MA570 Spring 2021 Stochastic Methods in OR Krigman
Mid-Term

Please show all your work in the blue book provided and $\boxed{\text{box off your answers.}}$
If you get stuck I recommend moving to the next problem and coming back to the one that gave you trouble later.

Some useful formulas:

- ***If***:
  the state k is an absorbing state, then the set of absorption probabilities f_{ik} satisfies the system of equations

$$f_{ik} = \sum_{j=0}^{M} p_{ij} f_{jk}, \qquad \text{for } i = 0, 1, \ldots, M,$$

subject to the conditions

$$f_{kk} = 1,$$
$$f_{ik} = 0, \qquad \text{if state } i \text{ is recurrent and } i \neq k.$$

- **Expected first passage times:**

$$\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} \mu_{kj}.$$

- **Geometric series sum:** $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ *if* $|x|<1$.

1. **Problem (35 pts total)**. Consider the Markov chain that has the following (one-step) transition matrix.

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{4}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

(a) (5 pts) Draw the transition diagram for this system
(b) (5 pts) Determine the classes of this Markov chain and, for each class, determine whether it is recurrent or transient.
(c) (5 pts) For each of the classes identified in part (b), determine the period of the states in that class.
(d) (10 pts) Determine the following three quantities: $p_{22}{}^{(17)}$; $\lim_{n\to\infty} p_{03}{}^{(n)}$; $\lim_{n\to\infty} p_{11}{}^{(n)}$
(e) (10 pts) Estimate the following ***probabilities*** of the first passage times: $f_{00}{}^{(1)}$; $f_{00}{}^{(2)}$; $f_{24}{}^{(1)}$; $f_{24}{}^{(2)}$; $f_{34}{}^{(19)}$. Give a brief explanation for your answers.

2. **Problem (15 pts total).** A transition matrix P is said to be *doubly stochastic* if the sum over each column equals 1; that is $\sum_{i=0}^{M} p_{ij} = 1$, for all j. If such a chain is irreducible, aperiodic, and consists of M+1 states show that $\pi_j = \frac{1}{M+1}$, for j=0,1,…,M.
Note: *You may assume that a chain consists of only 2 states (i.e. M=1) and show the above result for this case only.*
3. **Problem (25 pts).** Three players are tossing a ball to one another. Player A always tosses the ball to player B. Player B always tosses to C. And player C tosses to A two thirds of the times and to B one third of the times. The game may be described by a stochastic process where the states may be labeled as state 0 when Player A has the ball, state 1 when B has the ball and 2 when C has the ball.
   a) (5 pts) Write down the transition matrix corresponding to this system.
   b) (10 pts) Determine the steady state probabilities.
   c) (10 pts) Using your answer to part (a), give the expected recurrence times for the ball to return to Player A, Player B, and Player C.
4. **Problem (25 points).** Suppose that we modify the rules of the game above: Player A always tosses the ball to player B. Player B always tosses to C (same as before). But now, player C ***always*** tosses to player A.
   a. **(5 pts)** Give the new transition matrix.
   b. **(10 pts)** Determine the steady state probabilities.
   c. **(10 pts)** Estimate the following ***probabilities*** of the first passage times: ${f_{AA}}^{(1)}$; ${f_{AB}}^{(1)}$; ${f_{AC}}^{(2)}$; ${f_{BB}}^{(3)}$; ${f_{CC}}^{(6)}$. (Subscripts A, B, or C refer to the states as defined by which player has a ball). Give a brief explanation for your answers.

5. **(30 pts) Problem**: Your city has a serious flu virus going around. You are given the following information:

If you go around your regular daily activities as you normally would, the following odds hold.

- You would have close encounter with enough people to make your daily chances of contracting a virus and getting sick on any given day to be 0.01.
- ***If you get sick***, the following outcomes are possible on any given day:
    - Probability of remaining sick at the end of the day is 0.8
    - Probability of getting well is 0.195
    - Probability of death is 0.005
- If you get well (after having been sick) you will become immune to the flu and never get it again.

