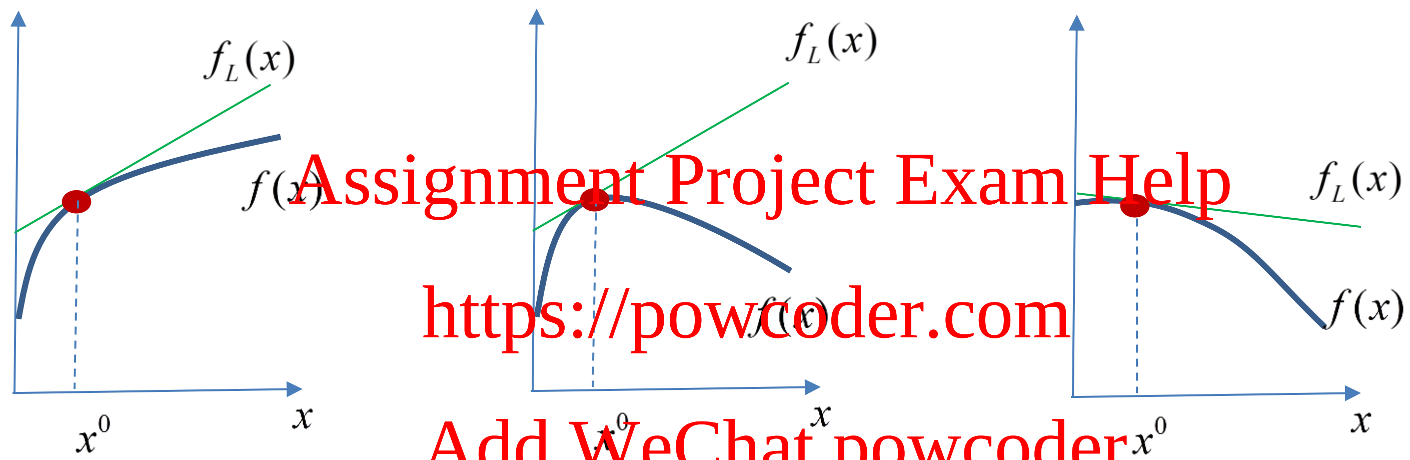
A differentiable function is concave if every tangent line is above the graph of the function. i.e.,

$$f(x) \leq f(x^0) + f'(x^0)(x^1 - x^0)$$

Definition 3: Concave Function

A differentiable function f defined on an interval X is concave if $f'(x)$, the derivative of $f(x)$ is decreasing.

The three types of differentiable concave function are depicted below.



Note that in each case the linear approximations at any point x^0 lie above the graph of the function.

Concave function of n variables

Definition 1: A function is concave if, for every x^0 and x^1,

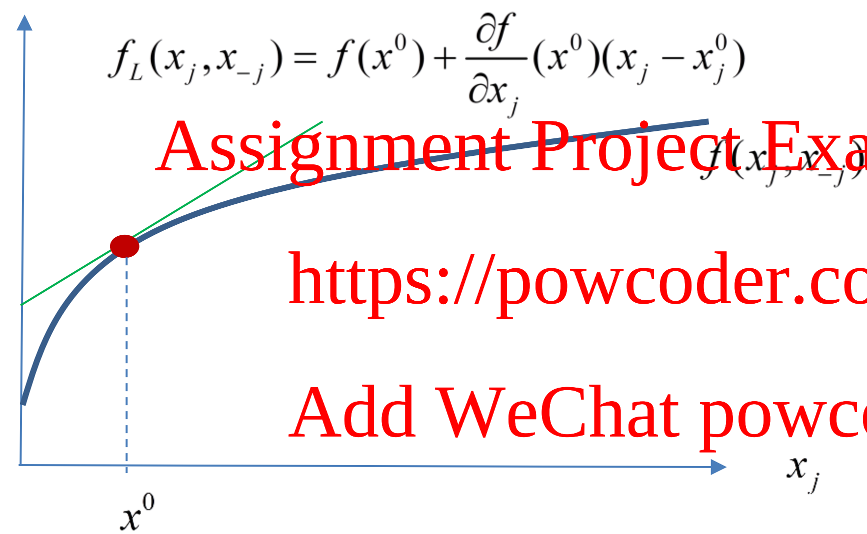
$$f(x^\lambda) \geq (1-\lambda) f(x^0) + \lambda f(x^1) \text{ for every convex combination } x^\lambda = (1-\lambda)x^0 + \lambda x^1,\ 0<\lambda<1$$

(Exactly the same as the definition when $n=1$)

Linear approximation of the function f at x^0

$$f_L(x) \equiv f(x^0) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x^0)(x_j - x_j^0) \ .$$

Note that for each x_j the linear approximation has the same value at x^0 and the same first derivative (the slope.)



Definition 2: Differentiable Concave function

For any x^0 and x^1

$$f(x^1) \leq f(x^0) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x^0)(x_j - x_j^0)$$

Group exercise: Appeal to one of these definitions to prove the first of the following important propositions.

Proposition

If $f(x)$ is concave, and $\bar{x}$ satisfies the necessary conditions for the maximization problem

$$\underset{x \geq 0}{Max}\{f(x)\}$$

then $\bar{x}$ solves the maximization problem.

Proposition

If $f(x)$ and $h(x)$ are concave, and $\bar{x}$ satisfies the necessary conditions for the maximization problem

$$\underset{x \geq 0}{Max}\{f(x) \,|\, h(x) \geq 0\}$$

then $\bar{x}$ is a solution of the maximization problem

Remark: This result continues to hold if there are multiple constraints $h_i(x) \geq 0$ and each function $h_i(x)$ is concave.

Concave functions of n variables

Proposition

1. The sum of concave functions is concave

2. If f is linear (i.e. $f(x) = a_0 + b \cdot x$) and g is concave then $h(x) = g(f(x))$ is concave.

3. An increasing concave function of a concave function is concave.

4. If $f(x)$ is homogeneous of degree 1 (i.e. $f(\theta x) = \theta f(x)$ for all $\theta > 0$) and for some increasing function $g(\cdot)$, $h(x) = g(f(x))$ is concave, then $f(x)$ is concave.

