Limits of Computation

3 - The `WHILE`-language
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Last time

- we discussed what problems are
- discussed that our first objective is to show that at least one of those problems cannot be "computed"
- defined what computable means in terms of "effective procedures"
- but did not commit to any specific kind of "effective procedures"

WHILE-programs as Effective Procedures

THIS TIME

- in this lecture we define a particular version of "effective procedure": WHILE-programs
- and how we use WHILE's data type

```
program reverse read X {
    Y := nil;                       (* initi
    while X {                       (* run t
      Y := cons hd X Y;             (* apper
      X := tl X                     (* remov
    }
  }
  write Y
```

a WHILE-program

WHILE

- Identify: 'effective procedure' = WHILE-program
- *"The WHILE language has just the right mix of expressive power and simplicity."* [N. Jones]
- WHILE-programs can be interpreted on any sufficiently rich machine model…
- …but, just like Alan Turing once did, we can define how to interpret WHILE-programs on paper (next time).
- Later we will use an interpreter.

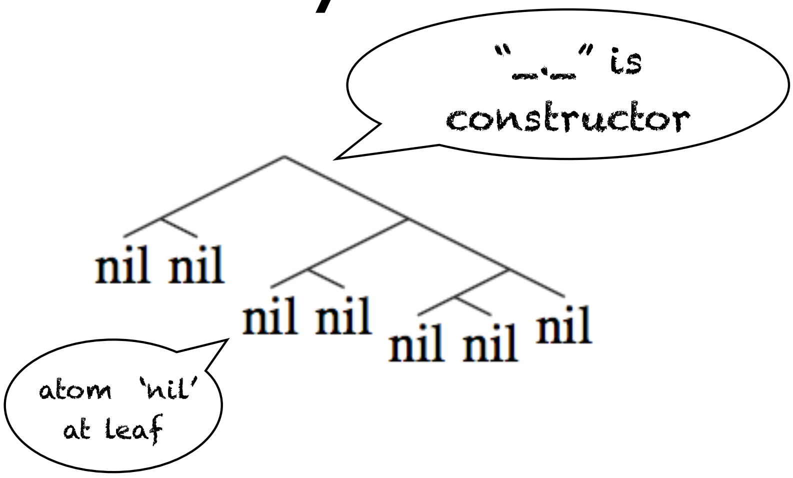
`WHILE`

- `WHILE`-programs will be much more easily understandable, and easier to write as well, than Turing machine programs (or RAM / MIPS machine programs) which we will see much later in the term.
- The idea is that this allows you to relate the concepts presented here to your perspective as programmers (and Computer Science students).

Data type: binary tree

- Our `WHILE`-language is *untyped*.
- Our `WHILE`-language has binary trees as only built-in datatype.
- allowing us to easily encode other data, including programs (!), as data values
- similar to LISP trees (or lists in other functional languages!)

Binary Trees



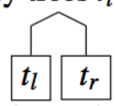
Binary Trees formally

Definition 3.1. The set of binary trees is given inductively. It contains

1. the *empty tree*:

   nil

2. any tree constructed from two binary trees t_l and t_r:

   and which is written $\langle t_l . t_r \rangle$ in textual notation.
3. and no other trees.

The set of binary trees is denoted $\mathbb{D}$ (short for "data").

$\boxed{\mathbb{D}}$

Other data types?

- We can encode easily other types, for instance,
  - booleans
  - natural numbers
  - lists
- How?

