Choice under uncertainty

Part 1

Part 2

4. Measure of risk aversion

Remark: Linear transformations of Von Neumann utility functions $v(x)$

Consider $u(x) = k_1 + k_2 v(x)$

$$\mathbb{E}[u(x)] = \pi_1 u(x_1) + \pi_2 u(x_2) = \pi_1 (k_1 + k_2 v(x_1)) + \pi_2 (k_1 + k_2 v(x_2))$$

$$= k_1 + k_2 (\pi_1 v(x_1)) + \pi_2 v(x_2)) = k_1 + k_2 \mathbb{E}[v(x)]$$

Thus the ranking of lotteries is identical under linear transformations

Absolute aversion to risk

The bigger is $ARA(w) \equiv -\dfrac{v''(w)}{v'(w)}$ the bigger is $\alpha = (-\dfrac{v''(w)}{v'(w)})\dfrac{x}{4} = ARA(w)\dfrac{x}{4}$.

Thus an individual with a higher $ARA(w)$ requires the odds of a favorable outcome to be moved more. Thus $ARA(w)$ is a measure of an individual's aversion to risk.

$ARA(w) \equiv$ degree of absolute risk aversion

Examples: $v(x) = 3x^{1/2}$, $v(x) = \ln x$, $v(x) = 6 - 2x^{-1}$

$$ARA(x) = \frac{1}{2x} \qquad = \frac{1}{x} \qquad = \frac{2}{x}$$

Relative risk aversion

Betting on a small percentage of wealth

New risky alternative: $(w_1,w_2;\pi_1,\pi_2)=(\hat{w}(1+\beta),\hat{w}(1-\beta);\tfrac{1}{2}+\alpha,\tfrac{1}{2}-\alpha)$.

Choose α so that the consumer is indifferent between gambling and not gambling.

Note that we can rewrite the risky alternative as follows:

$(w_1,w_2;\pi_1,\pi_2)=(\hat{w}+x,\hat{w}-x;\tfrac{1}{2}+\alpha,\tfrac{1}{2}-\alpha)$ where $x=\beta\hat{w}$.

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Relative risk aversion

Betting on a small percentage of wealth

New risky alternative: $(w_1, w_2; \pi_1, \pi_2) = (\hat{w}(1+\beta), \hat{w}(1-\beta); \tfrac{1}{2}+\alpha, \tfrac{1}{2}-\alpha)$.

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Note that we can rewrite the risky alternative as follows:

$(w_1, w_2; \pi_1, \pi_2) = (\hat{w}+x, \hat{w}-x; \tfrac{1}{2}+\alpha, \tfrac{1}{2}-\alpha)$ where $x = \beta\hat{w}$.

From our earlier argument,

$$\alpha = \left(-\frac{v''(\hat{w})}{v'(\hat{w})}\right)\frac{x}{4} = \left(-\frac{v''(\hat{w})}{v'(\hat{w})}\right)\frac{\beta\hat{w}}{4} = \left(-\frac{\hat{w}v''(\hat{w})}{v'(\hat{w})}\right)\frac{\beta}{4}.$$

*

Relative risk aversion

Betting on a small percentage of wealth

New risky alternative: $(w_1,w_2;\pi_1,\pi_2)=(\hat{w}(1+\beta),\hat{w}(1-\beta);\tfrac{1}{2}+\alpha,\tfrac{1}{2}-\alpha)$.

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From our earlier argument,

$$\alpha=\left(-\frac{v''(\hat{w})}{v'(\hat{w})}\right)\frac{x}{4}=\left(-\frac{v''(\hat{w})}{v'(\hat{w})}\right)\frac{\beta\hat{w}}{4}=\left(-\frac{\hat{w}v''(\hat{w})}{v'(\hat{w})}\right)\frac{\beta}{4}.$$

Relative aversion to risk

The bigger is $RRA(w)\equiv-\dfrac{wv''(w)}{v'(w)}$ the bigger is $\alpha=\left(-\dfrac{wv''(w)}{v'(w)}\right)\dfrac{\beta}{4}=RRA(w)\dfrac{\beta}{4}$.

Thus an individual with a higher $RRA(w)$ requires the odds of a favorable outcome to be moved more. Thus $RRA(w)$ is a measure of an individual's aversion to risk.

$RRA(w)\equiv$ degree of relative risk aversion

Remark on estimates of relative risk aversion

$RRA(w) \equiv -\dfrac{wv''(w)}{v'(w)}$. Typical estimate between 1 and 2

Remark on estimates of absolute risk aversion

$$ARA(w) \equiv -\frac{v''(w)}{v'(w)} = \frac{1}{w} RRA(w)$$

Thus ARA is very small for anyone with significant life-time wealth

5. Insurance

A consumer with a wealth $\hat{w}$ has a financial loss of L with probability π_1. We shall call this outcome the "loss state" and label it state 1. With probability $\pi_2 = 1-\pi_1$ the consumer is in the "no loss state" an label it state 2.

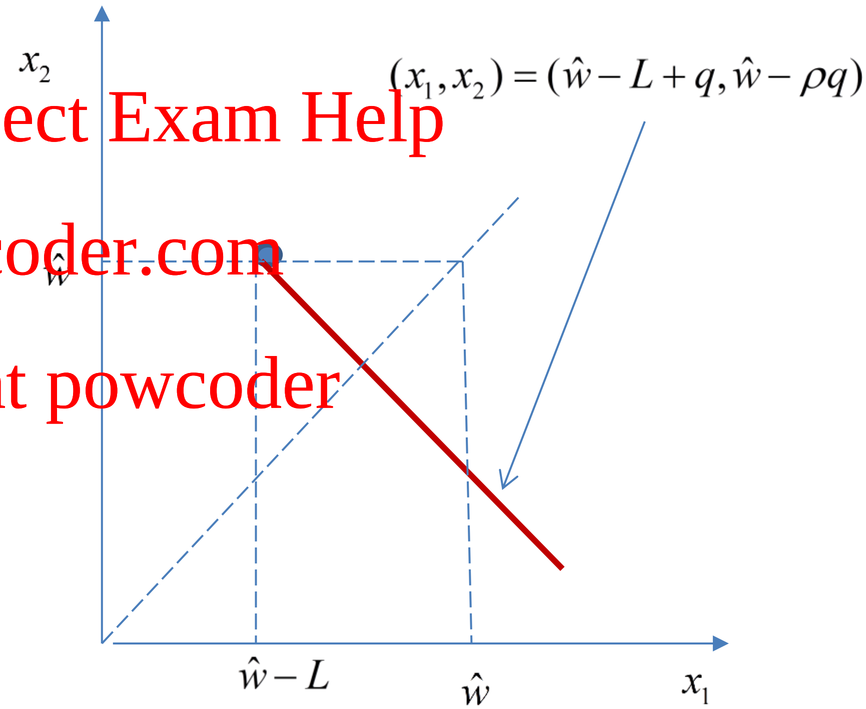
With no exchange the consumer's state contingent wealth is

$(x_1, x_2) = (\hat{w} - L, \hat{w})$.

