Lecture 15: Duality, Arbitrage, and Asset Pricing

ISyE 6673: Financial Optimization

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Midterm survey recap

- 36% response rate
- Thank you for all the feedback!
- **Top comments:**

  - Recording lectures: difficult to do without more AV tech support
  - Typos: I will do better, please keep me accountable on Piazza
  - Workload/difficulty: exam 1 was too long, some HWs are too long; I will try to shorten homeworks and create some extra credit opportunities

Duality Recap

- Lecture 10: Lagrangian duality
  - Lagrangian relaxation
  - Dual problem is finding the best Lagrangian relaxation
- Lecture 11: Sensitivity analysis
  - Origin of shadow prices
  - Generalized dependence on the right-hand side b
- Lecture 12: Application to A/B testing
- Lecture 13: Geometry of duality
  - Normal cones
  - Farkas lemma
- Lecture 14: Large-scale optimization
  - Column generation and cutting planes

Today

Applications of duality in asset pricing and relationship to arbitrage!

Today

- A market model
- What are desirable properties of a market?
  - Completeness
  - Law of one price
  - Fundamental theorem(s) of asset pricing

Two-period market model

Market setup

- We consider a portfolio of n *assets* (could be bonds, stocks, currencies, etc.)

- Two-period model:
  - At time 0 (today), we know the *price* of each asset, represented by a vector $\mathbf{p}$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

  - At time 1 (tomorrow), the asset prices are *uncertain*: the world could be in one of m *scenarios*

  - Scenario matrix:

$$\mathbf{S} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \\ S_{m1} & & S_{mn} \end{bmatrix}, \quad S_{ij} \text{ is the price of asset } j \text{ in scenario } i$$

Market setup

Example

Consider the following market:

- We have $n = 3$ assets
- Prices at time 0 (today) are given by $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$
- At time 1 (tomorrow) the world could be in one of 4 scenarios:

$$\mathbf{S} = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 1 & 5 \\ 1 & 5 & 10 \\ 1 & 2 & 8 \end{bmatrix}$$

Q: Are all these assets equally risky?

Constructing portfolios

- A portfolio $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ counts shares purchased at time 0
  - $x_i > 0$ means a long position in asset i
  - $x_i < 0$ means a short position in asset i
- *Price* of portfolio at time 0 is $\mathbf{p}^\intercal \mathbf{x} = \sum_{j=1}^n p_j x_j$
- *Payoff* of portfolio at time 1 is

$$\mathbf{S}\mathbf{x} = \sum_{i=1}^{n} \mathbf{S}_j x_j = \begin{bmatrix} \sum_{j=1}^n S_{1j} x_j \\ \vdots \\ \sum_{j=1}^n S_{mj} x_j \end{bmatrix} \in \mathbb{R}^m$$

  - One payoff value for each scenario

Constructing portfolios

Example

- Market setup: $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 1 & 5 \\ 1 & 5 & 10 \\ 1 & 2 & 8 \end{bmatrix}$
- Portfolio $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
- Price: $\mathbf{p}^T\mathbf{x} = 1 - 2 + 7 = 6$
- Payoff: $\mathbf{S}\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \times 1 + \begin{bmatrix} 3 \\ 1 \\ 5 \\ 2 \end{bmatrix} \times (-1) + \begin{bmatrix} 9 \\ 5 \\ 10 \\ 8 \end{bmatrix} \times 1 = \begin{bmatrix} 7 \\ 5 \\ 6 \\ 7 \end{bmatrix}$

Complete markets

Completeness

- A market is specified by prices $\mathbf{p}$ today and payoffs $\mathbf{S}$ tomorrow

Definition

A market is *complete* if any payoff $\mathbf{z} \in \mathbb{R}^m$ can be achieved by some portfolio $\mathbf{x}$.

- The **asset span** $\mathcal{M}$ is the set of payoffs achievable by some portfolio $\mathbf{x}$:

$$\mathcal{M} = \{\mathbf{z} \in \mathbb{R}^m : \mathbf{z} = \mathbf{S}\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

  - $\mathcal{M}$ is the column space of $\mathbf{S}$; completeness means $\mathcal{M} = \mathbb{R}^m$

Completeness

Example

Consider a market with 1 asset and 2 scenarios:

$$p = 1, \quad \mathbf{S} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Can we achieve the payoff $\mathbf{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?
- We cannot! The payoff in scenario 1 must always be the same as the payoff in scenario 2
- This market is not complete!
- The matrix does not have *full row rank*

Complete markets: linear algebra review!

