Models for Stationary Time Series

MAS 640

1/22/2018

Outline

- Residual Analysis and The Autocorrelation Function (ACF)
- Moving Average Models
- Autoregressive Models

Bigger Picture

- In Lecture 2 our focus was on detrending non-stationary time series
- Procedure:
  - Plot time series
  - Determine appropriate regression model
  - Fit that model
  - Diagnose it's goodness

- Now we have residuals
  - In a typical regression course, we stop here
  - In this course, we try to model those residuals

Comments

$$Y_t = \mu_t + \epsilon_t$$

- We're breaking this into pieces, slowly building up to a comprehensive class of Time Series models.
  - Model μ_t with a regression model
  - Model ϵ_t with a time series model

Residual Analysis

- Previously eyeballed residual scatterplot and said "looks ok" (not perfect but common practice)
- Two alternatives
  - Runs test
  - Estimate and plot autocorrelation of **standardized** residuals for a number of time lags

- Independence is particularly important to us now
  - If correlations remain in the residuals, we will model that correlation with a time series model
  - Plot of estimated autocorrelation particularly useful, because it will help determine which model to fit

Sample Autocorrelation Function (ACF)

- Estimating and plotting the ACF is essential
- When we detrend the data, we are hoping to get a stationary time series

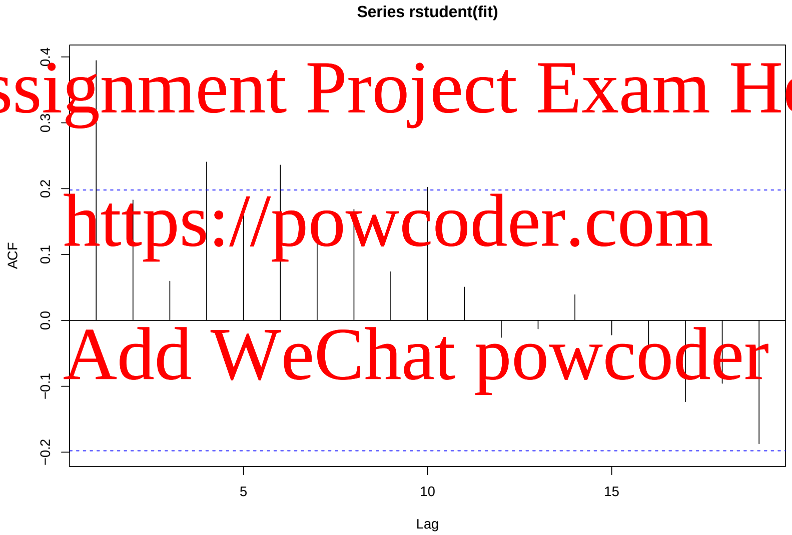
- It's hard to see if remaining correlations exist, but if they do, we want to model them
- The ACF will help determine how we model them

Sample Autocorrelation

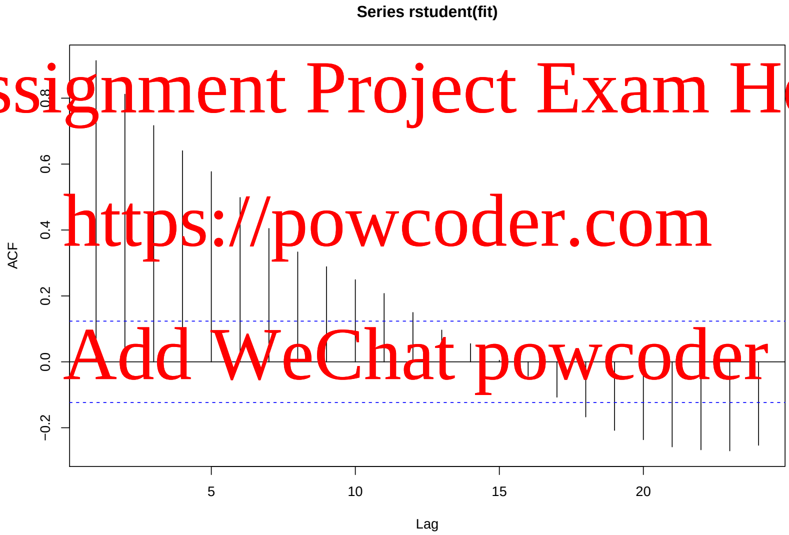
- Under independence, the sample ACF of the **standardized** residuals are approximately $N(0, 1/n)$
- Estimate and plot the sample ACF using the **standardized residuals**
  - If certain lags are above $1/n$, we suspect remaining correlation in the residuals