This consumer wishes to exchange wealth in state 2 for wealth in state 1.

Suppose there is a market in which such an exchange can take place. For each dollar of coverage in the loss state, the consumer must pay ρ dollars in the no loss state.

$(x_1, x_2) = (\hat{w} - L + q, \hat{w} - \rho q)$



The steepness of the line is rate at which the consumer

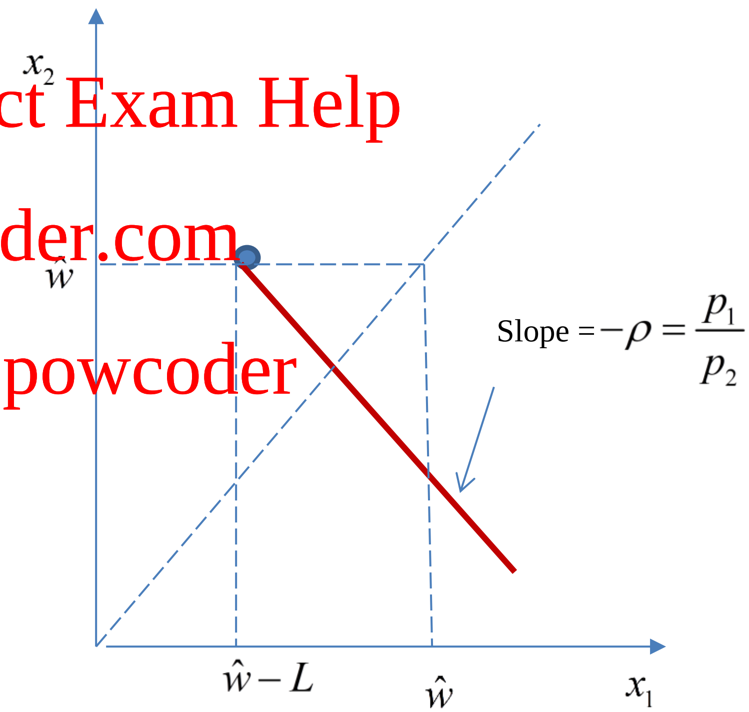
must exchange units in state 2 for units in state 1

So ρ is a market exchange rate.

Then if there were prices for units in each state

$$\rho = \frac{p_1}{p_2}$$

Suppose that the consumer purchases q units.



$$x_1 = \hat{w} - L + q$$

$$x_2 = \hat{w} - \rho q = \hat{w} - \frac{p_1}{p_2} q$$

$$p_1 x_1 = p_1(\hat{w} - L) + p_1 q$$

$$p_2 x_2 = p_2 \hat{w} - p_1 q$$

Adding these equations,

$$p_1 x_1 + p_2 x_2 = p_1(\hat{w} - L) + p_2 \hat{w}$$

The consumer's expected utility is
$U = \pi_1 v(x_1) + \pi_2 v(x_2)$

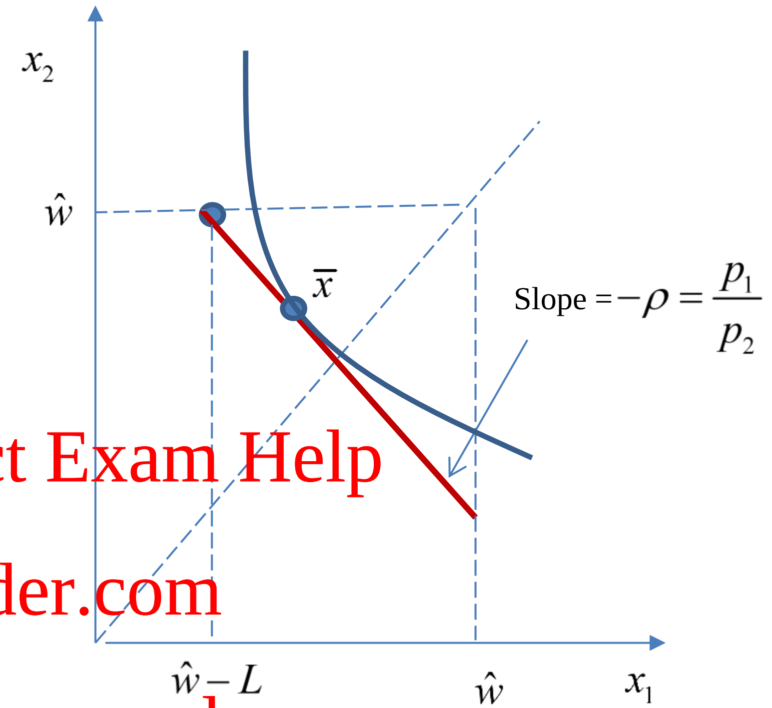
We have argued that the consumer's

choices are constrained to satisfy the

following budget constraint

$p_1 x_1 + p_2 x_2 = p_1(\hat{w} - L) + p_2 \hat{w}.$

This is the line depicted in the figure.



Implicit budget line though the "endowment"

Group Exercise: What must be the price ratio

if the consumer purchases full coverage? (i.e. $\bar{x}_1 = \bar{x}_2$)

6. Sharing the risk on a South Pacific Island

Alex lives on the west end of the island and has 600 coconut palm trees. Bev lives on the East end and has 800 coconut palm trees. If the typhoon approaching the island makes landfall on the west end it will wipe out 400 of Alex's palm trees. If instead the typhoon makes landfall on the East end of the island it will wipe out 400 of Bev's coconut palms. The probability of each event is 0.5.

Let the West end typhoon landfall be state 1 and let the East end landfall be state 2. Then the risk facing Alex is $(200,600;\frac{1}{2},\frac{1}{2})$ while the risk facing Bev is $(800,400;\frac{1}{2},\frac{1}{2})$.

What should they do?

What would be the WE prices if they could trade state "contingent claims" provided by competitive insurance companies (in effect, market makers)?

What would be the WE outcome?

Let $v_B(\cdot)$ be Bev's VNM utility function so that

her expected utility is

$$U^B(x^B) = \pi_1 v_B(x_1^B) + \pi_2 v_B(x_2^B).$$

where π_s is the probability of state s .

In state 1 Bev's "endowment " is $\omega_1^B = 800$

In state 2 the endowment is $\omega_2^B = 400$.

