Limits of Computation

8 - Our first non-computable problem
Bernhard Reus

A non-computable problem

THIS TIME

- we define formally what computability and decidability means (for `WHILE`)
- we consider a decision problem:
  the *Halting Problem*,
  and prove *it is* `WHILE`-undecidable!

Deep Thought

"What is the Ultimate Answer to Life, the Universe, and Everything?"

Problems Revisited

Remember

- we restricted to problems of the form:
  - can we compute a given function of type *L*-data→*L*-data$_\perp$?
  - can we decide membership in a set (i.e. can we compute whether a given element is in a given set – yes or no?)
- we now narrow this down to our chosen notion of computability:

`WHILE` Computability

Definition 8.1 A partial function $f : \mathbb{D} \to \mathbb{D}_\perp$ is `WHILE`-*computable* if there is a `WHILE`-program p such that $f = [\![p]\!]^{\mathtt{WHILE}}$, in other words if f is equal to the semantics of p (we can also say "if p implements f").

Slogan: a `WHILE`-computable function on trees is one that can be implemented in `WHILE`.

partial function $f : \mathbb{D} \to \mathbb{D}_\perp$ so

Notation

$f\,(\mathtt{d}) = \perp$	means that *f* is undefined at `d`
$f(a){\downarrow}$	means that *f* is defined at a

$\texttt{WHILE}$ decidability (formally)

Definition 8.2 A set $A \subseteq \mathbb{D}$ is WHILE-decidable if, and only if, there is a WHILE-program p such that $[\![p]\!]^{\texttt{WHILE}}(d)\downarrow$ (meaning $[\![p]\!]^{\texttt{WHILE}}(d)$ is defined) for all d in $\mathbb{D}$, and, moreover, $d \in A$ if, and only if, $[\![p]\!]^{\texttt{WHILE}}(d) = \text{true}$.

Slogan: a `WHILE`-decidable set or problem on trees is one for which the membership test can be implemented in `WHILE`.

Our first Non-computable Problem

A decision problem:

Definition 8.3 The *Halting problem*—as set HALT $\subseteq \mathbb{D}$—is defined as follows:

$$\text{HALT} = \{\, [p, d] \in \mathbb{D} \mid [\![p]\!]^{\texttt{WHILE}}(d)\downarrow \,\}$$

`WHILE`-program as data

`WHILE`-data

the list (and thus the program) are encoded but we drop the encoding brackets

Big Question:

is HALT `WHILE`-decidable?

About the Halting Problem

- Solving the Halting Problem would be most useful, e.g. a compiler could check for termination of function calls and warn about non-termination like a type checker warns about incompatible types.
- Note that simply interpreting the program on its input does not work: if the interpreter does not terminate, one cannot return the answer ‘no’.

Proof of the Undecidability of the Halting Problem

IDEA

- Assume a `WHILE`-program `h` exists that *DOES* solve the Halting Problem.
- With `h`’s help write a new `WHILE`-program `r`
- establish a contradiction
- so that the assumption that `h` exists must be wrong.

Establishing a contradiction to destroy a robot or computer is a very popular SciFi plot line (a.k.a. Logic Bombs).

The Barber of Seville Paradox

strictly speaking not a "paradox" as the contradiction can be resolved.

http://www.wno.org.uk/4299

The Barber of Seville says:

"In my town Seville, I shave all men who do ***not*** *shave themselves. Those who actually shave themselves, I do not shave."*

This is a version of Bertrand Russell's paradox (Famous Welsh logician, 1872-1970)

The Barber of Seville Paradox

http://www.wno.org.uk/4299

The Barber of Seville says:

"In my town Seville, I shave all men who do ***not*** *shave themselves. Those who actually shave themselves I do not shave."*

Does the barber shave himself?
(note that he lives in Seville and is a man):

No implies he shaves himself
Yes implies he does not shave himself

contradiction

"paradox" can be resolved by saying such a Barber does not exist :-). Does not contradict any laws of nature.

The Barber of Seville Paradox as Diagonalisation

diagonal

man of Seville no

shaves	1	2	3	4	5	6	...	3466	3467	...
1	yes	yes	no	yes	no	no		no	no	
2	no	no	yes	no	no	yes		no	yes	
3	no	yes	no	no	no	no		no	yes	
4	yes	yes	no	no	yes	yes		yes	yes	
5	no	no	no	no	no	no		no	no	
6	yes	yes	no	no	no	yes		yes	yes	
...	...	...	...	...	...	...		...	...	
3466	yes	yes	no	no	yes	yes		yes	yes	
3467	no	yes	yes	no	yes	yes		yes	no	
...										
Barber's row	no	yes	yes	yes	yes	no	...	no	yes	...

man of Seville no.

