Homework 2 Solutions

Proof of Riemann-Lebesgue

We first consider step functions $f = \sum_{k=1}^m c_k 1_{I_k}$, where the $I_k = [a_k, b_k)$ are pairwise-disjoint subintervals of $[-\pi, \pi]$, the c_k are some real numbers, and 1_A denotes the indicator function of the set A (which means $1_A(x) = 0$ for $x \notin A$ and $1_A(x) = 1$ for $x \in A$). In this case we compute

$$2\pi \hat{f}(n) = \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \sum_{k=1}^m c_k \int_{a_k}^{b_k} e^{-inx} dx = \sum_{k=1}^m c_k \frac{e^{-ina_k} - e^{-inb_k}}{in}.$$

Noting that $|e^{ina_k} - e^{inb_k}| \le |e^{ina_k}| + |e^{inb_k}| = 2$ and using the triangle inequality in the above finite sum, we find that $|2\pi \hat{f}(n)| \le \frac{2}{n} \sum_{k=1}^m |c_k|$, which clearly tends to 0 as $|n| \to \infty$. Note here that m is not related to n in any way.

Next, we use the given fact to extend the convergence to all f which are integrable. Let f be integrable, and take some $\epsilon > 0$. Then there exists some step function $g_\epsilon \le f$ such that $\frac{1}{2\pi} \int_{-\pi}^{\pi} (f - g_\epsilon) \le \epsilon$. Consequently, we find that

$$\begin{aligned} |\hat{f}(n)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) - g_\epsilon(x)) e^{-inx} dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} g_\epsilon(x) dx \right| \\ &\le \frac{1}{2\pi} \int |f(x) - g_\epsilon(x)| dx + |\widehat{g_\epsilon}(n)| \\ &\le \epsilon + |\widehat{g_\epsilon}(n)|. \end{aligned}$$

Letting $|n| \to \infty$ on both sides and noting that g_ϵ is a step function, we notice that $\limsup_{|n| \to \infty} |\hat{f}(n)| \le \epsilon$. Since ϵ is arbitrary, this gives the desired result.

The proof extends to complex-valued f by considering real and imaginary parts separately.

Chapter 2, Problem 2

Noting that $|\sin(x/2)| \leq |x/2|$ and making the substitution $u = (N + \frac{1}{2})\pi$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_N(x)|dx &= 2\int_0^{\pi} |D_N(x)|dx \\
&= 2\int_0^{\pi} \frac{|\sin((N+\frac{1}{2})x)|}{|\sin(x/2)|}dx \\
&\geq 2\int_0^{\pi} \frac{|\sin((N+\frac{1}{2})x)|}{|x/2|}dx \\
&= 4\int_0^{(N+\frac{1}{2})\pi} \frac{|\sin(u)|}{u}du \\
&\geq 4\sum_{k=0}^{N-1}\int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{u}du.
\end{aligned}$$

Now noting that $\frac{1}{u} \leq \frac{1}{(k+1)\pi}$ for $u \in [k\pi, (k+1)\pi]$ and then noting that $\int_{k\pi}^{(k+1)\pi} |\sin u|du = 2$, we see that the last expression is bounded below by

$$4\sum_{k=0}^{N-1} \frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi} |\sin u|du = \frac{8}{\pi}\sum_{k=0}^{N-1}\frac{1}{k+1}.$$

Now we take for granted that the partial sums of the harmonic series are asymptotic to $\log N$, finishing the proof. If we multiply by $\frac{1}{2\pi}$ then we get the constant $c = \frac{4}{\pi^2}$ mentioned in the problem.

Chapter 3, Exercise 12

We have that

$$2\pi = \int_{-\pi}^{\pi} D_N(t)dt = \int_{-\pi}^{\pi} \sin((N+1/2)t)\left(\frac{1}{\sin(t/2)} - \frac{1}{t/2}\right)dt + \int_{-\pi}^{\pi} \frac{\sin((N+\frac{1}{2})t)}{t/2}dt.$$

Let us call these two terms on the right hand side as A_N and B_N, respectively. Then letting $f(t) = \frac{1}{\sin(t/2)} - \frac{1}{t/2}$, which is an odd function that extends continuously to 0, we have

$$\begin{aligned} A_N &= \int_{-\pi}^{\pi} \sin((N+1/2)t)f(t)dt \\ &= -i\int_{-\pi}^{\pi} e^{i(N+\frac{1}{2})t}f(t)dt \\ &= -i\int_{-\pi}^{\pi} e^{iNt}\left[e^{\frac{1}{2}it}f(t)\right]dt \end{aligned}$$

which tends to 0 as $n \to \infty$ by the Riemann-Lebesgue lemma (applied to the function $t \mapsto e^{\frac{1}{2}it}f(t)$).

