F70TS2: Time Series

Exercises 4 - Solutions

1. The asymptotic normality of $\bar{x}$ is a very important property. By means of this property we can construct an approximate confidence interval for μ with a given confidence level α (the so-called interval estimation). For instance, the approximate 95% confidence interval for μ is

$$\mu \in [\bar{x} - 2\text{SD}_{\bar{x}}, \bar{x} + 2\text{SD}_{\bar{x}}].$$

For our question we have $\bar{x} = 1.65$ and $\text{SD}_{\bar{x}} = \sqrt{\text{Var}(\bar{x})} = 0.1$. Inserting these values into the above formula, we have

$$\mu \in [1.56 - 0.2, 1.56 + 0.2] = [1.36, 1.76].$$

2. The general formula for the asymptotic variance of $\bar{x}$ is

$$\text{Var}(\bar{x}) \approx \frac{1}{n} \frac{(\sum_{i=0}^{q} \psi_i)^2}{(1 - \sum_{i=1}^{p} \phi_i)^2},$$

because $\sigma_\epsilon^2 = 1$ by assumption.

For a) we have $\text{Var}(\bar{x}) \approx (1 + 0.4)^2/(1 - 0.5 - 0.2)^2/900 = 0.0242$.

For b) we have $\text{Var}(\bar{x}) \approx (1 + 0.3 + 0.6)^2/(1 - 0.2)^2/900 = 0.00627$.

For c) we have $\text{Var}(\bar{x}) \approx (1 - 0.2)^2/(1 + 0.6)^2/900 = 0.000625$.

For the above examples, n and σ_ϵ^2 are the same. But the asymptotic variances change from case to case. We can see that, for both MA and AR parts, the larger the sums $\sum_{i=1}^{q} \psi_i$ and/or $\sum_{i=1}^{p} \phi_i$ (without ψ_0 and ϕ_0), the larger the asymptotic variance of $\bar{x}$, and vice versa.

Remark: Again, it is easy to see that as $\sum_{i=1}^{q} \psi_i$ tends to -1 then $\text{Var}(\bar{x})$ vanishes. On the other hand, as $\sum_{i=1}^{p} \phi_i$ tends to 1, then $\text{Var}(\bar{x})$ diverges.

3. This is similar to the solution to the last question.

For a) we have $\text{Var}(\bar{x}) \approx (1 - 0.7)^{-2}/400 = 0.02778$.

For b) we have $\text{Var}(\bar{x}) \approx (1 + 0.3)^{-2}/400 = 0.00148$.

For c) we have $\text{Var}(\bar{x}) \approx (1 - 0.45 - 0.3)^{-2}/400 = 0.040$.

Furthermore, we have to calculate $\text{Var}(X_t) = \gamma(0)$ in each case to obtain $\text{Var}(\bar{y})$.

For a) (an AR(1)), $\gamma(0) = 1/(1 - \phi^2) = 1.961$ and $\text{Var}(\bar{y}) = \gamma(0)/n = 0.0049 << \text{Var}(\bar{x})$.

For b) (an AR(1)), $\gamma(0) = 1/(1 - \phi^2) = 0.1.099$ and $\text{Var}(\bar{y}) = \gamma(0)/n = 0.002754 > \text{Var}(\bar{x})$.

For c) (an AR(2))

$$\gamma(0) = \frac{1-\phi_2}{(1+\phi_2)[(1-\phi_2)^2-\phi_1^2]} = \frac{1-0.3}{(1+0.3)[(1-0.3)^2-0.45^2]} = 1.873$$

and $\text{Var}(\bar{y}) = \gamma(0)/n = 0.00468 << \text{Var}(\bar{x})$.

The above models show that the sample mean of a time series may have much larger variance compared with that obtained from independent data with the same variance. However, sometimes the variance of the sample mean of a time series can also be much smaller than that for corresponding i.i.d. data.

4. For $n = 400$, the $\pm\frac{2}{\sqrt{n}}$ bounds are ± 0.1. And 5% or more out of 20 are one or more estimates.

   Hence, in case a) we can say that X_t is probably not an i.i.d. white noise. *Indeed, X_t should also not be uncorrelated.*

   In case b), we can only say that X_t are probably uncorrelated. *However, whether or not they are i.i.d. is not clear. Further diagnosis is required to answer this question. For example, to display the ACF of x_t^2. If X_t are i.i.d., then X_t^2 are also i.i.d.. But, if X_t are only uncorrelated, X_t^2 may be correlated to each other.*

   For $n = 1600$ the above bounds are $\pm\frac{2}{\sqrt{n}} = \pm 0.05$. And 5% out of 40 are 2 estimates. Further decisions are similar to those given above. *Note that, if the process is stationary, the larger n, the better the estimation quality.*