5. If $f(x)$ is homogeneous of degree 1 (i.e. $f(\theta x) = \theta f(x)$ for all $\theta > 0$) and the superlevel sets of $f(x)$ are convex, then $f(x)$ is concave.

Remark: The proof of 1-3 follows directly from the definition of a concave function. The proofs of 4 and 5 are very similar and more subtle.

Group exercise: Prove that the sum of concave functions is concave.

Group Exercise: Prove the following result

Proposition: Concave functions have convex superlevel sets

If $f(x)$ is a concave function then the superlevel sets of $f(x)$ are convex sets. i.e.,

If x^0, x^1 are in the superlevel set $S = \{x \mid f(x) \geq k\}$ then every convex combination is in S .

Group Exercise: Output maximization with a fixed budget

A plant has the CES production function

$$F(z) = (z_1^{1/2} + z_2^{1/2})^2 .$$

The CEO gives the plant manager a budget B and instructs her to maximize output. The input price vector is $r = (r_1, r_2)$. Solve for the maximum output $q(r, B)$.

Class Exercise: What is the firm's cost function

2. Production sets and returns to scale (first 3 pages are a review)

Feasible plan

If an input-output vector (z,q) where $z=(z_1,....,z_m)$ and $q=(q_1,...,q_n)$ is a feasible plan if q can be produced using z.

Production set

The set of all feasible plans is called the firm's production set.

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Production sets

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The set of all feasible plans is called the firm's production set.

Production function

If a firm produces one commodity the maximum output for some input vector z,

$$q=G(z)$$

is called the firm the firm's production function

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Production sets

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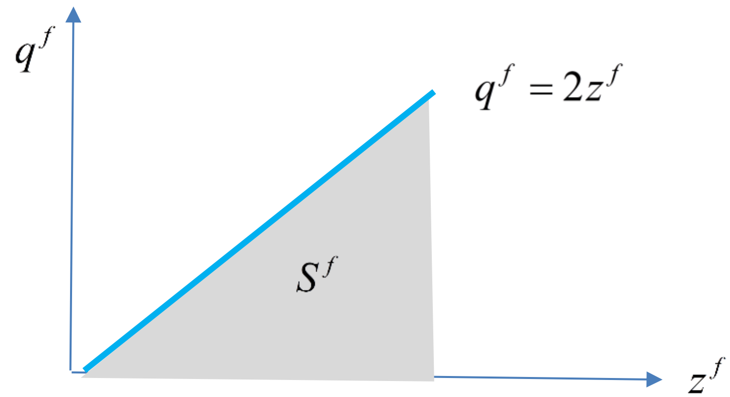
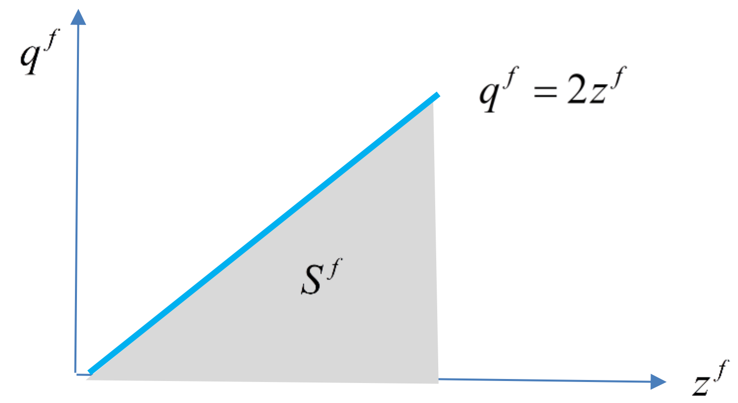
Example 1: One output and one input

$$S^f=\{(z^f,q^f)\}|0\le q_f\le 2z^f\}$$



Example 1: One output and one input

$$S^f = \{(z^f, q^f)\} \mid 0 \le q_f \le 2z^f\}$$



Note that the production function

$$q^f = G^f(z^f) = 2z_f$$

Is homogeneous of degree one

$$G^f(\theta z^f) = \theta^f G(z^f).$$

Such a firm is said to exhibit constant returns to scale

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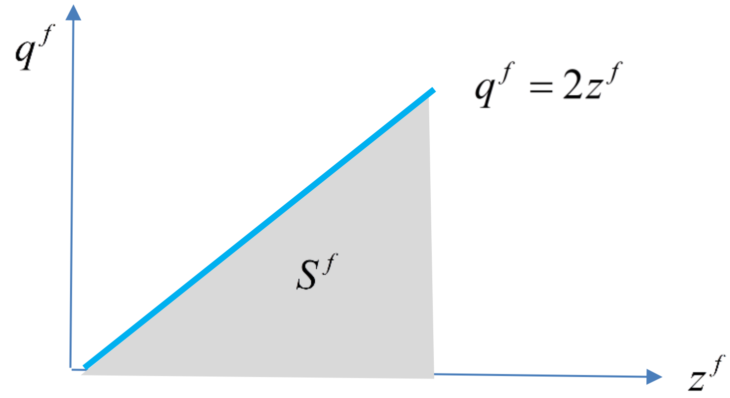
Example 1: One output and one input

$$S^f = \{(z^f, q^f)\} | 0 \le q_f \le 2z^f\}$$



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$$G^f(\theta z^f) = \theta^f G(z^f).$$