- Plot using `acf(rstudent(FITTED MODEL))`

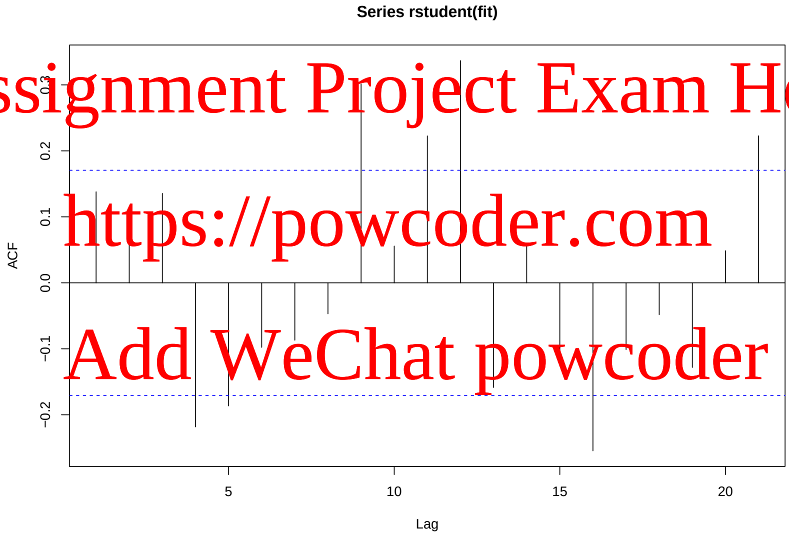
Autocorrelation Plot Example - `gtemp2` Data



Autocorrelation Plot Example - `gold` Data



Autocorrelation Plot Example - beersales Data



Moving Average and Autoregressive Processes

- Today we cover the two basic models for stationary time series

  - Moving averages
  - Autoregressive

Moving Average Processes

- Suppose e_t is a mean 0 white noise process with $var(e_t) = \sigma^2$.
- The process:

$$Y_t = e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q}$$

- is called a **moving average process of order q**, and is denoted MA(q)

- Todays value is a weighted average of the previous q error terms

Comment on coefficient signs...

- For technical reasons, this model is often written as:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

- R uses positive signs, so we'll stick with that.
- *Might* need to check on this if using other software.

MA(1) Process

The 1^{st} **order moving average process**, denoted MA(1) is

$$Y_t = e_t + \theta_1 e_t$$

Properties of an MA(1) Process

1. Mean: $E(Y_t) = 0$
2. Variance:

$$\begin{aligned} var(Y_t) &= var(e_t + \theta_1 e_{t-1}) \\ &= var(e_t) + \theta_1^2 var(e_{t-1}) + 2cov(e_t, e_{t-1}) \\ &= \sigma_e^2 + \theta_1^2 \sigma_e^2 = \sigma_e^2(1 + \theta_1^2) \end{aligned}$$

3. Autocorrelation function (ACF):

$$\rho_k = \begin{cases} 1 & k = 0 \\ \frac{\theta_1}{1+\theta_1^2} & k = 1 \\ 0 & k > 1 \end{cases}$$

Properties of an MA(1) Process

- This process has no correlation beyond lag 1!
- Observations 1 time unit apart are correlated, but observations more than 1 time unit apart are not

- Important to keep in mind when we consider models for real data using empirical evidence
  - i.e. when we look at ACF plots and see high correlation at lag 1 but low to no correlation at higher time lags

Properties of an MA(1) Process

The following theoretical properties apply to an MA(1) process

- When $\theta_1 = 0$, the MA(1) process reduces to white noise

$$Y_t = e_t + 0e_{t-1} = e_t$$

- θ_1 restricted to have absolute value less than 1 (for **invertibility**)

Properties of an MA(1) Process

- As θ_1 ranges from -1 to 1, the lag 1 autocorrelation ρ_1 ranges from -0.5 to 0.5
- For -1:

