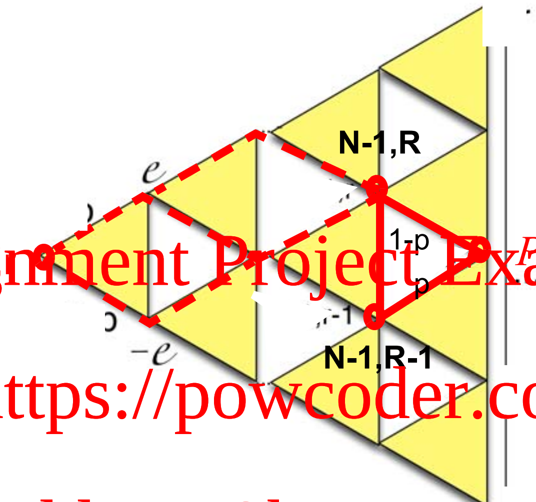
Theory: Back to The Continuum Limit of Many Small Binomial Steps Lead To: Arithmetic Brownian Motion

9.7 What is the transition probability distribution $P(x, t; X, T)$ given that

$$\overline{X-x} = \mu(T-t) \qquad \overline{(X-x)^2} - (\overline{X-x})^2 = \sigma^2(T-t)?$$



We will use the recursion relation:

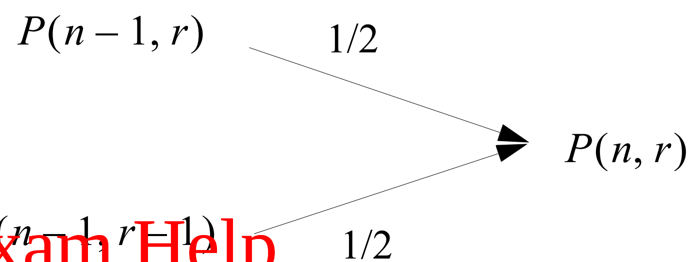
$$P(n, r, N, R) = pP(n, r, N-1, R-1) + (1-p)P(n, r, N-1, R)$$

to find the relation between time n and time $n+1$, and then let the time step become small and turn the relation into a PDE -- -Planck/Forward Kolmogorov Equation for the evolution of probabilities-- and solve it to find the distribution.

Continuous probability distribution starting from (0,0) using Jarrow-Rudd Calibration of Binomial Model

$P(0, 0; x, t) \equiv P(x, t)$ for convenient temporary notation

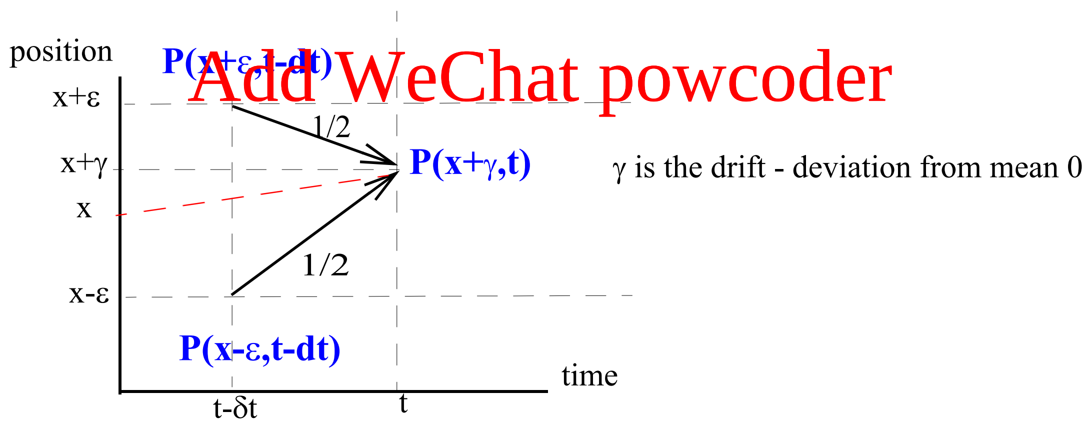
Conservation of Probability:



In discrete steps. Now go to the continuum limit to find and solve the pde for the continuous distribution:

The **Fokker-Planck/Forward Kolmogorov** Equation for evolution of probabilities:

Let $P(x, t)$ be the probability density of being between x and $x + dx$ at time t:



Then rewriting the recursion relation in terms of x and t rather than n and r:

$$P(x+\gamma, t) = 0.5P(x+\varepsilon, t-\delta t) + 0.5P(x-\varepsilon, t-\delta t)$$

Deriving the Fokker-Planck Equation for P(x,t) as $\delta t \to 0$

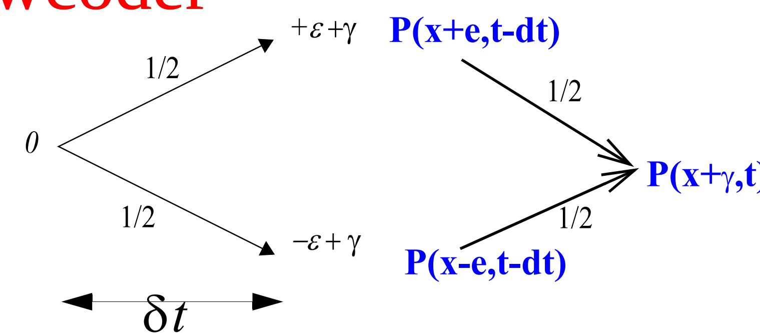
$$P(x+\gamma, t) = 0.5P(x+\varepsilon, t-\delta t) + 0.5P(x-\varepsilon, t-\delta t)$$

Work out Taylor series: $\frac{\partial P}{\partial x}\gamma = \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\varepsilon^2 - \frac{\partial P}{\partial t}\delta t$ to order δt

Then divide by δt $\frac{1}{2}\frac{\partial^2 P}{\partial x^2}\left(\frac{\varepsilon^2}{\delta t}\right) - \frac{\partial P}{\partial x}\frac{\gamma}{\delta t} = \frac{\partial P}{\partial t}$

But in limit $\delta t \to 0$ from Equations 9.5: $\frac{\gamma}{\delta t} = \mu$ drift and $\frac{\varepsilon^2}{\delta t} = \sigma^2$ variance

$$\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial x^2} - \mu\frac{\partial P}{\partial x} = \frac{\partial P}{\partial t}$$



This is the forward **Fokker-Planck equation** for $P(x, t)$

In the continuum limit only μ and σ^2 matter.
It's a **diffusion equation** for *x* as a function of time *t*.
When you see a differential equation like this, it *looks* impenetrable. Understand that it *really* only represents the **forward** recursive binomial process in the limit of continuous motion.