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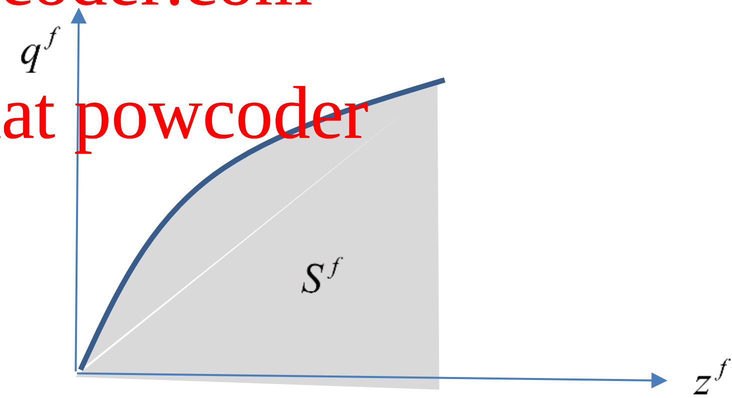
Example 2: One output and one input

$$S^f = \{(z^f, q^f) \ge 0 | q_f^{\,2} \le z^f\}$$

Equivalently

$$S^f = \{(z^f, q^f) \ge 0 | q_f \le (z^f)^{1/2}\}$$



Group Exercise: Show that S^f is convex

Example 3: two inputs and one output

$$S^f = \{(z,q) \geq 0 \mid h^f(z,q) = A(z_1)^{1/3}(z_2)^{2/3} - q \geq 0\}$$

Note that the production function is concave (why?)

Hence $h(z,q)$ is concave. (why)

Remark: We have proved that the superlevel set of a concave function are convex so S^f is a convex set

Example 4: one input and two outputs

$$S^f = \{(z,q) \geq 0 \mid h^f(z,q) = z^2 - a_1 q_1^2 - a_2 q_2^2 \geq 0\}$$

Equivalently

$$S^f = \{(z,q) \geq 0 \mid h^f(z,q) = z - (a_1 q_1^2 + a_2 q_2^2)^{1/2} \geq 0\}$$

Class Exercise: Explain why S^f is a convex set

Aggregate production set

Let $\{S^f\}_{f=1}^{F}$ be the production sets of the F firms in the economy.

The aggregate production set is

$$S = S^1 + ... + S^F$$

That is

$(z,q) \in S$ if there exist feasible plans $\{(z^f,q^f)\}_{f=1}^{F}$ such that $(z,q) = \sum_{f=1}^{F}(z^f,q^f)$.

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Example 1: $S^f = \{(z^f, q^f) \geq 0 \mid 2z^f - q^f \geq 0\}$

Exercise: Prove this using the methods from Example 2.

Example 2: $S^f = \{(z^f, q^f) \mid (q^f)^2 \le z^f\}$

(a) Show that with two firms the aggregate production set is $S = \{(z,q) \mid q^2 \le 2z\}$

(b) What is the industry production set if there are 4 firms?

Group Exercise

HINT: The maximum output of the two firms is

$$q = Max\{q^1 + q^2 \mid (q^1)^2 \le z^1,\ (q^2)^2 \le z^2,\ z^1 + z^2 \le z\}.$$

Rather than use the Lagrange method with 3 constraints, note that for any z^1 and z^2 output is maximized by choosing q^1 and q^2 so that the first two inequalities are binding.

Method 1: Then $z^f = (q^f)^2$ and so the problem is reduced to a one constraint problem.

$$q = \underset{q^1, q^2}{Max}\{q^1 + q^2 \mid (q^1)^2 + (q^2)^2 \le z\}$$

Method 2: Since $q^f = (z^f)^{1/2}$ it follows that maximized output is

$$q = \underset{q^1, q^2}{Max}\{q^1 + q^2 = (z^1)^{1/2} + (z^2)^{1/2} \mid z^1 + z^2 \le z\}$$

Remark: You might switch to subscripts to avoid confusion.

Aggregation Theorem for price taking firms

Proposition: If there are 2 firms in an industry, prices are fixed and $(\bar{z}^f, \bar{q}^f)$ is profit maximizing for firm f, $f = 1,2$ then $(z,q) = (\bar{z}_1 + \bar{z}_2, \bar{q}_1 + \bar{q}_2)$ is industry profit-maximizing.

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Proof: Let Π^f be maximized profit of firm f Since the industry can mimic the two firms, industry profit cannot be lower. Suppose it is higher. Then for some feasible $(\hat{z}^f, \hat{q}^f)$, $f = 1,2$,

$$p\cdot(\hat{q}^1 + \hat{q}^2) - r\cdot(\hat{z}^1 + \hat{z}^2) > \Pi^1 + \Pi^2 .$$

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$$p \cdot (\hat{q}^1 + \hat{q}^2) - r \cdot (\hat{z}^1 + \hat{z}^2) > \bar{\Pi}^1 + \bar{\Pi}^2 .$$

Rearranging the terms,

$$(p \cdot \hat{q}^1 - r \cdot \hat{z}^1 - \bar{\Pi}^1) + (p \cdot \hat{q}^2 - r \cdot \hat{z}^2 - \bar{\Pi}^2) > 0$$

Then either

$$p \cdot \hat{q}^1 - r \cdot \hat{z}^1 > \bar{\Pi}^1 \text{ or } p \cdot \hat{q}^2 - r \cdot \hat{z}^2 > \bar{\Pi}^2$$

But then $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^1, \bar{q}^1)$ cannot both be profit-maximizing.

QED

Remark: Arguing in this way we can aggregate to the entire economy.

7. Walrasian equilibrium (WE) with Identical homothetic preferences & constant returns to scale

Consumer h has utility function $U(x_1^h, x_2^h) = x_1^h x_2^h$. The aggregate endowment is $\omega = (a,1)$. All firms have the same linear technology. Firm f can produce 2 units of commodity 2 for every unit of commodity 1. That is the production function of firm f is $q^f = 2z^f$

Then the aggregate production function is $q = 2z$.

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Then the aggregate production function is $q = 2z$.

