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Stochastic Foundation of Finance (FIN 538)
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Problem Set 1

(*Please submit to Canvas on Wed, Sep 18 before class. TA: David Zhong, kemingzhong@wustl.edu*)

Problem 1 (An extension of Example 1 of Lecture 1) Suppose the the economy in Example 1, Lecture 1 lasts for three quarters. Similar to Example 4 of Lecture 1, consider a security that pays $d_t = \$1$ if the economy state in quarter t is G and $d_t = \$0$ if the economy state in quarter t is B. Let $X = d_1 + d_2 + d_3$ (assuming $d_0 = 0$).

1. What is the sample space?
   Answer: $\Omega = \{GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB\}$.

2. What is the filtration that corresponds to the σ-algebras $\mathcal{F}_t$ at $t = 0, 1, 2, 3$?
   Answer: $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \Omega, \{GGG, GGB, GBG, GBB\}, \{BGG, BGB, BBG, BBB\}\}$,$\mathcal{F}_2 = \{\emptyset, \Omega, \{GGG, GGB\}, \{GBG, GBB\}, \{BGG, BGB\}, \{BBG, BBB\}$, all possible unions of these sets$\}$, $\mathcal{F}_3 = 2^\Omega$, where 2^Ω is the set of all subsets of Ω.

3. Calculate the conditional expectations $Z_t = \mathbb{E}[X|\mathcal{F}_t]$ (note that Z_t's are random variables themselves).
   Answer: $Z_0 = \mathbb{E}[X|\mathcal{F}_0] = \mathbb{E}[X] = 3 * q^3 + 2 * 3 * q^2(1-q) + 1 * 3 * q(1-q)^2 + 0 * (1-q)^3 = 3q$
   $Z_1(GGG) = Z_1(GGB) = Z_1(GBG) = Z_1(GBB) = \mathbb{E}[X|\{GGG, GGB, GBG, GBB\}] = 1 + 2 * q^2 + 1 * 2q(1-q) = 1 + 2q$
   $Z_1(BGG) = Z_1(BGB) = Z_1(BBG) = Z_1(BBB) = \mathbb{E}[X|\{BGG, BGB, BBG, BBB\}] = 2q^2(1-q) + 2q(1-q)^2 = 2q$

$$Z_2(GGG) = Z_2(GGB) = \mathbb{E}[X|\{GGG, GGB\}] = 2 + q$$
$$Z_2(GBG) = Z_2(GBB) = \mathbb{E}[X|\{GBG, GBB\}] = 1 + q$$
$$Z_2(BGG) = Z_2(BGB) = \mathbb{E}[X|\{BGG, BGB\}] = 1 + q$$
$$Z_2(BBG) = Z_2(BBB) = \mathbb{E}[X|\{BBG, BBB\}] = q$$

   and $Z_3 = \mathbb{E}[X|\mathcal{F}_3] = \mathbb{E}[X|2^\Omega] = X$.

4. Find the σ-algebras generated by $Z_t,\ t = 0, 1, 2, 3$.
   Answer: $\sigma(Z_0) = \mathcal{F}_0$, $\sigma(Z_1) = \mathcal{F}_1$, $\sigma(Z_2) = \{\emptyset, \Omega, \{GGG, GGB\}, \{GBG, GBB, BGG, BGB\}, \{BBG, BBB\}$, all possible unions of these sets$\}$, note that $\sigma(Z_2) \neq \mathcal{F}_2$. $\sigma(Z_3)$ is generated by the following set

$$\{\{GGG\}, \{GGB, GBG, BGG\}, \{GBB, BGB, BBG\}, \{BBB\}\}$$

Problem 2 Let $(\Omega, \mathcal{F}, P)$ be a probability space where $\Omega = \{a, b, c, d, e, f\}$ and $\mathcal{F} = 2^{\Omega}$, P is uniform. Consider the following random variables

$$X(a) = X(b) = 1, X(c) = X(d) = 3, X(e) = X(f) = 5$$

$$Y(a) = Y(b) = Y(c) = 2, Y(d) = Y(e) = 4, Y(f) = 6$$

Calculate $Z_1 = \mathbb{E}[X|Y]$ and $Z_2 = \mathbb{E}[Y|X]$.

Answer.

$$\begin{aligned}
Z_1(a) &= \mathbb{E}[X|Y(a)] = \mathbb{E}[X|Y = 2] = \mathbb{E}[X|\{a, b, c\}] \\
&= P(a|\{a, b, c\}) * X(a) + P(b|\{a, b, c\}) * X(b) + P(c|\{a, b, c\}) * X(c) \\
&= \frac{5}{3}
\end{aligned}$$

Similarly, $Z_1(b) = Z_1(c) = \frac{5}{3}$. $Z_1(d) = Z_1(e) = 4$, $Z_1(f) = 5$.
$Z_2(a) = Z_2(b) = 2, Z_1(c) = Z_1(d) = 3, Z_1(e) = Z_1(f) = 5$.

Problem 3 Consider the geometric Brownian motion

$$X_t = e^{\mu t + \sigma B_t}$$

Let $\mathcal{F}_t$ be the natural filtration associated with B_t. Compute $\mathbb{E}[X_t]$, $Var\,[X_t]$, $\mathbb{E}\,[X_t|\mathcal{F}_s]$, $Var\,[X_t|\mathcal{F}_s]$ for $s < t$. Show your calculations.