… Fokker- Planck Equation in the Continuum Limit

The pde for the distribution $P(x, t)$ then becomes

$$\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial x^2} - \mu\frac{\partial P}{\partial x} = \frac{\partial P}{\partial t} \qquad \text{Eq.9.8}$$

In the continuum limit only μ and σ^2 matter.

This is called the *forward Fokker-Planck equation*. The diffusion equation.

It just says that probability is conserved through time as the particle moves through the tree.

When you see a differential equation like this, understand that it *really* only represents the **forward** recursive binomial process in the limit of continuous motion.

Forward because it tells you how probability propagates forward.

We showed above that the limit was chosen so that after time *t*,

$$\bar{x} = \mu t \qquad \overline{x^2} - (\bar{x})^2 = \sigma^2 t$$

The mean displacement is μt and the mean variance of displacement around the mean is $\sigma^2 t$.

This motion is called **Arithmetic Brownian motion**.

Arithmetic Brownian Motion

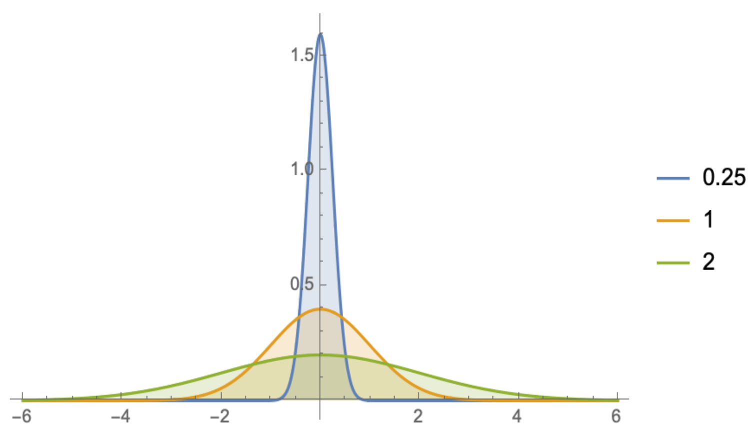
$$\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial x^2} - \mu\frac{\partial P}{\partial x} = \frac{\partial P}{\partial t} \qquad \text{PDE}$$

The solution to this differential equation for $t > 0$ is **the normal distribution,** subject to the initial condition of being at position $x = 0$ at time $t = 0$.

$$P(0, 0; x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right) \qquad \text{the probability density function (pdf)} \qquad \text{Eq.9.9}$$

$x \in N(\mu t, \sigma^2 t)$ **Prove** $P(x, t)$ satisfies the PDE

Check that the integral over all x from $-\infty$ to ∞ at any time t is always 1.



Some exercises:

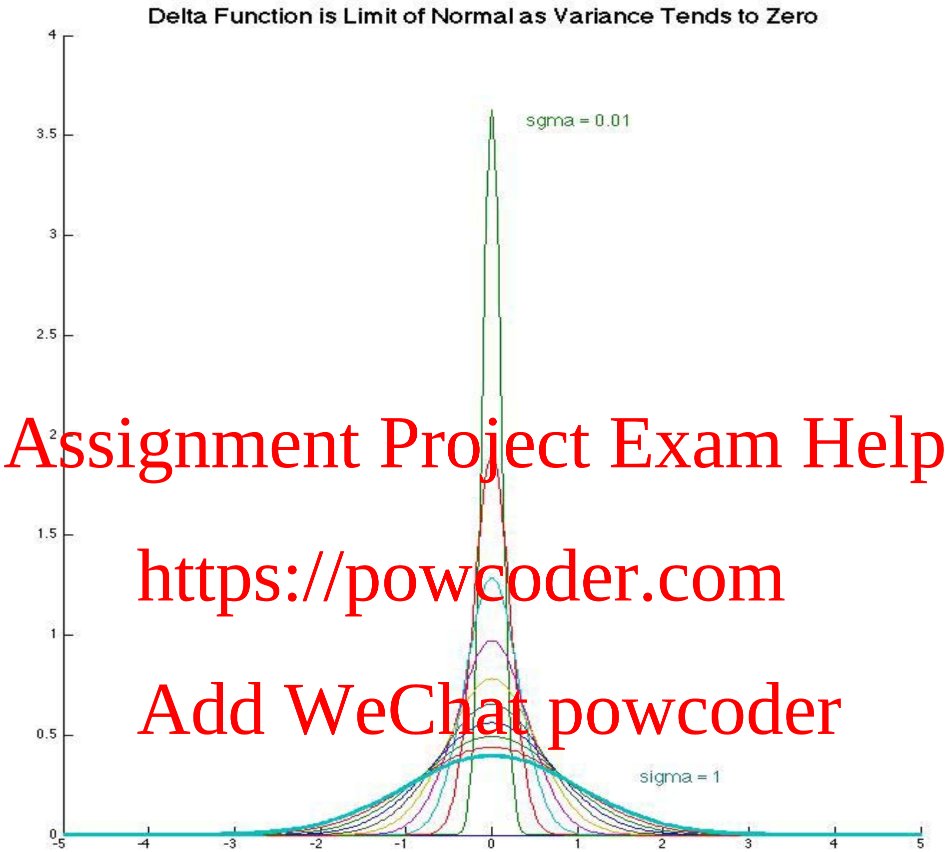
Prove $\bar{x} = \mu t \qquad \overline{x^2} - (\bar{x})^2 = \sigma^2 t.$

Think about where the exponential isn't very small.

