MAST30001 Stochastic Modelling

$\boxed{\textbf{Tutorial Sheet 4}}$

1. Let $p \in (0,1)$ and suppose that a time homogenous Markov chain on $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ has transition probabilities given by

$$p_{i,i+2} = p,\ p_{i,i-1} = 1-p \ \text{ and } p_{i,j} = 0 \ \text{ for all } i \in \mathbb{Z},\ j \neq i+2, i-1.$$

(a) Using Stirling's approximation, (for $n \geq 1$)

$$1 \leq \frac{n!e^n}{n^n\sqrt{2\pi n}} \leq 2,$$

determine for what values of p the chain is recurrent.

The chain is irreducible. According to the theorem in the lectures, we need only compute the expected number of visits to state 0 (starting from state 0 with probability 1), which is given by

$$\sum_{n=0}^{\infty} \mathbb{P}(S_{3n} = 0).$$

Now since the chain either increases by 2 or decreases by 1, the only way that it can be at 0 at time m is if it has taken twice as many -1 steps as +2 steps, so in particular m has to be a multiple of 3. In $3n$ steps there are $\binom{3n}{n}$ ways of choosing n upward jumps (and $2n$ downward jumps), and each of these paths has probability $p^n(1-p)^{2n} = (p(1-p)^2)^n$.

Therefore

$$\mathbb{P}(S_{3n} = 0) = \binom{3n}{n}(p(1-p)^2)^n$$

Using Stirling's approximation, we have that for $n \geq 1$ this is bounded above and below by a positive constant times

$$\frac{(3n)^{3n}\sqrt{n}}{n^n(2n)^{2n}\sqrt{n}\sqrt{n}}(p(1-p)^2)^n = \frac{(27p(1-p)^2/4)^n}{\sqrt{n}}.$$

The term in brackets can be at most 1 since otherwise $\mathbb{P}(S_{3n} = 0)$ diverges to $+\infty$. Thus these probabilities are summable if and only if $27p(1-p)^2/4 < 1$, i.e. if and only if $p \neq 1/3$.

So the chain is transient if $p \neq 1/3$ and is recurrent if $p = 1/3$.

(b) Compute the probability of reaching state -1, starting in state 0.

We have that $h_{0,-1} = 1 - p + ph_{2,-1}$. But in order to get from state 2 to state -1, the chain must pass through states 1 and 0 in that order (since the only downward steps are of size 1). Thus

$$h_{2,-1} = h_{2,1}h_{1,0}h_{0,-1}.$$

But all three terms on the right hand side are equal since the transitions are identical from every state (i.e. the chain is "translation invariant"), so

$$h_{2,-1} = h_{0,-1}^3.$$

Thus, $h_{0,-1} = 1-p+ph_{0,-1}^3$. *This is a cubic equation of the form* $px^3-x+1-p = 0$. *Clearly 1 is a solution, so we can write*

$$px^3 - x + 1 - p = (x-1)(px^2 + px - (1-p)) = 0.$$

The other two solutions are therefore

$$x = \frac{-p \pm \sqrt{p^2 + 4p(1-p)}}{2p}.$$

Clearly one of these solutions is negative. The discriminant is larger than p^2 *for all* $p \in (0,1)$ *so the other is positive. This other one is less than 1 if*

$$\sqrt{p^2 + 4p(1-p)} < 3p.$$

Squaring both sides and rearranging, we get that this solution is less than 1 if $1 < 3p$, *i.e.* $p > 1/3$.

Thus,

$$h_{0,-1} = \begin{cases} 1, & \text{if } p \leq 1/3 \\ \frac{-p+\sqrt{p^2+4p(1-p)}}{2p} & \text{if } p > 1/3. \end{cases}$$

2. Consider a Markov chain with state space $S = \{1, 2, \ldots, 6\}$ and transition matrix given by

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

(a) Find all stationary distributions for this chain
The full balance equations include $\pi_3 = \pi_4/2$ *and* $\pi_4 = \pi_3/4$, *so* $\pi_3 = 0 = \pi_4$. *The remaining equations then become*

$$\pi_2 = \pi_1/2, \qquad \pi_1 = \pi_1/2 + \pi_2, \qquad \pi_5 = 2\pi_6/3, \qquad \pi_6 = \pi_6/3 + \pi_5.$$