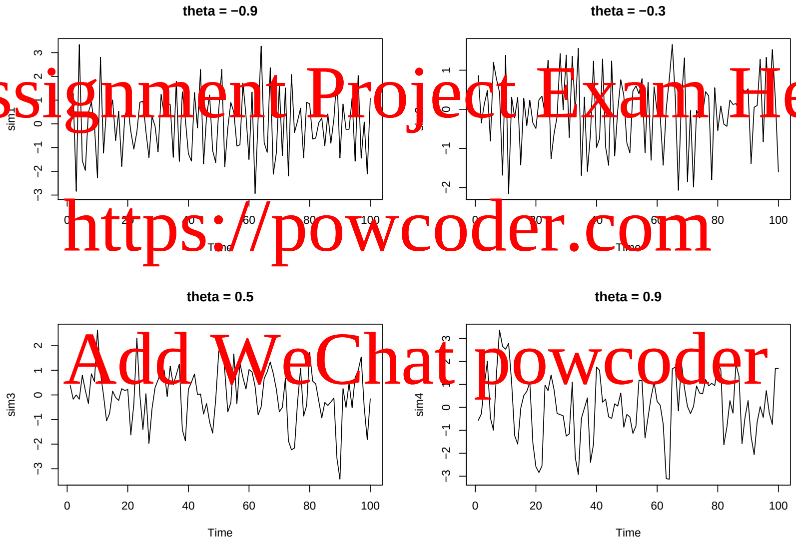
$$\rho_1 = \frac{-1}{1 + (-1)^2} = -0.5$$

- For 1:

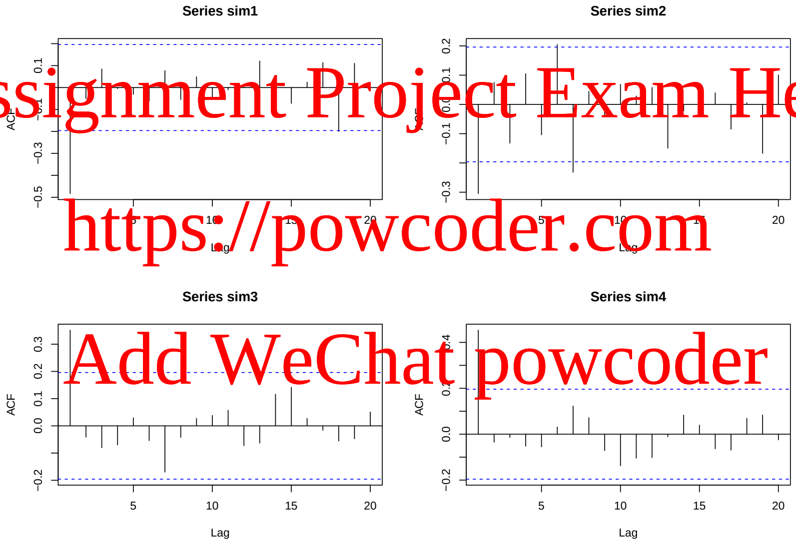
$$\rho_1 = \frac{1}{1 + 1^2} = 0.5$$

- Observing lag 1 autocorrelation well outside of this range is inconsistent with the MA(1) model

Simulated MA(1) Processes



ACF for Simulated MA(1) Processes



Example

- Try a few on your own.
- Vary the `ma` parameter between -1 and 1 and look at the resulting time series.

- Run each through the `acf()` function and look at results. What do you notice?
- We'll model these at the end today, but what would you expect your estimated coefficient to be in each case?

MA(2) Process

The 2^{nd} **order moving average** process, denoted by MA(2) is

$$Y_t = e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2}$$

Properties of an MA(2) Process

- Mean: $E(Y_t) = 0$
- Variance:

$$var(Y_t) = \sigma_e^2(1 + \theta_1^2 + \theta_2^2)$$

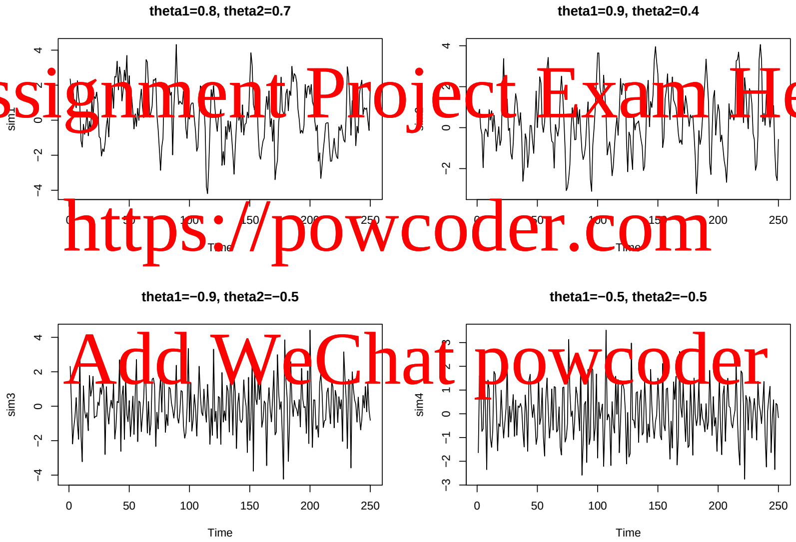
- Autocorrelation function (ACF):

$$\rho_k = \begin{cases} 1 & k = 0 \\ \frac{\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & k = 1 \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2} & k = 2 \\ 0 & k > 2. \end{cases}$$

