MP, MS, DT.

F70TS2 – Time Series

Solution Exercise Sheet 2 – Moving Average and Autoregressive Processes

Solution 1

1. The acfs for the two considered MA processes are given by

   (i) $\rho_1 = \frac{\beta_1+\beta_1\beta_2}{1+\beta_1^2+\beta_2^2}$, $\rho_2 = \frac{\beta_2}{1+\beta_1^2+\beta_2^2}$ and $\rho_k = 0$ for $k \geq 3$.

   (ii) $\rho_1 = \frac{\beta_1+\beta_1\beta_2+\beta_2\beta_3}{1+\beta_1^2+\beta_2^2+\beta_3^2}$, $\rho_2 = \frac{\beta_2+\beta_1\beta_3}{1+\beta_1^2+\beta_2^2+\beta_3^2}$, $\rho_3 = \frac{\beta_3}{1+\beta_1^2+\beta_2^2+\beta_3^2}$ and $\rho_k = 0$ for $k \geq 4$.

2. Figure 1 below shows the acfs for the coefficient values specified in the exercise sheet

   (i) $\rho_1 = \frac{120}{189} = \frac{40}{63}$, $\rho_2 = \frac{50}{189}$

   (ii) $\rho_1 = \frac{60}{189} = \frac{20}{63}$, $\rho_2 = -\frac{64}{189}$, $\rho_3 = -\frac{30}{189} = -\frac{10}{63}$

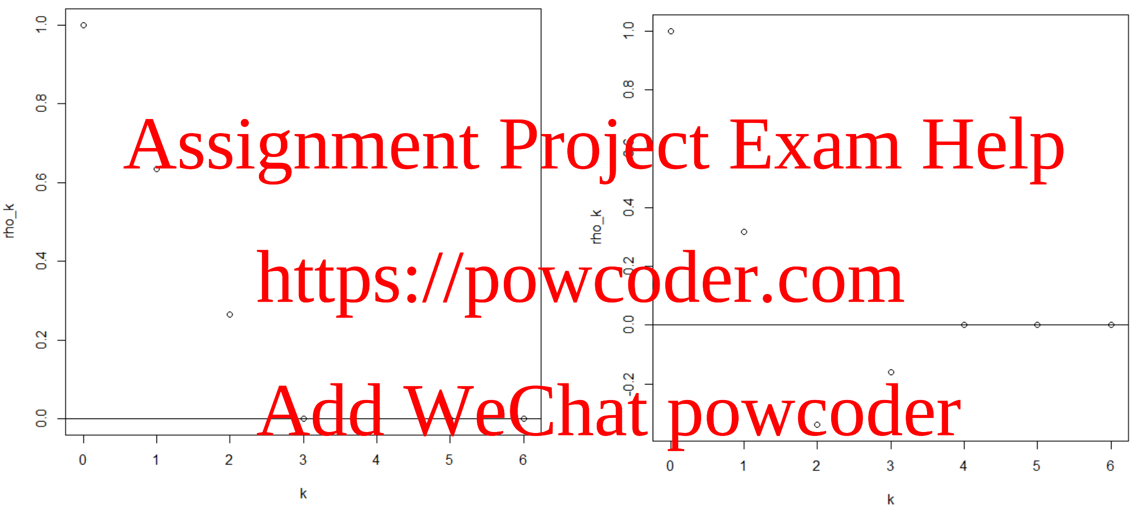

Figure 1: Autocorrelation functions for Q1(b).

Solution 2 For the MA(1) process $Y_t = Z_t + \beta Z_{t-1}$, we have $\rho_1 = \frac{\beta}{1+\beta^2}$. Hence

$$\frac{d\rho_1}{d\beta} = \frac{1-\beta^2}{(1+\beta^2)^2} = 0 \qquad \text{for } \beta = \pm 1 \qquad (\Rightarrow \rho_1 = \pm 1/2)$$

So $|\rho_1| \leq 1/2$.

