Homework 1 Solutions

Chapter 2, Exercise 12

We prove that if s_n converges to s, then $\frac{1}{n}\sum_1^n s_k$ also converges to s. So suppose $s_n \to s$. Then for any $\epsilon > 0$, we can choose $N = N(\epsilon)$ so that $|s_n - s| < \epsilon$ if $n \geq N$. Then we define $C := \sum_1^N |s_k - s|$. Then for $n > N$ we may write

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} s_k \quad - \quad s\right| &= \left|\frac{1}{n}\sum_{k=1}^{n}(s_n - s)\right| \\
&\leq \frac{1}{n}\sum_{k=1}^{n}|s_n - s| \\
&= \frac{1}{n}\sum_{k=1}^{N}|s_n - s| + \frac{1}{n}\sum_{k=N+1}^{n}|s_n - s| \\
&< \frac{C}{n} + \frac{1}{n}\sum_{k=N+1}^{n}\epsilon \\
&< \frac{C}{n} + \epsilon.
\end{aligned}$$

Letting $N \to \infty$ on both sides, we have

$$\limsup_{N\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n} s_k \quad - \quad s\right| \leq \epsilon,$$

which proves the claim since ϵ is arbitrary.

Chapter 2, Exercise 13

Part a: It suffices to prove the claim when $s = 0$, because otherwise one may define a new sequence by $\tilde{c}_0 = c_0 - s$, and $\tilde{c}_n = c_n$ for $n > 0$. Then $\sum c_n r^n - \sum_n \tilde{c}_n r^n = s$ for all r, and it is clear that $\sum \tilde{c}_n = 0$.

Now by telescoping, one has the identity

$$\begin{aligned}(1-r)\sum_{n=0}^{N} s_n r^n + s_N r^{N+1} &= \sum_0^N s_n r^n - \sum_0^N s_n r^{n+1} + s_N r^{N+1} \\ &= \sum_0^N (s_n - s_{n-1}) r^n - s_N r^{N+1} + s_N r^{N+1} \\ &= \sum_0^N c_n r^n.\end{aligned}$$

We can let $N \to \infty$ to obtain

$$\sum_0^\infty c_n r^n = (1-r)\sum_0^\infty s_n r^n, \tag{1}$$

whenever $|r| < 1$. Now assuming $s = 0$, we fix an $\epsilon > 0$, and we choose some M so that $n \geq M$ implies $|s_n| < \epsilon$. Then we can write

$$\sum_0^\infty c_n r^n = (1-r)\sum_0^M s_n r^n + (1-r)\sum_{n>M} s_n r^n.$$

The first term on the right side clearly aproaches 0 as $r \to 1$, and for the second one, we have

$$\left|(1-r)\sum_{n>M} s_n r^n\right| \leq (1-r)\sum_{n>M} \epsilon r^n = \epsilon r^{M+1}.$$

From the last expression, it is clear that

$$\limsup_{r\to 1^-} \left|\sum c_n r^n\right| \leq \epsilon,$$

which proves the claim since ϵ was arbitrary.

Part b: If $c_n = (-1)^n$, then s_n alternates between 0 and 1 depending on whether n is even or odd. So clearly s_n does not converge. However $\sum (-1)^n r^n = \frac{1}{1+r}$, which clearly approaches $1/2$ as $r \to 1$ from the left.

Part c: As before, we may assume that $\sigma = 0$. Replacing c_n with s_n in equation (1) it holds that

$$\sum_0^\infty s_n r^n = (1-r) \sum_0^\infty n\sigma_n r^n,$$

for all $|r| < 1$. Hence

$$\sum_0^\infty c_n r^n = (1-r) \sum_0^\infty s_n r^n = (1-r)^2 \sum_0^\infty n\sigma_n r^n.$$

Now we fix $\epsilon > 0$, and choose M so that $n \geq M$ implies $|\sigma_n| < \epsilon$. Then

$$\sum_0^\infty c_n r^n = (1-r)^2 \sum_{n \leq M} n\sigma_n r^n + (1-r)^2 \sum_{n > M} n\sigma_n r^n.$$

The first term on the right side clearly approaches 0 as $r \to 1$, and for the second one, we have

$$\left| (1-r)^2 \sum_{n>M} n\sigma_n r^n \right| \leq (1-r)^2 \sum_{n>M} n\epsilon r^n \leq \epsilon r,$$

where we used $\sum_{n>M} nr^n \leq \sum_{n \geq 0} nr^n = r/(1-r)^2$. From the last expression, it is clear that

$$\limsup_{r \to 1^-} \left| \sum c_n r^n \right| \leq \epsilon,$$

which proves the claim since ϵ was arbitrary.

Part d: We first show that if c_n is Cesaro summable to σ, then $c_n/n \to 0$. Indeed, let $\sigma_n = \frac{1}{n}\sum_1^n s_k$. Then $\sigma_n \to \sigma$, hence $\frac{c_n}{n} = \sigma_n - \frac{(n-1)}{n}\sigma_{n-1} \to \sigma - 1 \cdot \sigma = 0$, as desired. Therefore if $c_n = n(-1)^n$ then it cannot be Cesaro summable, since $c_n/n = (-1)^n$. However

$$\sum_{n=0}^\infty n(-1)^n r^n = r \frac{d}{dr}\left[\frac{1}{1+r}\right] = -\frac{r}{(1+r)^2},$$

which clearly approaches $-1/4$ as $r \to 1$ from the left.