Consider the following possible states of the discrete time Markov Chain where updates happen at the end of each day:

State = 0 (Healthy and had never had the flu, so being susceptible to getting flu)

State = 1 (Getting sick with the flu)

State = 2 (Getting healthy after flu, so being immune)

State = 3 (Death from the flu)

No other outcome is possible. If you get sick and then get well, consider that to be a final state, just like death.

(a) (10 pts) Draw a transition diagram and write down a transition matrix for these four states.

(b) (10 pts) Starting in State 0, compute the probabilities of final outcome being healthy and immune versus the final outcome being death. You need to use Markov chains formulation here.

Bonus: The system we described is simple enough that this problem has a very easy solution, which does not require using a Markov chain. Can you write it down? (You may use this to check your answer). ***If you solve this problem using both formulations, you get 10 bonus points***

(c) (5 pts) What is the expected length of time you will remain healthy but susceptible to getting sick (i.e. in State 0)?

(d) (5 pts) Given that you became sick, what is expected length of time you will remain sick?

6. **Problem. (30 pts)** Let's change assumptions in Problem 5 slightly. Suppose that you are encountering fewer people. So your daily chances of getting sick become 0.005. All other probabilities remain the same.
    a. (**10 pts**) Write down a transition matrix for these four states.
    b. (**10 pts**) Starting in State 0, compute the probabilities of final outcome being healthy and immune versus death. (Here you may use either the Markov chain formulation or the short-cut.)
    c. (**10 pts**) What is the expected length of time you will remain healthy but susceptible to getting sick?

7. **Problem**. **(10 pts)** Let's change assumptions in Problem 5 once again. Suppose that it is also possible to go directly from State 0 to State 2, meaning that you contact the disease but don't feel sick and don't show any symptoms and then get well with immunity. Because of this, some of the odds are modified in the following way:
    - When you go around your regular activities (not modifying behavior) your daily probability of getting the disease and becoming sick is still 0.01, when starting in State 0. However, your probability of contacting virus and becoming immune (without going through sickness) is also 0.01.
    - ***If you get sick***, all the outcomes listed in Problem 5 remain the same:
    a. **(10 pts)** Write down a transition matrix and draw a transition diagram for these four states.
    b. **(Bonus: 10 pts)** Starting in State 0, compute the probabilities of final outcome being healthy and immune versus death. (*Part (b) is a bonus problem: so, do this only if you have the time*)

Side Note regarding Problems 5-7 (You don't need this information to solve above problems, this is just a commentary):

These are clearly taken out of current headlines. So, I need to say a few caveats. The values provided in these problems ***are NOT representative***. They are just some numbers that I made up to make the "math" work out (kind of). Moreover, the system where the number of infections (and percentage of infected population) is exponentially rising would not be properly represented as Markov Chains with time invariant probabilities. Instead, you would need to solve a system of ODE's to get a much better representation of what might happen.

The formulation I give in problems below might be more representative of a system where the product of R0 (transmission factor) times fraction of susceptible population equals one. Meaning, enough people in the community are immune (either due to vaccination or because they had the illness and recovered), to make the product:

R0*S=1. (Where S stands for fraction of population that is susceptible).

The only thing that these problems are designed to do is to make you think about what we learned and exercise it while trying to make the material a bit more relevant.

Problem #	1	2	3	4	5	6	7	Total
Possible Points	35	15	25	25	30	30	10	170
Student Score								

Submission Instructions

Failure to follow these steps may result in your submission not being graded.

1. Write Down Your Name and your BU ID
2. Make a pdf file of your write-up and upload the file to BB Learn
3. Your file name MUST be in the form Last Name_First Name.pdf (I would submit as Krigman_Steven.pdf)

4. Email me a copy in a single file, with the subject: *MA 570; Midterm*. → Cut & Paste this subject title.
5. You must upload AND email as attachment in a single file.

a. Do not send multiple files!
b. Attach the single file to your email; do not "drag and drop" it into your email message body.