When σ or t get small, the curve gets higher and narrower, with area always equal to 1.

What does $P(x,t)$ look like when $t = 0$? It's a *Dirac Delta function*, width zero, height infinity, area 1, that has all the probability concentrated at one point, x = 0.

Delta Function as the Limit of a Normal Distribution with $\mu = 0$



Picture a **Dirac delta function** as the limit of $N(\mu, \sigma^2)$ with $\mu = 0$ zero drift and $\sigma^2 \to 0$

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ \infty \text{ if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)dx = 1$$

Picture a **Dirac delta function** $\delta(x)$ as the limit of $N(\mu, \sigma^2)$ with $\mu = 0$ zero drift and $\sigma^2 \to 0$

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ \infty \text{ if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) dx = 1$$

Moment Generating Function for a Normal Distribution

A random variable X follows a normal distribution with mean μ and variance σ^2 if its probability density function (pdf) is

$$p(x) = N(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad x \in [-\infty, \infty] \qquad \text{Eq.10.5}$$

We can obtain the moment generating function $M(a)$ by completing the square and integrating over x, for all real a:

$$M(a) = E[e^{aX}] = e^{a\mu + \frac{1}{2}a^2\sigma^2}$$

Differentiating n times w.r.t a and setting $a = 0$, we obtain the nth moment about the origin:

$$M(a) = E\left[1 + aX + \frac{1}{2}a^2X^2 + \ldots\right] = e^{a\mu + \frac{1}{2}a^2\sigma^2}$$

$$M'(a) = E[X + aX^2 + a^2\ldots] = (\mu + a\sigma^2)e^{a\mu + \frac{1}{2}a^2\sigma^2} \qquad \text{etc}$$

$$M''(a) = E[X^2 + a\ldots] = (\mu + a\sigma^2)^2 e^{a\mu + \frac{1}{2}a^2\sigma^2} + \sigma^2 e^{a\mu + \frac{1}{2}a^2\sigma^2}$$

$$M'(0) = E[X] = \mu \qquad M''(0) = E[X^2] = \mu + \sigma^2$$

Arithmetic vs Geometric Motion

An arithmetic series produces successive terms by adding. A geometric series produces successive terms by multiplication.

A random series of annually compounded returns 0.5, -0.5

If we take the additive average, the mean return is 0. This is arithmetic change.

But if these returns act on prices, starting with price 100, we get
$(100) \times (1.5) \times 0.5 = 100 \times 0.75 = 75$ and the return is not zero but rather -0.25.
This is geometric because successive terms multiply.

… Arithmetic Brownian Motion from the Central Limit Theorem

That the binomial distribution converges to a normal distribution also follows from the Central Limit Theorem, which says that the sum of N **iid** (independent identically distributed random variables) with mean m and variance σ^2 approaches, as the sum becomes large, a *normal distribution* with mean Nm and variance $N\sigma^2$.

We've already seen this from the binomial tree:

$$\bar{r} = pn$$

$$\overline{(r-\bar{r})^2} = \overline{r^2} - \bar{r}^2 = \text{variance} = np(1-p)$$

Stochastic Calculus of Brownian Motion

Why Stochastic Calculus?

Suppose, as an example, that you have an algorithmic trading strategy, a formula $n(S, t)$, that involves buying shares whenever the stock goes down and selling shares whenever the stock goes up, according to some formula. The rule is to sell $n(S, t)dt$ shares during the next instant of time dt when the stock price is S at time t, and S is random. If $n(S, t)$ is negative, that means we buy.

In that example, the cash available at the end is $\int_o^T n(S, t)Se^{r(T-t)}dt$ and the total number of shares sold is worth $S_T\int_o^T n(S, t)dt$ at time T. These integrals are path-dependent, and we are integrating a function of a stochastic variable S.

We need to be able to integrate over such Brownian motions in order to **model and evaluate trading strategies** that involve holding variable quantities of stock $n(S, t)$ as S changes randomly, to take averages, variances, etc.

10.5 Brownian Processes More Generally and Formally.

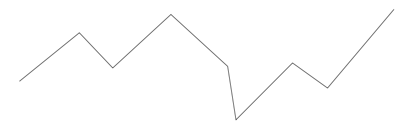
A *Brownian* ***process*** or a *Wiener* ***process***.

The binomial tree in the limit $\delta t \to 0$ is a **particular representation** of a more general and abstractly defined stochastic process.

Definition

A Brownian process is a stochastic process $\{X(t); t \geq 0\}$ with the properties

1. Every increment $X(t+s) - X(s)$ is **normally** distributed with mean μt and variance $\sigma^2 t$, with μ and σ as fixed parameters;
2. For every $t_1 < t_2 < \ldots < t_n$ the increments $X(t_i) - X(t_{i-1})$ are *independent* random variables with normal distributions as in 1. above.
3. $X(0) = 0$ and the paths emanating from there are continuous.



$$P(0, 0; x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x - \mu t)^2}{2\sigma^2 t}\right)$$

A *standard Brownian motion* or *standard Wiener process* $Z(t)$ has $\mu = 0$ and $\sigma^2 = 1$.

$$P(x, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$$

Some Properties

Show by integration for a standard Wiener process:

$$E[Z(t)] = 0$$

$$E[Z(t)^2] = var(Z) = t \quad \text{as we saw from the binomial distribution and its limit} \qquad \text{Eq.10.6}$$

For two independent Brownian motions X and Y: $E[XY] = E[X](E[Y]) = 0$

Another useful property is that $E[Z(t)Z(s)] = min(t, s)$

Proof

Assume $t > s$. Then this trick is used a lot, separating times:

