MAST30001 Stochastic Modelling

Tutorial Sheet 5

1. Fix $p \in (0,1)$. Consider the "Gambler's ruin" chain on $\mathcal{S} = \{0, 1, \ldots, k\}$ with $p_{i,i+1} = p$ and $p_{i,i-1} = 1-p$ for $1 \leq i \leq k-1$. Let $m_i := m_{i,\{0,k\}}$ denote the expected hitting time of $\{0,k\}$ starting from state i.

   (a) Show that if $p = 1/2$ then $m_i = i(k-i)$.

   (b) Show that if $p \neq 1/2$ then

   $$m_i = \frac{i}{1-2p} - \frac{k}{1-2p}\left[\frac{\alpha^i - 1}{\alpha^k - 1}\right],$$

   where $\alpha = (1-p)/p$.

2. In a tiny politically divided village there are 12 villagers living in a 3×4 grid as shown below. On each border, 5 always support party 1, and 5 always support party 0 as shown below.

1	1	1	1
1	?	?	0
0	0	0	0

The opinion of each of the two interior villagers is influenced by their neighbours and fluctuates over time as follows: at each discrete time $n \in \mathbb{N}$, one of the two interior voters (chosen uniformly at random) adopts the political preference of one of its 4 immediate neighbours (also chosen uniformly at random). Let L_n and R_n be the opinions carried by the left and right interior villagers (as in the picture) respectively.

   (a) What is the state-space of the Markov chain $X_n = (L_n, R_n)$?

   (b) Draw the transition diagram for the chain X_n.

   (c) Is the chain X_n:

      i. Irreducible?

      ii. Aperiodic?

      iii. Reversible?

   (d) What is the limiting proportion of time that the left interior villager supports party 1?

   (e) Suppose that at time $n = 1000000$ a vote is taken (and the whole village votes). Estimate the probability that party 1 wins the majority of votes, explaining your reasoning.

3. Each morning at 8am, it is raining with probability $r_m \in (0,1)$, and each afternoon at 4pm it is raining with probability $r_a \in (0,1)$ (both are independent of all previous weather conditions). Suppose that your MAST30001 lecturer has 2 umbrellas, and that he departs home for work at 8am each day and departs from work to home at 4pm each day (with a very short commute). Whenever it is raining at departure time s/he takes an umbrella on the trip if there was one available at the departure point. Find the long run proportion of trips for which it is raining on his departure but he has no umbrella available.