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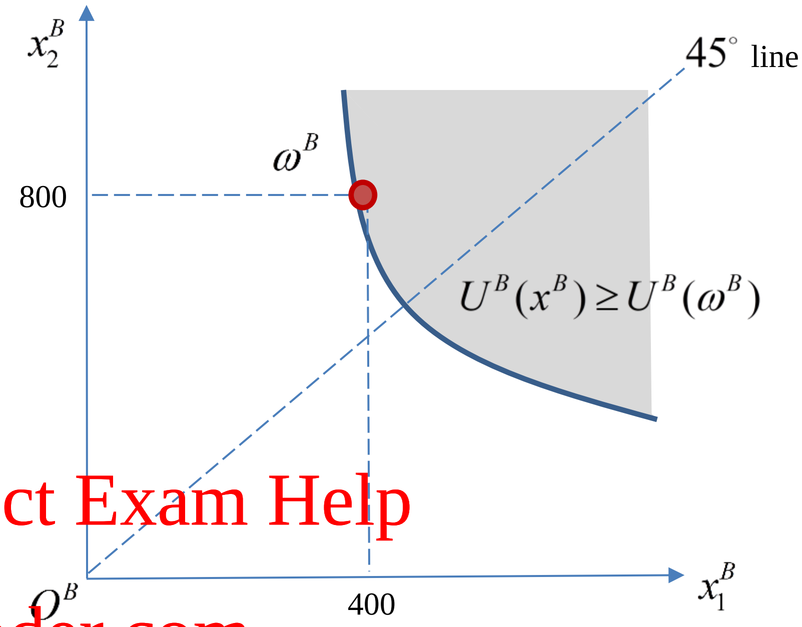


Let $v^h(\cdot)$ be Individual h's utility function so that

h's expected utility is

$$U^h(x^h) = \pi_1 v_h(x_1^h) + \pi_2 v_h(x_2^h).$$

where π_s is the probability of state s .

In state 1 Bev's "endowment " is $\omega_1^B = 800$

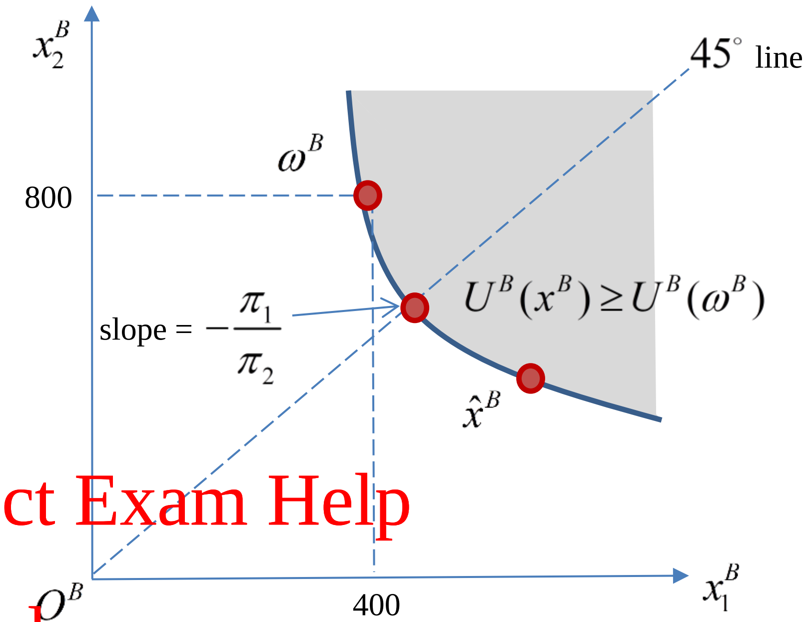
In state 2 the endowment is $\omega_2^B = 400$.

The level set for $U^B(x^B)$ through the

endowment point ω^B is depicted.

At a point $\hat{x}^B$ in the level set the steepness

of the level set is



$$MRS^B(\hat{x}^B) = \frac{MU_1}{MU_2} = \frac{\dfrac{\partial U^B}{\partial x_1^B}}{\dfrac{\partial U^B}{\partial x_2^B}} = \frac{\pi_1 v_B{}'(\hat{x}_1^B)}{\pi_2 v_B{}'(\hat{x}_2^B)} .$$

Note that along the 45° line the MRS is the probability ratio $\frac{\pi_1}{\pi_2}$ (equal probabilities so ratio is 1).

The level set for Alex is also depicted.

At each 45° line the steepness of the

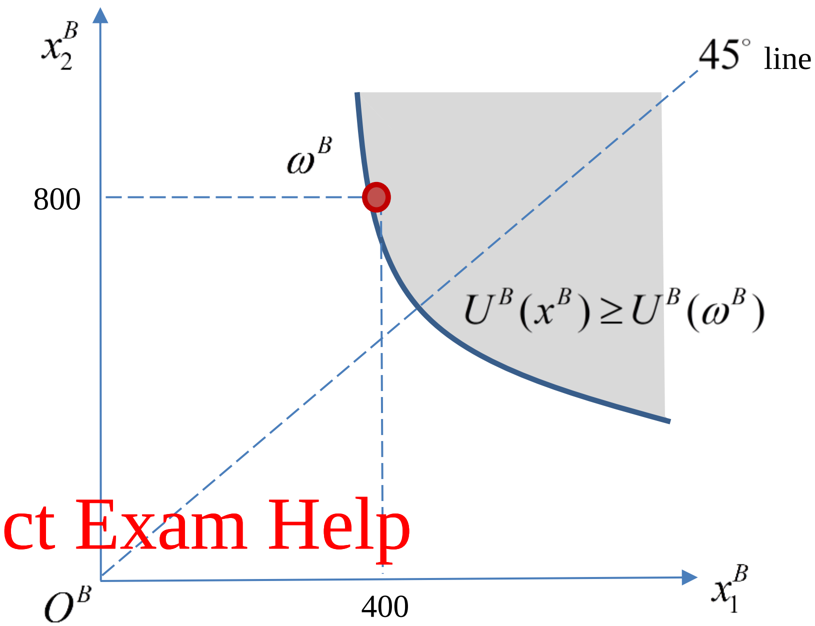
Respective sets are both 1.

Therefore

$MRS^B(\omega^B) > 1 > MRS^A(\omega^A)$

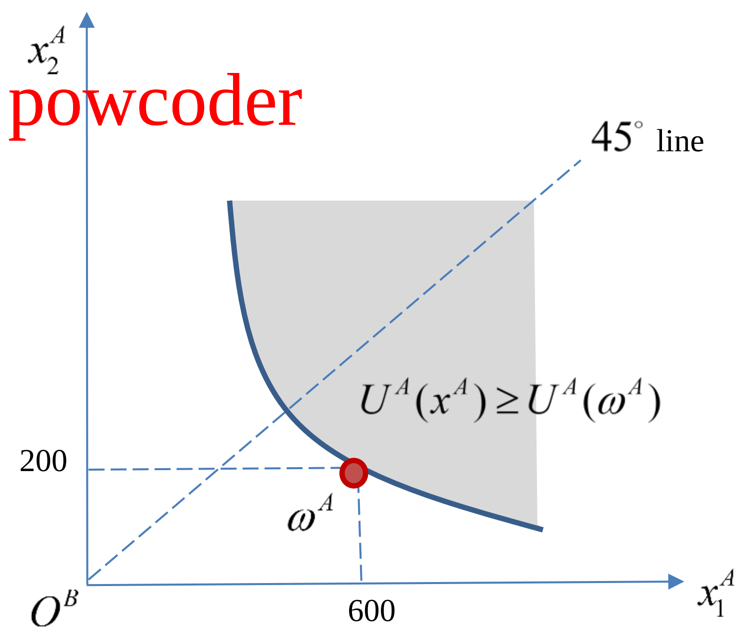
Therefore there are gains to be made from

trading state claims.