Aggregate production set

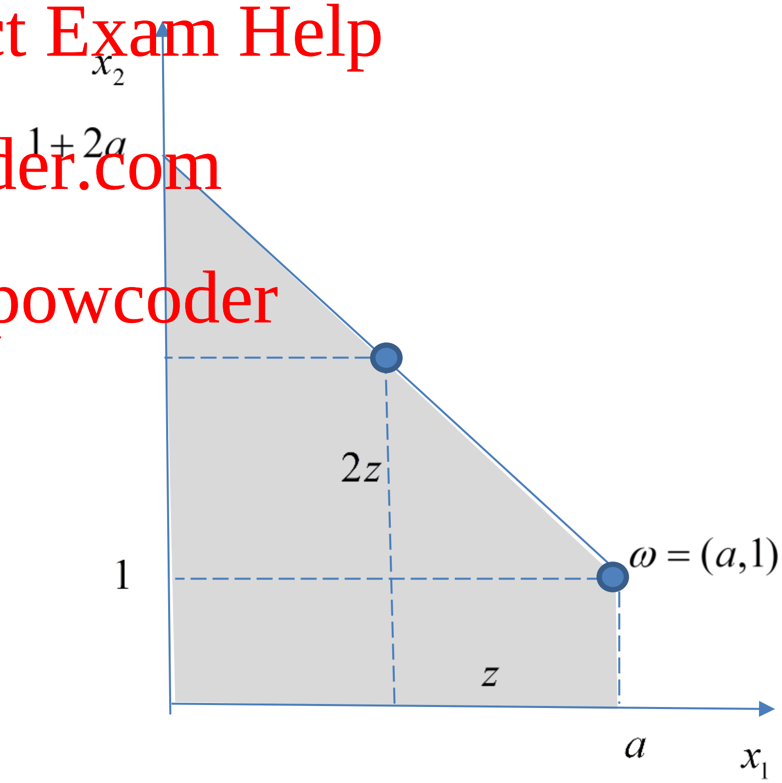
If the industry purchases z units of commodity 1

it can produce $q = 2z$ units of commodity 2.

Then total supply of each commodity is

$$(a - z, 1 + 2z) .$$

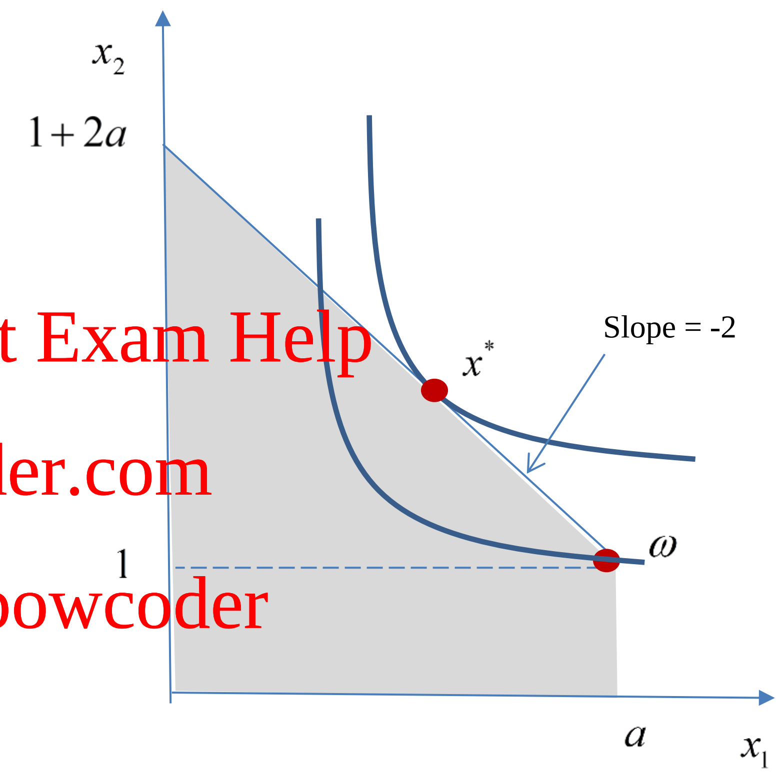
This is depicted opposite.



Maximizing the utility of the representative consumer

Simply solve for the utility maximizing point

In the aggregate production set.



Walrasian Equilibrium

First consider profit maximization

The profit of firm f is

$$\Pi^f = p_2 q_2^f - p_1 z_1^f = p_2 2z_1^f - p_1 z_1^f = z_1^f (2p_2 - p_1) .$$

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Walrasian Equilibrium

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$$\Pi^f = p_2 q_2^f - p_1 z_1^f = p_2 2 z_1^f - p_1 z_1^f = z_1^f (2p_2 - p_1) .$$

Case (i) $\frac{p_1}{p_2} > 2$: the profit maximizing firm will purchase no inputs and so produce no output.

Case (ii) $\frac{p_1}{p_2} < 2$: No profit maximizing plan

Case (iii) $\frac{p_1}{p_2} = 2$: any input-output vector $(z_1, q_2) = (z_1, 2z_1)$ is profit maximizing.

Note that in case (iii) the profit is zero.

Thus the is a WE with production the price ratio is $\frac{p_1}{p_2} = 2$ and maximized profit is zero.