Chapter 2, Exercise 15

Letting $\omega = e^{ix}$ and summing the geometric series, we have

$$\begin{aligned}
NF_N(x) &= \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1-\omega} \\
&= \frac{1}{1-\omega}\left[\sum_{n=0}^{N-1} \omega^{-n} - \sum_{k=0}^{N-1} \omega^{n+1}\right] \\
&= \frac{1}{1-\omega}\left[\frac{1-\omega^{-N}}{1-\omega^{-1}} - \frac{\omega(1-\omega^N)}{1-\omega}\right] \\
&= \frac{1}{1-\omega}\left[\frac{(1-\omega^{-N})(1-\omega) - \omega(1-\omega^N)(1-\omega^{-1})}{(1-\omega^{-1})(1-\omega)}\right] \\
&= \frac{1}{1-\omega}\left[\frac{\omega^{1-N} + \omega^{N+1} - 2\omega + 2 - \omega^{-N} - \omega^N}{2-\omega-\omega^{-1}}\right] \\
&= \frac{1}{1-\omega}\left[\frac{(2-\omega^N-\omega^{-N})(1-\omega)}{2-\omega-\omega^{-1}}\right] \\
&= \frac{2-\omega^N-\omega^{-N}}{2-\omega-\omega^{-1}} \\
&= \frac{-(\omega^{N/2}-\omega^{-N/2})^2}{-(\omega^{1/2}-\omega^{-1/2})^2},
\end{aligned}$$

which completes the proof after noting that $\omega^{k/2} - \omega^{-k/2} = e^{ikx/2} - e^{-ikx/2} = 2i\sin(kx/2)$.

Chapter 3, Exercise 19

Note that

$$\int_0^x D_N(t)dt = \int_0^x \sin((N+1/2)t)\left[\frac{1}{\sin(t/2)} - \frac{2}{t}\right]dt + \int_0^x \frac{\sin((N+1/2)t)}{t/2}dt.$$

Let's call the terms on the right side as A and B, respectively. Then

$$\begin{aligned} A &\leq \int_0^x |\sin((N+1/2)t)|\left|\frac{1}{\sin(t/2)} - \frac{2}{t}\right|dt \\ &\leq \int_0^\pi 1 \cdot \left|\frac{1}{\sin(t/2)} - \frac{2}{t}\right|dt, \end{aligned}$$

which proves the desired bound on A. Here we are using the fact that $|\sin((N+1/2)t)| \leq 1$, that $x \leq \pi$, and that the function $\frac{1}{\sin(t/2)} - \frac{2}{t}$ extends continuously to $[-\pi, \pi]$.

Now for B, we have by substitution $u = (N+1/2)t$:

$$\begin{aligned} B &= \int_0^x \frac{\sin((N+1/2)t)}{t/2}dt \\ &= 2\int_0^{(N+1/2)x} \frac{\sin u}{u}du \end{aligned}$$

which is of course uniformly bounded in N and x by $2\sup_{a>0} \int_0^a \frac{\sin u}{u}du$, which in turn is finite by the result of Exercise 12 of the same chapter.

Chapter 3, Exercise 20

Recall that there was a typo in the problem: there should be a "$+O(N^{-1})$ as $N \to \infty$" on the right-hand side of the last expression.

Setting $x = \pi/N$ and using the same chain of equalities as in the previous exercise, we have

$$\int_0^{\pi/N} D_N(t)dt = \int_0^{\pi/N} \sin((N+1/2)t)\left[\frac{1}{\sin(t/2)} - \frac{2}{t}\right]dt + 2\int_0^{\pi(1+\frac{1}{2N})} \frac{\sin u}{u}du.$$

Since $|\sin((N+1/2)t)| \leq 1$ and since $\frac{1}{\sin(t/2)} - \frac{2}{t}$ is bounded on $[0,\pi]$, the first term on the RHS is bounded by $C\pi/N = O(N^{-1})$. Similarly

$$\int_0^{\pi(1+\frac{1}{2N})} \frac{\sin u}{u}du = \int_0^{\pi} \frac{\sin u}{u}du + \int_{\pi}^{\pi(1+\frac{1}{2N})} \frac{\sin u}{u}du = \int_0^{\pi} \frac{\sin u}{u}du + O(N^{-1}).$$

This shows that

$$\frac{1}{2}\int_0^{\pi/N} (D_N(t)-1)dt = \int_0^{\pi} \frac{\sin u}{u}du + O(N^{-1}),$$

thus giving a lower bound for the desired maximum. To prove an upper bound, one may replace π/N with any sequence (x_n) such that $x_n \in [0,\pi/n]$, then go through this same chain of equalities and note that it is still bounded above by $\int_0^{\pi} \frac{\sin u}{u}du + O(N^{-1})$.