The consumers will reject any proposed exchange

that does not lie in their shaded superlevel sets.



Edgeworth Box diagram.

Bev will reject any proposed

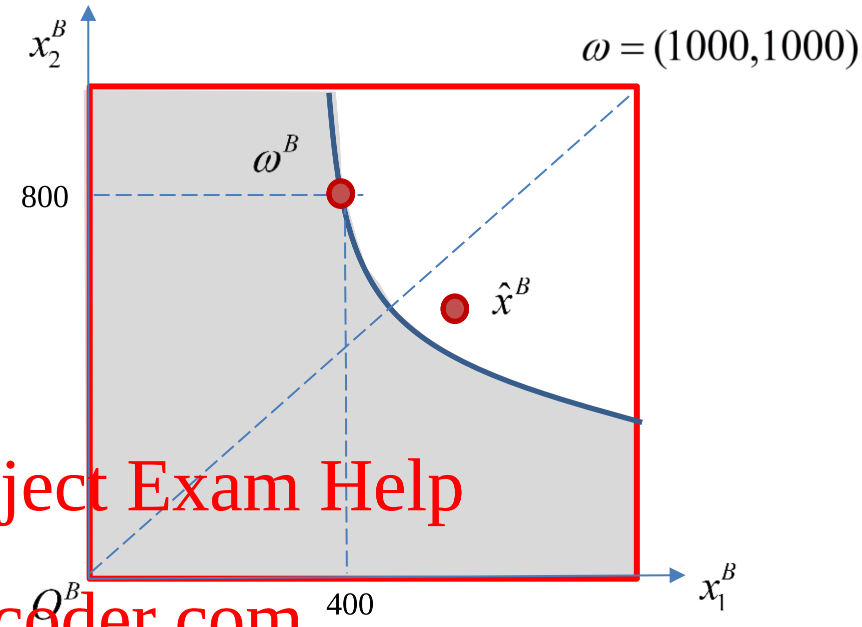
exchange that is in the shaded sublevel set.

Since the total supply of coconut palms is

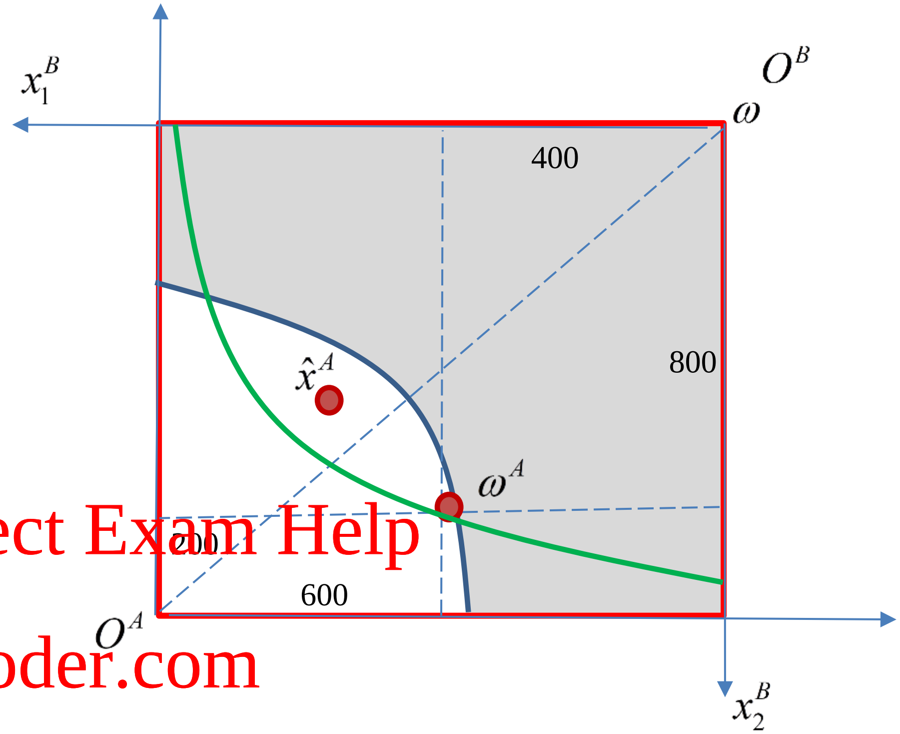
1000 in each state, the set of potentially

acceptable trades must be the unshaded

region in the square “Edgeworth Box”



The rotated Edgeworth Box



Note that $\omega^A = \omega - \omega^B$ and $\hat{x}^A = \omega - \hat{x}^B$

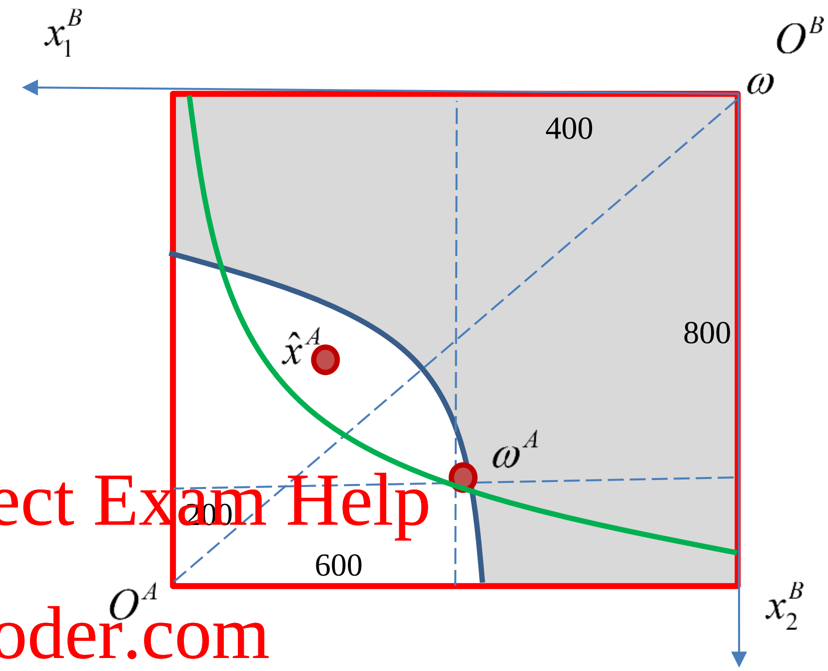
Also added to the figure is the green level set

for Alex's utility function through ω^A.

**

The rotated Edgeworth Box

Note that $\omega^A = \omega - \omega^B$ and $\hat{x}^A = \omega - \hat{x}^B$



Also added to the figure is the green level set

for Alex's utility function through ω^A.

Any exchange must be preferred by both consumers

over the no trade allocation (the endowments).

