Homework 5 Solutions

Chapter 4, Exercise 11

We will work over the interval $[-\pi, \pi]$ rather than $[-\frac{1}{2}, \frac{1}{2}]$. This slightly changes the formula for the heat kernel because $e^{-4\pi^2 n^2 t}$ becomes $e^{-n^2 t}$ and $e^{2i\pi n x}$ becomes e^{inx} (so the notation is more convenient but the constants change).

Note that $H_t * f - f$ has Fourier coefficients $\hat{f}(n)(e^{-n^2 t} - 1)$. Thus by Parseval

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (H_t * f(x) - f(x))^2 \, dx = \sum_{n \in \mathbb{Z}} |e^{-n^2 t} - 1|^2 |\hat{f}(n)|^2. \tag{1}$$

We want to show that the right-hand side tends to 0 as $t \to 0$. For this, we present two arguments.

As $t \to 0$, it is clear that $|e^{-n^2 t} - 1| \to 0$ for all $n \in \mathbb{Z}$. Fixing $\epsilon > 0$, we can truncate the series at some finite value $N = N(\epsilon)$ such that $\sum_{|n|>N} |\hat{f}(n)|^2 < \epsilon$, then take the limit as $t \to 0$ for the finite number of remaining terms (see the proof of Exercise 5.1b below which is very similar and more explicit). Less explicitly, this is basically an application the dominated convergence theorem applied to counting measure on $\mathbb{Z}$ with dominating function $g(n) = |\hat{f}(n)|^2$.

For the second proof, one may assume first that f is a trigonometric polynomial. Then using the identity $|e^{-q} - 1| \le q$ for $q > 0$, we see

$$\sum_{n \in \mathbb{Z}} |e^{-n^2 t} - 1|^2 |\hat{f}(n)|^2 \le t^2 \sum_{n \in \mathbb{Z}} n^4 |\hat{f}(n)|^2 \xrightarrow{t \to 0} 0.$$

Now one may use the fact that any Riemann-integrable function can be approximated arbitrarily closely in the mean-square-norm by trigonometric polynomials, together with a standard $\epsilon/3$-trick and the fact that convolution with the heat kernel contracts the mean-square-norm (by Parseval).

Remark: Assuming Corollary 5.3.4 in the next chapter: $\{H_t\}$ is a family of good kernels as $t \to 0$, from which the result may be deduced more directly.

Chapter 4, Exercise 13

Part a: Again, we work on $[-\pi, \pi]$ rather than $[-1/2, 1/2]$, so our constants will be different than those in the book. Noting that $\hat{H}_t(n) = e^{-n^2 t}$, we see by Parseval that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |H_t(x)|^2 dx = \sum_{n\in\mathbb{Z}} e^{-2n^2 t}.$$

Hence

$$t^{1/2}\int_{-\pi}^{\pi} |H_t(x)|^2 dx = 2\pi t^{1/2}\sum_{n\in\mathbb{Z}} e^{-2n^2 t} = 2\pi\delta \sum_{k\in\delta\mathbb{Z}} e^{-2k^2},$$

where $\delta := t^{1/2}$. Now by the result of Exercise 1b in Chapter 5 (see below), the RHS converges as $\delta \to 0$ to some nonzero constant $C = 2\pi\int_{\mathbb{R}} e^{-2u^2} du$.

Part b: We use the given hint that $x^2 \le C\sin^2(x/2)$. Then we fix a $t > 0$, and we define $f(x) = \sin^2(x/2)H_t(x) = (1-\cos x)H_t(x)$ and also define $g(x) = H_t(x)$. So we have

$$\int_{-\pi}^{\pi} x^2 H_t(x)^2 dx \le C\int_{-\pi}^{\pi} \sin^2(x/2)H_t(x)^2 dx = C\int_{-\pi}^{\pi} f(x)g(x)dx.$$

By Parseval $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)\hat{g}(n)$. It is clear that $\hat{g}(n) = e^{-n^2 t}$. Using the fact that the Fourier coefficients of a product of two functions is the convolution of the respective Fourier coefficients, one may see that $\hat{f}(n) = e^{-n^2 t} - \frac{1}{2}e^{-(n+1)^2 t} - \frac{1}{2}e^{-(n-1)^2 t}$. Now recall summation by parts: if (a_n) is square-summable then

$$\sum_{n\in\mathbb{Z}} a_n\left(a_n - \frac{1}{2}a_{n+1} - \frac{1}{2}a_{n-1}\right) = \frac{1}{2}\sum_{n\in\mathbb{Z}} (a_{n+1} - a_n)^2.$$

Applying this to $a_n = e^{-n^2 t}$, we find that

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\hat{g}(n) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \left(e^{-(n+1)^2 t} - e^{-n^2 t}\right)^2.$$

Letting $F(x) = e^{-x^2 t}$, we see that $F'(x) = -2xte^{-x^2 t}$, and so by the mean value theorem, we can always find constants $x_{n,t} \in [n, n+1]$ such that $F(n+1) - F(n) = F'(x_{n,t}) = -2x_{n,t}te^{-x_{n,t}^2 t}$. Summarizing, we have shown that for any $t > 0$, there exist points $x_{n,t} \in [n, n+1]$ such that

$$\int_{-\pi}^{\pi} x^2 H_t(x)^2 dx \le C\sum_{n\in\mathbb{Z}} x_{n,t}^2 t^2 e^{-2x_{n,t}^2 t}.$$

