Homework 7 Solutions

Chapter 5, Exercise 15

Part a: Let g be the same function from Exercise 2. By the Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2(\pi n + \pi\alpha)}{\pi^2(n+\alpha)^2} = \sum_{-\infty}^{\infty} \hat{g}(n+\alpha) = \sum_{-\infty}^{\infty} g(n) e^{2\pi i n\alpha} = 1.$$

Since $\sin(\pi n + \pi\alpha) = (-1)^n \sin(\pi\alpha)$, the desired identity follows immediately.

Part b: It suffices to prove the claim when $\alpha \in (0,1)$, since the sum on the left hand side only depends on the fractional part of α. So let $0 < \alpha < 1$.

First we note that for any $N \in \mathbb{N}$

$$\sum_{n=-N}^{N-1} \frac{1}{n+\frac{1}{2}} = 0$$

since the positive terms cancel the negative ones. Next note that for any $\alpha \in (0,1)$,

$$\frac{1}{n+\alpha} - \frac{1}{n+\frac{1}{2}} = \int_{\alpha}^{1/2} \frac{1}{(n+x)^2}\,dx,$$

and thus by combining the previous two identities we get

$$\sum_{n=-N}^{N-1} \frac{1}{n+\alpha} = \int_{\alpha}^{1/2} \sum_{n=-N}^{N-1} \frac{1}{(n+x)^2}\,dx.$$

Letting $N \to \infty$ on both sides and using uniform convergence on $[\alpha, 1/2]$ of the series inside the integral, we get

$$\lim_{N\to\infty} \sum_{n=-N}^{N-1} \frac{1}{n+\alpha} = \int_{\alpha}^{1/2} \frac{\pi^2}{\sin^2(\pi x)}\,dx = \frac{\pi}{\tan(\pi\alpha)},$$

where the RHS is interpreted as 0 for $\alpha = 1/2$, and we used the fact that the antiderivative of $\csc^2 u$ is $-\cot u$.

Chapter 5, Problem 1

Let us write $U(y,t) = u(e^{-y}, t)$. Then we see by the chain rule that

$$U_t(y,t) = u_t(e^{-y},t), \qquad U_y(y,t) = -e^{-y}u_x(e^{-y},t),$$

$$U_{yy}(y,t) = e^{-2y}u_{xx}(e^{-y},t) + e^{-y}u_x(e^{-y},t).$$

In particular, if $u_t = x^2 u_{xx} + axu_x$, then

$$\begin{aligned} U_t(y,t) &= u_t(e^{-y},t) \\ &= e^{-2y}u_{xx}(e^{-y},t) + ae^{-y}u_x(e^{-y},t) \\ &= U_{yy}(y,t) + (1-a)U_y(y,t). \end{aligned}$$

Thus U satisfies the constant-coefficient equation $U_t = U_{yy} + (1-a)U_y$. Taking Fourier transforms, we see that $\hat{U}$ satisfies the following ODE for fixed ξ:

$$\partial_t \hat{U}(\xi,t) = -4\pi^2\xi^2\hat{U}(\xi,t) + 2\pi i\xi(1-a)\hat{U}(\xi,t),$$

where $\hat{U}(\xi,t) = \int_{\mathbb{R}} U(y,t)e^{-2\pi i\xi y}dy$ is the Fourier transform in the y variable. Solving the ODE for $\hat{U}$ gives the solution

$$\hat{U}(\xi,t) = \hat{U}(\xi,0)e^{(-4\pi^2\xi^2+2(1-a)\pi i\xi)t},$$

so by inverting the Fourier transform,

$$U(y,t) = \int_{\mathbb{R}} \hat{U}(\xi,0)e^{(-4\pi^2\xi^2+2(1-a)\pi i\xi)t}e^{2\pi i\xi y}d\xi.$$

This last expression is simply the convolution of $U(y,0) = f(e^{-y}) =: F(y)$ with the kernel $p_t(y) := \int_{\mathbb{R}} e^{(-4\pi^2\xi^2+2(1-a)\pi i\xi)t}e^{2\pi i\xi y}d\xi$. By completing the square and then using the fact that $\int_{\mathbb{R}} e^{-(c\xi+d)^2}d\xi = \sqrt{\pi/c^2}$ for $c \in \mathbb{R}$, we obtain that

$$p_t(y) = e^{-(y+(1-a)t)^2/4t}\int_{\mathbb{R}} e^{-\left(2\pi\xi t^{1/2} - \frac{i}{2t^{1/2}}\left(y+(1-a)t\right)\right)^2}d\xi = \frac{1}{\sqrt{4\pi t}}e^{-(y+(1-a)t)^2/4t}.$$

Thus $U(y,t) = p_t * F(y) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} F(z)e^{-(y-z+(1-a)t)^2/4t}dz$, so that (transforming variables back to the original function),

$$\begin{aligned} u(x,t) &= U(-\log x, t) \\ &= \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} F(z)e^{-(-\log x - z + (1-a)t)^2/4t}dz \\ &= \frac{1}{\sqrt{4\pi t}}\int_0^\infty f(v)e^{-(\log(v/x)+(1-a)t)^2}\frac{dv}{v}, \end{aligned}$$

where we substituted $z = -\log v$ in the final line, so that dz became $-\frac{dv}{v}$, $F(z) = f(e^{-z})$ became $f(v)$, and the limits changed from $(-\infty,\infty)$ to $(0,\infty)$.