WE with no production

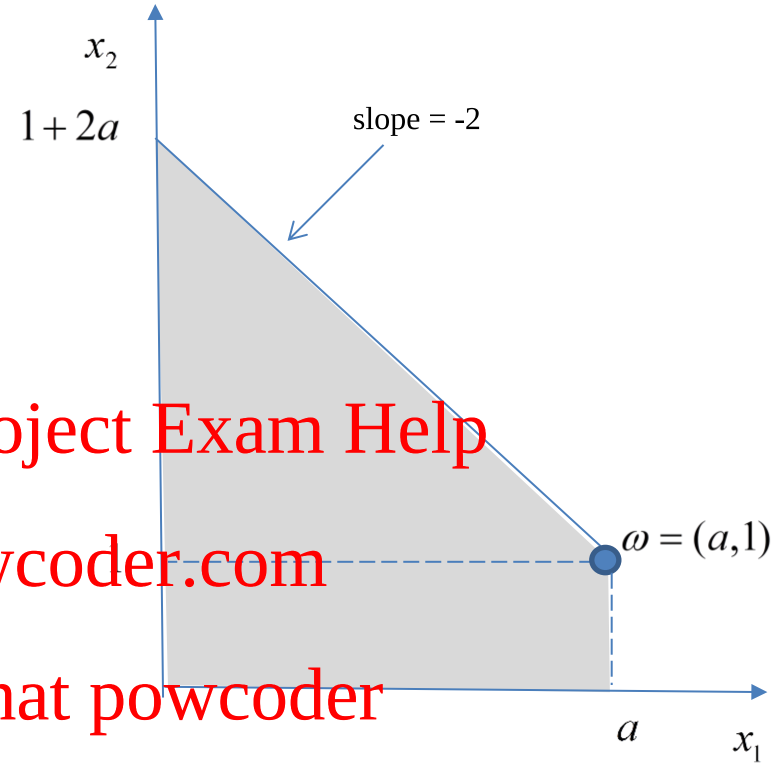
When is this the case?

If so the representative consumer

does not trade

FOC

$$\frac{\frac{\partial U}{\partial x_1}(\omega)}{p_1} = \frac{\frac{\partial U}{\partial x_2}(\omega)}{p_2}$$

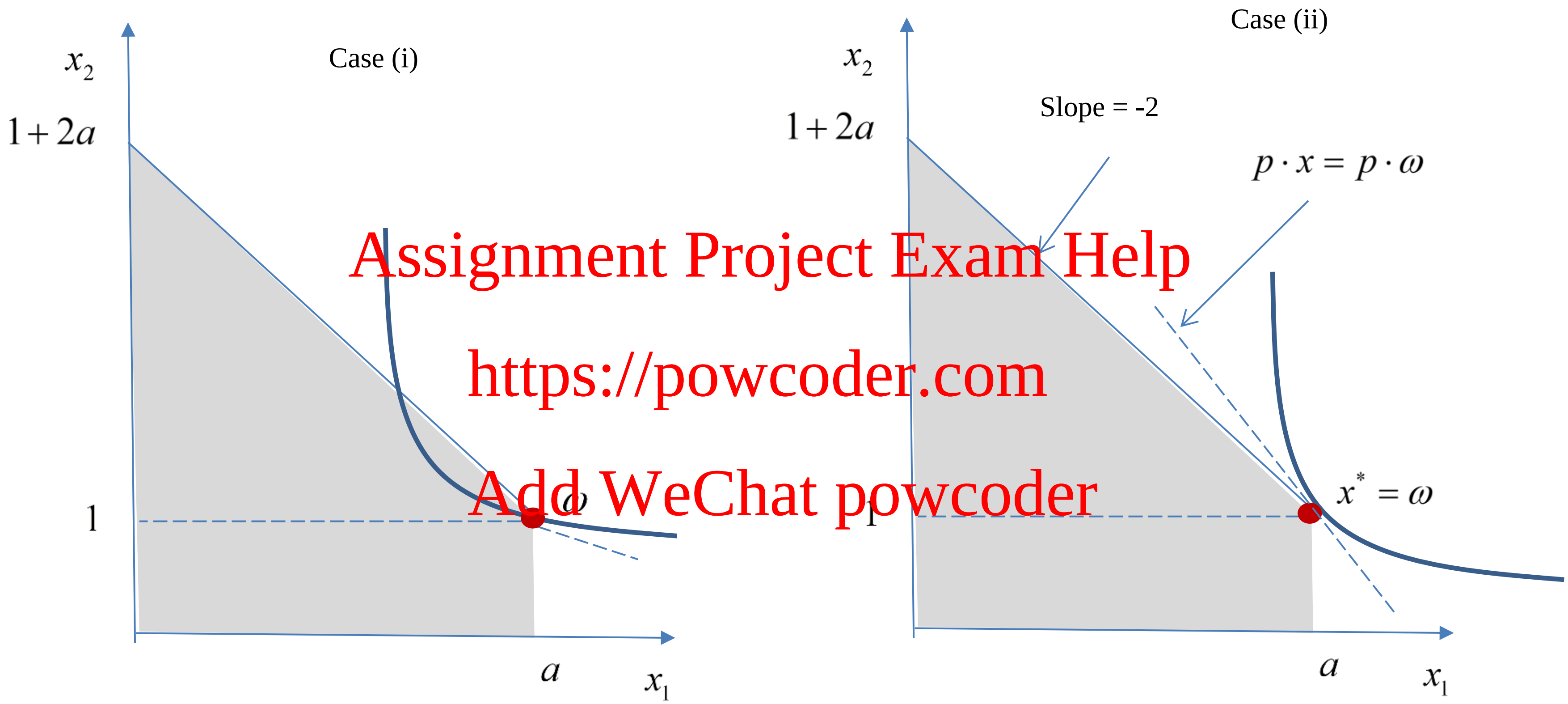


Therefore

$$\frac{p_1}{p_2} = \frac{\frac{\partial U}{\partial x_1}(\omega)}{\frac{\partial U}{\partial x_1}(\omega)} = \frac{\omega_2}{\omega_1} = \frac{1}{a}$$

There are two cases. Case (i) $a > \frac{1}{2}$ Case (ii): $a < \frac{1}{2}$

The indifference curve through ω is depicted below.



In the second case the no trade price ratio $\frac{p_1}{p_2} = \frac{1}{a}$ exceeds 2. As we have seen, production is not profitable at such a price ratio. Thus the right hand diagram depicts a WE.