For the MA(2) process $Y_t = Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}$, $\rho_1 = \frac{\beta_1(1+\beta_2)}{1+\beta_1^2+\beta_2^2}$ and $\rho_2 = \frac{\beta_2}{1+\beta_1^2+\beta_2^2}$. Hence setting $\frac{\partial\rho_1}{\partial\beta_1} = \frac{\partial\rho_1}{\partial\beta_2} = 0$ gives

$$0 = (1+\beta_2)(1-\beta_1^2+\beta_2^2) = \beta_1(1+\beta_1^2-\beta_2^2-2\beta_2)$$

The only valid solutions are

(i) $\beta_1 = 0$, $\beta_2 = -1$ which gives $\rho_1 = 0$

(ii) $1-\beta_1^2+\beta_2^2 = 0$, $1+\beta_1^2-\beta_2^2-2\beta_2 = 0$, i.e. $\beta_1 = \pm\sqrt{2}$ and $\beta_2 = 1$. This gives $\rho_1 = \pm\frac{1}{\sqrt{2}}$.

So $\sup \rho_1 = \frac{1}{\sqrt{2}}$, attained at $\beta_1 = \sqrt{2}$, $\beta_2 = 1$ (when we have $\rho_2 = \frac{1}{4}$). Also, $\inf \rho_1 = -\frac{1}{\sqrt{2}}$ attained at $\beta_1 = -\sqrt{2}$, $\beta_2 = 1$ (when we also have $\rho_2 = \frac{1}{4}$).

For ρ_2, setting $\frac{\partial \rho_2}{\partial \beta_1} = \frac{\partial \rho_2}{\partial \beta_2} = 0$ gives

$$\beta_1 \beta_2 = 1 + \beta_1^2 - \beta_2^2 = 0$$

which gives $\beta_1 = 0$, $\beta_2 = \pm 1$, and $\rho_2 = \pm\frac{1}{2}$.

So $\sup \rho_2 = \frac{1}{2}$, attained at $\beta_1 = 0$, $\beta_2 = 1$ (when we have $\rho_1 = 0$). Also, $\inf \rho_2 = -\frac{1}{2}$ attained at $\beta_1 = 0$, $\beta_2 = -1$ (when we also have $\rho_1 = 0$).

Solution 3 We have to check that the three conditions on α_1 and α_2, so that X_t is causal stationary, hold and then calculate $\rho(k)$ following the general formulas $\rho(0) = 1$, $\rho(1) = \alpha_1/(1-\alpha_2)$, $\rho(k) = \alpha_1 \rho(k-1) + \alpha_2 \rho(k-2)$ for $k \geq 2$. Note that, the causal stationary conditions are certainly fulfilled, if $|\alpha_1| + |\alpha_2| < 1$.

a) We have $\alpha_1 = -1.4$, $\alpha_2 = -0.65$, for which we have $\alpha_1 + \alpha_2 < 1$, $\alpha_2 - \alpha_1 < 1$ and $-1 < \alpha_2 < 1$. Hence this process is causal stationary. Using the above formulas we obtain $\rho(k)$ for $k = 0, 1, ..., 9$:

Lag k	0	1	2	3	4	5	6	7	8	9
$\rho(k)$	1.000	-0.848	0.537	-0.201	-0.067	0.225	-0.271	0.233	-0.150	0.059

b) We have $\alpha_1 = 0.45$, $\alpha_2 = 0.25$. This process is causal stationary, because $|\alpha_1| + |\alpha_2| < 1$. Using the above formulas we obtain $\rho(k)$ for $k = 0, 1, ..., 9$:

Lag k	0	1	2	3	4	5	6	7	8	9
$\rho(k)$	1.000	0.600	0.520	0.384	0.302	0.232	0.180	0.139	0.107	0.083

c) We have $\alpha_1 = 1.2$, $\alpha_2 = -0.75$, for which we have $\alpha_1 + \alpha_2 < 1$, $\alpha_2 - \alpha_1 < 1$ and $-1 < \alpha_2 < 1$. Hence this process is causal stationary. Using the above formulas we obtain $\rho(k)$ for $k = 0, 1, ..., 9$:

Lag k	0	1	2	3	4	5	6	7	8	9
$\rho(k)$	1.000	0.685	0.072	-0.426	-0.566	-0.360	-0.006	0.261	0.319	0.186