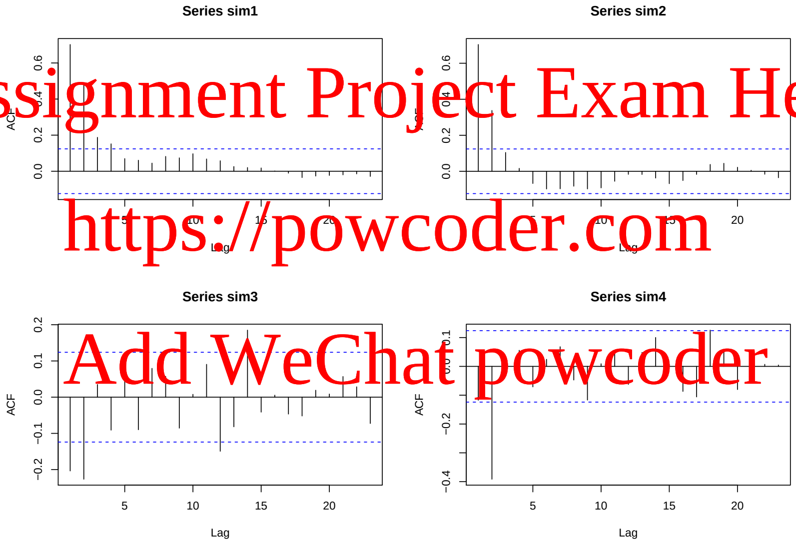
Properties of an MA(2) Process

- The MA(2) process has zero correlation beyond lag 2
- So, an ACF plot with spikes at lags 1 and 2 only would indicate a possible MA(2) model.

Simulated MA(2) Processes



ACF for Simulated MA(2) Processes



General MA(q) Process

The **qth order moving average** process, denoted by MA(q) is

$$Y_t = e_t + \theta_1 e_{t-1} + \ldots + \theta_q e_{t-q}$$

Properties of an MA(q) Process

- Mean: $E(Y_t) = 0$
- Variance:

$$var(Y_t) = var(Y_t) = \sigma_e^2(1 + \theta_1^2 + \ldots + \theta_q^2)$$

- Autocorrelation function (ACF):

$$\rho_k = \begin{cases} 1 & k = 0 \\ \frac{\theta_1 + \theta_1\theta_2 + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2} & k = 1, \ldots, q-1 \\ \frac{\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2} & k = q \\ 0 & k > q. \end{cases}$$

MA(q) Process

- Key feature of MA(q) models:

  - Nonzero autocorrelations for the first q lags
  - Autocorrelations $= 0$ for all lags $> q$

Comment

- q is *always* used to denote the order of an MA process
  - All the literature
  - All the software
  - When you see q, it's referring to the MA order

- We'll use functions that require us to specify p, d, q, P, D, Q and S so it's important to keep track

Autoregressive (AR) Process

Suppose $\{e_t\}$ is a zero mean white noise process with $var(e_t) = \sigma_e^2$. The process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

is called an autoregressive process of order p, denoted by AR(p).

- Todays value is a linear function of the previous p values, plus some error.

AR(1) Process

The AR(1) process is

$$Y_t = \phi_1 Y_{t-1} + e_t$$

- Note that if $\phi_1 = 1$, then the process reduces to a random walk.
- If $\phi_1 = 0$, this process reduces to a white noise.