Case (i)

Demand

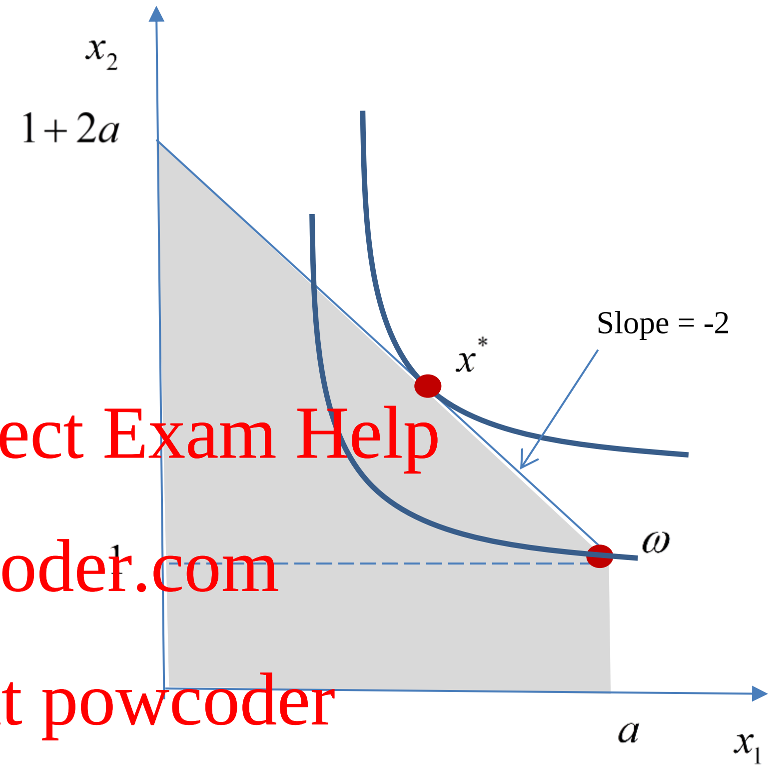
Consider the representative consumer.

The optimal consumption is depicted.

We know from the analysis of the

Firm that the price ratio is

$$\frac{p_1}{p_2} = 2$$



Also the profit is zero (no dividends)

Therefore the boundary of the feasible set

of outcomes is the budget line of the

representative consumer.

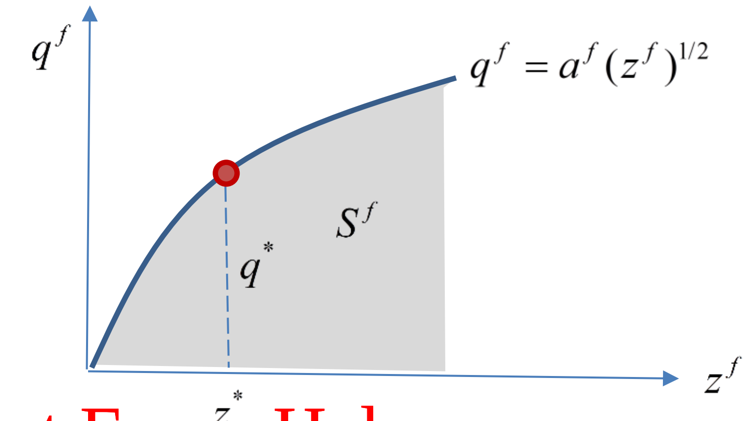
$$p \cdot x = p \cdot \omega .$$

The representative consumer's best choice is then x^*

Second example:

One output and one input

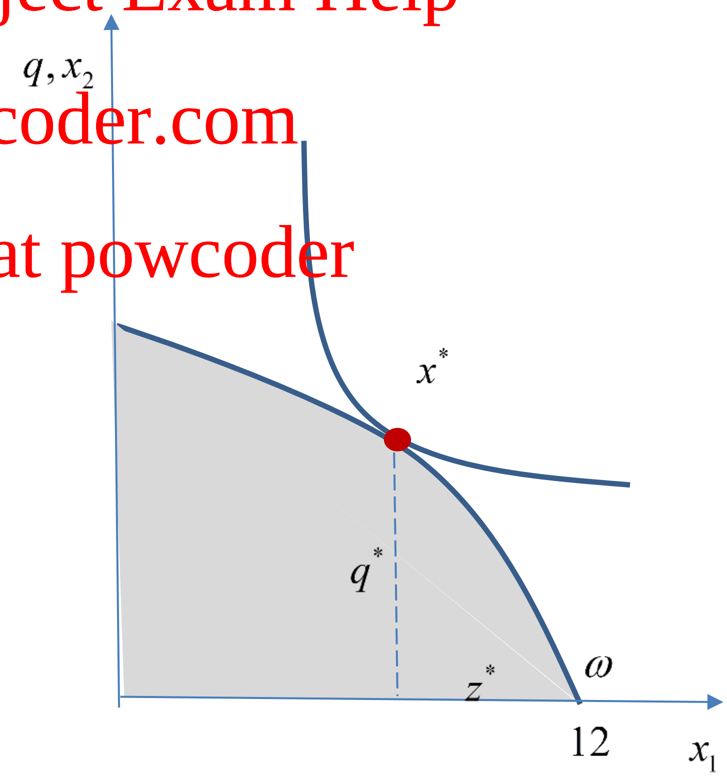
$$S^f = \{(z^f, q^f) \geq 0 \,|\, q^f \leq a^f (z^f)^{1/2}\}$$



There are two firms $(a^1, a^2) = (3, 4)$

The aggregate endowment is $\omega = (12, 0)$

Consumer preferences are as in the previous example.