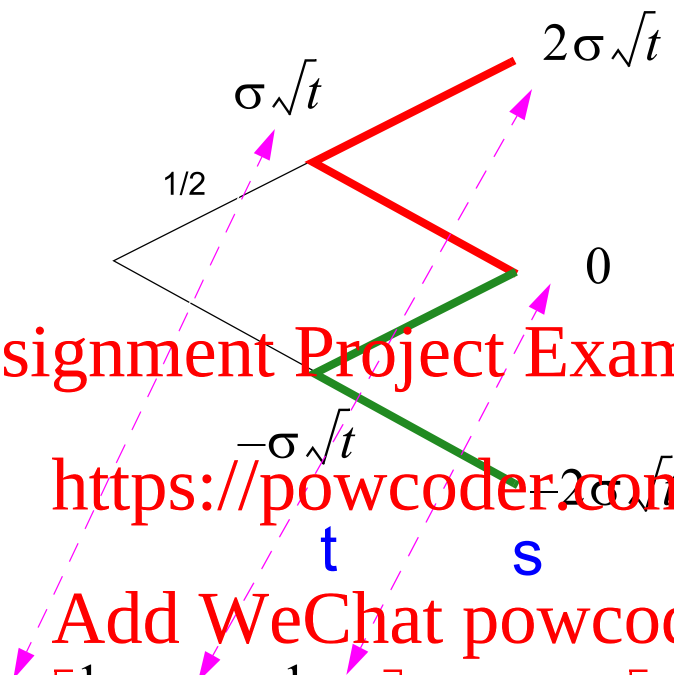
$$\begin{aligned} E[Z(t)Z(s)] &= E[\{Z(t) - Z(s)\}Z(s) + Z(s)^2] \\ &= E[\{Z(t) - Z(s)\}Z(s)] + E[Z(s)^2] \qquad \text{Eq.10.7} \\ &= 0 + s \qquad \text{(first term is zero because independent increments)} \\ &= min(s, t) \end{aligned}$$

The overlap in the processes gives variance proportional to the time that the processes overlap.

Brownian motions have no autocorrelation.

Illustration of $E[Z(t)Z(s)] = min(t, s)$ for the Binomial Tree

Let $s = 2t$:

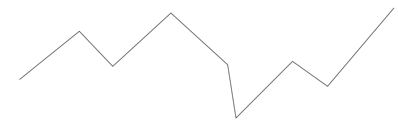


$$[Z(t)Z(s)] = \sigma\sqrt{t}\left[\frac{1}{4}2\sigma\sqrt{t} + \frac{1}{2}(0)\right] + (-\sigma\sqrt{t})\left[-\frac{1}{4}2\sigma\sqrt{t} + \frac{1}{2}(0)\right] = \sigma^2$$

10.6 Stochastic Calculus (a Heuristic Treatment)

Calculus, differential and integral, lets you handle the motion of differentiable variables.

Brownian motions are **not really differentiable** because they jump around, continuously but abruptly. They are stochastic processes. Nevertheless one can **integrate** Brownian motions too, and treating them in this way is called *stochastic calculus*.



We're interested in integration of random variables but we will often write derivatives **as a short-hand** for the inverse process of integration.

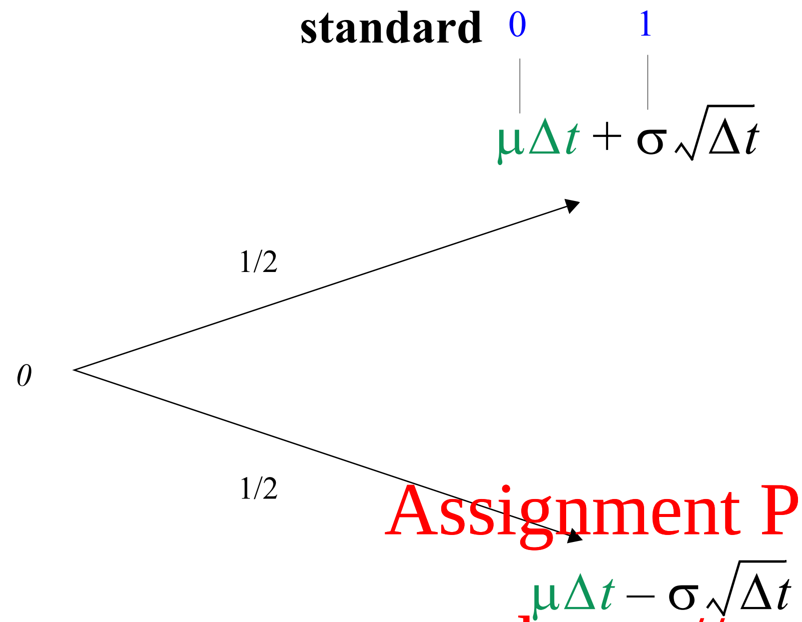
Thus in ordinary calculus, $\frac{df(x)}{dx} = x + 3$ also means $f(y) - f(0) = \int_0^y (x+3)dx$

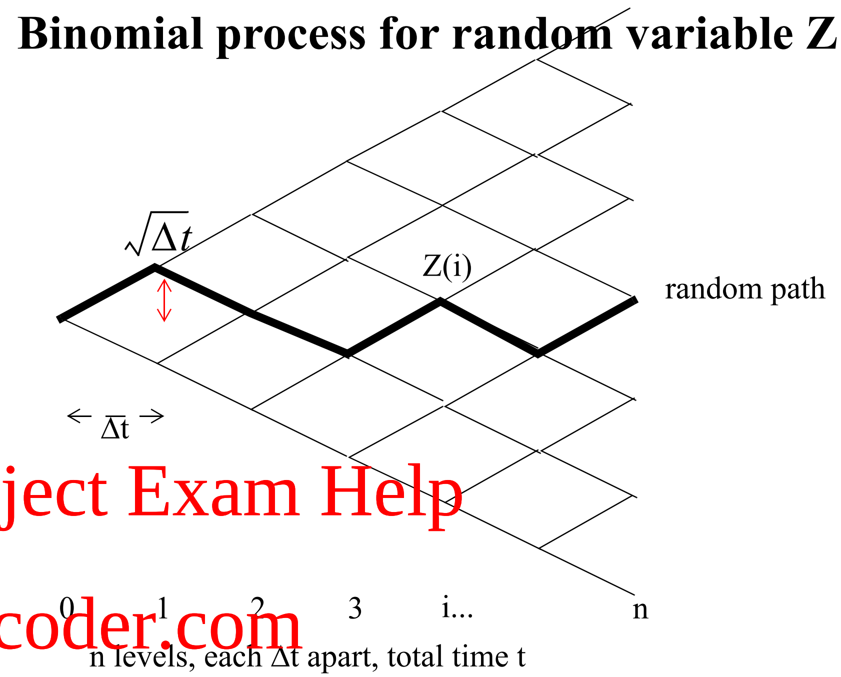
You can think of a Brownian motion as **the limit** of a discrete random walk process on a binomial tree with n periods over time t, with **the limit** taken as $n \to \infty$ for fixed t.