Properties of an AR(1) Process

- Variance:

$$var(Y_t) = \frac{\sigma_e^2}{1 - \phi_1^2}$$

Because $var(Y_t) > 0$, this implies that $-1 < \phi_1 < 1$

- The correlation between observations k time periods apart is

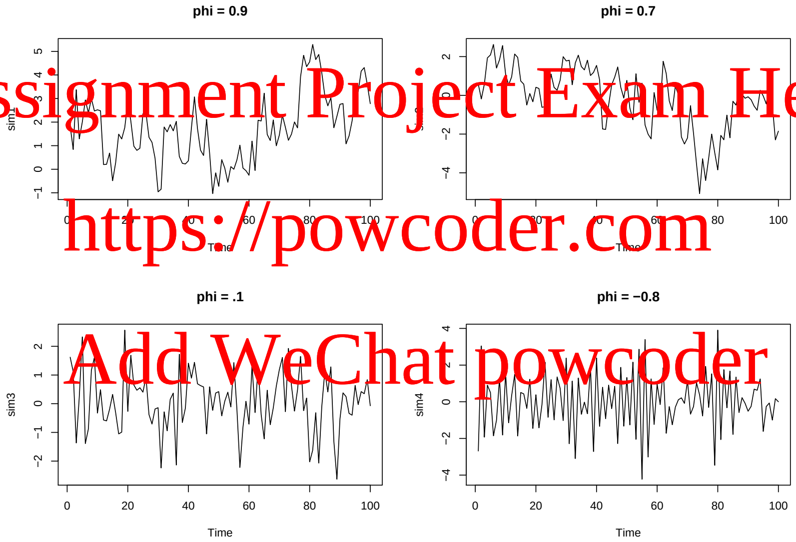
$$\rho_k = \phi_1^k$$

Properties of an AR(1) Process

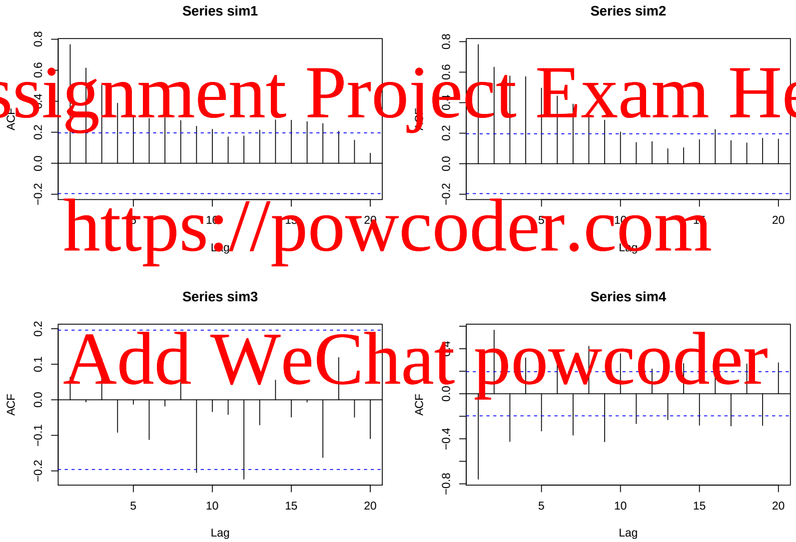
Because $-1 < \phi_1 < 1$ the ACF ρ_k decays exponentially as k increases

- If ϕ_1 is close to ± 1, ACF decays slowly
- If ϕ_1 is closer to 0, ACF decays rapidly
- If $\phi_1 > 0$, then all of the ACFs will be positive
- If $\phi_1 < 0$, the ACF alternates between positive and negative
- Remember these theoretical patterns so that when we see sample ACFs (from real data) we can make sensible decisions about potential model selection