Exercise

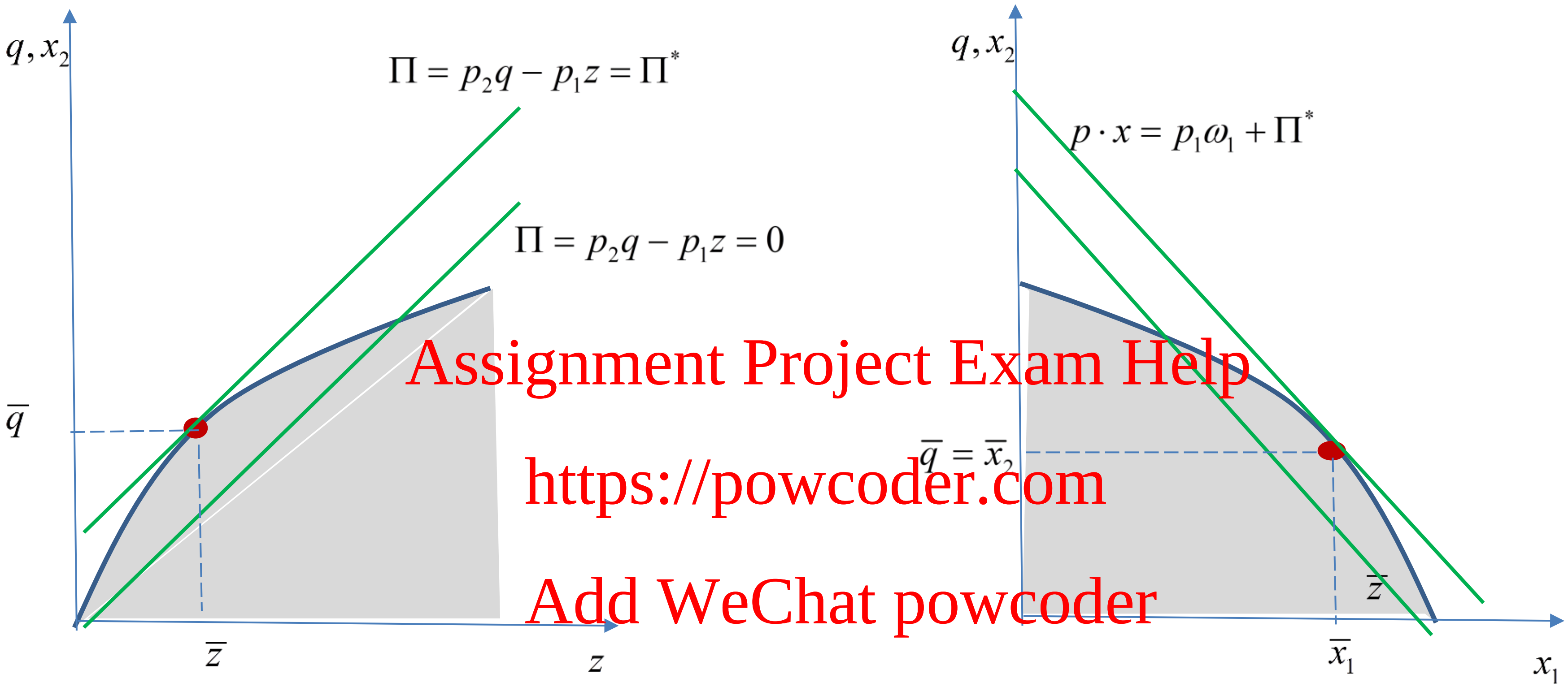
(a) Show that the aggregate production set

Can be written as follows:

$$S = \{(z, q) \geq 0 \,|\, q \leq 5z^{1/2}\}$$

(b) What is the best consumption vector x^*

Profit maximizing firm



Note that $\bar{z} = \omega_1 - \bar{x}_1$ so the level set for maximized profit can be rewritten as follows:

$$\bar{\Pi} = p_2 q - p_1 z = p_2 x_2 - p_1 z = p_2 x_2 - p_1(\omega_1 - x_1) = p \cdot x - p_1 \omega_1$$

Rearranging terms, the maximum profit level set is

$$p \cdot x = p_1 \omega_1 + \bar{\Pi}$$

Note that the maximum profit level set

$$p \cdot x = p_1\omega_1 + \overline{\Pi}$$

Is also the consumer's budget set.

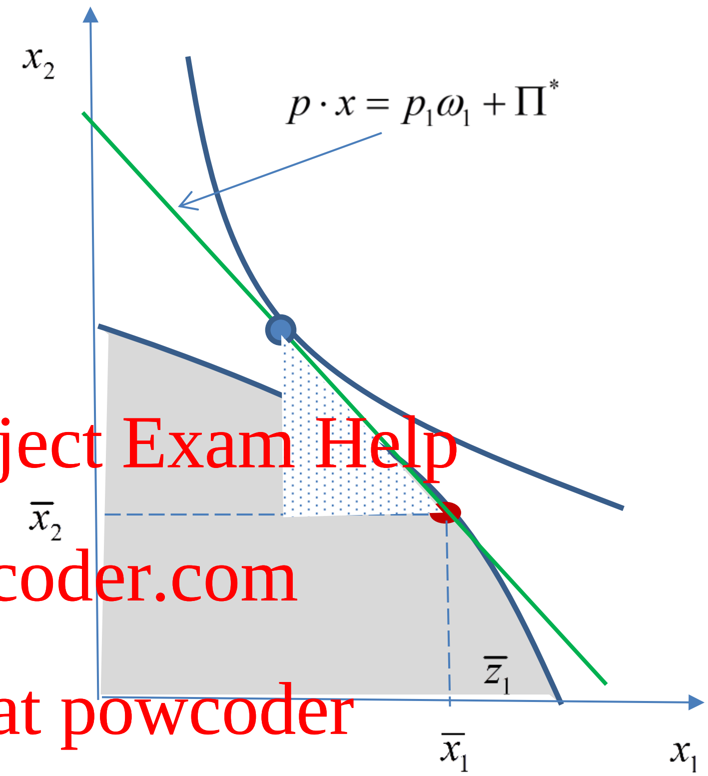
Adding the indifference curves, for any

price vector we can solve for the demands.

For the price vector shown there is

Excess demand for commodity 2 and

So excess supply of commodity 1.



Prices adjust.

Walrasian Equilibrium

Class Exercise:

For the following economy,

solve for the WE allocation and prices.