The paths are non-smooth and obviously not differentiable in the normal sense, but **they are continuous** and therefore integration can make sense. Differentiation makes things more spiky, but integration averages over spikes.

So, in order to approach integration, let's look at a binomial tree with n periods and discover some interesting and important properties of such paths and integrals of functions over them.

A Random Walk for Z(t) Has Known Finite Quadratic Variation





Quadratic Variation Q( ): The **square** of the changes along the path

$$Q(0, t) = \sum_{i=1}^{n} [Z(i) - Z(i-1)]^2 = \sum_{i=1}^{n} (\sqrt{\Delta t})^2 = n\Delta t = n\frac{t}{n} \to t \qquad \text{Eq.10.8}$$

This will hold even in the limit as $n \to \infty$ for fixed t.

The absolute variation is infinite: $n\left|\pm\sqrt{\Delta t}\right| = \left(\frac{t}{\Delta t}\right)\sqrt{\Delta t} = \frac{t}{\sqrt{\Delta t}} \to \infty$ as $\Delta t \to 0$

In Contrast, An Ordinary Differential Variable Has Zero Quadratic Variation. A Brownian Motion is Not Differentiable.

- For a **differentiable function** $X(t)$, $\dfrac{\partial X}{\partial t}$ is finite, and the quadratic variation is zero.

$$X(t+dt) - X(t) = \frac{\partial X}{\partial t} dt + \text{... higher order terms in } dt$$

$$Q(0,t) = \sum_{i=1}^{n} [X(i) - X(i-1)]^2 \approx \sum_{i=1}^{n} \left(\frac{\partial X}{\partial t} \Delta t\right)^2 \sim n(\Delta t)^2 = n\left(\frac{t}{n}\right)^2 \to 0 \quad \text{as } n \to \infty$$

$Q(0,t) = t$ means that *Z(t)* is not a differentiable function of time, though it is continuous.

- $dZ \sim \sqrt{dt}$

  $$\frac{dZ}{dt} \sim \frac{\sqrt{dt}}{dt} \sim \frac{1}{\sqrt{dt}} \to \infty \text{ as } dt \to 0$$

In finance we are interested in integrating or summing over stochastic paths to see what happens to the profit and loss as we trade, or to understand the statistics of random prices.

Thus we will want to look at the **stochastic Ito integral.**

10.7 The Ito Process

An *Ito process* describes a random path $X(t)$ generated with a variable volatility:

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dZ(s) \qquad \text{Eq.10.9}$$

As $dZ(s)$ is a random Wiener process, this generates many paths through time.

The differential form that corresponds to the integral which is the standard way of writing it is

$$dX(t) = \mu(t)dt + \sigma(t)dZ(t)$$

We must explain what we mean by the integral $\int_0^t \sigma(s)dZ(s)$ in the equation above.

And if we define a random path then we want to define an integral along the path.

The Ito Integral

$$\int_0^T f(t)dZ(t) \equiv \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\underbrace{[Z(t_i) - Z(t_{i-1})]}_{\text{forward differential}} \qquad \text{Eq.10.10}$$

$f(t)$ is **adapted** or **non-anticipative**: it depends **only on the path history** of $Z(t)$ up to time t.

In finance, the function $f(\ )$ usually reflects what we trade, e.g. how many shares $\Delta(S, t)$ we buy in a trading strategy when the stock is at price S at time t. Then the subsequent change in $Z(t)$ reflects what happens to the price S after that.

Let's see what an Ito integral looks like in a particular case. Let's integrate $Z(t)$ by evaluating

$$\int_0^T Z(t)dZ(t) = \lim_{n \to \infty} \sum_{i=1}^{n} Z(t_{i-1})[Z(t_i) - Z(t_{i-1})]$$

where $Z(t)$ is a standard Wiener process. Let's take $n = 3$ first as a simple example.

For ordinary variables, $\int_0^T Z(t)dZ(t) = \dfrac{Z(T)^2 - Z(0)^2}{2}$ The Chain Rule $ZdZ = \frac{1}{2}dZ^2$

… Now Let's Look at the Binomial Sum for n = 3 out to Time *t*

$$\sum_{i=1} Z(t_{i-1})[Z(t_i)-Z(t_{i-1})]$$

$$= \sum_{i=1}^{3} \frac{[\{Z(t_{i-1})+Z(t_i)\}-\{Z(t_i)-Z(t_{i-1})\}]}{2}[Z(t_i)-Z(t_{i-1})]$$

rearrange terms to get squares which we know how to handle via defn of Brownian motion quadratic variation.

$$= \sum_{i=1}^{3} \frac{1}{2}\left[Z^2(t_i)-Z^2(t_{i-1})\right]-\frac{1}{2}[Z(t_i)-Z(t_{i-1})]^2$$

$$= \frac{1}{2}\{Z(3)^2-Z(0)^2\} - \sum_{i=1}^{3}\frac{1}{2}[Z(t_i)-Z(t_{i-1})]^2$$

$dZ^2 = dt = t/3$

$$= \frac{1}{2}\{Z(3)^2-(Z(0))^2\}-\frac{1}{2}\left[3\frac{t}{3}\right]$$

$$\sum^{3} Z(t_{i-1})[Z(t_i)-Z(t_{i-1})] = \frac{1}{2}\{Z(3)^2-(Z(0))^2\}-\frac{1}{2}t$$

It's as though $\int ZdZ = \frac{1}{2}\int d(Z^2)-\frac{t}{2}$ or in differential form $dZ(t)^2 = 2Z(t)dZ(t)+dt$.

