THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2022

School of Mathematics and Statistics

MAST20030 Differential Equations

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 8 pages (including this page) with 7 questions and 105 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader and blank loose-leaf paper.
- One double sided A4 page of notes (handwritten only).
- No calculators are permitted. No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.
- You must not write your answers on an iPad or other electronic device.

- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- Marks are awarded for:

  - Using appropriate mathematical techniques,
  - Showing full working, including results used,
  - Accuracy of the solution,
  - Using correct mathematical notation.
- Write your answers on A4 paper. Page 1 should only have your student number, the subject code and the subject name. Write on one side of each sheet only. Each question should be on a new page. The question number must be written at the top of each page.

Scanning and Submitting

- **You must not leave Zoom supervision to scan your exam.** Put the pages in question order and all the same way up. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4.
- Submit your scanned exam as a single PDF file and carefully review the submission in Gradescope. Scan again and resubmit if necessary. Do not leave Zoom supervision until you have confirmed orally with the supervisor that you have received the Gradescope confirmation email.
- **You must not submit or resubmit after having left Zoom supervision.**

Question 1 (8 marks)

Consider the ordinary differential equation (ODE)

$$x^2 y''(x) + xy'(x) - n^2 y(x) = 0, \quad n = 0, 1, 2, 3, \ldots$$

(a) Given $n > 0$:

(i) Solve the ODE by trialling $y(x) = x^\lambda$.

(ii) Use the Wronskian to prove that the solutions from part (i) are linearly independent.

(b) Given $n = 0$, use reduction of order to find the general solution to the ODE.

Question 2 (15 marks)

(a) Use the Weierstrass test to determine if the following series converges uniformly

$$\sum_{n=0}^{\infty} \frac{2^{-n} \cos(1 + n\pi x)}{e^n}.$$

(b) Consider

$$(1 - x^2)y''(x) - xy'(x) + y(x) = 0.$$

(i) Solve the ODE using a power series expanded about $x = 0$

$$\sum_{n=0}^{\infty} c_n x^n.$$

(ii) State the interval of convergence of any infinite series solutions you find.

(iii) Suppose $y(0.2) = 1, y'(0.2) = 0$. Without solving the initial value problem (IVP), what is the largest interval in x in which a unique solution is guaranteed for the IVP?

Question 3 (15 marks)

Consider the following system of ODEs

$$\begin{aligned} x'(t) + 2x(t) - y(t) - 3 &= 0, \qquad & x(0) &= 0, \\ y'(t) + 2y(t) - x(t) + 1 &= 0, \qquad & y(0) &= 0. \end{aligned}$$

(a) Convert the system into a vector-valued IVP.

(b) Solve the system by matrix diagonalisation.

Question 4 (18 marks)

(a) Find the inverse Laplace transform of

(i) $\dfrac{s+2}{s^2+2s+10}$

(ii) $\dfrac{1}{s(s^2+2s+10)}$

(b) The displacement of a damped mass-spring system, $y(t)$, is governed by the ODE

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = u(t) - u(t-5)$$

subject to

$$y(0) = 0, \quad y'(0) = 0.$$

(i) Give an interpretation of the forcing term and the initial conditions.

(ii) Use Laplace transform to solve for $y(t)$. *Express your solution in explicit form.*

Question 5 (13 marks)

Consider the function defined over $[-\pi, \pi]$.

$$f(x) = \begin{cases} -\cos\left(\dfrac{x}{2}\right) & -\pi \leq x < 0 \\ \cos\left(\dfrac{x}{2}\right) & 0 \leq x \leq \pi. \end{cases}$$

(a) Sketch $f(x)$.

(b) Is $f(x)$ even, odd or neither?

(c) Find $g(x)$, the Fourier series representation of $f(x) = f(x + 2\pi)$?

(d) Given your answer to part (c):

  (i) How do the Fourier coefficients in $g(x)$ decay for large n?

  (ii) Would you expect $g(x)$ to converge uniformly? Briefly explain your answer.

  (iii) For which x in $[-\pi, \pi]$ does $g(x)$ exhibit Gibbs phenomenon? What is the error of the truncated Fourier approximation at this x value?

(e) Consider the function

$$h(x) = \cos\left(\frac{x}{2}\right), \quad x \in [-\pi, \pi].$$

Without calculations, does the Fourier series of $h(x)$ converge uniformly?

Question 6 (8 marks)

Consider the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c \in \mathbb{R}_{>0}, \quad x \in (-\infty, \infty), \quad t \in [0, \infty)$$

where

$$u(x,t) \to 0 \quad \text{as} \quad x \to \pm\infty$$

and

$$u(x,0) = \delta_2(x) + \delta_{-2}(x) \qquad \text{and} \qquad \partial_t u(x,0) = 0.$$

(a) From first principles, find the Fourier transform of

  (i) $f(x) = \delta_2(x)$

  (ii) $g(x) = \delta_{-2}(x)$

(b) Use your results in part (a) to find the inverse Fourier transform of $\widehat{h}(k) = \cos 2k$.

(c) Find the IVP for the Fourier transform of u.

(d) **Without calculations,** briefly describe the remaining steps to find u.