Such an exchange must lie in the lens shaped

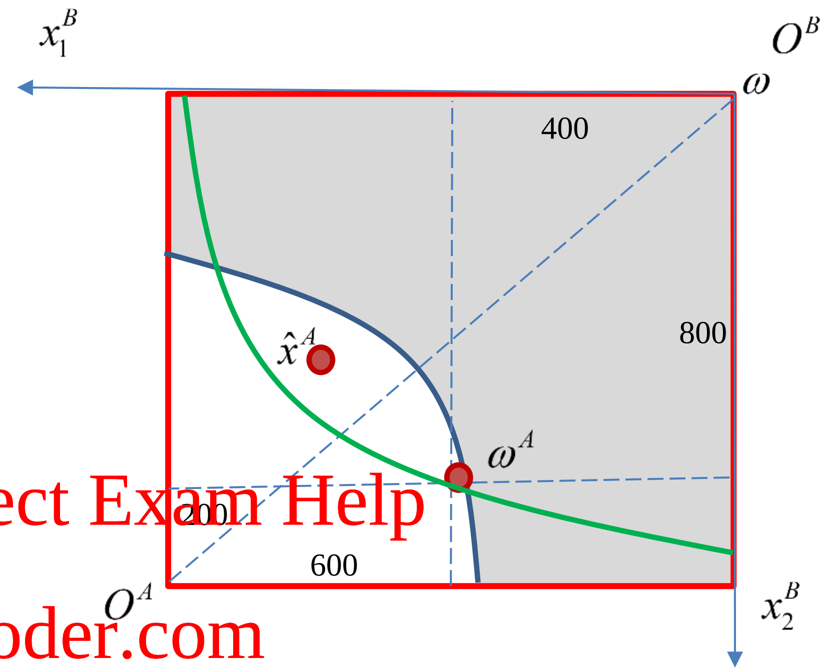
region to the right of Alex's level set and to the left

of Bev's level set.

*

The rotated Edgeworth Box

Note that $\omega^A = \omega - \omega^B$ and $\hat{x}^A = \omega - \hat{x}^B$



Also added to the figure is the green level set

for Alex's utility function through ω^A.

Any exchange must be preferred by both consumers

over the no-exchange allocation (the endowment).

Such an exchange must lie in the lens shaped

region where both are better off.

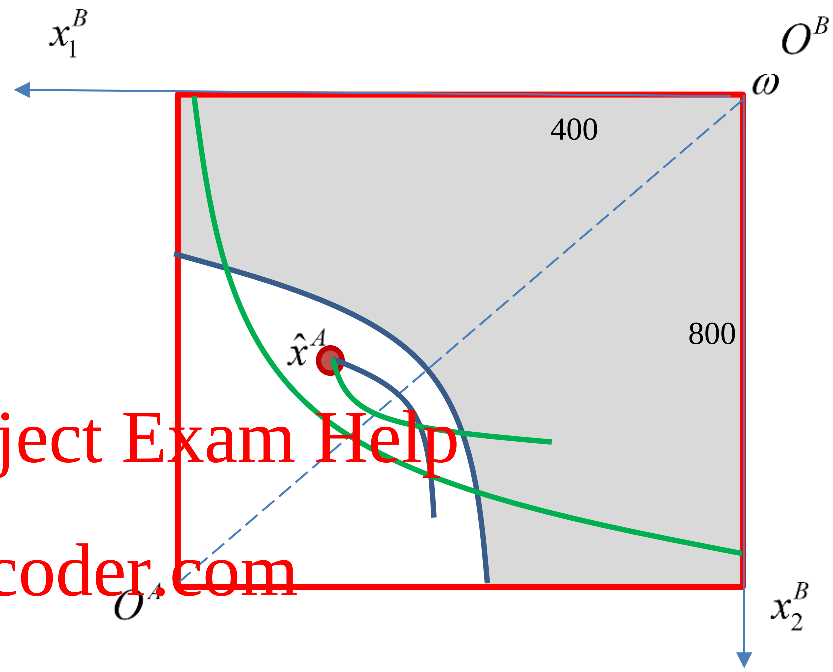
Pareto preferred allocations

If the proposed allocation is weakly preferred by both consumers and strictly preferred by at least one of the two consumers the new allocation is said to be Pareto preferred.

In the figure $\hat{x}^A$ (in the lens shaped region) is Pareto preferred to ω^A since Alex and Bev are both strictly better off.

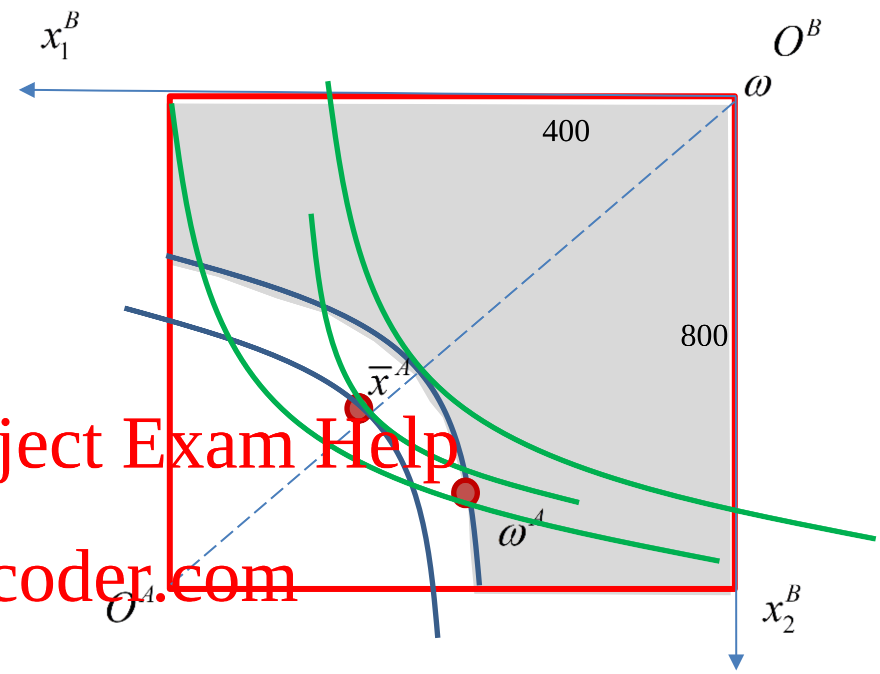
Consider any allocation such as $\hat{x}^A$

Where the marginal rates of substitution differ. From the figure there are exchanges that the two consumers can make and both have a higher utility.



Consider any allocation such as $\hat{\hat{x}}^A$

Where the marginal rates of substitution differ. From the figure there are exchanges that the two consumers can make and both have a higher utility.



Pareto Efficient Allocations

It follows that for an allocation

$$x^A \text{ and } x^B = \omega - x^A$$

to be Pareto efficient (i.e. no Pareto improving allocations)

$$MRS^A(x^A) = MRS^B(x^B)$$

Along the 45° line $MRS^A(\bar{x}^A) = \dfrac{\pi_1}{\pi_2} = MRS^B(\bar{x}^B)$.

Thus the Pareto Efficient allocations are all the allocations along 45° degree line.

Pareto Efficient exchange eliminates all individual risk.

Walrasian Equilibrium?

Suppose that insurance companies act as competitive intermediaries (effectively market makers) for people who want to trade the commodity in one state for more of the commodity in the other state. Let p_s be the price that a consumer must pay for delivery of a unit in state s, i.e. the price of "claim" in state s.