Theorem

The following statements are equivalent:

- *A market is complete:* $\mathcal{M} = \mathbb{R}^m$
- *The column space of* $\mathbf{S}$ *is* $\mathbb{R}^m$
- $\mathbf{S}$ *has* m *linearly independent columns*
- $\mathbf{S}$ *has* m *linearly independent rows (full row rank)*
- $\mathbf{S}$ *has rank* m

Number of assets

A complete market must have more assets than scenarios, i.e. $m \leq n$. Otherwise, $\mathbf{S}$ has rank at most $n < m$

The Law of One Price

Law of One Price

Theorem

If two portfolios $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same payoff in all scenarios at time 1, they must have the same price at time 0.

$$\mathbf{S}\mathbf{x}_1 = \mathbf{S}\mathbf{x}_2 \Rightarrow \mathbf{p}^\top \mathbf{x}_1 = \mathbf{p}^\top \mathbf{x}_2$$

Equivalently

$$\mathbf{S}\mathbf{x} = 0 \Rightarrow \mathbf{p}^\top \mathbf{x} = 0$$

(an asset with 0 payoff in all scenarios should have price 0)

Implications of the Law of One Price

- $\mathbf{S}\mathbf{x} = 0 \Leftrightarrow \mathbf{x} \in \text{null}(\mathbf{S})$
  - $\mathbf{x}$ is orthogonal to each row $\mathbf{s}_i$ of $\mathbf{S}$
  - $\mathbf{x} \perp \text{span}(\mathbf{s}_1, \ldots, \mathbf{s}_m)$

- $\mathbf{p}^\intercal \mathbf{x} = 0 \Leftrightarrow \mathbf{p} \perp \mathbf{x}$
  - Therefore $\mathbf{p} \in \text{null}(\mathbf{S})^\perp$ (orthogonal complement)
  - $\mathbf{p} \in \text{span}(\mathbf{s}_1, \ldots, \mathbf{s}_m) = \text{span}(\mathbf{S}^\intercal)$
  - There exists some $\mathbf{q} \in \mathbb{R}^m$ such that $\mathbf{S}^\intercal \mathbf{q} = \mathbf{p}$

Example in an incomplete market

Example

- $p = 1$, $\mathbf{S} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Law of one price holds trivially: $\mathbf{S}^\intercal \mathbf{x} = 0 \Rightarrow \mathbf{x} = 0 \Rightarrow \mathbf{p}^\intercal \mathbf{x} = 0$
- $\mathbf{q}^\intercal = \begin{bmatrix} 1 & 0 \end{bmatrix}$ or $\mathbf{q}^\intercal = \begin{bmatrix} 0 & 1 \end{bmatrix}$ both satisfy $\mathbf{q}^\intercal \mathbf{S} = \mathbf{p}^\intercal$

Why do we care about the law of one price?

- Consider a market where the law doesn't hold
- There exists a portfolio $\mathbf{x}$ such that
  - $\mathbf{S}\mathbf{x} = 0$ (0 payoff in all scenarios)
  - $\mathbf{p}^\intercal \mathbf{x} = \epsilon > 0$ (nonzero price)

- Consider the portfolio $\alpha\mathbf{x}$ (scale $\mathbf{x}$ by α)
  - $\mathbf{S}\alpha\mathbf{x} = \alpha\mathbf{S}\mathbf{x} = 0$ (still 0 payoff in all scenarios)
  - $\mathbf{p}^\intercal \alpha\mathbf{x} = \alpha\epsilon$ (arbitrary price)

- Now take an arbitrary portfolio $\mathbf{y} \in \mathbb{R}^n$ with payoff $\mathbf{S}\mathbf{y} = \mathbf{z} \in \mathbb{R}^m$, and augment it by $\alpha\mathbf{x}$
  - $\mathbf{S}(\mathbf{y} + \alpha\mathbf{x}) = \mathbf{S}\mathbf{y} = \mathbf{z}$ (same payoff as $\mathbf{y}$)
  - $\mathbf{p}^\intercal(\mathbf{y} + \alpha\mathbf{x}) = \mathbf{p}^\intercal\mathbf{y} + \alpha\epsilon$ (arbitrary price)
- Any portfolio can be purchased at any price!

