MAST30001 Stochastic Modelling

Tutorial Sheet 7

1. Let $(N_t)_{t\ge 0}$ be a Poisson process with rate λ and let $0 < T_1 < T_2 < \cdots$ be the times of "arrivals" or jumps of $(N_t)_{t\ge0}$. Compute:

   (a) $\mathbb{P}(N_3 \le 2, N_1 = 1)$,
   The name of the game for all of these problems is to recast the event you want to compute in terms of independent variables or events which are amenable to the Poisson process description. Here, N_1 and $N_3 - N_1$ are independent, Poisson with respective means λ and 2λ, so that

$$\begin{aligned}\mathbb{P}(N_3 \le 2, N_1 = 1) &= \mathbb{P}(N_3 - N_1 \le 1, N_1 = 1) = \mathbb{P}(N_3 - N_1 \le 1)\mathbb{P}(N_1 = 1)\\ &= \left(e^{-2\lambda} + 2\lambda e^{-2\lambda}\right)\lambda e^{-\lambda}.\end{aligned}$$

   (b) $\mathbb{P}(N_3 \le 2, N_1 \le 1)$,
   Similar to (a),

$$\begin{aligned}\mathbb{P}(N_3 \le 2, N_1 \le 1) &= \mathbb{P}(N_3 \le 2, N_1 = 1) + \mathbb{P}(N_3 \le 2, N_1 = 0)\\ &= [\textit{Ans. to (a)}] + e^{-2\lambda}\left(1 + 2\lambda + (2\lambda)^2/2\right)e^{-\lambda}.\end{aligned}$$

   (c) $\mathbb{P}(N_2 = 2, N_1 = 2, N_{1/2} = 0)$,

   Use that $N_2 - N_1, N_1 - N_{1/2}$, and $N_{1/2}$ are independent and Poisson and that

$$\mathbb{P}(N_2 = 2, N_1 = 2, N_{1/2} = 0) = \mathbb{P}(N_2 - N_1 = 0, N_1 - N_{1/2} = 2, N_{1/2} = 0)$$

   to find

$$\mathbb{P}(N_2 = 2, N_1 = 2, N_{1/2} = 0) = e^{-\lambda}\frac{(\lambda/2)^2}{2}e^{-\lambda/2}e^{-\lambda/2}.$$

   (d) $\mathbb{P}(N_7 - N_3 = 2 | N_5 - N_2 = 2)$,

   Again, $N_7 - N_5$, $N_5 - N_3$, and $N_3 - N_2$ are independent and Poisson and

$$\mathbb{P}(N_7 - N_3 = 2 | N_5 - N_2 = 2) = \frac{\mathbb{P}(N_7 - N_3 = 2, N_5 - N_2 = 2)}{\mathbb{P}(N_5 - N_2 = 2)}.$$

   The denominator of the fraction on the right hand side is easy. The numerator equals

$$\begin{aligned}&\sum_{i=0}^{2}\mathbb{P}(N_7 - N_5 = 2 - i, N_5 - N_3 = i, N_3 - N_2 = 2 - i)\\ &\quad = \sum_{i=0}^{2}\frac{(2\lambda)^{2-i}}{(2-i)!}e^{-2\lambda}\frac{(2\lambda)^i}{(i)!}e^{-2\lambda}\frac{(\lambda)^{2-i}}{(2-i)!}e^{-\lambda}.\end{aligned}$$

   (e) the (joint) distribution function of (T_1, T_2),
   The joint cdf $\mathbb{P}(T_1 \le s, T_2 \le t)$ is 0 if either t or $s \le 0$. Otherwise if $t \le s$ this is

$$\mathbb{P}(T_2 \le t) = \mathbb{P}(N_t \ge 2) = 1 - e^{-\lambda t}[1 + \lambda t].$$

   Otherwise $0 \le s < t$ and we have

$$\begin{aligned}\mathbb{P}(T_1 \le s, T_2 \le t) &= \mathbb{P}(N_s \ge 1, N_t \ge 2) = \mathbb{P}(N_t \ge 2) - \mathbb{P}(N_t \ge 2, N_s = 0)\\ &= \mathbb{P}(N_t \ge 2) - \mathbb{P}(N_t - N_s \ge 2)\mathbb{P}(N_s = 0)\\ &= 1 - e^{-\lambda t}[1 + \lambda t] - \left(1 - e^{-\lambda(t-s)}[1 + \lambda(t-s)]\right)e^{-\lambda s}.\end{aligned}$$

(f) the joint density of (T_1, T_2),
We differentiate the joint cdf with respect to s and t. The derivative with respect to s is 0 for $t < s$, while for $0 < s < t$ it is equal to $-\lambda e^{-\lambda t}$. Differentiating with respect to t gives $\lambda^2 e^{-\lambda t}$ on $0 < s < t < \infty$.