Modified Chain Rule. We can generalize this as follows for arbitrary n periods covering time t.

… The (Forward) Ito Integral for n > 3

$$\int_0^t Z(t)dZ(t) = \lim_{n \to \infty} \sum_{i=1} Z(t_{i-1})[Z(t_i) - Z(t_{i-1})]$$

$$= \sum_{i=1}^{n} \frac{[\{Z(t_{i-1}) + \cancel{Z(t_i)}\} - \{\cancel{Z(t_i)} - Z(t_{i-1})\}]}{2}[Z(t_i) - Z(t_{i-1})]$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} \{Z(t_{i-1}) + Z(t_i)\}\{Z(t_i) - Z(t_{i-1})\} - \{Z(t_i) - Z(t_{i-1})\}^2\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} \left\{Z^2(t_i) - Z^2(t_{i-1})\right\} - \{Z(t_i) - Z(t_{i-1})\}^2\right)$$

$$= \frac{1}{2}\left[Z^2(t_n) - Z^2(0) - \sum_{i=1}^{n} dt\right]$$

In the limit of an infinitely fine grid over time t, with $n\Delta t = T$ as $n \to \infty$, we obtain

$$\int^T Z(t)dZ(t) = \frac{1}{2}[Z^2(T) - Z^2(0) - T]$$

$$2\int_0^T Z(t)dZ(t) = -T + \{Z(T)^2 - Z(0)^2\} \qquad \text{Eq.10.11}$$

This only has real meaning as an integral, but we can write the **symbolic differential form** that represents Equation 10.11:

$$2Z(t)dZ(t) = d[Z^2(t)] - dt \qquad \text{or}$$

$$d[Z^2(t)] = 2Z(t)dZ(t) + dt$$

$$df = \frac{\partial f}{\partial Z}dZ + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}dt \qquad \text{Eq.10.12}$$

The extra *dt* term comes from the Brownian motion/quadratic variation of moves $\sim \sqrt{dt}$.

We will often (always) write equations like Equations 10.12, which look like differential calculus, but they are really a short way of specifying what happens when you integrate. They describe the stochastic process of the function $[Z^2(t)]$

Equation 10.12 **is an instance of *Ito's Lemma*** for Wiener processes. It is a version of the chain rule for differentiation, but for stochastic processes.

11.6 Ito's Lemma:
A Statement About How To Integrate Functions Of A Stochastic Variable $dX(t) = \mu(t)dt + \sigma(t)dZ(t)$

$$df(X,t) = \left[\frac{\partial}{\partial t}f(X,t) + \frac{\sigma^2(t)}{2}\frac{\partial^2}{\partial X^2}f(X,t)\right]dt + \frac{\partial}{\partial X}f(X,t)dX \qquad \text{Eq.11.8}$$

A heuristic proof follows from doing the Taylor expansion of $f(X, t)$ up to second order in X and t, keeping all terms of order dt, and using the rules

$$dX^2(t) = \sigma^2(t)dt$$

"Box algebra"

$$dXdt = O((dt)^{3/2}) = 0$$

$$dt^2 = 0$$

·	dt	$dW(t)$
dt	0	0
$dW(t)$	0	dt

Box algebra for a standard Wiener process keeping leading orders of dt in the Taylor expansion:

11.7 Geometric Brownian Motion for Stock Prices

Log returns $X(t)$ are modeled as normal. Stock prices are modeled as exponentiated returns.

$$S(t) = S_0 e^{X(t)} \qquad \ln\frac{S(t)}{S_0} = X(t)$$

This means returns can be positive or negative, normally distributed, and the exponentiated return is always positive, so the stock price never goes negative. (The original application of Brownian motion to stocks by Bachelier used ABM and allowed stock prices to go negative.)

Calibration of the Drift of the Return: suppose we observe that the stock price compounds at an average rate μ so that $E[S(t)] = S_0 e^{\mu t}$ with volatility of returns σ.

How must we choose *X(t)* to grow so that $S(t)$ increases at rate μ?

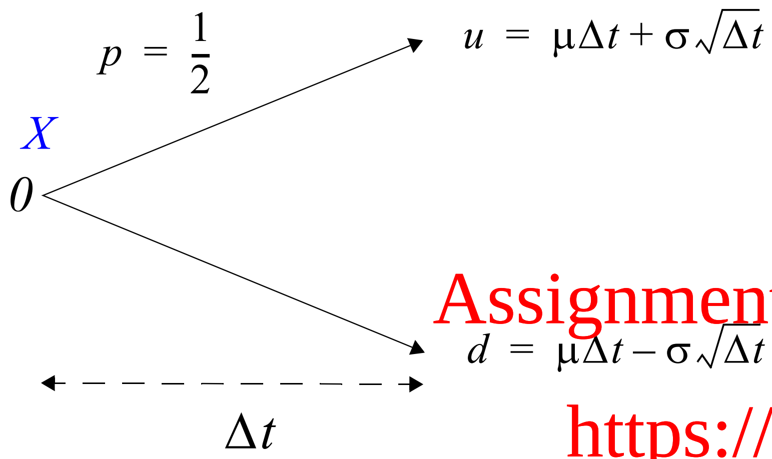
X is an arithmetic Brownian motion for returns. so that $dX = \text{something}\ dt + \sigma dZ$.

