Homework 1 Solutions

Chapter 2, Exercise 1

Part 1: By substituting $u = x + 2\pi$ and then using periodicity of f, we see that

$$\int_{a+2\pi}^{b+2\pi} f(x)dx = \int_a^b f(u-2\pi)du = \int_a^b f(u)du,$$

which proves the first equality. For the second equality, we can use a similar argument, or we can simply apply the first equality with (a, b) replaced by $(a - 2\pi, b - 2\pi)$.

Assignment Project Exam Help

Part 2: Note that

https://powcoder.com

$$\int_{-\pi+a}^{\pi+a} f(x)dx = \int_{-\pi+a}^{-\pi} f(x)dx + \int_{-\pi}^{\pi} f(x)dx + \int_{\pi}^{\pi+a} f(x)dx.$$

By Part 1 of the problem, the first and third terms cancel out, thus proving that

Add WeChat powcoder

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi+a}^{\pi+a} f(x)dx = \int_{-\pi}^{\pi} f(u+a)du,$$

where we made a substitution $u = x - a$ in the final equality.

Remark: In the case that f is continuous, there is an alternative solution to Part 2, namely we consider the function $F(r) = \int_{-\pi+r}^{\pi+r} f(x)dx$, then note that $F'(r) = f(\pi + r) - f(-\pi + r) = 0$, hence F must be constant.

Chapter 2, Exercise 4b

Since f is **odd** by definition, it follows that

$$\hat{f}(n) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta = \frac{i}{\pi}\int_0^{\pi}\theta(\pi-\theta)\sin(n\theta)d\theta,$$

where we used the fact that $f(\theta)\sin(n\theta)$ is an even function in the last equality. When n is even, the substitution $x = \pi - \theta$ reveals that the above integral is its own negative, hence 0. Thus we now suppose that n is **odd**. Using integration-by-parts:

$$\begin{aligned}
\int_0^{\pi}\theta\sin(n\theta)d\theta &= -\frac{\theta}{n}\cos(n\theta)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(n\theta)d\theta = \frac{\pi}{n},\\
\int_0^{\pi}\theta^2\sin(n\theta)d\theta &= -\frac{\theta^2}{n}\cos(n\theta)\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi}\theta\cos(n\theta)d\theta\\
&= \frac{\pi^2}{n} + \frac{2}{n}\left[\frac{\theta}{n}\sin(n\theta)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(n\theta)d\theta\right]\\
&= \frac{\pi^2}{n} + \frac{4}{n^3}.
\end{aligned}$$

Assignment Project Exam Help

https://powcoder.com

Subtracting π times the first integral minus the second one, then multiplying by i/π, we obtain that when n is odd,

$$\hat{f}(n) = \frac{4i}{\pi n^3}.$$

Add WeChat powcoder

Using the fact that f is odd, we know that $\hat{f}(n) = -\hat{f}(-n)$ so that

$$f(\theta) = \sum_k \hat{f}(k)e^{ik\theta} = \sum_{k>0,odd} 2i\hat{f}(k)\sin(k\theta) = \frac{8}{\pi}\sum_{k>0,odd}\frac{\sin(k\theta)}{k^3}.$$

Chapter 2, Exercise 6

Part b: Let $f(\theta) = |\theta|$, and let g be the function from Exercise 4. Now note that g is a C^1 function whose (periodic) derivative is $g'(\theta) = \pi - 2|\theta|$. Now recall from class that for a 2π-periodic C^1 function, the fourier coefficients of the derivative are related to the fourier coefficients of the original function by $\widehat{g'}(n) = in\hat{g}(n)$.

Since the addition of constant terms only affect the zeroth fourier coefficient, and since $g' = \pi - 2f$, it follows that for $n \neq 0$ one has $\hat{f}(n) = -\frac{1}{2}\widehat{g'}(n) = -\frac{1}{2}in\hat{g}(n)$, which is $-2/(\pi n^2)$ for odd n and zero for even $n \neq 0$. When $n = 0$, it is easily computed that $\hat{f}(0) = \pi/2$.

Part c:

$$f(\theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k>0,odd} \frac{\cos(k\theta)}{k^2}.$$

Assignment Project Exam Help

Part d: Plugging in $\theta = 0$, we see

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k>0,odd} \frac{1}{k^2} \implies \sum_{k>0,odd} \frac{1}{k^2} = \frac{\pi^2}{8}.$$

https://powcoder.com

Now let $S = \sum_{k>0} k^{-2}$. We find that

Add WeChat powcoder

$$S = \sum_{k>0,odd} \frac{1}{k^2} + \sum_{k>0,even} \frac{1}{k^2} = \frac{\pi^2}{8} + \frac{1}{4}S \implies S = \frac{\pi^2}{6}.$$

Chapter 2, Exercise 8

The computation of the Fourier coefficients can be done using repeated integration-by-parts as in exercise 4.

So we still need to verify that the given series of functions satisfies the conditions of the Dirichlet test at each point x. Since $\frac{1}{n}$ is a monotone decreasing sequence, we just need to show that for every $x \neq 0$, there exists a constant $C(x)$ such that

$$\sup_{N \in \mathbb{N}} \left| \sum_{|n| \leq N} e^{-inx} \right| \leq C(x).$$

To this end, we note by the geometric series identity that

$$\sum_{|n| \leq N} e^{inx} = \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}}.$$

Assignment Project Exam Help

By the triangle inequality, the numerator is always bounded-in-magnitude by 2, meanwhile the denominator is simply $2i\sin(x/2)$, and hence we find that if $x \neq 0$ then

https://powcoder.com

$$\sup_{N \in \mathbb{N}} \left| \sum_{|n| \leq N} e^{-inx} \right| \leq \frac{1}{|\sin(x/2)|},$$

thus proving the claim.

Add WeChat powcoder

When $x = 0$, the positive partial sums cancel out the negative partial sums, hence the limit clearly exists and is zero.

Chapter 2, Exercise 10

Recall from class that if f is of class C^1, then we have that $\widehat{f'}(n) = in\hat{f}(n)$. This was proved using integration-by-parts.

Now if f is of class C^k, then induction on the above fact reveals that

$$\widehat{f^{(k)}}(n) = i^k n^k \hat{f}(n).$$

Let $g = f^{(k)}$. By assumption g is continuous (and thus bounded) on $[-\pi, \pi]$, therefore $|\hat{g}(k)| \leq \int |g(x)|dx \leq C$, for some constant C. Hence we find that $\hat{f}(n) = \frac{\hat{g}(n)}{i^k n^k}$ is bounded in magnitude by C/n^k.

Assignment Project Exam Help

https://powcoder.com

Add WeChat powcoder