MA570 Spring 2017 Stochastic Methods in OR Krigman

Mid-Term

Please show all your work in the blue book provided and $\boxed{\text{box off your answers.}}$

<u>If you get stuck I recommend moving to the next problem and coming back to the one that gave you trouble later.</u>

1 **Problem (<u>25 pts total</u>)**. Consider the Markov chain that has the following (one-step) transition matrix.

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.2 & 0.5 & 0.3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.2 & 0 & 0.1 & 0.7 \\ 0 & 1 & 0 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0.1 & 0 \end{bmatrix} \end{array};$$

You may also use the following information if you need it:

$$\mathbf{P*P} = \begin{bmatrix} 0 & 0.4 & 0 & 0.25 & 0.35 \\ 0 & 1 & 0 & 0 & 0 \\ 0.56 & 0.17 & 0 & 0.27 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.26 & 0.4 & 0.34 & 0 \end{bmatrix}; \quad \mathbf{P*P*P} = \begin{bmatrix} 0.28 & 0.285 & 0 & 0.435 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.382 & 0.28 & 0.338 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0.42 & 0 & 0.3 & 0.28 \end{bmatrix}.$$

(a) **(15 pts)** Indicate which states are accessible from which other states. From this determine which state communicate and determine the classes of this Markov chain. For each class, determine whether it is recurrent or transient.

(b) **(10 pts)** For each of the classes identified in part (*a*), determine the period of the states in that class**.**

2. **Problem (15 pts total).** A transition matrix P is said to be doubly stochastic if the sum over each column equals 1; that is $\sum_{i=0}^{M} p_{ij} = 1$, for all j. If such a chain is irreducible, aperiodic, and consists of M+1 states show that $\pi_j = \dfrac{1}{M+1}$, for j=0,1,…,M.

Note: <u>*You may assume that a chain consists of only 2 states (i.e. M=1) and show the above result for this case only.*</u>

3. **Problem (15 pts total).** A particle moves on a circle through points that have been marked 0, 1, 2, 3, and 4 (in a clockwise order). The particle starts at point 0. At each step it has probability 0.5 of moving one point clockwise (0 follows 4) and one point counterclockwise.

Let $X_n \ (n \geq 0)$ denote its location on the circle after step n. $\{X_n\}$ is a Markov chain.

a. (5 pts) Construct the one-step transition matrix.

b. (10 pts) Use results given in problem 2 to find steady-state probabilities for this Markov Chain.

4. **Problem (30 pts total).** A particle moves on a circle through points that have been marked 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (in a clockwise order). The particle may move clockwise, counter clockwise, or remain in the same spot (note the particle now has a third option). At each step it has probability α of moving one point clockwise (0 follows 9) and a probability of α to move one point counterclockwise. (Assume that 0 < α < 0.5).

a. (10 pts) Construct a transition matrix P.

b. (10 pts) Find the steady state probabilities.

c. (10 pts) For a transition matrix P in part (a), write down the matrix which corresponds to the following limit: $\lim_{n \to \infty} P^{(n)}$

5. **Problem (20 points).** Prove the following property (called transitivity property) of stationary Markov Chains: *If state i communicates with state j and state j communicates with state k, then state i communicates with state k.*

6. **Problem** (**<u>30 pts total</u>**). Sales of a company product may be characterized by two levels: "Low" and "High". Their sales depend upon whether or not they advertise.
   - The company is trying to determine what its advertising policy should be.
   - Advertising in any quarter of a year has its primary impact on sales in the *following* quarter. Therefore, at the beginning of each quarter, the needed information is available to forecast accurately whether sales will be low or high that quarter and to decide whether to advertise that quarter.
   - The cost of advertising is $1 million for each quarter of a year in which it is done.
   - When advertising is done during a quarter, the probability of having high sales the next quarter is 1/2 when the current quarter's sales are low, and 3/4 when the current quarter sales are high.
   - When advertising is not done during the current quarter these probabilities go down to 1/4 and 1/2 respectively.
   - The company's quarterly profits (excluding advertising costs) are $4 million when sales are high but only $2 million when sales are low. (Hereafter, use units of millions of dollars.)

(a) **(10 pts.)** Construct the (one-step) transition matrix for each of the following advertising strategies: (*i*) never advertise, (*ii*) always advertise. *(Hint: Let state 0 indicate the "Low" level of sales and state 1 indicate the "High" level of sales during the current quarter, where each transition of the process goes from one quarter to the next.)*

(b) **(10 pts.)** Determine the steady-state probabilities manually for each of the two cases in part (*a*).

(c) **(10 pts.)** Find the long run expected average profit (including a deduction for advertising costs) per quarter for each of the two advertising strategies in part (*a*). Which of these strategies is best according to this measure of performance?

Problem #	1	2	3	4	5	6	Total
Possible Points	25	15	15	30	20	30	135
Student Score							