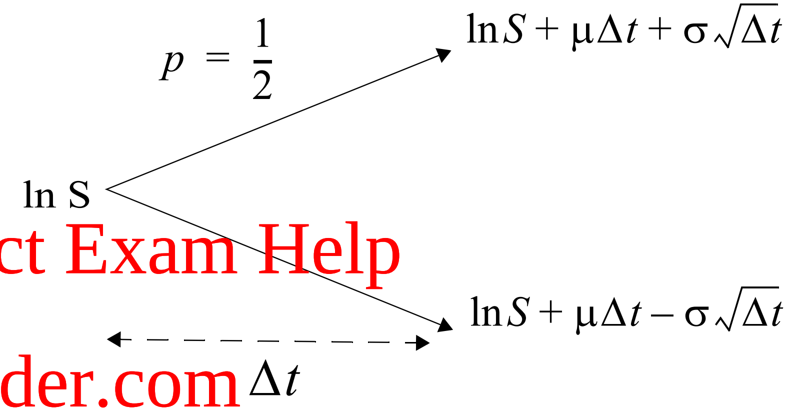
What is the required drift something? It is not exactly μ.

What happens when you exponentiate an arithmetic Brownian motion? Well, we'll see that if the ABM has zero drift, the GBM has a greater positive drift. Let's look at it binomially first.

11.7.1 From Normal/Arithmetic to Lognormal/Geometric Binomially

We have earlier modeled the binomial distribution of an Arithmetic random variable $X = \ln S$ as involving infinitesimal increments $\mu\Delta t \pm \sigma\sqrt{\Delta t}$





Repeated $\frac{t}{\Delta t}$ times, we showed this converges to a normal distribution $N(\mu t, \sigma^2 t)$ with drift μ.

Now let's see what happens when we exponentiate the arithmetic Brownian motion.

$$p = \frac{1}{2}$$

$\ln S \;\to\; \ln S + \mu\Delta t + \sigma\sqrt{\Delta t}$

$\ln S \;\to\; \ln S + \mu\Delta t - \sigma\sqrt{\Delta t}$

Δt

$\longleftrightarrow$

$S \xrightarrow{1/2} S\exp(\mu\Delta t + \sigma\sqrt{\Delta t})$

$S \xrightarrow{1/2} S\exp(\mu\Delta t - \sigma\sqrt{\Delta t})$

Δt

$$\exp(\mu\Delta t + \sigma\sqrt{\Delta t}) \approx \left(1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t\right)e^{\mu\Delta t}$$

$$\exp(\mu\Delta t - \sigma\sqrt{\Delta t}) \approx \left(1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t\right)e^{\mu\Delta t}$$

$$E[S] = \frac{1}{2}S \times \left(e^{\mu\Delta t + \sigma\sqrt{\Delta t}} + e^{\mu\Delta t - \sigma\sqrt{\Delta t}}\right) = Se^{\mu\Delta t}\frac{\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right)}{2}$$

$$= Se^{\mu\Delta t}\left(1 + \frac{\sigma^2\Delta t}{2}\right) \approx Se^{\mu\Delta t}e^{\frac{\sigma^2\Delta t}{2}} \approx Se^{\left(\mu + \frac{\sigma^2}{2}\right)\Delta t}$$

The stock on average grows at an exponential rate $\mu + \dfrac{\sigma^2}{2}$

If we want the stock to grow at average rate μ then the $\ln(\text{stock})$ must grow at $\mu - \sigma^2/2$

Now let’s look at this more elegantly using Ito’s Lemma

11.7.2 Using Ito's Lemma for Geometric Brownian Motion

$$\text{Assume } dX(t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ(t)$$

For S as $S = f(X, t) = S_0 e^{X(t)}$ $\quad \frac{\partial S}{\partial t} = 0, \quad \frac{\partial S}{\partial X} = S, \quad \frac{\partial^2 S}{\partial X^2} = S \quad$ geometric

$$dS(t) = \cancel{\frac{\partial S}{\partial t}dt} + \frac{\partial S}{\partial X}dX + \frac{1}{2}\frac{\partial^2 S}{\partial X^2}(\sigma^2 dt)$$

$$dS(t) = S(t)dX(t) + S(t)\frac{\sigma^2}{2}dt = S(t)\left[\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ(t)\right] + S(t)\frac{\sigma^2}{2}dt$$

$$dS(t) = S(t)[\mu dt + \sigma dZ(t)]$$

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dZ(t)$$

$$\text{take Z averages} \quad d\langle S\rangle = \mu\langle S\rangle dt$$

$$\langle S(t)\rangle = S(0)\exp(\mu t)$$

If $\ln S$ has drift $\mu - \sigma^2/2$ then S has exponential growth rate μ.

Ito's Lemma is really nothing more than analogous to a nonlinear function Taylor- expanded on a binomial tree.

Summarizing Geometric Brownian Motion for the Stock Price

Stochastic Differential Equation

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dZ(t) \qquad \text{or} \qquad dS(t) = \mu S(t) dt + \sigma S(t) dZ(t)$$

$$d\ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dZ(t) \qquad \text{Eq.11.9}$$

integrate

$$\ln\frac{S(t)}{S(0)} = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma Z(t)$$

μ is called the **expected growth rate** or **drift**. It's the geometric rate of growth.

σ is called the **volatility**. It's the volatility of the log of the stock price, i.e. of the returns.

Statistics of the Stock Price

$$S(t) = S(0) e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma Z(t)}$$

We can calculate the first and second moments of $S(t)$ for any future time t.

$$E[S(t)] = S(0) e^{\left(\mu - \frac{\sigma^2}{2}\right) t} E[e^{\sigma Z(t)}] = S(0) e^{\left(\mu - \frac{\sigma^2}{2}\right) t} e^{\frac{1}{2}\sigma^2 t} = S(0) e^{\mu t}$$

Homework: What is the variance of $S(t)$? It's a little different from the variance of $dS(t)$.

More Ito

- Ito for $S^{1/2}$

  If S is geometric, then $S^{1/2}$ is too, but with different drift and half the volatility.