A consumer's endowment $\omega = (\omega_1, \omega_2)$, thus has a market value of $p \cdot \omega = p_1\omega_1 + p_2\omega_2$. The consumer can then choose any outcome (x_1, x_2) satisfying

$$p \cdot x \leq p \cdot \omega$$

Given a utility function $u_h(x_s)$, the consumer chooses $\bar{x}^h$ to solve

$$\underset{x}{Max}\{U_h(x^h, \pi) \mid p \cdot x \leq p \cdot \omega^h\}$$

i.e.

$$\underset{x^h}{Max}\{\pi_1 v_h(x_1^h) + \pi_2 v_h(x_2^h) \mid p \cdot x^h \leq p \cdot \omega^h\}$$

FOC: $MRS_h(\bar{x}^h) = \dfrac{MU_1}{MU_2} = \dfrac{\pi_1 v_h{}'(\bar{x}_1^h)}{\pi_2 v_h{}'(\bar{x}_2^h)} = \dfrac{p_1}{p_2}$

Example: $\omega^A = (200,600)$, $\omega^B = (800,400)$, $\pi = (\pi_1, \pi_2) = (\frac{1}{5}, \frac{4}{5})$

Group exercises

1. What is the WE price ratio?
2. What is the WE allocation?
3. Normalizing so the sum of the price is 1, what is the value of each plantation?
4. What is the profit of the insurance companies?

Class exercise

1. What ownership of plantations would give the two consumers the WE outcome?
2. Could they trade shares in their plantations and achieve this outcome?

Class (or group) Exercise: What if the loss in state 1 is bigger than in state 2

Suppose $v(x^h) = \ln x^h$ and the aggregate endowment is $\omega = (\omega_1, \omega_2)$. Then $\omega_1 < \omega_2$.

What are the PE allocations?

Hint: Consider the Edgeworth-Box.

Class Question: What does the First Welfare Theorem tell us about the WE allocation?

Given this, what must be the WE price ratio.

6. Portfolio choice

An investor with wealth $\hat{W}$ chooses how much to invest in a risky asset and how much in a riskless asset. Let $1+r_0$ be the return on each dollar invested in the riskless asset and let $1+\tilde{r}$ be the return on the risky asset (a random variable.) If the investor spends x on the risky asset (and so $\hat{W}-x$ on the riskless asset) her final wealth is

$$\tilde{W}=(\hat{W}-x)(1+r_0)+x(1+\tilde{r})$$

**

7. Portfolio choice

An investor with wealth $\hat{W}$ chooses how much to invest in a risky asset and how much in a riskless asset. Let $1+r_0$ be the return on each dollar invested in the risky asset and let $1+\tilde{r}$ be the return on the risky asset (a random variable.) If the investor spends q on the risky asset (and so $\hat{W}-q$ on the riskless asset) her final wealth is

$$\begin{aligned}\tilde{W} &= (\hat{W}-q)(1+r_0)+q(1+\tilde{r}) \\ &= \hat{W}(1+r_0)+q(\tilde{r}-r_0)\end{aligned}$$

*

7. Portfolio choice

An investor with wealth $\hat{W}$ chooses how much to invest in a risky asset and how much in a riskless asset. Let $1+r_0$ be the return on each dollar invested in the risky asset and let $1+\tilde{r}$ be the return on the risky asset (a random variable.) If the investor spends q on the risky asset (and so $\hat{W}-q$ on the riskless asset) her final wealth is

$$\begin{aligned}\tilde{W} &= (\hat{W}-q)(1+r_0)+q(1+\tilde{r}) \\ &= \hat{W}(1+r_0)+q(\tilde{r}-r_0) \\ &= \hat{W}(1+r_0)+q\tilde{\theta} \quad \text{where } \tilde{\theta}=\tilde{r}-r_0 .\end{aligned}$$

Class exercise:

What is the simplest possible model that we can use to analyze the investor's decision?

Two state model

Wealth in state s, $s = 1,2$

$$W_s = (\hat{W} - q)(1+r_0) + q(1+r_s)$$

$$= \hat{W}(1+r_0) + q(r_s - r_0)$$

$= \hat{W}(1+r) + q\theta_s$ where $\theta_s \equiv r_s - r_0$.

**

Two state model

Wealth in state s, $s = 1,2$

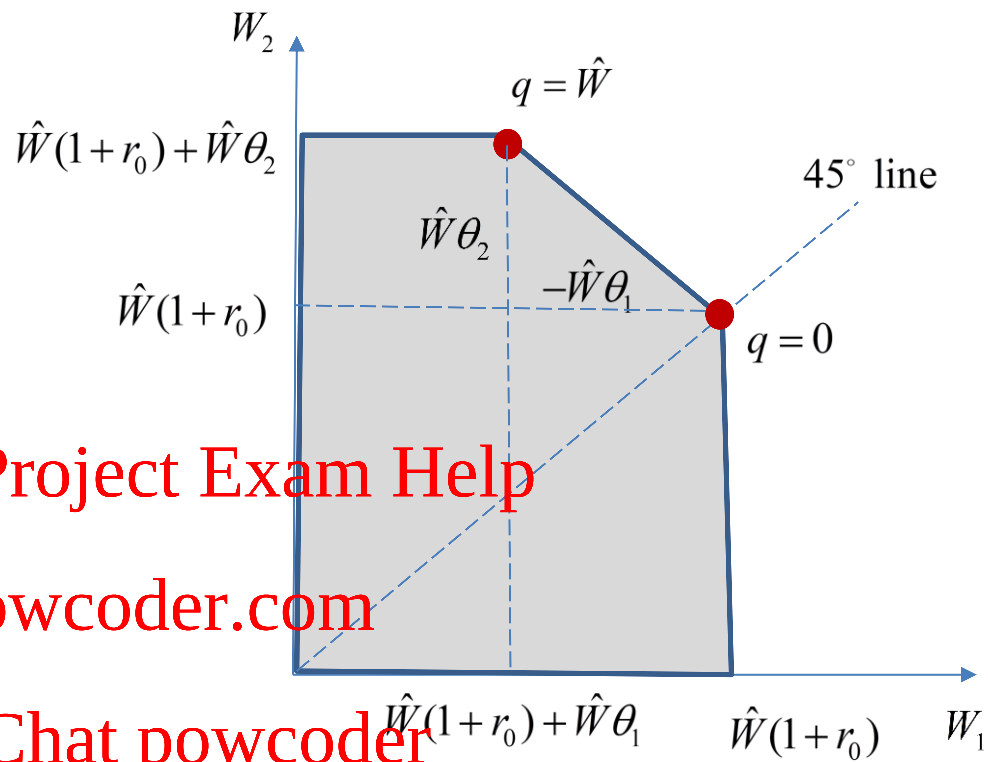
$$W_s = (\hat{W} - q)(1 + r_0) + q(1 + r_s)$$

