F70TS2: Time Series

Exercise Sheet 5

1. We consider three time series. In each case we assume that the unknown model is an AR(1) with unknown mean. We obtain the following statistics:

   (i) $\sum_{t=1}^{n}(x_t-\bar{x})^2 = 1224.79$ and $\sum_{t=1}^{n-1}(x_t-\bar{x})(x_{t+1}-\bar{x}) = 995.11$,

   (ii) $\sum_{t=1}^{n}(x_t-\bar{x})^2 = 703.52$ and $\sum_{t=1}^{n-1}(x_t-\bar{x})(x_{t+1}-\bar{x}) = 459.05$,

   (iii) $\sum_{t=1}^{n}(x_t-\bar{x})^2 = 662.33$ and $\sum_{t=1}^{n-1}(x_t-\bar{x})(x_{t+1}-\bar{x}) = -446.17$.

   In each case $n = 400$. Carry out the following:

   (a) Estimate the unknown ϕ in each case and write down your estimated AR(1) model (by means of a symbol μ for the unknown mean).

   Assume now that $\bar{x} = 10.25$.

   (b) Estimate $\gamma(0)$, σ_ϵ^2, Var$(\bar{x})$ and the approximate 95%-CI of μ for the model in (i).

   (c) Estimate $\gamma(0)$, σ_ϵ^2, Var$(\bar{x})$ and the approximate 95%-CI of μ for the model in (iii).

2. From two time series $x_1,\ldots,x_{500}$ and $y_1,\ldots,y_{500}$ we obtain

   (a) $\sum\limits_{t=1}^{n} x_t = 9677.84$, $\sum\limits_{t=1}^{n}(x_t-\bar{x})^2 = 3630.92$,
   $\sum\limits_{t=1}^{n-1}(x_t-\bar{x})(x_{t+1}-\bar{x}) = 2516.80$ and $\sum\limits_{t=1}^{n-2}(x_t-\bar{x})(x_{t+2}-\bar{x}) = 2183.88$.

   (b) $\sum\limits_{t=1}^{n} y_t = 20050.48$, $\sum\limits_{t=1}^{n}(y_t-\bar{y})^2 = 8515.04$,
   $\sum\limits_{t=1}^{n-1}(y_t-\bar{y})(y_{t+1}-\bar{y}) = 5591.87$ and $\sum\limits_{t=1}^{n-2}(y_t-\bar{y})(y_{t+2}-\bar{y}) = 2211.04$.

   Assume that the unknown models are AR(2) with unknown means. For both time series, estimate and write down the models. Then estimate σ_ϵ^2, Var$(\bar{x})$, Var$(\bar{y})$ and calculate the approximate 95%-CI of μ_X and μ_Y.

3. Suppose we are given time series data $x_1, \ldots, x_{100}$ which satisfies

$$\begin{aligned} \sum_{t=1}^{100} (x_t - \bar{x})^2 &= 247.8\,, \\ \sum_{t=1}^{99} (x_t - \bar{x})(x_{t+1} - \bar{x}) &= -189.1\,, \end{aligned}$$

where $\bar{x}$ is the mean of $x_1, \ldots, x_{100}$. The last eight values of this time series data are shown in the table below.

t	93	94	95	96	97	98	99	100
x_t	1.993	-2.750	3.458	-2.850	3.743	-3.910	2.118	-1.844

a) Calculate the value of the fitted parameters $\widehat{\phi}$ and $\widehat{\sigma}_\epsilon^2$ when an AR(1) model is fitted to the data $x_1, \ldots, x_{100}$.

b) Using this fitted model, calculate 95% forecasting intervals for X_{101} and X_{102}, assuming that the white-noise terms ϵ_t have a Normal distribution.

c) Estimate the limit of the width of the 95% forecasting interval for X_{100+k} as $k \to \infty$.

4. Represent the following one-dimensional AR(2) process in a VAR(1) form:

$$X_t = -0.2X_{t-1} + 0.6X_{t-2} + \epsilon_t.$$

5. Let the ARCH(1) process (X_t) be defined by

$$X_t = \epsilon_t \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}\,,$$

where $\alpha_0 > 0$, $0 < \alpha_1 < 1$, and (ϵ_t) is a white noise process with variance $\sigma_\epsilon^2 = 1$.

Let $\rho(k)$ be the autocorrelation at lag k of the process (X_t).

Throughout this question you may assume that the process (X_t) is weakly stationary and that $E(X_t^4) < \infty$.

a) Calculate $\mathrm{Var}(X_t)$.

b) Find $\rho(k)$ for all k.

c) Show that $\mathrm{Cov}(X_{t-1}^2, X_t^2) = \alpha_1 \mathrm{Var}(X_{t-1}^2)$.