Olin Business School
Washington University in St. Louis
Stochastic Foundation of Finance (FIN 538)
Master of Finance Program
Summer 2021
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Student's name: **Student's ID:** **Section:**

FINAL EXAM (Duration: 120 min)

1. (30 points) Let $X_i, i = 1, 2$ be independent random variables that have the following distribution: $X_i = 1$ with probability $\frac{3}{4}$ and $X_i = 0$ with probability $\frac{1}{4}$:

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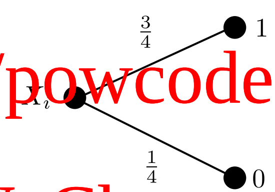

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Define

$$B_1 = X_1$$
$$B_2 = X_1 + X_2$$

Denote an upward move of X_i by H and a downward move by L. Consider the following sample space for the above random variables:

$$\Omega = \{HH, HL, LH, LL\}$$

Let $\mathcal{F}_i,\ i = 1, 2$ be the natural filtration associated with the process B_i.

(a) (15 points) Describe $Z_1 = \mathbb{E}[B_2|B_1]$ as a map from Ω into the set of real numbers $\mathbb{R}$. Is the process $B_i, i = 1, 2$ a martingale? Why or why not?

Answer. We have

$$
\begin{aligned}
Z_1(HH) &= \mathbb{E}[B_2|B_1](HH) = \mathbb{E}[B_2|B_1 = 1] = \mathbb{E}[B_2|\{HH, HL\}] \\
&= 1 + \frac{3}{4} * 1 + \frac{1}{4} * 0 \\
&= \frac{7}{4} \\
Z_1(HL) &= Z_1(HH) = \frac{7}{4}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Z_1(LH) &= \mathbb{E}[B_2|B_1](LH) = \mathbb{E}[B_2|B_1 = 0] = \mathbb{E}[B_2|\{LH, LL\}] \\
&= 0 + \frac{3}{4} * 1 + \frac{1}{4} * 0 \\
&= \frac{3}{4} \\
Z_1(LL) &= Z_1(LH) = \frac{3}{4}
\end{aligned}
$$

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$B_i, i \in \{1, 2\}$ is not a martingale because $B_1 \neq \mathbb{E}[B_2|B_1]$.

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(b) (15 points) Suppose you observed that $B_2 = 1$ but did not observe either X_1 or X_2. What are the expected values of X_1 and X_2 given this information?

Answer. We have

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$$
\begin{aligned}
\mathbb{E}[X_1|B_2 = 1] &= \mathbb{E}[X_1|\{HL, LH\}] \\
&= Prob(HL|\{HL, LH\}X_1(HL) + Prob(LH|\{HL, LH\}X_1(LH) \\
&= \frac{1}{2} * 1 + \frac{1}{2} * 0 \\
&= \frac{1}{2}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}[X_2|B_2 = 1] &= \mathbb{E}[X_2|\{HL, LH\}] \\
&= Prob(HL|\{HL, LH\}X_2(HL) + Prob(LH|\{HL, LH\}X_2(LH) \\
&= \frac{1}{2} * 0 + \frac{1}{2} * 1 \\
&= \frac{1}{2}
\end{aligned}
$$

2. (30 points in total):

(a) (10 points) Find a such that the process $X_t = e^{-t+aB_t}$ is a martingale where B_t is a standard Brownian motion. **Answer**. By Ito's lemma:

$$dX_t = X_t\Big((-1+\frac{a^2}{2})dt + adB_t\Big)$$

X_t is a martingale if and only if the drift term is zero: $a = \sqrt{2}$ or $a = -\sqrt{2}$.

(b) (10 points) Calculate the (unconditional) expectation $\mathbb{E}[(B_s + B_t)B_u]$, for $s \le t \le u$ where B_t is a standard Brownian motion.

Answer. We have $\mathbb{E}[(B_s + B_t)B_u] = \mathbb{E}[B_sB_u] + \mathbb{E}[B_tB_u]$, where

$$\begin{aligned}\mathbb{E}[B_sB_u] &= \mathbb{E}[B_s(B_u - B_s) + B_s^2] \\ &= \mathbb{E}[B_s(B_u - B_s)] + \mathbb{E}[B_s^2] \\ &= \mathbb{E}[B_s]\mathbb{E}[(B_u - B_s] + s \\ &= \boxed{s}\end{aligned}$$

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Similarly,

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$$\begin{aligned}\mathbb{E}[B_tB_u] &= \mathbb{E}[B_t(B_u - B_t) + B_t^2] \\ &= \mathbb{E}[B_t(B_u - B_t)] + \mathbb{E}[B_t^2] \\ &= \mathbb{E}[B_t]\mathbb{E}[(B_u - B_t] + t \\ &= \boxed{t}\end{aligned}$$

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Thus,

$$\boxed{\mathbb{E}[(B_s + B_t)B_u] = s + t}$$

(c) (10 points) Apply Ito's lemma to calculate dX_t for $X_t = B_t^2 e^{\int_0^t B_s^{-2}ds}$ where B_t is a standard Brownian motion. Calculate the instantaneous expected rate of return $\frac{1}{dt}\mathbb{E}_t\left[\frac{dX_t}{X_t}\right]$.

