Final Exam Solutions
FIN 538 Fall 2019 Mini A
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Question 1 (40 points in total): 1a. (10 points) Apply Ito's lemma to calculate $\sqrt{t}B_t$. Is $\sqrt{t}B_t$ a martingale? Explain.
Answer. By Ito's lemma,

$$d(t\sqrt{B_t}) = \frac{1}{2\sqrt{t}}B_t dt + \sqrt{t}dB_t$$

$t\sqrt{B_t}$ is not a martingale because the drift term of its differential form is not equal to zero.
1b. (10 points) Suppose that the process $X_t = B_t^3 - atB_t$ is a martingale. Find a and calculate $\mathbb{E}[X_t|\mathcal{F}_s],\ s < t$ for that value of a.
Answer. By Ito's lemma

$$dX_t = (-aB_t + 3B_t)dt + (3B_t^2 - at)dB_t$$

X_t is a martingale if the drift term is zero: $-aB_t + 3B_t = B_t(3 - a) = 0$. Thus $a = 3$. Since $X_t = B_t^3 - 3tB_t$ is a martingale, $\mathbb{E}[X_t|\mathcal{F}_s] = X_s = B_s^3 - 3sB_s$.

1c. (10 points) Calculate the (unconditional) expectation $\mathbb{E}[B_sB_tB_u]$, for $s \leq t \leq u$.
Hint. The following formulas might be useful: If X has the distribution $N(\mu, \sigma^2)$ then $\mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2$.

Answer.

$$\begin{aligned}
\mathbb{E}[B_sB_tB_u] &= \mathbb{E}[B_sB_t(B_u - B_t) + B_sB_t^2] \\
&= \mathbb{E}[B_sB_t(B_u - B_t)] + \mathbb{E}[B_sB_t^2] \\
&= \mathbb{E}[B_sB_t]\mathbb{E}[B_u - B_t] + \mathbb{E}[B_s(B_t - B_s)^2 + 2B_s^2B_t - B_s^3] \\
&= \mathbb{E}[B_s(B_t - B_s)^2] + \mathbb{E}[2B_s^2B_t - B_s^3] \\
&= \mathbb{E}[2B_s^2(B_t - B_s) + B_s^3] \\
&= \mathbb{E}[B_s^3] \\
&= 0
\end{aligned}$$

1d. (10 points) Apply Ito's lemma to write the differential form for $X_t = B_t e^{\int_0^t B_s ds}$. Calculate the expected rate of return $\frac{1}{dt}\mathbb{E}_t[\frac{dX_t}{X_t}]$.
Answer. Let $Y_t = \int_0^t B_s ds$, and $Z_t = e^{Y_t} = e^{\int_0^t B_s ds}$ so that $X_t = B_tZ_t$. We have $dY_t = B_t dt$ and so $dZ_t = B_tZ_t dt$. Therefore

$$\begin{aligned}
dX_t = d(B_tZ_t) &= B_t dZ_t + Z_t dB_t + dZ_t dB_t \\
&= B_t^2 Z_t dt + Z_t dB_t + B_tZ_t dt dB_t \\
&= Z_t(B_t^2 dt + dB_t)
\end{aligned}$$

This implies $\frac{dX_t}{X_t} = B_t dt + \frac{1}{B_t}dB_t$, so that

$$\frac{1}{dt}\mathbb{E}_t[\frac{dX_t}{X_t}] = B_t$$

Question 2 (20 points in total): Let's consider a world with only two dates: Today and Tomorrow. There are three possible states tomorrow: Burst, Normal, and Boom. We have three risky stocks X, Y, Z traded in the market. The current prices and future possible payoffs of these risky stocks, if they are known, are reported in the following table

Asset	Today's price	Tomorrow payoff		
		Burst	Normal	Boom
X	\$ 2	\$ 1	\$ 2	\$ 3
Y	\$ 2	\$ 4	\$ 0	\$ 0
Z	???	\$ 0	\$ 1	\$ 2