Barber's row is the negated diagonal. It thus can't be one of the rows of the table, so Barber can't be a male from Seville, but he is!

contradiction

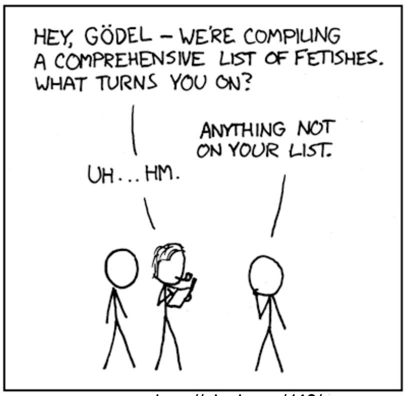
What is the table entry (x,y) ?
At row x and column y we put yes *if man # x shaves man # y and* false *otherwise.*

More on that theme

The problem is the implicit self-reference.

AUTHOR KATHARINE GATES RECENTLY ATTEMPTED TO MAKE A CHART OF ALL SEXUAL FETISHES.

LITTLE DID SHE KNOW THAT RUSSELL AND WHITEHEAD HAD ALREADY FAILED AT THIS SAME TASK.



http://xkcd.com/468/

Proof of HALT's Undecidability

Assume a program deciding the Halting Problem existed:
`h read A {B} write C`
then define `r` as:

```
r read X {
A:= [ X, X ];
B;
Y:= C;
while Y { Y:=Y }
}
write Y
```

and derive a **contradiction**.
Then `h` cannot exist.

Does $[\![\mathtt{r}]\!](\mathtt{r})\downarrow$ hold?

in other words: does program `r` terminate when run with r as input?

$\mathtt{Y} = \mathtt{C} = [\![h]\!]^{\mathtt{WHILE}}\ [\,r, r\,]$

`Y` = true means *h* says termination but *r* behaves otherwise.
`Y` = false means *h* says non-termination but *r* behaves otherwise.

`if` $[\![\mathtt{r}]\!](\mathtt{r})\downarrow$ `then`
`r doesn't terminate`
`else r terminates`

`if` $[\![\mathtt{r}]\!](\mathtt{r})\downarrow$ `then` $[\![\mathtt{r}]\!](\mathtt{r}) = \perp$
`if` $[\![\mathtt{r}]\!](\mathtt{r}) = \perp$ `then` $[\![\mathtt{r}]\!](\mathtt{r})\downarrow$

Proof of HALT's Undecidability

- The proof was using the Barber paradox technique.
- Can we also understand (reformulate) this as a proof by ***diagonalisation***?
- In order to do that, first note that we can enumerate all `WHILE`-programs (like we could enumerate all men in Seville). Why is that?

The Halting Problem as Diagonalisation

- How does `r` behave for arbitrary input programs `X`:
- If `X` run on input `X` terminates, `r` does not terminate.
- If `X` run on input `X` does not terminate, `r` does terminate.
- So `r` behaves a bit like the Barber.

```
r read X {
A:= [X,X];
B;
Y:= C;
while Y {
    Y:=Y
   }
}
write Y
```

`read A {B} write C`
assumed to decide the Halting Problem

The Halting Problem as Diagonalisation

does program A terminate when input is program B? / `WHILE` program A no \ `WHILE` program B no	1	2	3	4	5	6	…	3466	3467	…
1	yes	yes	no	yes	no	no		no	no	
2	no	no	yes	no	no	yes		no	yes	
3	no	yes	no	no	no	no		no	yes	
4	yes	yes	no	no	yes	yes		yes	yes	
5	no	no	no	no	no	no		no	no	
6	yes	yes	no	no	no	yes		yes	yes	
…	…	…	…	…	…	…		…	…	
3466	yes	yes	no	no	yes	yes		yes	yes	
3467	no	yes	yes	no	yes	yes		yes	no	
…										
r's row	no	yes	yes	yes	yes	no	…	no	yes	…

diagonal

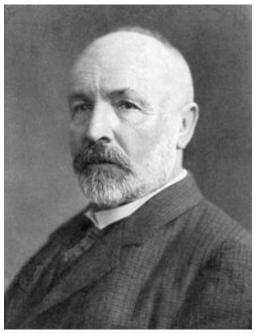
`r`'s row is the *negated diagonal. It thus can't be one of the rows of the table, so* `r` *can't be a* *WHILE* *program, but it is!*

"`r` terminates if its argument (program) does not terminate when it is given itself as input; otherwise `r` does not terminate."

contradiction

Diagonalisation Idea

- ... is very clever
- ... needs items of interest to be enumerable
- ... was discovered by Cantor in 1891 to show the existence of sets that are larger than the set of natural numbers.



Georg Cantor
1845-1918

END

Next time:
More on *semi*-decidability