Payoff pricing

Payoff pricing

How much do we need to pay at time 0 to secure a particular payoff $\mathbf{z}$ at time 1?

- Only defined if $\mathbf{z} \in \mathcal{M} = \text{col}(\mathbf{S})$
- Same payoff can be achieved by multiple portfolios
- Set of prices determined by function $q : \mathbb{R}^m \to P(\mathbb{R})$:

$$q(\mathbf{z}) = \{\mathbf{p}^\intercal \mathbf{x} : \mathbf{S}\mathbf{x} = \mathbf{z}, \mathbf{x} \in \mathbb{R}^n\}$$

- In general, $q(\mathbf{z})$ is a set
- If the Law of One Price holds:

$$\mathbf{S}\mathbf{x}_1 = \mathbf{S}\mathbf{x}_2 \Rightarrow \mathbf{p}^\intercal \mathbf{x}_1 = \mathbf{p}^\intercal \mathbf{x}_2 \Rightarrow q(\mathbf{z}) \text{ has a unique value}$$

Payoff pricing function

- When $q(\mathbf{z})$ is unique for all $\mathbf{z} \in \mathcal{M}$, we call q a *payoff pricing function*.
- $q(\mathbf{z})$ unique $\Leftrightarrow$ Law of One Price holds $\Leftrightarrow \mathbf{p}^\intercal = \mathbf{q}^\intercal \mathbf{S}$
- Putting both together for a portfolio $\mathbf{x}$ achieving payoff $\mathbf{z}$:

$$q(\mathbf{z}) = \mathbf{p}^\intercal \mathbf{x} = \mathbf{q}^\intercal \mathbf{S} \mathbf{x} = \mathbf{q}^\intercal \mathbf{z}$$

Theorem

The Law of One Price holds in a market if and only if the payoff pricing function is linear.

Example in a complete market

Example

- $p = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- Check LOP holds: $\mathbf{S}^\intercal \mathbf{x} = 0 \Rightarrow \mathbf{x} = 0 \Rightarrow \mathbf{p}^\intercal \mathbf{x} = 0$
- $\mathbf{q}^\intercal = \begin{bmatrix} 1 & 1 \end{bmatrix}$ satisfies $\mathbf{q}^\intercal \mathbf{S} = \mathbf{p}^\intercal$ so $q(\mathbf{z}) = \mathbf{q}^\intercal \mathbf{z}$
- If we want to achieve payoff $\mathbf{z} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ then we need to buy a portfolio worth

$$\mathbf{q}^\intercal \mathbf{z} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3$$

Example in an incomplete market

Example

- $p = 1, \quad \mathbf{S} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Check LOP holds: $\mathbf{S}^\intercal \mathbf{x} = 0 \Rightarrow \mathbf{x} = 0 \Rightarrow \mathbf{p}^\intercal \mathbf{x} = 0$

- $\mathbf{q}_1^\intercal = \begin{bmatrix} 1 & 0 \end{bmatrix}$ or $\mathbf{q}_2^\intercal = \begin{bmatrix} 0 & 1 \end{bmatrix}$ both satisfy $\mathbf{q}^\intercal \mathbf{S} = \mathbf{p}^\intercal$
- We cannot achieve a payoff $\mathbf{z} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ at any price

- To achieve a payoff $\mathbf{z} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, we must pay

$$\mathbf{q}_1^\intercal \mathbf{z} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \mathbf{q}_2^\intercal \mathbf{z}$$

Arbitrage

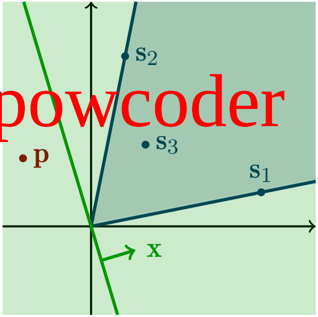
Strong arbitrage

Definition

In a market with price vector $\mathbf{p}$ and payoff matrix $\mathbb{S}$, a strong arbitrage is a portfolio $\mathbf{x}$ that verifies:

- $\mathbf{p}^\intercal \mathbf{x} < 0$ (guaranteed gain at time 0)
- $\mathbf{S}\mathbf{x} \geq 0$ (guaranteed no losses at time 1)