Multiplying by $t^{-1/2}$, we see that

$$t^{-1/2}\int_{-\pi}^{\pi} x^2 H_t(x)^2 \le Ct^{1/2}\sum_{n\in\mathbb{Z}} \left(t^{1/2}x_{n,t}\right)^2 e^{-2\left(t^{1/2}x_{n,t}\right)^2}.$$

Since $x_{n,t} \in [n, n+1]$, a close inspection of the right-hand side shows that it is basically a Riemann sum approximation of $\int_{\mathbb{R}} u^2 e^{-2u^2} du$, with mesh size $t^{1/2}$. Thus as $t \to 0$, the right side converges to that integral (which can be made rigorous by adapting the proof of Exercise 1b in Chapter 5 below, or just by applying the dominated convergence theorem in a slightly clever way). This proves the desired upper bound, but we remark that a lower bound may also be proved in exactly the same way by noting that $x^2 \geq c \sin^2(x/2)$ for $x \in [-\pi, \pi]$. Hence $t^{1/2}$ is really the true order of this integral as $t \to 0$.

Chapter 5, Exercise 1

Part a: Since $\hat{f}$ is of moderate decrease, it follows that $\sum_{n \in \mathbb{Z}} |\hat{f}(n/L)| < \infty$ for any $L > 0$. This means that $\sum_n |a_n(L)| < +\infty$. Noting that $a_n(L)$ is just the Fourier coefficient of f viewed as a function on the bounded interval $[-L/2, L/2]$, we can apply Corollary 2.3 in chapter 2 to see that $\sum_n a_n(L) e^{2\pi i n x/L}$ actually converges uniformly to f on $[-L/2, L/2]$.

Part b: Suppose F is continuous and $|F(x)| \leq A(1+x^2)^{-1}$. Let $\epsilon > 0$ and set $M = M(\epsilon) := 4A\epsilon^{-1}$. Then we see that

$$\int_{\{|x|>M\}} |F(x)| dx \leq 2A \int_M^\infty x^{-2} dx = \epsilon/2,$$

and similarly, for any $\delta > 0$,

$$\delta \sum_{|n|>1+M/\delta} |F(\delta n)| \leq 2A\delta \sum_{n \geq 1+M/\delta} \delta^{-2} n^{-2} \leq 2A\delta^{-1} \int_{M/\delta}^\infty x^{-2} dx = \epsilon/2.$$

Noting that F is uniformly continuous on $[-M, M]$, it holds that

$$\lim_{\delta \to 0} \delta \sum_{|n| \leq 1+M/\delta} F(\delta n) = \int_{\{|x| \leq M\}} F(x) dx.$$

Now by the triangle inequality,

$$\left| \delta \sum_{n \in \mathbb{Z}} F(\delta n) - \int_{\mathbb{R}} F(x) dx \right| \leq \epsilon/2 + \epsilon/2 + \left| \delta \sum_{|n| \leq 1+M/\delta} F(\delta n) - \int_{\{|x| \leq M\}} F(x) dx \right|.$$

Now letting $\delta \to 0$ on both sides (while keeping ϵ, M fixed) shows that

$$\limsup_{\delta \to 0} \left| \delta \sum_{n \in \mathbb{Z}} F(\delta n) - \int_{\mathbb{R}} F(x) dx \right| \leq \epsilon.$$

As ϵ was arbitrary, we are done.

Part c: This follows immediately by letting $\delta \to 0$ (i.e., $L \to \infty$) in the formula from Part a, then applying the result of Part b.

Chapter 5, Exercise 2

We have

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx = \int_{-1}^{1} e^{-2\pi i\xi x}dx = -\frac{1}{2\pi i\xi}(e^{-2\pi i\xi} - e^{2\pi i\xi}) = \frac{\sin(2\pi\xi)}{\pi\xi}.$$

Now a quick computation reveals that $(f*f)(x) = 2g(x/2)$, which means $g(x) = \frac{1}{2}(f*f)(2x)$, hence

$$\hat{g}(\xi) = \frac{1}{2}\left[\frac{1}{2}\hat{f}(\xi/2)^2\right] = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2.$$

Chapter 5, Exercise 8

Let $g(x) = e^{-x^2}$. Under the given condition on f, we see that

$$(f*g)(x) = \int_{\mathbb{R}} f(y)e^{-(x-y)^2}dy = e^{-x^2}\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}dy = e^{-x^2}\cdot 0 = 0.$$

Thus,

$$\frac{1}{\sqrt{2}}e^{-\xi^2/4}\hat{f}(\xi) = \hat{f}(\xi)\hat{g}(\xi) = \widehat{(f*g)}(\xi) = 0.$$

Hence $\hat{f}(\xi) = 0$ for all ξ, and we conclude that

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}d\xi = 0.$$