Homework 4 Solutions

Laplacian in Polar Coordinates

Suppose that we have a C^2 function $u : \mathbb{R}^2 \to \mathbb{R}$, and we define a new function $v : [0, \infty) \times [0, 2\pi) \to \mathbb{R}$ by $v(r, \theta) = u(r\cos\theta, r\sin\theta)$. The question is then asking us to prove the identity

$$(u_{xx} + u_{yy})(r\cos\theta, r\sin\theta) = v_{rr}(r,\theta) + \frac{1}{r}v_r(r,\theta) + \frac{1}{r^2}v_{\theta\theta}(r,\theta).$$

To prove this, we compute using the chain rule:

$$v_r(r,\theta) = u_x(r\cos\theta, r\sin\theta)\cos\theta + u_y(r\cos\theta, r\sin\theta)\sin\theta,$$

$$v_\theta(r,\theta) = -u_x(r\cos\theta, r\sin\theta)r\sin\theta + u_y(r\cos\theta, r\sin\theta)r\cos\theta.$$

Differentiating again using the chain rule, we find:

$$\begin{aligned} v_{rr}(r,\theta) &= u_{xx}(r\cos\theta, r\sin\theta)\cos^2\theta + 2u_{xy}(r\cos\theta, r\sin\theta)\cos\theta\sin\theta \\ &\quad + u_{yy}(r\cos\theta, r\sin\theta)\sin^2\theta, \\ v_{\theta\theta}(r,\theta) &= u_{xx}(r\sin\theta, r\sin\theta)r^2\sin^2\theta - u_x(r\cos\theta, r\sin\theta)r\cos\theta \\ &\quad - 2u_{xy}(r\cos\theta, r\sin\theta)r^2\sin\theta\cos\theta \\ &\quad + u_{yy}(r\cos\theta, r\sin\theta)r^2\cos^2\theta - u_y(r\cos\theta, r\sin\theta)r\sin\theta. \end{aligned}$$

Now just add the expressions for v_{rr}, $\frac{1}{r}v_r$, and $\frac{1}{r^2}v_{\theta\theta}$ and check that it equals $(u_{xx} + u_{yy})(r\cos\theta, r\sin\theta)$. The identity $\sin^2\theta + \cos^2\theta = 1$ will be useful.

This proves the formula for the laplacian in polar coordinates. The separation of variables is done quite explicitly on page 22 of the book.

Chapter 2, Exercise 19

Separation of variables leads to the solutions $u(x,y) = f(x)g(y)$, where $f(x) = c_1 \sin(ax) + c_2 \cos(ax)$ and $g(y) = d_1 e^{ay} + d_2 e^{-ay}$ (the other solution where f has an exponential form and g has a sinusoidal form is not possible unless $f = 0$ due to the boundary conditions). The boundary conditions say $f(0) = f(1) = 0$ which force $c_2 = f(0) = 0$ and hence $a \in \pi\mathbb{Z}$. Assuming without loss of generality that $a > 0$, we throw out the unbounded ("unreasonable") part of the solution for g, corresponding to $d_1 e^{ay}$. So we have showed that any bounded solution $u(x,y) = f(x)g(y)$ has the form $ce^{-n\pi y} \sin(n\pi x)$.

Then the general solution is given by superposition:

$$u(x,y) = \sum_{n=1}^{\infty} a_n e^{-n\pi y} \sin(n\pi x).$$

If we define

$$P_y(x) = \sum_{n=-\infty}^{\infty} e^{-|n|\pi y} e^{i\pi x},$$

then we claim that

$$u(x,y) = (f * P_y)(x) = \frac{1}{2} \int_{-1}^{1} f(z) P_y(x-z)\,dz,$$

where f is defined on $[-1,0]$ by odd reflection. Indeed, f has Fourier coefficients

$$\hat{f}(n) = \frac{1}{2} \int_{-1}^{1} f(x) e^{-i\pi x}\,dx = -i \operatorname{sign}(n) a_{|n|}/2,$$

while P_y has Fourier coefficients

$$\hat{P}_y(n) = \frac{1}{2} \int_{-1}^{1} P_y(x) e^{-i\pi x}\,dx = e^{-|n|\pi y},$$

and therefore $f * P_y$ has Fourier coefficients which are the product of those of the individual functions, namely $-i \operatorname{sign}(n) a_{|n|} e^{-|n|\pi y}/2$, which agree with those of $u(\cdot, y)$.

Chapter 3, Exercise 3

Let us enumerate a sequence of intervals $\{I_k\}_{k=1}^{\infty}$ as

$$I_1 = [0, 1/2], \quad I_2 = [1/2, 1],$$

$$I_3 = [0, 1/3], \quad I_4 = [1/3, 2/3], \quad I_5 = [2/3, 1],$$

$$I_6 = [0, 1/4], \quad I_7 = [1/4, 1/2], \quad I_8 = [1/2, 3/4], \quad I_9 = [3/4, 1], \ldots.$$

Then let f_k denote the indicator function of the interval I_k, i.e., $f_k(x) = 1$ if $x \in I_k$ and $f_k(x) = 0$ otherwise.