Chapter 5, Problem 5

Part a: Proceed by showing the contrapositive (i.e., prove that if u solves the heat equation but violates the stated maximum principle, then we can find a solution to the heat equation which violates the statement made in part a). To this end, let u solve the heat equation, then let $A := \max_{\partial' R} u$, and suppose u has a strict maximum at $(x_0, t_0) \in R \backslash \partial' R$. Now consider the function $v := A - u$, and check that this also solves the heat equation. Clearly, $v \geq 0$ on $\partial' R$, but $v(x_0, t_0) < 0$, since $u(x_0, t_0) > A$.

Part b: Suppose that v (as defined in the book) has a minimum at $(x_1, t_1) \in R$, where $x_1 \notin \{a, b\}$ and $t_1 \neq 0$. Since v has a minimum at (x_1, t_1),

$$v_t(x_1, t_1) \leq 0 \quad \text{and} \quad v_{xx}(x_1, t_1) \geq 0.$$

Here we are using the fact that $t_1 > 0$ which ensures that v is actually differentiable at (x_1, t_1), and we are also using $x_1 \notin \{a, b\}$ (since boundary minima need not satisfy the second derivative test). The above expression implies that $v_{xx}(x_1, t_1) - v_t(x_1, t_1) \geq 0$. On the other hand, since u solves the heat equation, $v_t = u_t + \epsilon = u_{xx} + \epsilon = v_{xx} + \epsilon$, and thus $v_{xx}(x_1, t_1) - v_t(x_1, t_1) = -\epsilon < 0$, contradiction.

Part c: Assume (see part a) that $u \geq 0$ on $\partial' R$. For $\epsilon > 0$, we know from part b that $v^\epsilon := u + \epsilon t$ achieves its minimum at some point $(x_1^\epsilon, t_1^\epsilon) \in \partial' R$, which means that

$$u(x, t) + \epsilon t = v^\epsilon(x, t) \geq v^\epsilon(x_1^\epsilon, t_1^\epsilon) = u(x_1^\epsilon, t_1^\epsilon) + \epsilon t_1^\epsilon \geq \epsilon t_1^\epsilon,$$

where we used the fact that $u \geq 0$ on $\partial' R$ in the final inequality. Consequently, we find that $u(t, x) \geq \epsilon(t_1^\epsilon - t)$ for every $\epsilon > 0$, as desired. The claim now follows by letting $\epsilon \to 0$.

Chapter 6, Problem 8

Part a: We set $\beta = 2\pi|x|$ on both sides of the given expression. In order to prove the expression, we need to take the Fourier transform (in the x variable) of both sides and check they are equal. First the left side:

$$\begin{aligned}\int_{\mathbb{R}} e^{-2\pi|x|} e^{-2\pi i\xi x} dx &= \int_0^\infty e^{-2\pi x(1+i\xi)} dx + \int_{-\infty}^0 e^{2\pi x(1-i\xi)} dx \\ &= \frac{1}{2\pi}\left[\frac{1}{1+i\xi} + \frac{1}{1-i\xi}\right] \\ &= \frac{1}{\pi(1+\xi^2)}.\end{aligned}$$

Now the right side: we recall that $\int_{\mathbb{R}} e^{-cx^2} e^{-2\pi i\xi x} dx = \sqrt{\frac{\pi}{c}} e^{-\pi^2\xi^2/c}$, and so by changing the order of integration, we get

$$\begin{aligned}\int_{\mathbb{R}}\left[\int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-4\pi^2 x^2/4u} du\right] e^{-2\pi i\xi x} dx &= \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}}\left[\int_{\mathbb{R}} e^{-\pi^2 x^2/u} e^{-2\pi i\xi x} dx\right] du \\ &= \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}}\left[\sqrt{\frac{u}{\pi}} e^{-u\xi^2}\right] du \\ &= \frac{1}{\pi}\int_0^\infty e^{-(1+\xi^2)u} du \\ &= \frac{1}{\pi(1+\xi^2)},\end{aligned}$$

thus proving the claim.

Part b: Using the subordination principle, we may write

$$e^{-2\pi|\xi|y} = \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-\pi^2|\xi|^2 y^2/u} du,$$

where $\xi \in \mathbb{R}^d$ and $|\xi| = (\xi_1^2 + ... + \xi_d^2)^{1/2}$. Thus we find that

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\begin{aligned}
P_y^{(d)}(x) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-2\pi|\xi|y} d\xi \\
&= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \left[\int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-\pi^2|\xi|^2 y^2/u} du\right] d\xi \\
&= \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} \left[\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-\pi^2|\xi|^2 y^2/u} d\xi\right] du \\
&= \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} \left[\frac{u^{d/2}}{\pi^{d/2} y^d} e^{-u\frac{|x|^2}{y^2}}\right] du \\
&= \pi^{-(d+1)/2} y^{-d} \int_0^\infty u^{(d-1)/2} e^{-u(1+\frac{|x|^2}{y^2})} du \\
&= \frac{\pi^{-(d+1)/2} y^{-d}}{(1+\frac{|x|^2}{y^2})^{(d+1)/2}} \int_0^\infty v^{(d-1)/2} e^{-v} dv \\
&= \frac{\Gamma((d+1)/2)\ \ y}{\pi^{(d+1)/2}(y^2+|x|^2)^{(d+1)/2}}.
\end{aligned}
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In the second-to-last equality, we made the substitution $v = u(1+\frac{|x|^2}{y^2})$, and in the final line we used the definition of the gamma function: $\Gamma(z) = \int_0^\infty v^{z-1}e^{-v}dv$, and also multiplied the numerator and denominator by y^{d+1}.