Homework 6 Solutions

Chapter 5, Exercise 9

If $\mathcal{F}_R$ is defined as given, then one may check that $\widehat{\mathcal{F}_R}(\xi) = \left(1 - \frac{|\xi|}{R}\right) \cdot 1_{[-R,R]}(\xi)$. This is easily derived from the result of Exercise 2 of the same chapter. Consequently, we see by inversion that

$$\begin{aligned}(\mathcal{F}_R * f)(x) &= \int_{\mathbb{R}} (\widehat{\mathcal{F}_R * f})(\xi) e^{2\pi i \xi x} dx \\ &= \int_{\mathbb{R}} \widehat{\mathcal{F}_R}(\xi) \hat{f}(\xi) e^{2\pi i \xi x} dx \\ &= \int_{-R}^{R} \left(1 - \frac{|\xi|}{R}\right) \hat{f}(\xi) e^{2\pi i \xi x} dx.\end{aligned}$$

To see that $\{\mathcal{F}_R\}$ is a family of good kernels as $R \to \infty$, note that

$$\int_{\mathbb{R}} |\mathcal{F}_R(x)| dx = \int_{\mathbb{R}} \mathcal{F}_R(x) dx = \widehat{\mathcal{F}_R}(0) = 1,$$

and using the bound $\sin^2(\pi t R) \leq 1$, we also have that

$$R \int_{\{|x|>\delta\}} \frac{\sin^2(\pi t R)}{(\pi t R)^2} dt = 2R \int_{\delta}^{\infty} \frac{\sin^2(\pi t R)}{(\pi t R)^2} dt \leq 2R \int_{\delta}^{\infty} \frac{1}{(\pi t R)^2} dt = \frac{2}{\delta \pi^2} R^{-1},$$

and the right side clearly tends to 0 as $R \to \infty$.

Chapter 5, Exercise 11

The fact that u is continuous on the closure of the upper half plane follows immediately from Theorem 5.2.1(ii) in the same chapter.

To show that $u(t,x) \to 0$ as $|x| + t \to \infty$, we apply the hint given to write

$$u(t,x) = \int_{\mathbb{R}} f(x-y)H_t(y)dy = \int_{\{|y|>|x|/2\}} f(x-y)H_t(y)dy + \int_{\{|y|\le|x|/2\}} f(x-y)H_t(y)dy.$$

Let's call the terms on the right side as A, B, respectively. To bound A, note that f is bounded (say $|f| \le C\sqrt{4\pi}$), and $1 < 2y/|x|$ if $|y| > |x|/2$, so

$$\begin{aligned}
|A| &\le \int_{\{|y|>|x|/2\}} |f(x-y)|H_t(y)dy \\
&\le \frac{2C}{t^{1/2}} \int_{|x|/2}^{\infty} e^{-y^2/4t}dy \\
&\le \frac{2C}{t^{1/2}} \int_{|x|/2}^{\infty} \frac{2y}{|x|} e^{-y^2/4t}dy \\
&= \frac{8Ct^{1/2}}{|x|} e^{-x^2/16t}.
\end{aligned}$$

To bound B, note that $|x-y| \ge \frac{1}{2}|x|$ if $|y| \le \frac{1}{2}|x|$, thus $\frac{1}{1+(x-y)^2} \le \frac{1}{1+x^2/4}$. But since $f \in \mathcal{S}$, it follows that $|f(x)| \le D/(1+x^2)$ for some $D > 0$. Thus,

$$\begin{aligned}
|B| &\le \int_{\{|y|\le|x|/2\}} |f(x-y)|H_t(y)dy \\
&\le \frac{D}{1+x^2/4} \int_{\mathbb{R}} H_t(x)dx \\
&= \frac{D}{1+x^2/4}.
\end{aligned}$$

On the other hand, $|H_t(y)| \le t^{-1/2}$, and hence

$$|u(t,x)| \le \int_{\mathbb{R}} |f(x-y)|H_t(y)dy \le t^{-1/2} \int_{\mathbb{R}} |f(x)|dx =: Kt^{-1/2}.$$

Thus, given $\epsilon > 0$, we choose $T = T(\epsilon)$ large enough so that $KT^{-1/2} < \epsilon$, then we choose $M = M(\epsilon)$ large enough so that $\frac{D}{1+x^2/4} + \frac{8CT^{1/2}}{|x|} e^{-x^2/16T} < \epsilon$ whenever $|x| > M$. Then it is true that $|u(t,x)| < \epsilon$ whenever $|x| + t \ge M + T$.