(g) the distribution of $T_1 | \{T_2 = t_2\}$.
We have seen in class that $T_1|\{T_2 = t\}$ is uniform on $[0, t]$. Another way to see this is that T_1 must be less than T_2, but the joint density above does not depend on s, so the conditional distribution of T_1 given $\{T_2 = t\}$ must not depend on the value of $s \in [0, t]$ (so it must be uniform). More explicitly, we can verify this here by dividing the joint density above by the density of T_2 (which, as the sum of two independent exponential(λ) random variables is $f_{T_2}(t) = \lambda^2 t e^{-\lambda t}$ at t to give the conditional density $\lambda^2 e^{-\lambda t}/(\lambda^2 t e^{-\lambda t}) = 1/t$.

2. Yeast microbes from the air outside of a culture float by according to a Poisson process with rate 2 per minute. Each microbe that floats by joins the population of the culture with probability p and with probability $1 - p$ the microbe doesn't join the culture, and this choice is made independent from the times of arrival and choice to join of all other microbes.

(a) Find the probability that exactly four outside microbes float by in the first 3 minutes.

Let N_t be the number of microbes that float by up to time to t and let M_t be the number that join the colony up to time t. Then N_t is Poisson with mean $2t$ and M_t is Poisson with mean $2pt$, independent of the process $N_t - M_t$ which is Poisson with mean $2(1-p)t$. Also conditional on a Poisson process being equal to k at time t, the distribution of the k points in the interval $(0, t)$ are the same as k i.i.d. variables that are uniform on $(0, t)$. These facts and the description of the process imply we have the following answers.....

$$\frac{e^{-6}6^4}{4!} \qquad \textit{since } N_t \textit{ is a Poisson process.}$$

(b) Find the probability that exactly four outside microbes join the culture in the first 3 minutes.

$$\frac{e^{-6p}(6p)^4}{4!} \qquad \textit{since } M_t \textit{ is a Poisson process.}$$

(c) Given that 7 outside microbes have floated by the culture in first 3 minutes, what is the probability that at least two of the seven join the culture?
$1 - 7p(1-p)^6 - (1-p)^7$ *since the description of the M_t process as a thinning of N_t implies that M_t conditional on $\{N_t = 7\}$ is Binomial.*

(d) Given that 7 outside microbes have floated by the culture in first 3 minutes, what is the probability that exactly 3 float by in the first 1 minute?
$\binom{7}{3}(1/3)^3(2/3)^4$ *from the conditional Poisson process description. You can also see this by writing it as*

$$\frac{\mathbb{P}(N_1 = 3, N_3 - N_1 = 4)}{\mathbb{P}(N_3 = 7)},$$

and using the independence of the two events in the numerator etc.

(e) What is the probability that in the first 3 minutes, exactly four microbes join the culture and 3 float by that don't join the culture?
$\frac{e^{-6p}(6p)^4}{4!} \cdot \frac{e^{-6(1-p)}(6(1-p))^3}{3!}$ *(M_t and $N_t - M_t$ are independent Poissons).*

Assume now that a second strain of yeast microbes independently float by the culture according to a Poisson process with rate 1, and each microbe joins the culture with probability q, analogous to the previous process.

(f) What is the probability that exactly four yeast microbes (from either strain) float by in the first 3 minutes?
Let N_t and M_t be respectively, the number of the first strain that float by, and the number of this strain that float by and join the culture up to time t. Let K_t and L_t be the analogous processes for the second strain. Then as before, all of these processes are Poisson processes and with N_t having rate 2, M_t having rate $2p$, K_t having rate 1, L_t having rate q, and N_t, M_t are independent of K_t, L_t. Because of independence, superposition of Poisson processes implies that $N_t + K_t$ and $M_t + L_t$ are Poisson processes with rates 3 and $2p + q$. Using these facts we find that

$$\mathbb{P}(N_3 + K_3 = 4) = \frac{e^{-9}9^4}{4!}.$$

(g) What is the probability that exactly four yeast microbes (from either strain) join the culture in the first 3 minutes?

$$\mathbb{P}(L_3 + M_3 = 4) = \frac{e^{-3(2p+q)}(3(2p+q))^4}{4!}.$$

3. Let $U_{(1)}, \ldots, U_{(n)}$ be order statistics of independent variables, uniform on the interval $(0,1)$. For $0 < x < y < 1$ find:

(a) $\mathbb{P}(U_{(1)} > x, U_{(n)} < y)$,
The event $\{U_{(1)} > x, U_{(n)} < y\}$ is the same as all the U_i's are between x and y which occurs with probability $(y - x)^n$.

(b) $\mathbb{P}(U_{(1)} < x, U_{(n)} < y)$,
$\mathbb{P}(U_{(1)} < x, U_{(n)} < y) + \mathbb{P}(U_{(1)} > x, U_{(n)} < y) = \mathbb{P}(U_{(n)} < y) = y^n$ *and then use the previous answer to get $y^n - (y - x)^n$.*

(c) $\mathbb{P}(U_{(k)} < x, U_{(k+1)} > y)$.
The event $\{U_{(k)} < x, U_{(k+1)} > y\}$ is the same as k of the U_i's are smaller than x and the rest are larger than y, which occurs with probability

$$\binom{n}{k} x^k (1 - y)^{n-k}.$$