Data in List Form

```
(scientist
  (id "ATM")
  (firstName "Alan")
  (midInitial "M")
  (lastName "Turing")
  (famousFor
     (achievement "crack Enigma code")
     (achievement "define computability")
  )
)
```

LISP S-expressions

```
{
  "scientist": {
   "id": "ATM",
   "firstName": "Alan",
   "midInitial": "M",
   "lastName": "Turing",
   "famousFor": {
        { "achievement" : "crack Enigma code" },
        { "achievement": "define computability" }
        }
   }
}
```

JSON

```
<scientist id="AMT">
  <firstName>"Alan"</firstName>
  <midInitial>"M"</midInitial>
  <lastName>"Turing"</lastName>
  <famousFor>
     <achievement>"crack Enigma code"</achievement>
     <achievement>"define computability"</achievement>
   </famousFor>
</scientist>
```

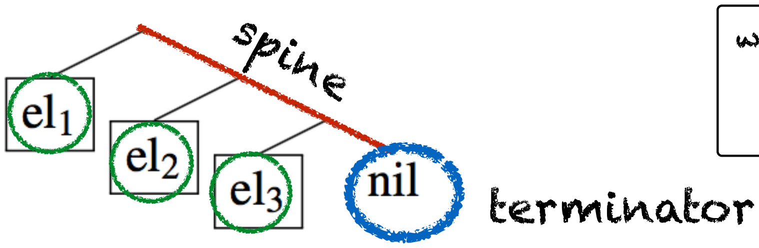
XML

Lists

Definition 3.4. The empty list is encoded by the empty tree nil and appending an element at the front of the list is modelled by $\langle \_.\_ \rangle$. More formally we define:

$$\ulcorner [\,] \urcorner = \text{nil} \tag{3.1}$$

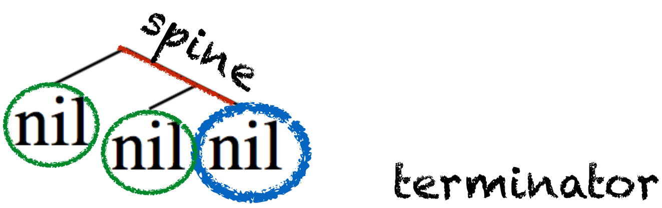
$$\ulcorner [a_1, a_2, \ldots, a_n] \urcorner = \langle \ulcorner a_1 \urcorner . \langle \ulcorner a_2 \urcorner . \, \langle \cdots \langle \ulcorner a_n \urcorner . \text{nil} \rangle \rangle \cdots \rangle \rangle \tag{3.2}$$



we use $\ulcorner \_ \urcorner$ to denote encodings

Example

$$\ulcorner [\,[\,], [\,]\,] \urcorner = \langle \text{nil}. \langle \text{nil}.\text{nil} \rangle \rangle$$



Booleans and Numbers

Definition 3.3. We encode Boolean values as follows:

we use $\ulcorner\_\urcorner$ to denote encodings

$$\ulcorner \text{false} \urcorner = \text{nil}$$
$$\ulcorner \text{true} \urcorner = \langle \text{nil.nil} \rangle$$

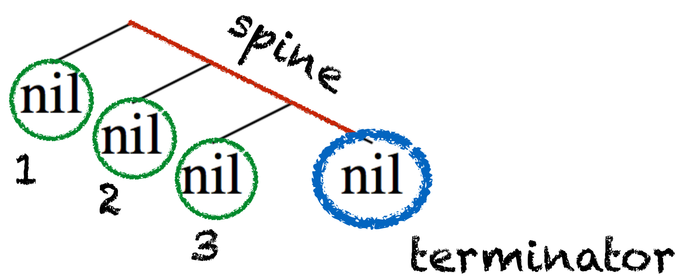
Definition 3.5. We encode numbers inductively as follows:

$$\ulcorner 0 \urcorner = \text{nil}$$
$$\ulcorner n+1 \urcorner = \langle \text{nil.} \ulcorner n \urcorner \rangle$$

Examples

$$\ulcorner 1 \urcorner = \langle \text{nil.} \ulcorner 0 \urcorner \rangle = \langle \text{nil.nil} \rangle$$