$$= \hat{W}(1 + r_0) + q(r_s - r_0)$$

$$= \hat{W}(1 + r_0) + q\theta_s \quad \text{where } \theta_s = r_s - r_0 \ .$$

$$q = 0$$

$$W_s = \hat{W}(1 + r)$$



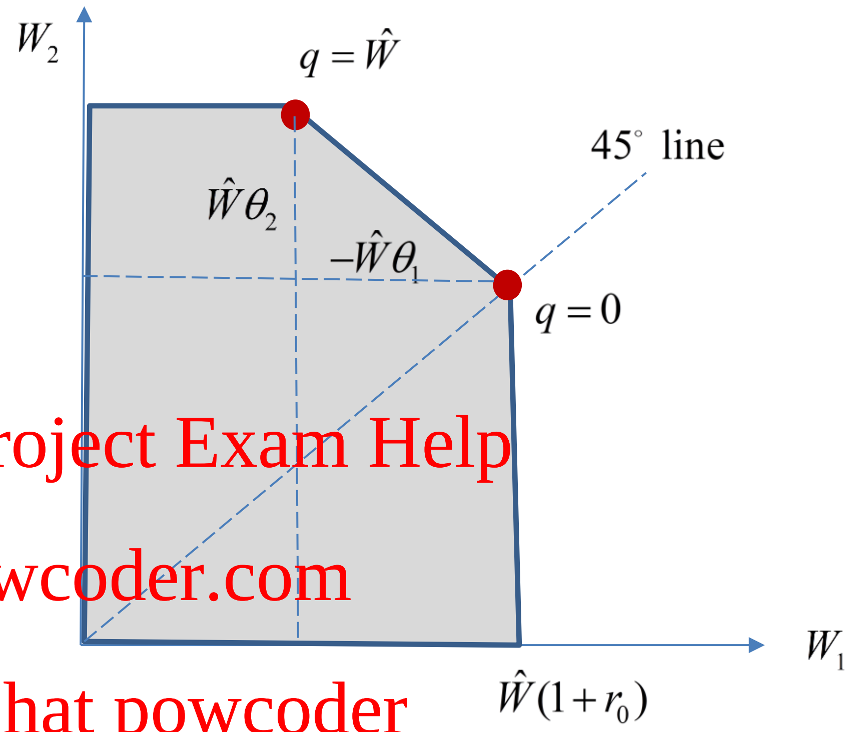
$$q = \hat{W}$$

$$W_s = \hat{W}(1 + r) + \hat{W}\theta_s$$

*

Two state model

Wealth in state s, $s=1,2$



$$W_s = (\hat{W}-q)(1+r_0)+q(1+r_s)$$
$$=\hat{W}(1+r_0)+q(r_s-r_0)$$
$$=\hat{W}(1+r)+q\theta_s$$

where $\theta_1 = r_1 - r_0 < 0 < r_2 - r_0 = \theta_2$.

$$q=0$$

$$W_s = \hat{W}(1+r)$$

$$q=\hat{W}$$

$$W_s = \hat{W}(1+r)+\hat{W}\theta_s$$

Expected utility of the investor

$$U(w,\pi)=\pi_1 u(W_1)+\pi_2 u(W_2)$$

When will the investor purchase some of the risky asset?

Two state model

$W_s = \hat{W}(1+r) + q\theta_s$ where $\theta_s = r_s - r_0$.

The steepness of the boundary of the

set of feasible outcomes is $-\dfrac{\theta_2}{\theta_1}$

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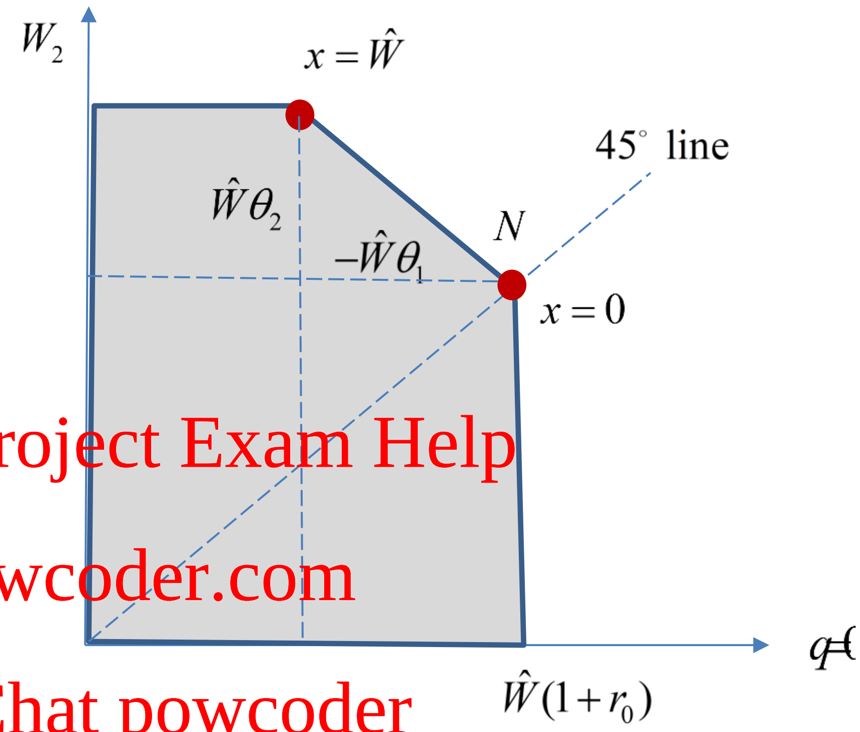
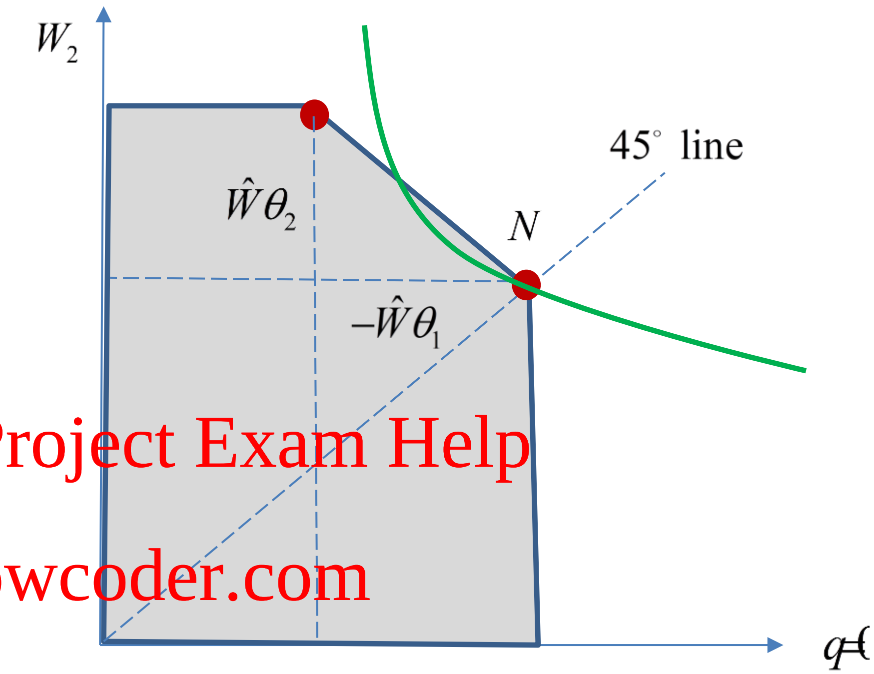


Two state model

$W_s = \hat{W}(1+r) + q\theta_s$ where $\theta_s = r_s - r_0$.



