# F70TS2 – Time Series
## Exercise Sheet 3 – MA($\infty$), AR($\infty$), ARMA and ARIMA

**Solution 1** $Y_t = 0.6Y_{t-1} + Z_t - 0.3Z_{t-1} - 0.1Z_{t-2}$. Take $\sigma_Z^2 = 1$ for convenience. Note that $E[Y_t Z_t] = \sigma_Z^2 = 1$. We multiply the defining equation of the model by each of $Y_t$, $Y_{t-1}$ and $Y_{t-2}$ in turn and take expectations to obtain:

- Multiplying by $Y_t$: $\gamma_0 = 0.6\gamma_1 + 1 - 0.3E[Y_t Z_{t-1}] - 0.1E[Y_t Z_{t-2}]$.
- Multiplying by $Y_{t-1}$: $\gamma_1 = 0.6\gamma_0 - 0.3 - 0.1E[Y_{t-1}Z_{t-2}]$.
- Multiplying by $Y_{t-2}$: $\gamma_2 = 0.6\gamma_1 - 0.1$.

Now, multiplying the equation of the model by $Z_{t-1}$ and $Z_{t-2}$ and taking expectations, we have

$$\begin{aligned} E[Y_t Z_{t-1}] &= 0.6 - 0.3 = 0.3 \\ E[Y_t Z_{t-2}] &= 0.6(0.3) - 0.1 = 0.08 \end{aligned}$$

Solving the above equations gives

$$\gamma_0 = 1.1, \qquad \gamma_1 = 0.33, \qquad \gamma_2 = 0.098$$

Hence, $\rho_1 = 0.3$ and $\rho_2 = 0.089$.



Now, multiplying the equation of our model by $Y_{t-k}$ (for $k \geq 3$) and taking expectations, we get that $\gamma_k = 0.6\gamma_{k-1}$. So, $\rho_k = 0.6\rho_{k-1}$ for $k \geq 3$. Solving this gives $\rho_k = a(0.6)^k$ for $k \geq 2$. The value of $a$ may be found using $\rho_2 = 0.089$ (so that $0.36a = 0.089$). This gives $a = \frac{89}{360}$.



Hence we have $\rho_1 = 0.3$ and $\rho_k = \frac{89}{360}(0.6)^k$ for $k \geq 2$. A plot of the autocorrelation function is given in Figure 1.



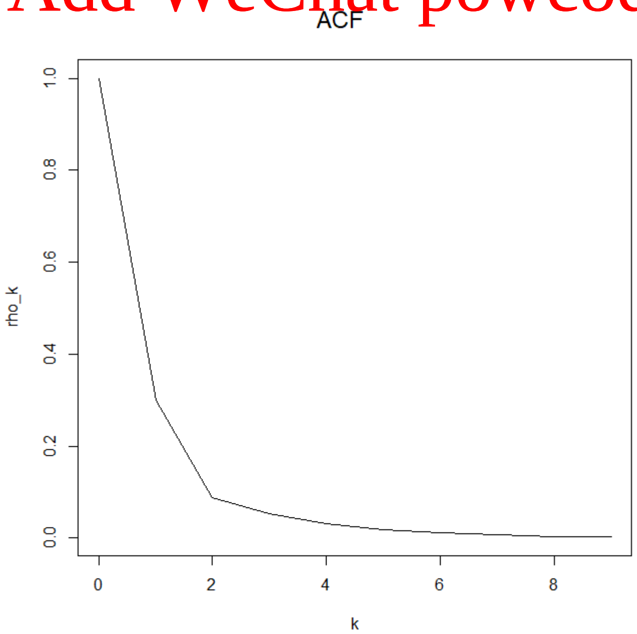


Figure 1: ACF for Q1

**Solution 2** We consider the process

$$Y_t = Z_t + a(Z_{t-1} + Z_{t-2} + \cdots)\,.$$

Clearly $E[Y_t] = 0$ for all $t$. We have

$$\mathrm{Var}(Y_t) = \sigma_Z^2(1 + a^2 + a^2 + a^2 + \cdots)$$

which diverges as $t$ increases. Hence, $\mathrm{Var}(Y_t)$ increases with time. So, the model is non–stationary.