- $\mathbf{S}\mathbf{x} \geq 0 \Leftrightarrow \mathbf{x}^\intercal \mathbf{s}_i \geq 0 \; \forall i \in [m]$
- $\mathbf{p}^\intercal \mathbf{x} < 0 \Leftrightarrow \mathbf{x}^\intercal \mathbf{p} < 0$
- Equivalently, $\mathbf{p} \notin \text{cone}(\mathbf{s}_1, \ldots, \mathbf{s}_m)$

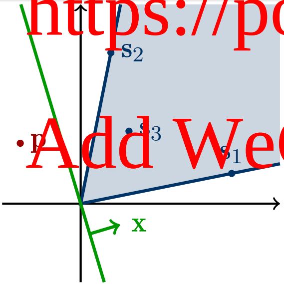


Strong arbitrage and the Farkas lemma

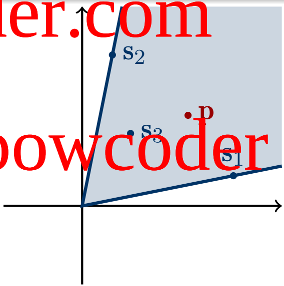
Theorem (Farkas Lemma)

Let $\mathbf{S} \in \mathbb{R}^{m \times n}$ *and* $\mathbf{p} \in \mathbb{R}^n$*. Exactly one of the following is true:*

1. *There exists some* $\mathbf{x}$ *such that* $\mathbf{S}\mathbf{x} \geq 0$ *and* $\mathbf{p}^\intercal \mathbf{x} < 0$
2. *There exists some* $\mathbf{q} \geq 0$ *such that* $\mathbf{S}^\intercal \mathbf{q} = \mathbf{p}$



Strong arbitrage:
$\mathbf{p} \notin \text{cone}(\mathbf{s}_1, \ldots, \mathbf{s}_m)$



No strong arbitrage:
$\mathbf{p} \in \text{cone}(\mathbf{s}_1, \ldots, \mathbf{s}_m)$

Fundamental theorem of asset pricing

- Farkas' lemma gives us two alternatives
  - Either the market has a strong arbitrage
  - Or there exists $\mathbf{q} \geq 0$ such that $\mathbf{S}^\intercal \mathbf{q} = \mathbf{p}$
- Any portfolio $\mathbf{x}$ yielding payoff $\mathbf{S}\mathbf{x} = \mathbf{z}$ will have price

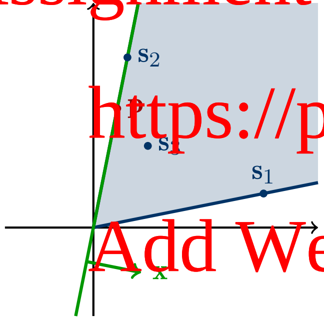
$$\mathbf{p}^\intercal \mathbf{x} = \mathbf{q}^\intercal \mathbf{S}\mathbf{x} = \mathbf{q}^\intercal \mathbf{z}$$

- We have a linear pricing rule again (so the LOP holds!) but this time it is non-negative

Theorem (Fundamental theorem of asset pricing)

A market is strong-arbitrage-free if and only if there exists a linear and non-negative payoff pricing function.

An edge case

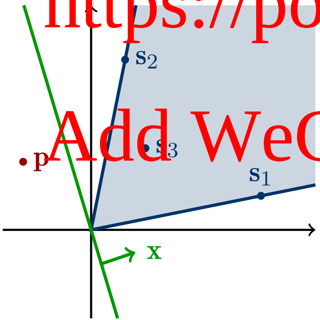


- $\mathbf{p} \in \text{cone}(\mathbf{s}_1, \ldots, \mathbf{s}_m) \Leftrightarrow$ no strong arbitrage
- Consider the portfolio $\mathbf{x}$
  - $\mathbf{p}^\intercal \mathbf{x} = 0$ (no cost at time 0)
  - $\mathbf{x}^\intercal \mathbf{s}_i \geq 0 \ \forall i \in [m] \Leftrightarrow \mathbf{S}\mathbf{x} \geq 0$ (no losses at time 1)
  - $\mathbf{s}_1^\intercal \mathbf{x} > 0$ (at least one scenario has positive payoff)
- Portfolio $\mathbf{x}$ is called a *weak arbitrage*

Strong vs. weak arbitrage

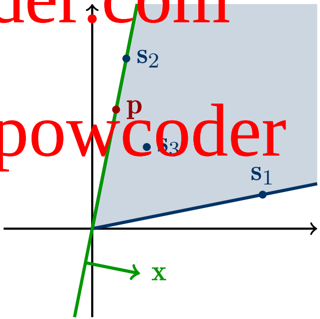
Strong arbitrage

- $\mathbf{p}^\intercal \mathbf{x} < 0$
- $\mathbf{S}\mathbf{x} \geq 0$



Weak arbitrage

- $\mathbf{p}^\intercal \mathbf{x} = 0$
- $\mathbf{S}\mathbf{x} \geq 0$
- $\exists i \in [m], \mathbf{s}_i^\intercal \mathbf{x} > 0$



Ruling out weak arbitrage

How to rule out weak arbitrage?