The (net) risk-free rate in the market is given to be $r = 0\%$. Assume that there are no arbitrage opportunities in the market.
2.a (10 points) What are the risk neutral probabilities of the states Burst, Normal, Boom?
Answer. Let p_1, p_2, p_3 be the risk neutral probabilities of the states Burst, Normal, Boom correspondingly. Since these are the only possible states tomorrow, $p_1 + p_2 + p_3 = 1$. The risk neutral pricing formula says

$$2 = p_1 + 2p_2 + 3p_3$$
$$2 = 4p_1$$

Solving these equations yield $p_1 = \frac{1}{2}, p_2 = 0, p_3 = \frac{1}{2}$.

2.b (10 points) What is the current price of stock Z?
Answer. The price of the stock Z is equal to

$$0 * p_1 + 1 * p_2 + 2 * p_3 = \$1$$

Question 3 (40 points in total): Suppose that the risk-free rate follows the following stochastic process:

$$dr_t = (1 - r_t)dt + e^{-\frac{t}{2}} dB_t$$

3a. (10 points) Denote $R_t = e^t r_t$. Using Ito's lemma, find the expression for dR_t.
Answer.

$$dR_t = (e^t r_t + e^t(1 - r_t))dt + e^t e^{-\frac{t}{2}} dB_t = e^t dt + e^{\frac{t}{2}} dB_t$$

3b. (10 points) Solve for R_t and then r_t. In your answer, R_t and r_t should be written as a sum of a deterministic term and an Ito integral.
Answer.

$$R_t = r_0 + e^t - 1 + \int_0^t e^{\frac{s}{2}} dB_s$$

So

$$r_t = r_0 e^{-t} + 1 - e^{-t} + e^{-t} \int_0^t e^{\frac{s}{2}} dB_s$$

3c. (10 points) Calculate $\mathbb{E}[r_t]$. When t approaches infinity, what does $\mathbb{E}[r_t]$ approach to? Please show your work.

Answer.

$$\mathbb{E}[r_t] = r_0 e^{-t} + 1 - e^{-t}, \lim_{t\to\infty} \mathbb{E}[r_t] = 1$$

3d. (10 points) Calculate $Var(r_t)$. When t approaches infinity, what does $Var(r_t)$ approach to? Please show your work.

Answer.

$$Var(r_t) = e^{-2t} Var\Big[\int_0^t e^{\frac{s}{2}} dB_s\Big]$$

where

$$Var\Big[\int_0^t e^{\frac{s}{2}} dB_s\Big] = \int_0^t e^s ds = e^t - 1$$

Therefore

$$\lim_{t\to\infty} Var[r_t] = \lim_{t\to\infty} e^{-2t}(e^t - 1) = 0$$

Question 4 (5 points): Calculate $\mathbb{E}[\big(\int_0^T s dB_s\big) * \big(\int_0^T s^2 dB_s\big)]$, where $T > 0$ and B_t is a standard Brownian motion.

Answer. We have $\mathbb{E}[\big(\int_0^T s dB_s\big) * \big(\int_0^T s^2 dB_s\big)] =$

$$= \lim_{n\to\infty} \mathbb{E}[\big(\sum_{i=1}^n s_{i-1}(B_{s_i} - B_{s_{i-1}})\big) * \big(\sum_{j=1}^n s_{j-1}^2 (B_{s_j} - B_{s_{j-1}})\big)]$$

$$= \lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^n s_{i-1}^3 (B_{s_i} - B_{s_{i-1}})^2 + 2 \sum_{1\le i<j\le n} s_{i-1} s_{j-1}^2 (B_{s_i} - B_{s_{i-1}})(B_{s_j} - B_{s_{j-1}})]$$

$$= \lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^n s_{i-1}^3 (B_{s_i} - B_{s_{i-1}})^2]$$

$$= \lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^n s_{i-1}^3 (s_i - s_{i-1})]$$

$$= \int_0^T s^3 ds$$

$$= \frac{T^4}{4}$$