$$\begin{aligned} DY_t &= Y_t - Y_{t-1} \\ &= Z_t + a(Z_{t-1} + Z_{t-2} + \cdots) - Z_{t-1} - a(Z_{t-2} + Z_{t-3} + \cdots) \\ &= Z_t + (a-1)Z_{t-1} \end{aligned}$$

This is an MA(1) process, and is hence stationary. For $\{DY_t\}$, $\rho_1 = \frac{a-1}{1+(a-1)^2}$ and $\rho_k = 0$ for $k \geq 2$.

**Solution 3** We have to check that the three conditions on the MA coefficients $\psi_1$ and $\psi_2$, so that $X_t$ is invertible, hold. Again, these conditions are certainly fulfilled, if $|\psi_1| + |\psi_2| < 1$.

a) We have $\psi_1 = -0.9$, $\psi_2 = 0.2$. And $\psi_1 + \psi_2 > -1$, $\psi_1 - \psi_2 < 1$ and $-1 < \psi_2 < 1$. Hence this process is invertible.



b) We have $\psi_1 = 0.3$, $\psi_2 = -0.6$. Hence this process is clearly invertible, because $|\psi_1| + |\psi_2| < 1$.

c) Now we have an MA(3) process. The explicit conditions for the invertibility in this case are not given. Hence we have to try to find a factorisation of $\psi(z)$. For this special example we have



$$\psi(z) = 1 - 1.5z + 0.75z^2 - 0.125z^3 = (1 - 0.5z)^3$$

with the roots $z_1 = z_2 = z_3 = 2$. This process is invertible.



**Solution 4** We have to check whether the AR part is causal stationary ($\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$ and $-1 < \phi_2 < 1$) and whether the MA part is invertible ($\psi_1 + \psi_2 > -1$, $\psi_1 - \psi_2 < 1$ and $-1 < \psi_2 < 1$).

a) This process is causal stationary and invertible, because all of the above mentioned conditions are fulfilled.

b) This process is causal stationary, because $\phi_1$ and $\phi_2$ satisfy the required conditions. But it is non-invertible, because the MA(1) part is with coefficient $\psi = 1.2$.

c) This process is invertible, because $|\psi_1| + |\psi_2| < 1$ for the MA(2) part. But it is not causal stationary, because $\phi_1 + \phi_2 > 1$ for the AR(2) part.

d) This process is neither causal stationary, because $\phi_1 + \phi_2 > 1$ for the AR(2) part, nor invertible, because $\psi_1 - \psi_2 > 1$ for the MA(2) part.

**Solution 5** For $X_t = \phi_1 X_{t-1} + \epsilon_t$ with $|\phi_1| > 1$ we have $X_{t+1} = \phi_1 X_t + \epsilon_{t+1}$. It follows that

$$X_t = \phi_1^*(X_{t+1} + \epsilon_{t+1}^*),$$

where $\phi_1^* = 1/\phi_1$ with $|\phi_1^*| < 1$ and $\epsilon_{t+1}^* = -\epsilon_{t+1}$. (Note that $\epsilon_{t+i}^*$ is also an i.i.d. series with $E(\epsilon_t^*) = 0$ and $\mathrm{Var}(\epsilon_t^*) = \sigma_\epsilon^2/\phi_1^2$). Further expansion leads to

$$\begin{aligned} X_t &= \phi_1^*(\phi_1^*(X_{t+2} + \epsilon_{t+2}^*) + \epsilon_{t+1}^*) \\ &= \ldots = \sum_{i=1}^{\infty} (\phi_1^*)^i \epsilon_{t+i}^* = \sum_{i=1}^{\infty} \alpha_i \epsilon_{t+i}, \end{aligned}$$

where $\alpha_i = -(\phi_1^*)^i = -(1/\phi)^i$ for $i = 1, 2, \ldots$, which are absolutely summable. Hence $X_t$ is stationary, but non-causal, since it depends on future information ($\epsilon_{t+i}$).