- $\mathbf{p} \in \text{cone}(\mathbf{s}_1, \ldots, \mathbf{s}_m)$ rules out strong arbitrage
- To rule out weak arbitrage, $\mathbf{p}$ cannot be on the boundary of the cone:

$$\mathbf{p} \in \text{int}(\text{cone}(\mathbf{s}_1, \ldots, \mathbf{s}_m))$$

- Equivalently, there exists $\mathbf{q} > 0$ such that $\mathbf{S}^\intercal \mathbf{q} = \mathbf{p}$

- We need a *stronger* fundamental theorem of asset pricing to rule out *weak* arbitrage
- Ironic, I know!

Fundamental theorem of asset pricing

Theorem (Weak form)

A market is strong-arbitrage-free if and only if there exists a linear and non-negative payoff pricing function.

Theorem (Strong form)

A market is arbitrage-free if and only if there exists a linear and positive payoff pricing function.

Payoff pricing summary

Linear positive Payoff Pricing $\Leftrightarrow$ No Arbitrage $\Leftrightarrow$ $p \in \mathrm{int}(\mathrm{cone}(S^\top))$

$\Downarrow$ $\Downarrow$ $\Downarrow$

Linear Nonnegative Payoff Pricing $\Leftrightarrow$ No Strong Arbitrage $\Leftrightarrow$ $p \in \mathrm{cone}(S^\top)$

$\Downarrow$ $\Downarrow$ $\Downarrow$

Linear Payoff Pricing $\Leftrightarrow$ Law of One Price $\Leftrightarrow$ $p \in \mathrm{span}(S^\top)$

Why do we care so much about linea pricing rules?

Because it gives us a way to price new assets (e.g. derivatives!)

Risk-neutral probabilities

Risk-neutral probabilities

- A market is arbitrage-free if and only if there exists a positive linear pricing function $\mathbf{q} > 0$, i.e.

$$\mathbf{p} = \mathbf{S}^\intercal \mathbf{q}$$

- For each asset j,

$$p_j = \mathbf{S}_j^\intercal \mathbf{q} = \sum_{i=1}^{m} q_i S_{ij}$$

- Let $\delta = \sum_{i=1}^{m} q_i$:

$$p_j = \sum_{i=1}^{m} \underbrace{\frac{q_i}{\delta}}_{\tilde{q}_i \in (0,1]} \delta S_{ij} = \sum_{i=1}^{m} \tilde{q}_i \delta S_{ij} = \mathbb{E}_{\tilde{q}}[\delta S_j],$$

where we notice that $\tilde{q}$ defines a valid probability distribution, which we call a *risk-neutral probability distribution*

The meaning of δ

- In most markets, there is at least one *risk-free asset* (e.g., US Treasury bonds); assume our first asset is risk-free with interest rate r
  - Price is p_1
  - Future price is $S_1 = \begin{bmatrix} (1+r)p_1 \\ \vdots \\ (1+r)p_1 \end{bmatrix}$ for all scenarios
- Arbitrage-free market:

$$p_1 = \sum_{i=1}^{m} \tilde{q}_i \delta S_{i1} = \sum_{i=1}^{m} \tilde{q}_i \delta (1+r) p_1 = \delta (1+r) p_1 \Rightarrow \delta = \frac{1}{1+r}$$

- δ is a discount rate! Risk-neutral pricing is thus: $p_j = \mathbb{E}_{\tilde{q}}\left[\frac{S_j}{1+r}\right]$

Arbitrage-free and complete markets

- Arbitrage-free means there exists a positive linear pricing function, equivalently a risk-neutral probability distribution
- Complete means $\mathbf{S}$ has rank $m \leq n$
- If $\mathbf{S}$ has rank m, then so does $\mathbf{S}^\intercal$, and $\mathbf{S}^\intercal \mathbf{q} = \mathbf{p}$ has a unique solution

Theorem (Second fundamental theorem of asset pricing)

An arbitrage-free market is complete if and only if the risk-neutral probability distribution is unique.