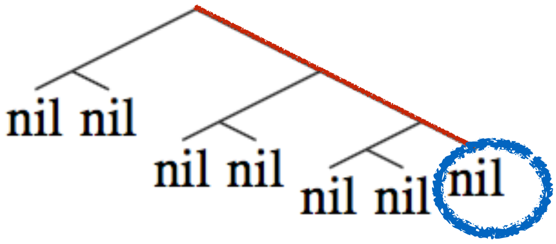
$$\ulcorner 3 \urcorner = \langle \text{nil.} \ulcorner 2 \urcorner \rangle = \langle \text{nil.} \langle \text{nil.} \ulcorner 1 \urcorner \rangle \rangle = \langle \text{nil.} \langle \text{nil.} \langle \text{nil.} \ulcorner 0 \urcorner \rangle \rangle \rangle = \langle \text{nil.} \langle \text{nil.} \langle \text{nil.nil} \rangle \rangle \rangle$$



Trees as Lists

- **Any** tree can be interpreted as a list (of something). Why?

There is always a spine & terminator!



= [1,1,1]

`WHILE` Syntax

BNF Grammar for WHILE

Expressions

⟨*expression*⟩	::=	⟨*variable*⟩	(variable expression)
	|	`nil`	(atom nil)
	|	`cons` ⟨*expression*⟩ ⟨*expression*⟩	(construct tree)
	|	`hd` ⟨*expression*⟩	(left subtree)
	|	`tl` ⟨*expression*⟩	(right subtree)
	|	(⟨*expression*⟩)	(right subtree)

Statement (lists)

⟨*block*⟩	::=	{ ⟨*statement-list*⟩ }	(block of commands)
	|	{ }	(empty block)
⟨*statement-list*⟩	::=	⟨*command*⟩	(single command list)
	|	⟨*command*⟩ ; ⟨*statement-list*⟩	(list of commands)
⟨*elseblock*⟩	::=	`else` ⟨*block*⟩	(else-case)
⟨*command*⟩	::=	⟨*variable*⟩ := ⟨*expression*⟩	(assignment)
	|	`while` ⟨*expression*⟩ ⟨*block*⟩	(while loop)
	|	`if` ⟨*expression*⟩ ⟨*block*⟩	(if-then)
	|	`if` ⟨*expression*⟩ ⟨*block*⟩ ⟨*elseblock*⟩	(if-then-else)

Programs

⟨*program*⟩	::=	⟨*name*⟩ `read` ⟨*variable*⟩
		⟨*block*⟩
		`write` ⟨*variable*⟩

BNF: Expressions

identifier

⟨*expression*⟩	::=	⟨*variable*⟩	(variable expression)
	|	`nil`	(atom nil)
	|	`cons` ⟨*expression*⟩ ⟨*expression*⟩	(construct tree)
	|	`hd` ⟨*expression*⟩	(left subtree)
	|	`tl` ⟨*expression*⟩	(right subtree)
	|	(⟨*expression*⟩)	(right subtree)

BNF: Statement (Blocks)

$\langle block \rangle$	::=	{ $\langle statement\text{-}list \rangle$ }	(block of commands)
	|	{ }	(empty block)
$\langle statement\text{-}list \rangle$	::=	$\langle command \rangle$	(single command list)
	|	$\langle command \rangle$; $\langle statement\text{-}list \rangle$	(list of commands)
$\langle elseblock \rangle$	:=	`else` $\langle block \rangle$	(else-case)
$\langle command \rangle$	::=	$\langle variable \rangle$:= $\langle expression \rangle$	(assignment)
	|	`while` $\langle expression \rangle$ $\langle block \rangle$	(while loop)
	|	`if` $\langle expression \rangle$ $\langle block \rangle$	(if-then)
	|	`if` $\langle expression \rangle$ $\langle block \rangle$ $\langle elseblock \rangle$	(if-then-else)

BNF: Programs

identifier

$\langle program \rangle$ ::= $\langle name \rangle$ `read` $\langle variable \rangle$ one input

$\langle block \rangle$

`write` $\langle variable \rangle$ one output

this is where the magic happens – "main"

END

Next time:
the semantics and
extensions of `WHILE`