Answer. By the product rule:

$$dX_t = d(B_t^2)e^{\int_0^t B_s^{-2}ds} + B_t^2 d(e^{\int_0^t B_s^{-2}ds}) + d(B_t^2)d(e^{\int_0^t B_s^{-2}ds}) \tag{1}$$

We have

$$d(B_t^2) = 2B_t dB_t + dt$$

Set $Y_t = \int_0^t B_s^{-2}ds$, then $dY_t = B_t^{-2}dt$. Thus

$$d(e^{\int_0^t B_s^{-2}ds}) = de^{Y_t} = B_t^{-2}e^{Y_t}dt = B_t^{-2}e^{\int_0^t B_s^{-2}ds}dt$$

Plug these into (1) we get

$$
\begin{aligned}
dX_t &= (2B_t dB_t + dt)e^{\int_0^t B_s^{-2} ds} dt + B_t^2 B_t^{-2} e^{\int_0^t B_s^{-2} ds} dt + (2B_t dB_t + dt) B_t^{-2} e^{\int_0^t B_s^{-2} ds} dt \\
&= (2B_t dB_t + 2dt) e^{\int_0^t B_s^{-2} ds} dt
\end{aligned}
$$

which implies

$$\frac{dX_t}{X_t} = \frac{2}{B_t} dB_t + \frac{2}{B_t^2} dt$$

Therefore

$$\frac{1}{dt}\mathbb{E}_t\left[\frac{dX_t}{X_t}\right] = \mathbb{E}_t\left[\frac{2}{B_t} dB_t + \frac{2}{B_t^2} dt\right] = \boxed{\frac{2}{B_t^2}}$$

3. (30 points) Consider the price of a security that follows the process in the figure below. At each date $t = 0$ and $t = 1$, the price jumps up with a risk-neutral probability of q and jumps down with a risk-neutral probability of $1 - q$. The risk-free rate is the same across the two periods and the states.

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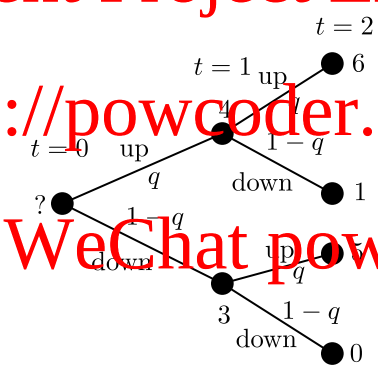

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(a) (15 points) Find the risk-neutral probability q, the risk free rate and the security price at $t = 0$.

Answer. Let r denote the risk free rate. We have from the second period prices:

$$
\begin{aligned}
\frac{4}{1+r} &= 6q + 1 - q = 5q + 1 \\
\frac{3}{1+r} &= 5q
\end{aligned}
$$

Solving the above yields

$$
\begin{aligned}
r &= 0 \\
q &= \frac{3}{5}
\end{aligned}
$$

The security's price at $t = 0$ is

$$S_0 = \frac{3}{5} * 4 + \frac{2}{5} * 3 = \frac{18}{5}$$

(b) (15 points) Find the date-0 price of a call option that matures at the end of the second period and that has a strike price of 4. The call option's payoffs are:

$$\text{If the state is} \begin{cases} \text{upup} : 6 - 4 = 2 \\ \text{updown} : \max(1 - 4, 0) = 0 \\ \text{downup} : 5 - 4 = 1 \\ \text{downdown} : \max(0 - 4, 0) = 0 \end{cases} \tag{2}$$

The date-0 price of this call option is therefore:

$$\frac{3}{5} * \frac{3}{5} * 2 + \frac{2}{5} * \frac{3}{5} * 1 = \frac{24}{25}$$

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4. (10 points) Calculate $Var\Big(\int_0^t e^{B_s} dB_s\Big)$ where B_t is a standard Brownian motion.
Answer. We have

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$$Var\Big(\int_0^t e^{B_s} dB_s\Big) = \mathbb{E}[\Big(\int_0^t e^{B_s} dB_s\Big)^2] - \Big(\mathbb{E}[\int_0^t e^{B_s} dB_s]\Big)^2 = \mathbb{E}[\Big(\int_0^t e^{B_s} dB_s\Big)^2]$$

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Consider a partition $\{t_0 = 0, t_1, \cdots, t_n = t\}$ of the interval $[0, t]$. We have

$$
\begin{aligned}
\mathbb{E}[\Big(\int_0^t e^{B_s} dB_s\Big)^2] &= \lim_{n\to\infty} \mathbb{E}[\Big(\sum_{i=1}^n e^{B_{t_{i-1}}}(B_{t_i} - B_{t_{i-1}})\Big)^2] \\
&= \lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^n e^{2B_{t_{i-1}}}(B_{t_i} - B_{t_{i-1}})^2 + 2\sum_{1\le i<j\le n} e^{B_{t_{i-1}}}(B_{t_i} - B_{t_{i-1}})e^{B_{t_{j-1}}}(B_{t_j} - B_{t_{j-1}})] \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}[e^{2B_{t_{i-1}}}(B_{t_i} - B_{t_{i-1}})^2] \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}[\mathbb{E}[e^{2B_{t_{i-1}}}(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}[e^{2B_{t_{i-1}}}\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}[e^{2B_{t_{i-1}}}(t_i - t_{i-1})] \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}[e^{2t_{i-1}}(t_i - t_{i-1})] \\
&= \int_0^t e^{2s} ds \\
&= \frac{e^{2t} - 1}{2}
\end{aligned}
$$

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