MP, MS, DT.

F70TS2 – Time Series

Solution to Exercise Sheet 1 – Stationarity and the autocorrelation function

Solution 1 To calculate the autocorrelation function we first calculate the autocovariance function

$$\begin{aligned}\gamma(0) &= \mathrm{Var}(X_t) = \mathrm{E}[X_t^2] \\ &= \mathrm{E}[(\beta\epsilon_{t-1} + \epsilon_t)^2] \\ &= \mathrm{E}[\beta^2\epsilon_{t-1}^2 + \epsilon_t^2 + 2\beta\epsilon_{t-1}\epsilon_t] \\ &= \beta^2\mathrm{E}[\epsilon_{t-1}^2] + \mathrm{E}[\epsilon_t^2] \\ &= (1+\beta^2)\sigma_\epsilon^2\,,\end{aligned}$$

and

$$\begin{aligned}\gamma(\pm 1) &= \mathrm{E}[(\beta\epsilon_{t-2} + \epsilon_{t-1})(\beta\epsilon_{t-1} + \epsilon_t)] \\ &= \mathrm{E}[\beta\epsilon_{t-1}^2] \\ &= \beta\sigma_\epsilon^2\,,\end{aligned}$$

and

$$\begin{aligned}\gamma(\pm k) &= \mathrm{E}[(\beta\epsilon_{t-k-1} + \epsilon_{t-k})(\beta\epsilon_{t-1} + \epsilon_t)] \\ &= 0\,,\end{aligned}$$

for $k \geq 2$ because of $\{\epsilon_t\}$ is i.i.d with mean zero. Hence, the autocorrelation function is $\rho(1) = 1$, $\rho(\pm 1) = \frac{\beta}{1+\beta^2}$, and $\rho(\pm k) = 0$ for $k \geq 2$. For $|\beta| < 1$ it holds $1+\beta^2 > 2\beta$ and $1+\beta^2 > -2\beta$. So $-\frac{1}{2} < \frac{\beta}{1+\beta^2} < \frac{1}{2}$.

Solution 2

We present a detailed solution for (c) which is the most difficult case. The solutions to (a) and (b) are derived by following the same steps, and are omitted as a result.

Note that ϵ_t are iid with mean zero.

$$\begin{aligned}\gamma(0) &= \mathrm{var}\,(X_t) = \mathrm{var}\,(\epsilon_t + 0.6\epsilon_{t-1} + 0.3\epsilon_{t-2} - 0.2\epsilon_{t-3}) \\ &= \mathrm{var}\,(\epsilon_t) + \mathrm{var}\,(0.6\epsilon_{t-1}) + \mathrm{var}\,(0.3\epsilon_{t-2}) + \mathrm{var}\,(-0.2\epsilon_{t-3}) = 1.49\sigma_\epsilon^2.\end{aligned}$$

$$\begin{aligned}\gamma(\pm 1) &= \mathrm{cov}\,(X_t, X_{t+1}) = \mathrm{cov}\,(\epsilon_t + 0.6\epsilon_{t-1} + 0.3\epsilon_{t-2} - 0.2\epsilon_{t-3}, \epsilon_{t+1} + 0.6\epsilon_t + 0.3\epsilon_{t-1} - 0.2\epsilon_{t-2}) \\ &= 1 * 0.6\mathrm{var}\,(\epsilon_t) + 0.6 * 0.3\mathrm{var}\,(\epsilon_{t-1}) - 0.3 * 0.2\mathrm{var}\,(\epsilon_{t-2}) = 0.72\sigma_\epsilon^2.\end{aligned}$$

$$\begin{aligned}\gamma(\pm 2) &= \mathrm{cov}\,(X_t, X_{t+2}) = \mathrm{cov}\,(\epsilon_t + 0.6\epsilon_{t-1} + 0.3\epsilon_{t-2} - 0.2\epsilon_{t-3}, \epsilon_{t+2} + 0.6\epsilon_{t+1} + 0.3\epsilon_t - 0.2\epsilon_{t-1}) \\ &= 1 * 0.3\mathrm{var}\,(\epsilon_t) - 0.6 * 0.2\mathrm{var}\,(\epsilon_{t-1}) = 0.18\sigma_\epsilon^2.\end{aligned}$$

$$\begin{aligned}\gamma(\pm 3) &= \mathrm{cov}\,(X_t, X_{t+3}) = \mathrm{cov}\,(\epsilon_t + 0.6\epsilon_{t-1} + 0.3\epsilon_{t-2} - 0.2\epsilon_{t-3}, \epsilon_{t+3} + 0.6\epsilon_{t+2} + 0.3\epsilon_{t+1} - 0.2\epsilon_t) \\ &= -1 * 0.2\mathrm{var}\,(\epsilon_t) = -0.2\sigma_\epsilon^2.\end{aligned}$$

$\gamma(\pm k) = 0$ for $k > 3$.

And $\rho(0) = 1$, $\rho(\pm 1) = 0.483$, $\rho(\pm 2) = 0.121$, $\rho(\pm 3) = -0.134$, $\rho(\pm k) = 0$ for $k > 3$.