Question 7 (28 marks)

The temperature $u(r,\theta,t)$ of a metal semi-annulus plate is modelled by the heat equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial u}{\partial t}, \qquad 1<r<2, \qquad 0\le\theta\le\pi, \qquad t>0.$$

(a) Consider the *steady state* temperature $u(r,\theta)$ subject to the boundary conditions

$$\begin{aligned} u(r,0)&=0, && 1<r<2,\\ u(r,\pi)&=0, && 1<r<2,\\ u(1,\theta)&=0, && 0\le\theta\le\pi,\\ u(2,\theta)&=\frac{\pi}{2}\sin 4\theta-\sin 7\theta, && 0\le\theta\le\pi. \end{aligned}$$

(i) Use separation of variables to show that the steady state heat equation reduces to:

$$\begin{aligned} r^2R''(r)+rR'(r)-\lambda R(r)&=0\\ S''(\theta)+\lambda S(\theta)&=0, \end{aligned}$$

where $\lambda\in\mathbb{R}$ is the separation constant.

(ii) Solve for $u(r,\theta)$. *Assume that the cases $\lambda=0$ and $\lambda<0$ lead to trivial solutions: do not work through these cases. Also you may quote any relevant parts of your solution to Question 1 to avoid repeating any working.*

(b) The condition $\dfrac{\partial^2 u}{\partial \theta^2}=0$ leads to a time-evolving temperature $u(r,t)$ governed by

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{\partial u}{\partial t}, \qquad 1<r<2, \qquad 0\le\theta\le\pi.$$

Assume all four boundaries of the annulus plate are insulated and the initial condition is

$$u(r,0)=r, \qquad 1\le r\le 2.$$

(i) Find the total heat of the annulus plate.

(ii) **Without solving the governing PDE** for $u(r,t)$, sketch $u(r,t)$ versus r, $1\le r\le 2$, $0\le t<\infty$ for the following four time states t:

$$0<t_1<t_2<\lim t\to\infty.$$

(c) Suppose the formal series solution $u(\theta,t)=\displaystyle\sum_{n=0}^{\infty}u_n(\theta,t)$ is a genuine solution to

$$\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial u}{\partial t}, \qquad 0\le\theta\le\pi, \qquad t>0.$$

Which of the series $\displaystyle\sum_{n=0}^{\infty}u_n(\theta,t)$ and its derivatives converge uniformly in the given domain?

End of Exam — Total Available Marks = 105

MAST20030 Differential Equations Formulae Sheet

1) Laplace Transforms

1.	$f(t)$	$(\mathcal{L}f)(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$	(Definition of Transform)
2.	1	$\frac{1}{s}$	
3.	t^n	$\frac{n!}{s^{n+1}}$	
4.	e^{at}	$\frac{1}{s-a}$	
5.	$\sin(at)$	$\frac{a}{s^2+a^2}$	
6.	$\cos(at)$	$\frac{s}{s^2+a^2}$	
7.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	
8.	$\cosh(at)$	$\frac{s}{s^2-a^2}$	
9.	$\delta(t-a)$	e^{-as}	$(a \geq 0)$
10.	$f'(t)$	$sF(s) - f(0)$	
11.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
12.	$f^{(n)}(t)$	$s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$	
13.	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	
14.	$e^{-at}f(t)$	$F(s+a)$	(s-Shifting Theorem)
15.	$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	$(a > 0,$ t-Shifting Theorem$)$
16.	$\int_0^t f(\tau)\,g(t-\tau)\,d\tau$	$F(s)\,G(s)$	(Convolution)

2) Standard Limits

(i) $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0)$

(ii) $\displaystyle\lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$

(iii) $\displaystyle\lim_{n\to\infty} a^{1/n} = 1 \quad (a > 0)$

(iv) $\displaystyle\lim_{n\to\infty} n^{1/n} = 1$

(v) $\displaystyle\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (\text{all } a)$

(vi) $\displaystyle\lim_{n\to\infty} \frac{\log_e n}{n^p} = 0 \quad (p > 0)$

(vii) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \mathrm{e}^a \quad (\text{all } a)$

(viii) $\displaystyle\lim_{n\to\infty} \frac{n^p}{a^n} = 0 \quad (\text{all } p, a > 1)$

3) The Generalised Harmonic Series (p-Series)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } p > 1 \\ \text{divergent} & \text{if } p \leq 1 \end{cases}$$

4) Geometric Series

$$\sum_{n=0}^{\infty} ar^n \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

5) Fourier Series Formulae

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos k_n x + b_n \sin k_n x\right), \quad k_n = \frac{n\pi}{L} \\ a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\, dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x) \cos k_n x \, dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x) \sin k_n x \, dx \end{aligned}$$

6) Fourier Transforms Formulae

$$\widehat{f}(k) = (\mathscr{F} f)(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ikx}\, dx$$

$$f(x) = (\mathscr{F}^{-1}\widehat{f})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(k)\mathrm{e}^{ikx}\, dk$$

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = (ik)^n \widehat{f}(k), \qquad x^n f(x) = i^n \frac{\mathrm{d}^n \widehat{f}}{\mathrm{d}k^n}$$

7) **Trigonometric and Hyperbolic Formulae**

$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x \qquad \coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \qquad \sinh x \sinh y = \frac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \qquad \cosh x \cosh y = \frac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$$

$$\sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right] \qquad \sinh x \cosh y = \frac{1}{2}\left[\sinh(x+y) + \sinh(x-y)\right]$$

$$\sin x = \frac{i}{2}\left(\mathrm{e}^{-ix} - \mathrm{e}^{ix}\right) \qquad \sinh x = \frac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right)$$

$$\cos x = \frac{1}{2}\left(\mathrm{e}^{-ix} + \mathrm{e}^{ix}\right) \qquad \cosh x = \frac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)$$