Now for B_N, we substitute $u = (N+\frac{1}{2})t$ to obtain that

$$\begin{aligned} B_N &= 2\int_0^{\pi} \frac{\sin((N+\frac{1}{2})t)}{t/2}dt \\ &= 4\int_0^{(N+\frac{1}{2})\pi} \frac{\sin u}{u}du. \end{aligned}$$

Combining the results of the past several paragraphs, we find that

$$\lim_{N\to\infty} \int_0^{(N+\frac{1}{2})\pi} \frac{\sin u}{u}du = \lim_{N\to\infty} \frac{1}{4}B_N = \lim_{N\to\infty} \frac{1}{4}(2\pi - A_N) = \frac{\pi}{2}.$$

The proof is finished by noting that if $|x - (N+\frac{1}{2})\pi| \le \frac{\pi}{2}$, then $x \ge N\pi$, so

$$\left| \int_x^{(N+\frac{1}{2})\pi} \frac{\sin u}{u}du \right| \le \frac{1}{2N}.$$

Chapter 3, Exercise 13

Let $f \in C^k(\mathbb{T})$, and let $g := f^{(k)}$. From Homework 1 (or directly by repeated integration-by-parts), we know that $\hat{g}(n) = i^k n^k \hat{f}(n)$. Since g is integrable (in fact continuous) by assumption, we also know from Riemann-Lebesgue that $\hat{g}(n) \to 0$, hence $|n^k \hat{f}(n)| = |i^{-k}\hat{g}(n)| \to 0$ as $|n| \to \infty$ (since $|i^{-k}| = 1$).

Chapter 3, Exercise 15

Part a: Fixing $n \neq 0$ and making the substitution $\theta = x + \frac{\pi}{n}$, we find that

$$2\pi \hat{f}(n) = \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta = \int_{-\pi-\pi/n}^{\pi+\pi/n} f(x+\pi/n) e^{-in(x+\frac{\pi}{n})} dx.$$

Now using the result of (the second part of) Exercise 1 of Chapter 2 with $a = -\pi/n$ and then noting that $e^{-in(x+\frac{\pi}{n})} = -e^{-inx}$, we find that the last expression is just

$$\int_{-\pi}^{\pi} f(x+\pi/n) e^{-in(x+\frac{\pi}{n})} dx = -\int_{-\pi}^{\pi} f(x+\pi/n) e^{-inx} dx,$$

proving the first part. For the second part, note that

$$\begin{aligned} 4\pi \hat{f}(n) &= 2\pi \hat{f}(n) + 2\pi \hat{f}(n) \\ &= \int_{-\pi}^{\pi} f(x) e^{-inx} dx - \int_{-\pi}^{\pi} f(x+\pi/n) e^{-inx} dx \\ &= \int_{-\pi}^{\pi} \big[f(x) - f(x+\pi/n)\big] e^{-inx} dx. \end{aligned}$$

Part b: If $|f(x)-f(y)| \leq C|x-y|^\alpha$, then clearly $|f(x)-f(x+\pi/n)| \leq C(\pi/n)^\alpha = C' n^{-\alpha}$, and therefore

$$2\pi|\hat{f}(n)| \leq \int_{-\pi}^{\pi} \big|[f(x) - f(x+\pi/n)]e^{-inx}| dx \leq 2\pi C' n^{-\alpha}.$$

Part c: We write

$$f(x) - f(y) = \sum_{2^k < |x-y|^{-1}} 2^{-k\alpha}(e^{i2^k x} - e^{i2^k y}) \; + \sum_{2^k \geq |x-y|^{-1}} 2^{-k\alpha}(e^{i2^k x} - e^{i2^k y}).$$

Let's call the terms on the right-hand side as A and B respectively. Note that these are functions of x and y. We define the integer quantity

$$K := \min\{k \in \mathbb{N} : 2^k \geq |x-y|^{-1}\} = \lceil -\log_2 |x-y| \rceil,$$

which is also a function of x and y. To solve the problem, the first observation we make is that if $|z|, |w| \leq 1$ then $|e^{2^k z} - e^{2^k w}| \leq 2^k |z - w|$, which is given as a hint. In particular, applying this to A and applying a geometric sum identity gives

$$\begin{aligned} A &\leq \sum_{2^k < |x-y|^{-1}} 2^{-k\alpha} |e^{i2^k x} - e^{i2^k y}| \\ &\leq \sum_{2^k < |x-y|^{-1}} 2^{(1-\alpha)k} |x - y| \\ &= \frac{2^{(1-\alpha)K} - 1}{2^{1-\alpha} - 1} |x - y|. \end{aligned}$$

Using the fact that $2^K = 2 \cdot 2^{K-1} \leq 2|x-y|^{-1}$, we see that $2^{(1-\alpha)K} - 1 \leq 2^{(1-\alpha)K} \leq 2^{1-\alpha}|x - y|^{\alpha-1}$. So the last expression is bounded above by

$$\frac{2^{1-\alpha}|x-y|^{\alpha-1}}{2^{1-\alpha}-1} |x - y| = \frac{2^{1-\alpha}}{2^{1-\alpha}-1} |x-y|^{\alpha},$$

which proves the desired bound for A. Now for B, we use that $|e^{i2^k x} - e^{i2^k y}| \leq 2$ and $2^{-K} \leq |x - y|$ so that

$$\begin{aligned} B &\leq \sum_{2^k \geq |x-y|^{-1}} 2^{-k\alpha} \\ &= \frac{2^{-\alpha K}}{1 - 2^{-\alpha}} \\ &\leq \frac{1}{1 - 2^{-\alpha}} |x - y|^{\alpha}. \end{aligned}$$

which proves the claim.