The steepness of the boundary of the

set of feasible outcomes is $-\dfrac{\theta_2}{\theta_1}$

$$U(W,\pi) = \pi_1 v(W_1) + \pi_2 v(W_2)$$

The steepness of the level set through

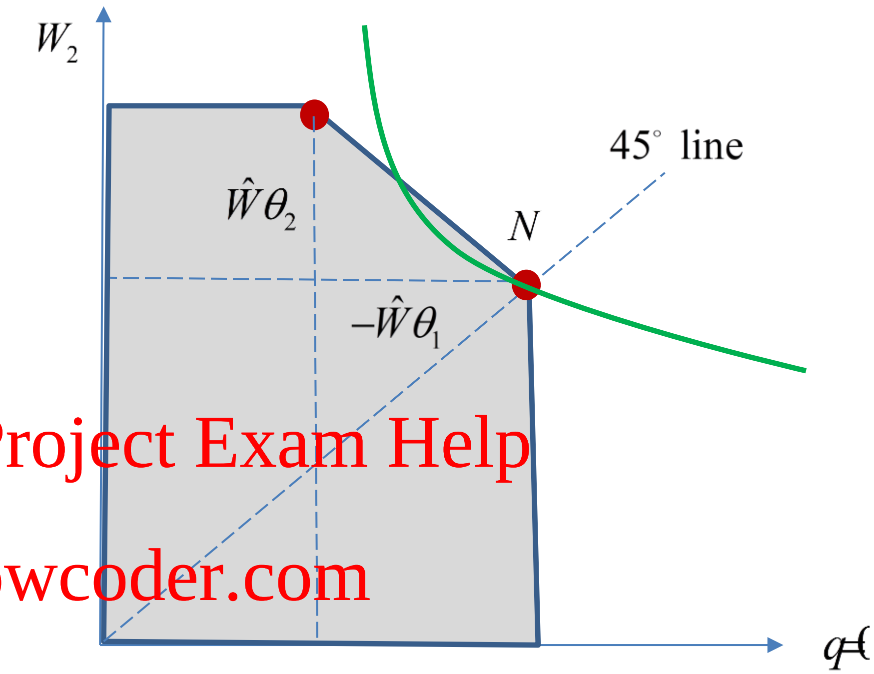
the no risk portfolio is

$$MRS^N = \frac{\pi_1}{\pi_2}$$

*

Two state model

$W_s = \hat{W}(1+r) + q\theta_s$ where $\theta_s = r_s - r_0$.



The steepness of the boundary of the

set of feasible outcomes is $-\dfrac{\theta_2}{\theta_1}$

$$U(W,\pi) = \pi_1 v(W_1) + \pi_2 v(W_2)$$

The steepness of the level set through

the no risk portfolio is

$$MRS^N = \frac{\pi_1}{\pi_2}$$

Purchase some of the risky asset as long as $-\dfrac{\theta_2}{\theta_1} > \dfrac{\pi_1}{\pi_2}$ i.e. $\pi_1\theta_1 + \pi_2\theta_2 > 0$.

The risky asset has a higher expected return

Calculus approach

$$U(q)=\pi_1 v(W_1)+\pi_2 v(W_2)=\pi_1 v(W(1+r_0)+\theta_1 q)+\pi_2 v(W(1+r_0)+\theta_2 q)$$

Where

$\theta_1 = r_1 - r_0$ and $\theta_2 = r_2 - r_0$

$$U'(q)=\pi_1\theta_1 v'(W(1+r_0)+\theta_1 q)+\pi_2\theta_2 v'(W(1+r_0)+\theta_2 q)$$

*

Calculus approach

$$U(q) = \pi_1 v(W_1) + \pi_2 v(W_2) = \pi_1 v(W(1+r_0) + \theta_1 q) + \pi_2 v(W(1+r_0) + \theta_2 q)$$

Where

$\theta_1 = r_1 - r_0$ and $\theta_2 = r_2 - r_0$

$$U'(q) = \pi_1 \theta_1 v'(W(1+r_0) + \theta_1 q) + \pi_2 \theta_2 v'(W(1+r_0) + \theta_2 q)$$

Therefore

$$U'(0) = \pi_1 \theta_1 v'(W(1+r_0)) + \pi_2 \theta_2 v'(W(1+r_0) = (\pi_1 \theta_1 + \pi_2 \theta_2) v'(W(1+r_0))$$

Thus if $q = 0$, then the marginal gain to investing in the risky asset is strictly positive if and only if

$\pi_1 \theta_1 + \pi_2 \theta_2 > 0$,

i.e.

$$\pi_1 (r_1 - r_0) + \pi_2 (r_2 - r_0) = \pi_1 r_1 + \pi_2 r_2 - r_0 > 0$$

i.e. the expected payoff is strictly greater for the risky asset

Class Exercise: Is this still true with more than two states

Exercises (for the TA session)

1. Consumer choice

(a) If $u(x_s) = \ln x_s$ what is the consumer's degree of relative risk aversion?

(b) If there are two states, the consumer's endowment is ω and the state claims price vector is p, solve for the expected utility maximizing consumption.

(c) Confirm that if $\dfrac{p_1}{p_2} > \dfrac{\pi_1}{\pi_2}$ then the consumer will purchase more state 2 claims than state 1 claims.

2 . Consumer choice

(a), (b), (c) as in Exercise 1 except that $u(x_s) = x_s^{1/2}$.

(d) Try to compare the state claims consumption ratio in Exercise 1 with that in Exercise 2.

(e) Provide the intuition for your conclusion.

3. Equilibrium with social risk.

Suppose that both consumers have the same expected utility function

$$U_h(x,\pi) = \pi_1 \ln x_1^h + \pi_2 \ln x_2^h \, .$$

The aggregate endowment is $\omega = (\omega_1, \omega_2)$ where $\omega_1 > \omega_2$.

(a) Solve for the WE price ratio $\frac{p_1}{p_2}$.

(b) Explain why $\frac{p_1}{p_2} < \frac{\pi_1}{\pi_2}$.

4. Equilibrium with social risk.

Suppose that both consumers have the same expected utility function

$$U_h(x,\pi) = \pi_1 (x_1^h)^{1/2} + \pi_2 (x_2^h)^{1/2} .$$

The aggregate endowment is $\omega = (\omega_1, \omega_2)$ where $\omega_1 > \omega_2$.

(a) Solve for the WE price ratio $\frac{p_1}{p_2}$.

(b) Compare the equilibrium price ratio and allocations in this and the previous exercise and provide some intuition.