Solution 4 The first two Yule–Walker equations are $\rho_1 = \alpha_1 + \alpha_2 \rho_1$ and $\rho_2 = \alpha_1 \rho_1 + \alpha_2$. Hence

$$\rho_1 = \frac{\alpha_1}{1-\alpha_2}, \qquad \rho_2 = \alpha_2 + \frac{\alpha_1^2}{1-\alpha_2}.$$

Similarly,

$$\alpha_1 = \frac{\rho_1(1-\rho_2)}{1-\rho_1^2}, \qquad \alpha_2 = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}.$$

Solution 5

1. We have $0.14z^2 + 0.5z - 1 = 0$, so we have the roots $z_1 = -5$ and $z_2 = 10/7$.

   $|z_i| > 1$ for $i = 1, 2$ and so the process is stationary.

   Hence, from the hint, the solution of the Yule–Walker equations is of the form $\rho_k = a z_1^{-k} + b z_2^{-k}$, subject to $\rho_0 = 1$ and $\rho_1 = -0.5 + 0.14\rho_1$ (from the first Yule–Walker equation). Hence $\rho_1 = -\frac{25}{43}$.

   Solving $a + b = 1$ and $0.2a - 0.7b = -\frac{25}{43}$ gives $a = \frac{17}{129}$ and $b = \frac{112}{129}$. Therefore

   $$\rho_k = \frac{17}{129}(0.2)^k + \frac{112}{129}(-0.7)^k$$

   for $k \geq 0$. Plotting this we obtain the correlogram in Figure 2.

2. We have $-0.6z^2 - 1 = 0$ which has roots $z = \pm i\sqrt{5/3}$ outside the unit circle. Hence we have

$$\rho_k = a(i\sqrt{0.6})^k + b(-i\sqrt{0.6})^k = (0.6^{k/2})i^k\{a + b(-1)^k\}$$

subject to $a+b = 1$ (since $\rho_0 = 1$) and $\rho_1 = -0.6\rho_1$ (from the first Yule–Walker equation), so $\rho_1 = 0$. Hence $a - b = 0$ and so $a = b = \frac{1}{2}$. Therefore

$$\rho_k = \frac{1}{2}i^k(0.6)^{k/2}\{1 + (-1)^k\}$$

for $k \geq 0$. A plot of the acf is given in Figure 3.

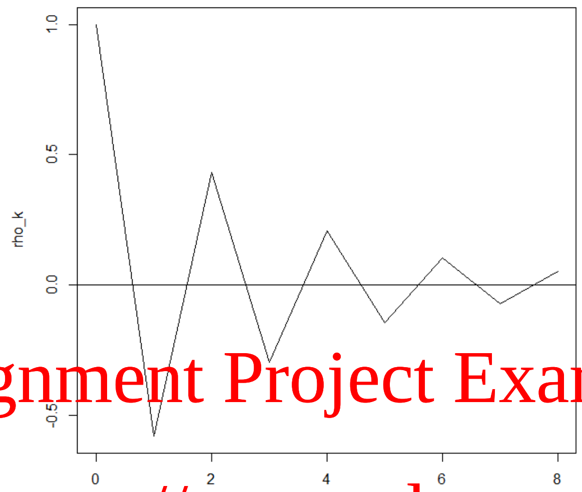

Figure 2: ACF for Q4(a)

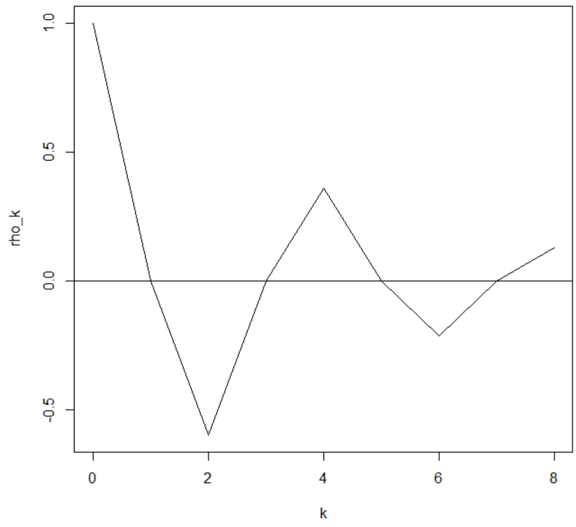

Figure 3: ACF for Q4(b)