It is clear that any point x lies in I_k infinitely often, but x also lies in the complement of I_k infinitely often. Hence $f_k(x)$ alternates between 0 and 1 infinitely often, for any x.

On the other hand, the lengths of the intervals I_k are clearly tending to 0 as $k \to \infty$; in fact $|I_k| \le 1/n$ as soon as $k > n(n+1)/2$. Hence $\int f_k(x)^2 dx = |I_k| \to 0$.

Correction: Our example was constructed on the interval $[0, 1]$. The book actually asks to construct an example on $[0, 2\pi]$, which is very similar (just consider $f_k(\frac{x}{2\pi})$).

Chapter 3, Exercise 5

One may solve the problem by using the hint to find an explicit antiderivative for $(\log x)^2$. However, we use a different approach.

We claim that for any $\alpha > 0$, there exists some constant $C = C(\alpha)$ such that $|\log(1/x)| \leq Cx^{-\alpha}$ for every $x \in (0, 2\pi]$. Indeed, L'Hopital's rule shows that

$$\lim_{x\to 0+} \frac{\log(x)}{x^{-\alpha}} = \lim_{x\to 0+} \frac{1/x}{-\alpha x^{-\alpha-1}} = -\alpha^{-1} \lim_{x\to 0+} x^{\alpha} = 0.$$

Using this bound, we get that $|\log(1/x)|^2 \leq C^2 x^{-2\alpha} = C' x^{-2\alpha}$. Letting $\alpha = 1/4$ gives $|\log(1/x)|^2 \leq C' x^{-1/2}$. Consequently, if $0 < a < b$ then

$$\int_a^b |\log(1/x)|^2 dx \leq C' \int_a^b x^{-1/2} dx \leq 2C' b^{1/2} = C'' b^{1/2}.$$

In particular, f_n must be a Cauchy sequence since if $n > m$ then

$$\int_0^{2\pi} (f_n(x) - f_m(x))^2 dx = \int_{1/n}^{1/m} |\log(1/x)|^2 dx \leq C'' m^{-1/2},$$

which can clearly be made smaller than some predefined ϵ if m is large enough.

Nevertheless, f_n does not converge in $\mathcal{R}$ because the limit would be f which is unbounded, hence not in $\mathcal{R}$. Indeed, any unbounded function has infinite upper Darboux sums, hence is not Riemann integrable.

Chapter 3, Exercise 8

Recall Parseval's identity which says

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

Part a: Using Part b of Exercise 6 of Chapter 2, we see that $\hat{f}(0) = \pi/2$, $\hat{f}(n) = -\frac{2}{\pi n^2}$ for n odd, and $\hat{f}(n) = 0$ otherwise. Consequently, Parseval gives

$$\frac{\pi^2}{4} + \frac{4}{\pi^2}\sum_{n\in\mathbb{Z},\ odd} n^{-4} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{3}\pi^2.$$

In particular,

$$\frac{8}{\pi^2}\sum_{n=0}^{\infty}(2n+1)^{-4} = \frac{4}{\pi^2}\sum_{n\in\mathbb{Z},\ odd} n^{-4} = \left[\frac{1}{3} - \frac{1}{4}\right]\pi^2 = \frac{1}{12}\pi^2.$$

Hence $\sum_0^\infty (2n+1)^{-4} = \pi^4/96$. Letting $S = \sum_1^\infty n^{-4}$, we see that

$$S = \sum_{n\geq 1,\ even} n^{-4} + \sum_{n\geq 1,\ odd} n^{-4} = \frac{S}{2^4} + \frac{\pi^4}{96} \Longrightarrow S = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90}.$$

Part b: Using Exercise 4b of Chapter 2, the Fourier coefficients are $\hat{f}(n) = \frac{4}{i\pi n^3}\,\mathrm{sign}(n)$, if n is odd and 0 otherwise. Then applying Parseval,

$$\frac{32}{\pi^2}\sum_{n=0}^{\infty}(2n+1)^{-6} = \frac{16}{\pi^2}\sum_{n\in\mathbb{Z},\ odd} n^{-6} = \frac{1}{\pi}\int_0^{\pi} x^2(\pi - x)^2 dx = \frac{1}{30}\pi^4.$$

This gives the desired identity $\sum_{n=0}^\infty (2n+1)^{-6} = \pi^6/960$, from which the other identity may be derived using the same odd-even decomposition as in part a (after noting that $960 \cdot \frac{63}{64} = 945$).