- $dX = \alpha(m - X)dt + \sigma dZ(t)$ Ornstein Uhlenbeck

- Shortly: Ito for two stochastic variables

11.8 Some Tricky Stuff About Ensemble Average vs. Time Average

$$d\log\frac{S(t)}{S(0)} = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ(t)$$

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t)$$

now take average over all paths dZ, ENSEMBLE average < >for all investors

$$\langle dS(t)\rangle = \langle \mu S(t)dt\rangle + \langle \sigma S(t)d\cancel{Z(t)}\rangle$$

$$d\langle S\rangle = \mu\langle S\rangle dt$$

$$\langle S(t)\rangle = S(0)\exp(\mu t)$$

So average stock price for all investors in parallel grows at rate μ

What about one investor who simply holds on to one stock forever?

$$d\log\frac{S(t)}{S(0)} = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ(t)$$

At ANY time in the future the growth rate is $\left(\mu - \frac{\sigma^2}{2}\right) \pm$ random dZ

So half the time it is $+dZ$ and the other half it is $-dZ$

So TIME AVERAGE individual growth rate for log S is $\left(\mu - \frac{\sigma^2}{2}\right)$

So $\log\frac{S(T)}{S(0)} \to \left(\mu - \frac{\sigma^2}{2}\right)T$ as $T \to \infty$ with $S(T) \to S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)T}$ very slow convergence

… Difference between Ensemble Average and Time Average

As an example, let's take a dramatic example with large volatility $\mu = 0.1$ and $\sigma = 0.5$

$$\mu - \frac{\sigma^2}{2} = -0.025$$

So the stock price ensemble average (for all possible paths) grows at rate 10%

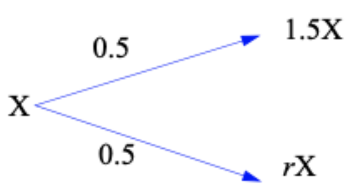
But any one path in the long run has an average log return of -2.5% and so in the long run

$$\log\frac{S(100)}{S(0)} \to -0.025(100) = -2.5$$

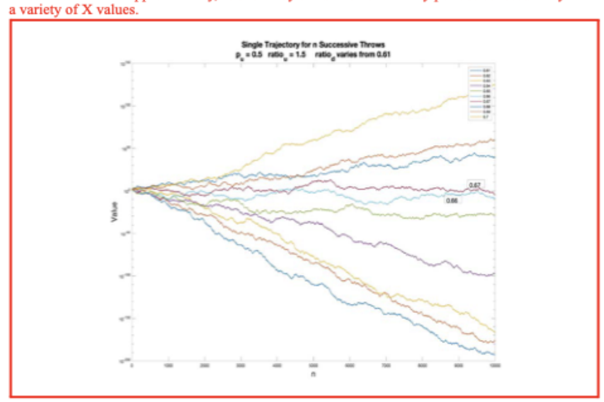
$$S(100) \to S(0)\exp(-2.5) = S(0)0.082\ldots$$

very slow convergence

Recall HW 1.

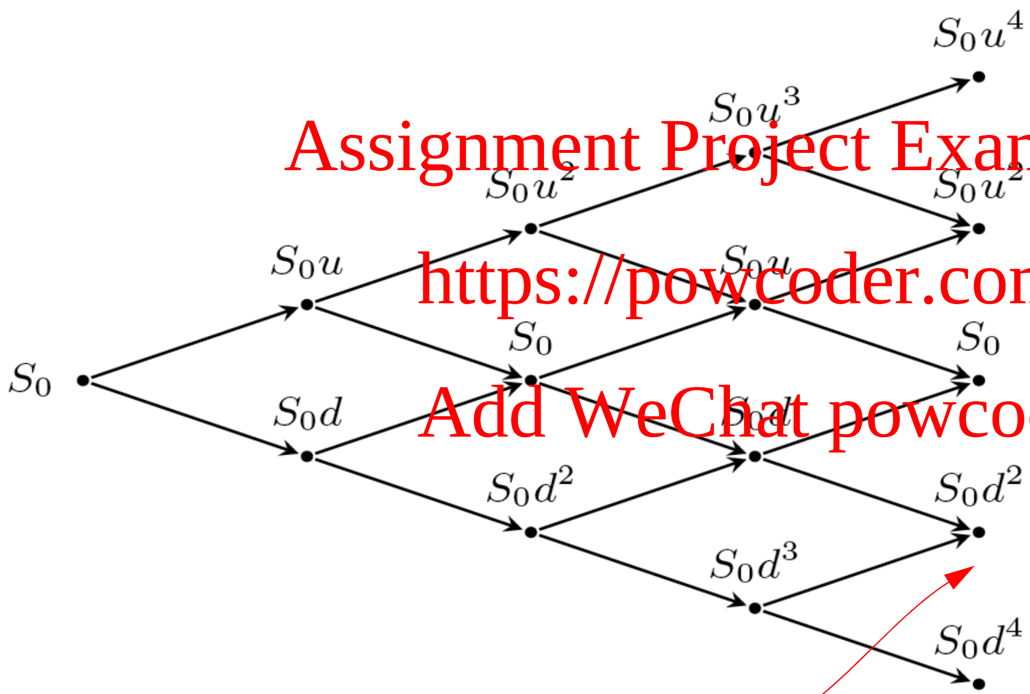


Answer: X = 0.667 approximately, as shown by simulation over many periods consecutively with a variety of X values.



11.9 Look Ahead: Building A Binomial Lattice (CRR, ud=1) For The Stock Price Thru Time - We'll Use This for Option Pricing Later

$$u = \exp(\sigma\sqrt{\Delta t})$$



$ud^3 = d^2$

Multi-period Binomial Tree. At time zero, initial stock price $S(0) = S_0$ is known. After each period, price either goes up by u or goes down by d. We set $ud = 1$. At time $i\Delta t$ (end of i-th period), asset price takes $i+1$ possible values:

$$S_0 u^j d^{i-j}, \quad j = 0, 1, \ldots, i.$$