Derivative pricing

Derivatives

- Let's say we have a new payoff vector $\mathbf{z}$ (also called a contingency claim)
- If $\mathbf{z}$ is a deterministic function of the existing n asset payoffs, it is called a *derivative security*:

$$\mathbf{z} = f(\mathbf{S}_1, \ldots, \mathbf{S}_n)$$

Examples

Derivative	Outcome (at time 1)	Payoff
Forward contract	Must buy asset j at price K	$\mathbf{S}_j - K$
Call option	Can buy asset j at price K	$\max(\mathbf{S}_j - K, 0)$
Put option	Can sell asset j at price K	$\max(K - \mathbf{S}_j, 0)$

Put-call parity

$$\begin{aligned}
\mathbf{z}_{\text{call}} - \mathbf{z}_{\text{put}} &= \max(\mathbf{S}_j - K, 0) - \max(K - \mathbf{S}_j, 0) \\
&= \mathbf{S}_j - K \\
&= \mathbf{z}_{\text{forward}}
\end{aligned}$$

In a complete and arbitrage-free market

$$\begin{aligned}
q(\mathbf{z}_{\text{call}} - \mathbf{z}_{\text{put}}) &= q(\mathbf{z}_{\text{call}}) - q(\mathbf{z}_{\text{put}}) && \text{(unique linear pricing)} \\
&= q(\mathbf{z}_{\text{forward}}) \\
&= \mathbb{E}_{\tilde{q}}\left[\frac{\mathbf{S}_j - K}{1 + r}\right] && \text{(risk-neutral probability)} \\
&= p_j - \frac{K}{1 + r}
\end{aligned}$$

Arbitrage-free derivative pricing

How to price derivatives?

Given a derivative with payoff/contingency claim $\mathbf{z}$, how to determine the current price $q(\mathbf{z})$?

Basic idea

- Start with a market $M_1 = (\mathbf{p}, \mathbf{S})$ with n assets
- Create a new market $M_2 = \left(\begin{bmatrix} \mathbf{p} \\ p_{n+1} \end{bmatrix}, \begin{bmatrix} \mathbf{S} & \mathbf{z} \end{bmatrix} \right)$ with $n+1$ assets (the new contingency claim is the $n+1$-th asset)

- Does there always exist an arbitrage-free price p_{n+1} for $\mathbf{z}$?
- Is the arbitrage-free price of $\mathbf{z}$ unique?

Arbitrage-free derivative pricing

Theorem

Suppose the derivative has non-negative payoff. If M_1 is arbitrage-free and complete, there exists a unique price of the derivative such that M_2 is also arbitrage-free and complete. Moreover, M_1 and M_2 have the same risk-neutral probability distribution.

- If M_1 is complete, then M_2 is also complete (adding an asset/column cannot decrease the column space)
- M_1 is arbitrage-free and complete, so there is a unique risk-neutral probability distribution $\tilde{q}$
- We can use this distribution to price $\mathbf{z}$, i.e.

$$q(\mathbf{z}) = \mathbb{E}_{\tilde{q}}\left[\frac{z}{(1+r)}\right]$$

Example

Example

Consider the following complete arbitrage-free market:

$$\mathbf{p} = \begin{bmatrix} 1 \\ 50 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 1 & 60 \\ 1 & 20 \end{bmatrix}$$

What is the price of a put option with strike price $K = 40$?

- First work out the risk-neutral probabilities (notice asset 1 is risk-free with 0 interest):

$$50 = q \cdot 60 + (1 - q) \cdot 20 \Rightarrow q = \frac{3}{4}$$

- The risk-neutral probabilities are $\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}$
- The payoff is $\mathbf{z} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$, so $q(\mathbf{z}) = \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 20 = 5$.

Arbitrage-free pricing in incomplete markets

- As long as the market is arbitrage-free, we can always find arbitrage-free prices for derivatives
- If the market is not complete, the price may not be unique
- This is the topic of Problem 3 on HW4!

Summary

- Farkas' lemma underpins fundamental theorems in asset pricing
- Markets that obey the law of one price have linear pricing rules
- Complete markets have a unique pricing rule
- Arbitrage-free markets have positive pricing rules