MAST30001 Stochastic Modelling

Tutorial Sheet 8

1. A CTMC $(X_t)_{t\geq 0}$ with state space $\mathcal{S} = \{1,2,3,4\}$ has non-zero transition rates $q_{1,2} = 4$, $q_{2,1} = 1 = q_{2,4}$ and $q_{2,3} = 3$. Suppose that $\mathbb{P}(X_0 = 1) = 1$ (i.e. the chain starts in state 1), and let $T_1 = \inf\{t > 0 : X_t \neq X_0\}$ be the first jump time of $(X_t)_{t\geq 0}$, and $T_2 = \inf\{t > T_1 : X_t \neq X_{T_1}\}$ denote the time of the second jump.

   (a) Draw the transition diagram for the CTMC $(X_t)_{t\geq 0}$

   (b) Describe the communicating classes of $(X_t)_{t\geq 0}$

   (c) Find $h_{1,3}$, the probability of ever reaching state 3.

   (d) What is the distribution of T_1?

   (e) What is the distribution of X_{T_1}?

   (f) Find $\mathbb{E}[T_2]$.

   (g) What is the distribution of X_{T_2}?

   (h) Find $m_{1,\{3,4\}}$, which is the expected time until $(X_t)_{t\geq 0}$ reaches state 3 or 4.

   (i) Find the limiting proportion of time spent in each state.

2. Let $\lambda, \mu > 0$, and consider a CTMC with state space $\mathcal{S} = \mathbb{Z}_+$ whose non-zero transition rates are $q_{i,i+1} = \lambda$ and $q_{i+1,i} = (i+1)\mu$ for each $i \in \mathbb{Z}_+$.

   (a) Explain intuitively why this CTMC is positive recurrent.

   (b) Find the stationary distribution.

   (c) Find the limiting distribution, starting from initial distribution a.

   (d) Find the limiting proportion of time spent in each state.

3. (CTMCs as limits of DTMCs) Let P be a stochastic matrix with i,j-th entry $p_{i,j}$, and such that $p_{i,i} = 0$ for all i. For $(\lambda_i)_{i\in\mathcal{S}}$ and for each integer $m > \sup_{i\in\mathcal{S}} \lambda_i$, define a DTMC $(Y_n^{(m)})_{n\in\mathbb{Z}_+}$ by

$$\mathbb{P}(Y_{n+1}^{(m)} = i | Y_n^{(m)} = i) = \left(1 - \frac{\lambda_i}{m}\right),$$

   and for $i \neq j$

$$\mathbb{P}(Y_{n+1}^{(m)} = j | Y_n^{(m)} = i) = \frac{\lambda_i}{m} p_{ij}.$$

   Define a continuous time process (not a CTMC though) by

$$X_t^{(m)} = Y_{\lfloor mt \rfloor}^{(m)},$$

   where $\lfloor a \rfloor$ is the greatest integer not bigger than a.

   (a) What does a typical trajectory of $X^{(m)}$ look like? At what times does it jump?

   (b) Given $X_0^{(m)} = i$, what is the distribution of the random time

$$T^{(m)}(i) = \min\{t \geq 0 : X_t^{(m)} \neq i\}$$

   (c) As $m \to \infty$, to what distribution does that of the previous item converge?

   (d) It turns our that $X^{(m)}$ converges (in a certain sense) as $m \to \infty$ to a continuous time Markov chain. What are its transition rates?