Chapter 5, Exercise 12

For notational convenience, we write $H(t, x)$ for the heat kernel, and subscripts will always indicate partial derivatives throughout this problem.

Let $u(t, x) = xt^{-1}H(t, x)$. Then it is clear that $u(t, x) = -2H_x(t, x)$ if $t > 0$ and $x \in \mathbb{R}$, thus we have

$$u_t(t, x) = -2\frac{\partial}{\partial t}\frac{\partial}{\partial x}H(t, x) = -2\frac{\partial}{\partial x}\frac{\partial}{\partial t}H(t, x) = -2\frac{\partial^3}{\partial x^3}H(t, x) = u_{xx}(t, x).$$

If $x \neq 0$, then $\lim_{t\to 0} xt^{-3/2}e^{-x^2/4t} = 0$; this follows immediately from the fact that $e^{x^2/4t} \geq \frac{1}{2}(x^2/4t)^2$, which in turn follows from the fact that $e^u = 1 + u + \frac{1}{2}u^2 + ... \geq \frac{1}{2}u^2$ for $u \geq 0$. If $x = 0$, then obviously $\lim_{t\to 0} xt^{-3/2}e^{-x^2/4t} = 0$. This proves that $u(t, x) \to 0$ as $t \to 0$ along lines parallel to the axis $\{x = 0\}$.

On the other hand, if we let $(t, x) \to 0$ along the parabola $x^2 = 4t$, then we get $u(t, x(t)) = (4\pi)^{-1/2}(4t)^{1/2}t^{-3/2}e^{-1} = ct^{-1/2} \to +\infty$ as $t \to 0$, hence u is not continuous at 0.

Chapter 5, Exercise 13

Suppose u is harmonic on the strip $\{(x,y) : -\frac{\pi}{2} < y < \frac{\pi}{2}\}$, and u extends continuously to the boundary, with $u(x,-\pi/2) = u(x,\pi/2) = 0$. Also assume that u vanishes at infinity. We want to show $u = 0$. We give two solutions.

Solution 1: Given $\epsilon > 0$, we can choose M large enough so that $|x| > M$ implies $|u(x,y)| \leq \epsilon$. Then the boundary condition on u and the maximum principle imply that $|u(x,y)| \leq \epsilon$ whenever $|x| \leq M$ (the maximum principle is an easy corollary of the mean value property, and says that any harmonic function achieves its maximum on the boundary of a connected domain). Since we can choose M as large as we'd like, this actually shows that $|u(x,y)| < \epsilon$ everywhere on the strip. Since ϵ is arbitrary, $u = 0$.

Solution 2: Let's define $H = \{(x,y) : x > 0\}$, and let us define a new function

$$v(x,y) = u\left(\frac{1}{2}\log(x^2+y^2),\ \tan^{-1}(y/x)\right).$$

We claim that v is a harmonic function on H, and extends continuously to the closure of H, where it is zero on the boundary and vanishes at ∞. If we can prove this then we will be done, because then we would have $v = 0$ by theorem 5.2.7, from which it follows that $u(x,y) = v(e^x \cos y,\ e^x \sin y) = 0$.

To prove that v is harmonic, we write v in polar coordinates $v(r,\theta) = u(\log r, \theta)$, from which we may compute the three quantities

$$v_r = \frac{1}{r}u_x(\log r, \theta), \qquad v_{rr} = \frac{1}{r^2}(u_{xx} - u_x)(\log r,\ \theta), \qquad v_{\theta\theta} = u_{yy}(\log r, \theta).$$

Consequently we find using the formula for the laplacian in polar coordinates:

$$\Delta v = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = \frac{1}{r^2}(u_{xx} + u_{yy}) = 0,$$

so that v is harmonic. Showing that v does indeed satisfy the zero boundary conditions is then an easy consequence of the boundary conditions on u.

Chapter 5, Exercise 14

Note from exercise 9 that $\widehat{\mathcal{F}_R}(\xi) = \left(1 - \frac{|\xi|}{R}\right) \cdot 1_{[-R,R]}(\xi)$. Hence by the Poisson summation formula, we find that

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = \sum_{n=-\infty}^{\infty} \widehat{\mathcal{F}_N}(n) e^{2\pi i n x} = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}.$$