Answer. $X_t = e^{N(\mu t, \sigma^2 t)}$.

$$\begin{aligned}
\mathbb{E}[X_t] &= \int_0^{\infty} e^x e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} dx = \int_0^{\infty} e^{-\frac{x^2 - 2(\mu+\sigma^2)tx + \mu^2 t^2}{2\sigma^2 t}} dx = \int_0^{\infty} e^{-\frac{x^2 - 2(\mu+\sigma^2)tx + \mu^2 t^2}{2\sigma^2 t}} dx \\
&= \int_0^{\infty} e^{-\frac{(x-(\mu+\sigma^2)t)^2 + \mu^2 t^2 - (\mu+\sigma^2)^2 t^2}{2\sigma^2 t}} dx = e^{(\mu + \frac{\sigma^2}{2})t} \\
Var(X_t) &= \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = \mathbb{E}[e^{2\mu t + 2\sigma B_t}] - e^{(2\mu+\sigma^2)t} = e^{(2\mu + 2\sigma^2)t} - e^{(2\mu+\sigma^2)t} \\
\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}[e^{\mu t + \sigma(B_t - B_s) + \sigma B_s}|\mathcal{F}_s] = e^{\mu s + \sigma B_s}\mathbb{E}[e^{\mu(t-s) + \sigma(B_t - B_s)}] = X_s e^{(\mu + \frac{\sigma^2}{2})(t-s)} \\
Var[X_t|\mathcal{F}_s] &= \mathbb{E}[X_t^2|\mathcal{F}_s] - \mathbb{E}[X_t|\mathcal{F}_s]^2 = X_s^2\{e^{(2\mu+2\sigma^2)(t-s)} - e^{(2\mu+\sigma^2)(t-s)}\}
\end{aligned}$$

Problem 4 Which of the following three processes is adapted to the natural filtration of the Brownian motion: $X_t = B_t^2 + t + 1, Y_t = B_t + B_{t+\frac{1}{2}}, Z_t = \min_{s \le t} B_s$?

Answer. X_t, Z_t.

Problem 5 Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_t$ be a filtration. Define a continuous-time stochastic process Y_t by $Y_t = \mathbb{E}[X|\mathcal{F}_t]$. Show that Y_t is a martingale with respect to $\mathcal{F}_t$.

Answer. Assuming $\mathbb{E}[|X|] < \infty$, then $\mathbb{E}[|Y_t|] = \mathbb{E}[|\mathbb{E}[X|\mathcal{F}_t]|] \le \mathbb{E}[\mathbb{E}[|X||\mathcal{F}_t]] = \mathbb{E}[|X|] < \infty$ (law of iterated expectation). Y_t is adapted to $\mathcal{F}_t$ by definition and by law of iterated expectation,

$$\mathbb{E}[Y_t|\mathcal{F}_s] = \mathbb{E}[X_t|\mathcal{F}_s] = Y_s$$

Problem 6 1. Show that if X, Y are independent random variables then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

2. Calculate $\mathbb{E}[B_t B_s]$ for $t \geq s$ where B_t is the standard Brownian motion.

3. Calculate $Var[B_t + B_s]$ for $t \geq s$ where B_t is the standard Brownian motion.

Answer.

1. We showed this in class.

2. $\mathbb{E}[B_t B_s] = \mathbb{E}[(B_t - B_s)B_s + B_s^2] = \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}[B_s^2] = 0 + s = s.$

3. $Var(B_t + B_s) = Var(B_t) + Var(B_s) + 2Cov(B_t, B_s) = t + s + 2(\mathbb{E}[B_t B_s] - \mathbb{E}[B_t]\mathbb{E}[B_s]) = t + 3s.$

Problem 7 1. Calculate $\mathbb{E}[B_t B_s | \mathcal{F}_s]$ for $t \geq s$ where B_t is the standard Brownian motion..

2. Calculate $\mathbb{E}[B_t^2 | \mathcal{F}_s]$ for $t \geq s$ where B_t is the standard Brownian motion..

3. Suppose that $B_t^2 - ct$ is a martingale with respect to the natural filtration of the Brownian motion B_t. Find c.

Answer.

1. $\mathbb{E}[B_t B_s | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)B_s + B_s^2 | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)B_s | \mathcal{F}_s] + B_s^2 = B_s \mathbb{E}[(B_t - B_s | \mathcal{F}_s] + B_s^2 = B_s^2.$

2. $\mathbb{E}[B_t^2 | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2 + 2B_t B_s - B_s^2 | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2 | \mathcal{F}_s] + 2\mathbb{E}[B_t B_s | \mathcal{F}_s] - B_s^2 = \mathbb{E}[(B_t - B_s)^2] + B_s^2 = Var(B_t - B_s) + B_s^2 = t - s + B_s^2.$

3. $\mathbb{E}[B_t^2 - ct]$ is a martingale if $\mathbb{E}[B_t^2 - ct | \mathcal{F}_f] = B_s^2 - cs$. Consider $\mathbb{E}[B_t^2 - ct | \mathcal{F}_f] = B_s^2 + t - s - ct$, which is equal to $B_s^2 - cs$ if $c = 1$.