Simulated AR(1) Processes



ACF for Simulated AR(1) Processes



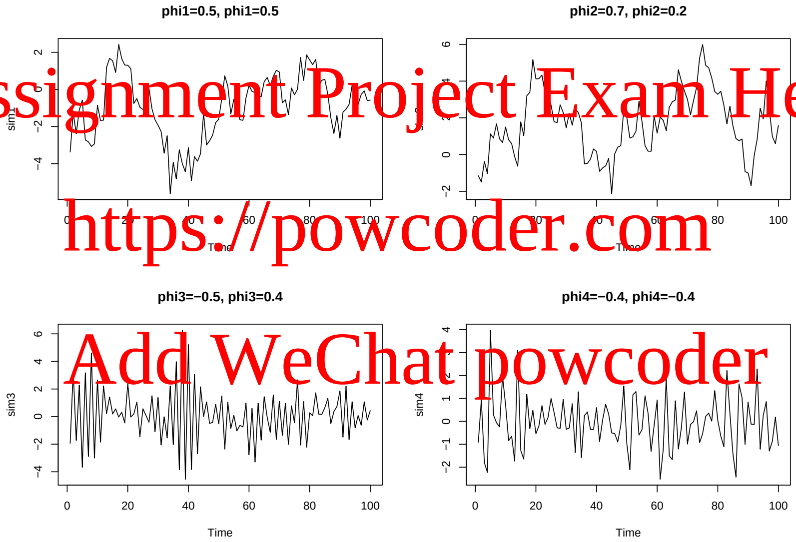
AR(2) Process

The AR(2) process is

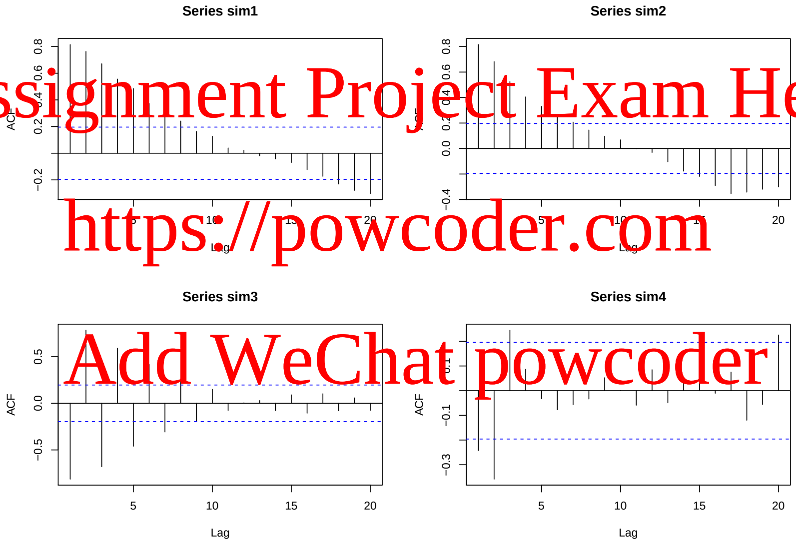
$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

- Todays value is a linear function of the previous two values, plus some error
- ACF gets quite involved, so we leave it out here
- But ACF continues to trail off similar to the AR(1)

Simulated AR(2) Processes



ACF for Simulated AR(2) Processes



AR(p) Process

The general AR(p) process is given by

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

- Note that p is *always* used to denote the order of an AR process, just as q is *always* used to denote the order of an MA process.

ACF for AR Processes

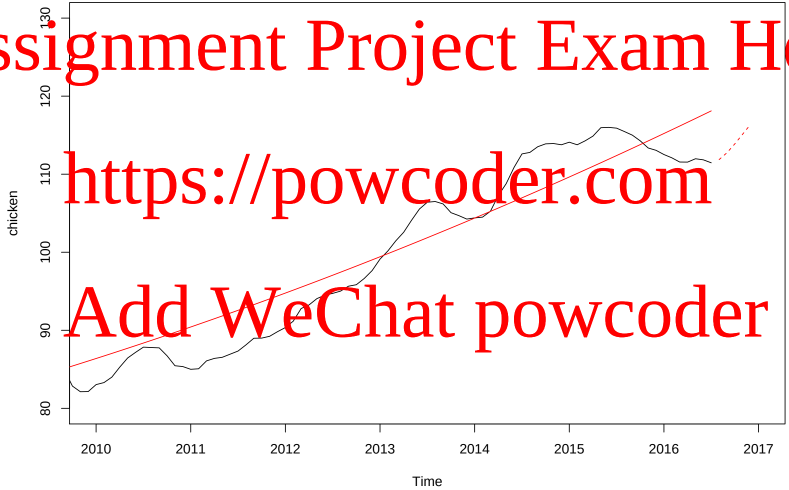
- Looking at the ACF plots for the AR(1) and the AR(2) process...
  - Nothing to indicate the order in the plots
  - They both trail off to 0

- Need something else to help determine order p
  - Partial Autocorrelation Function
  - Discuss this next class

Fitting MA(q) or AR(p) Models in R

- Many functions for fitting and forecasting time series
- Google "fitting an MA or AR model in R"
- Some examples:
  - `ar(x)` - Determines order p via AIC, can set order.max
  - `arma(x, order=c(p,q))` - No quick function for forecasts, pretty worthless IMO
  - `arima(x, order=c(p,d,q))` - Use `predict(.., n.ahead=)` for quick forecasts
  - `sarima(x, p, d, q, P, D, Q, S)` - Must specify at least p, d, q, the others default to 0.
  - `sarima.for(x, n.ahead=..., p, d, q, P, D, Q, S)` - Gives forecasts, outputs several useful plots
    - sarima appears to be the latest and greatest...

Combining today with the last class



Example 2