Solution 3

(i) Since $E[Z_t] = 0$, $E[Y_t] = E[Y_{t-1}]$ for all t, and so the mean of the process is constant. But

$$\mathrm{Var}(Y_t) = \mathrm{Var}(Y_{t-1}) + \mathrm{Var}(Z_t) + 2\mathrm{Cov}(Y_{t-1}, Z_t) = \mathrm{Var}(Y_{t-1}) + \mathrm{Var}(Z_t)$$

So, for stationarity we would need $\mathrm{Var}(Z_t) = 0$, which is not the case. Hence, the process is not stationary. It is, in fact, a random walk.

(ii) $E[Y_t] = E[Y_{t-1}] + \alpha$ so for the mean to be constant we would need $\alpha = 0$, which is not the case. Hence, the process is not stationary. It is a random walk with deterministic drift.

(iii) $E[Y_t] = \alpha E[Y_{t-1}]$, so for stationarity in mean we have $\mu = \alpha\mu$, so $\mu = 0$.

$$\mathrm{Var}(Y_t) = \alpha^2 \mathrm{Var}(Y_{t-1}) + \mathrm{Var}(Z_t) + 2\mathrm{Cov}(Y_{t-1}, Z_t) = \alpha^2 \mathrm{Var}(Y_{t-1}) + \mathrm{Var}(Z_t)$$

So for stationarity we have have $\gamma_0 = \alpha^2\gamma_0 + \sigma_Z^2$, so $\gamma_0 = \sigma_Z^2/(1-\alpha^2)$. Note that the variance γ_0 increases as $|\alpha|$ increases and $\gamma_0 \to \sigma_Z^2$ as $|\alpha| \to 0$. With $\alpha = 0$, Y_t is itself just a noise process.

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t+k}) &= \mathrm{Cov}(Y_t, \alpha Y_{t+k-1} + Z_{t+k}) \\ &= \alpha\mathrm{Cov}(Y_t, Y_{t+k-1}) + \mathrm{Cov}(Y_t, Z_{t+k}) = \alpha\mathrm{Cov}(Y_t, Y_{t+k-1})\end{aligned}$$

for $k \geq 1$. So, for stationarity we have $\gamma_k = \alpha\gamma_{k-1}$ for $k \geq 1$. That is, $\gamma_1 = \alpha\gamma_0$, $\gamma_2 = \alpha\gamma_1 = \alpha^2\gamma_0$ and generally $\gamma_k = \alpha^k\gamma_0$. This gives γ_k which depends only on k, not on t, so the process is stationary.

In fact, this is a 'Markov' time series process, an autoregressive process of order 1.

(iv) $E[Y_t] = E[Z_{t-1}]E[Y_{t-2}] + E[Z_t] = 0$, so the process is stationary in mean.

We have $Y_t^2 = Z_{t-1}^2 Y_{t-2}^2 + Z_t^2 + 2Z_t Z_{t-1} Y_{t-2}$, and so

$$E[Y_t^2] = E[Z_{t-1}^2]E[Y_{t-2}^2] + E[Z_t^2] + 2E[Z_t]E[Z_{t-1}]E[Y_{t-2}]$$

So for stationarity we would have $\gamma_0 = \gamma_0 + 1$, which is not possible for finite γ_0. Hence, the process is not stationary.

Solution 4 $E[Y_t] = E[U_t] + E[V_t] = \mu_U + \mu_V$. This does not depend on t and so Y is stationary in mean.

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t+k}) &= \mathrm{Cov}(U_t + V_t, U_{t+k} + V_{t+k}) \\ &= \mathrm{Cov}(U_t, U_{t+k}) + \mathrm{Cov}(U_t, V_{t+k}) + \mathrm{Cov}(V_t, U_{t+k}) + \mathrm{Cov}(V_t, V_{t+k}) \\ &= \gamma_k^U + 0 + 0 + \gamma_k^V\end{aligned}$$

since the U and V processes are independent. So, $\{Y_t\}$ is stationary in second–order moments and hence (weakly) stationary.

Solution 5 From Question 4, $\{Y_t\}$ is stationary with $\gamma_k^Y = \gamma_k^U + \gamma_k^Z$. Recall that $\gamma_k^Z = 0$ for $k \geq 1$. Hence, $\sigma_Y^2 = \gamma_0^Y = \gamma_0^U + \gamma_0^Z = \sigma_U^2 + \sigma_Z^2$ and $\gamma_k^Y = \gamma_k^U$ for $k \geq 1$. Therefore

$$\rho_k^Y = \frac{\gamma_k^Y}{\gamma_0^Y} = \frac{\gamma_k^U}{\sigma_U^2 + \sigma_Z^2} = \frac{\rho_k^U}{1 + \frac{\sigma_Z^2}{\sigma_U^2}} = \frac{\rho_k^U}{1 + \frac{1}{\mathrm{SNR}}}$$

for $k \geq 1$. In practice we see that $\rho_k^Y < \rho_k^U$, i.e. the autocorrelation of the observed signal is less than that of the pure signal.

If the noise variance is small compared to that of the signal, the signal is not disturbed very much: the SNR is large and we have $\rho_k^Y \approx \rho_k^U$ (the observed signal has much the same autocorrelation structure as the pure signal). If the noise variance is of the same order as that of the signal then $\rho_k^Y \approx 0.5\rho_k^U$. If the noise variance is large compared to that of the signal, the signal is seriously disturbed: the SNR is small and ρ_k^Y is much smaller than ρ_k^U (the observed signal has much weaker autocorrelation structure than the pure signal). In general, increasing the variance of the noise imposed on the pure signal weakens the autocorrelation structure of the observed signal.