Aggregate production set

$$S = \{(z_1, x_2) \geq 0 \mid x_2 \leq 5z_1^{1/2}\}$$

Utility function

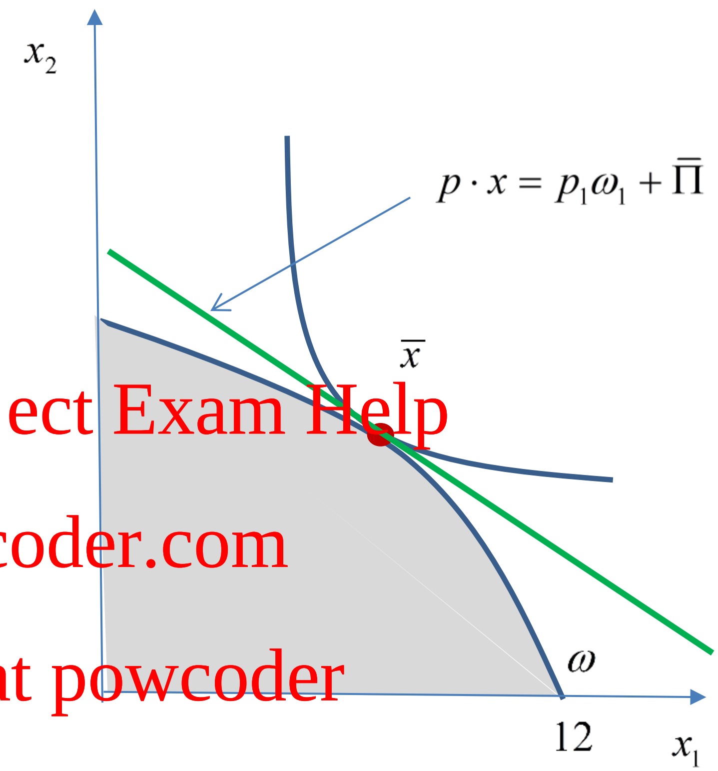
$$U(x_1^h, x_2^h) = x_1^h x_2^h$$

Homothetic hence we consider the

Representative consumer

$$U(x_1, x_2) = x_1 x_2$$

Aggregate endowment $\omega = (12, 0)$.



Answer to exercise

Example 1: $S^f = \{(z^f, q^f) \geq 0 \mid 2z^f - q^f \geq 0\}$

Exercise: Prove this using the methods from Example 2.

Proof:

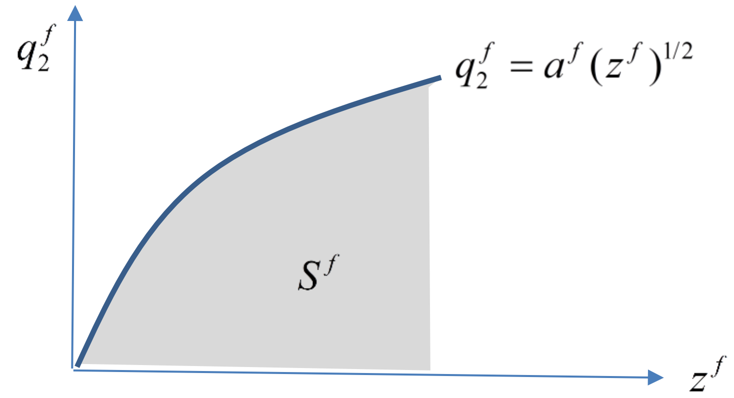
With firm inputs z^1 and z^2, maximized outputs of the two firms are $q^1 = 2z^1$ and $q^2 = 2z^2$. Therefore maximized to total output is $q = 2(z^1 + z^2)$. If the total input available is z then $z^1 + z^2 \leq z$ and so $q = 2(z^1 + z^2) \leq 2z$.

Therefore the aggregate production set is $S = \{(z,q) \mid 2z - q \geq 0\}$

Answer to exercise:

One output and one input

$$S^f = \{(z^f, q^f) \geq 0 \mid q^f \leq a^f (z^f)^{1/2}\}$$



There are two firms $(a^1, a^2) = (3,4)$

(a) Show that the aggregate production set

can be written as follows:

$$S = \{(z,q) \geq 0 \mid q \leq 5z^{1/2}\}$$

If the allocation of the input to firm 1 is z^1, then maximized output is $q^1 = 3(z^1)^{1/2}$. Similarly $q^2 = 4(z^2)^{1/2}$ and so

$$q^1 + q^2 = 3(z^1)^{1/2} + 4(z^2)^{1/2}$$

Maximized industry output is therefore

$$q = Max\{q^1 + q^2 = 3(z^1)^{1/2} + 4(z^2)^{1/2} \mid z^1 + z^2 \leq z\}$$

The problem is concave so the necessary condition are sufficient. We look for a solution with

$(z^1, z^2) >> 0$. The Lagrangian is

$$L = 3(z^1)^{1/2} + 4(z^2)^{1/2} + \lambda(z - z^1 - z^2)$$

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$$\frac{\partial L}{\partial q^1} = \frac{3}{2}(z^1)^{-1/2} - \lambda = 0\,, \quad \frac{\partial L}{\partial q^1} = \frac{4}{2}(z^1)^{-1/2} - \lambda = 0$$

Therefore

$$\frac{3}{(z^1)^{1/2}} = \frac{4}{(z^2)^{1/2}}$$

Squaring and appealing to the Ratio Rule,

$$\frac{9}{z^1} = \frac{16}{z^2} = \frac{25}{z^1 + z^2} = \frac{25}{z}$$

Therefore

$z^1 = \frac{9}{25}z$ and $z^2 = \frac{16}{25}z$ and so $q^1 = 3(z^1)^{1/2} = \frac{9}{5}z^{1/2}$ and $q^2 = 4(z^2)^{1/2} = \frac{16}{5}z^{1/2}$.

So $q = q^1 + q^2 = 5z^{1/2}$