MP, MS, DT.

F70TS2 – Time Series
Exercise Sheet 1 – Stationarity and the autocorrelation function

Question 1 *Let $\{\epsilon_t\}$ be iid random variables with $E(\epsilon_t) = 0$ and $var(\epsilon_t) = \sigma_\epsilon^2$ (White Noise). We define a process X by*

$$X_t = \beta\epsilon_{t-1} + \epsilon_t \text{ with } |\psi| < 1.$$

Show that X is weakly stationary and that the autocorrelation function of X has the form

$$\rho_X(k) = \begin{cases} 1, & k = 0, \\ \rho(\pm 1), & k = \pm 1, \\ 0, & |k| > 1. \end{cases} \tag{1}$$

Calculate $\rho_X(\pm 1)$ in terms of β and show that $-\frac{1}{2} < \rho_X(\pm 1) < \frac{1}{2}$.

Question 2 *Let $\{\epsilon_t\}$ be a white noise. Calculate the acf of the following processes:*

a) $X_t = 0.5\epsilon_{t-1} + 0.4\epsilon_{t-2} + \epsilon_t$,

b) $X_t = 0.8\epsilon_{t-1} - 0.2\epsilon_{t-2} + \epsilon_t$,

c) $X_t = 0.6\epsilon_{t-1} + 0.3\epsilon_{t-2} - 0.2\epsilon_{t-3} + \epsilon_t$.

Question 3 *By considering first and second order moments (i.e. means, variances, covariances), investigate whether or not each of the following processes is (weakly) stationary. As always, $\{Z_t\}$ is a white noise process.*

(i) $Y_t = Y_{t-1} + Z_t$

(ii) $Y_t = Y_{t-1} + \alpha + Z_t \quad (\alpha \neq 0)$

(iii) $Y_t = \alpha Y_{t-1} + Z_t \quad (|\alpha| < 1)$

(iv) $Y_t = Z_{t-1}Y_{t-2} + Z_t$, *where* $\sigma_Z^2 = 1$

Question 4 *Consider a time series process $\{Y_t\}$ that is the sum of two independent stationary processes $\{U_t\}$ and $\{V_t\}$, so $Y_t = U_t + V_t$. Show that $\{Y_t\}$ is a stationary process.*

Question 5 *(Hard question) Consider a stationary process $\{U_t\}$ with acf $\{\rho_k^U\}$. U_t represents a "signal", but because of noise/measurement error/interference on U_t, what we actually observe is $\{Y_t\}$, where $Y_t = U_t + Z_t$. It may be assumed that the signal and the noise are independent processes. Let the "signal to noise ratio" be defined as $SNR = \sigma_U^2/\sigma_Z^2$.*

Show that $\{Y_t\}$ is a stationary process with acf $\{\rho_k^Y\}$ given by:

$$\rho_k^Y = \left(1 + \frac{1}{SNR}\right)^{-1} \rho_k^U, \quad k = 1, 2, 3, \ldots$$

and comment on the result.