We also have that $(\pi_1 + \pi_2) + (\pi_3 + \pi_4) = 1$. *Once we fix* $a = (\pi_1 + \pi_2)$ *this determines both* π_1 *and* π_2. *It also determines* $(\pi_3 + \pi_4)$ *and hence* π_3 *and* π_4. *Any vector of the form* $\pi = (\frac{2}{3}a, \frac{1}{3}a, 0, 0, \frac{2}{5}(1-a), \frac{3}{5}(1-a))$ *with* $a \in [0,1]$ *is therefore an equilibrium distribution for this chain (and there are no others).*

(b) Find the probability that we ever reach state 1, starting from state 3
Note that $h_{2,1} = 1$ *and* $h_{5,1} = 0$ *(it's obvious, but you can also get this from the complete set of equations),*

$$h_{3,1} = \frac{3}{4}h_{2,1} + \frac{1}{4}h_{4,1} = \frac{3}{4} + \frac{1}{4}h_{4,1}$$
$$h_{4,1} = \frac{1}{2}h_{5,1} + \frac{1}{2}h_{3,1} = \frac{1}{2}h_{3,1}$$

Solving gives $h_{3,1} = \frac{6}{7}$, $h_{4,1} = \frac{3}{7}$.

(c) Find the expected time that we first reach the set of states $A = \{2, 5\}$, starting from state 3.

$$m_{3,A} = 1 + \frac{1}{4} m_{4,A}$$
$$m_{4,A} = 1 + \frac{1}{2} m_{3,A}$$

Solving gives

$$m_{3,A} = \frac{10}{7}.$$

(d) Starting from state 3, find the limiting proportion of time spent in state 1.
This is random. If the chain reaches state 5, (which has probability $1 - h_{3,1}$*) then the limiting proportion of time spent in state 1 is 0.*

If the chain reaches state 1 (which has probability $h_{3,1}$*) then it moves between states 1 and 2 only, visiting (on average) state 1 twice as much as state 2 (so the limiting proportion of time spent in state 1 is* $2/3$ *in this case. Therefore the limit* $Y(1)$ *is a random variable with*

$$\mathbb{P}(Y(1) = 2/3) = 6/7 = 1 - \mathbb{P}(Y(1) = 0).$$

(e) Starting from state 3, find the expected limiting proportion of time spent in state 1.
From above this is $\mathbb{E}[Y(1)] = \frac{4}{7}$.

(f) Repeat the above two questions, starting from state 4 instead of state 3.
Again $Y(1)$ *is random, we get the same as above but with* $h_{3,1}$ *replaced with* $h_{4,1} = \frac{3}{7}$. *In this case* $\mathbb{E}[Y(1)] = 2/3 * 3/7 = 2/7$.

3. Assume that we have an infinite supply of a certain electrical component and that the lifetimes (measured from the beginning of use) of the components are i.i.d. $\mathbb{N}$-valued random variables (so time is measured in discrete units) with

$$\mathbb{P}(T_i = k) = q_k > 0, \quad k = 1, 2, \dots,$$

and $\mathbb{E}[T_1] < \infty$.

At time $n = 0$, the first component begins use and when it fails (at time T_1), it is immediately replaced by the second component, and when it fails (at time $T_1 + T_2$), it is replaced, and so on. Let X_n be the age of the component in use at time n and say $X_n = 0$ at times n where there is a failure.

(a) Show that X_n is a Markov chain and write down its transition probabilities.
Let F *denote the cdf of* T_1. *If* $X_n = j$, *then either* $X_{n+1} = j + 1$ *or* $X_{n+1} = 0$ *since the component either fails or goes another time unit. We compute:*

$$\begin{aligned}
&\mathbb{P}(X_{n+1} = j + 1 | X_n = j, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) \\
&= \mathbb{P}(T_1 > j + 1 | T_1 > j) \\
&= \frac{1 - F(j + 1)}{1 - F(j)} =: p_j,
\end{aligned}$$

which only depends on j, so this quantity equals $\mathbb{P}(X_{n+1} = j+1 | X_n = j)$ and the chain has the Markov property with the transition probabilities given by that above plus

$$\mathbb{P}(X_{n+1} = 0 | X_n = j) = 1 - p_j = \frac{q_{j+1}}{1 - F(j)}.$$

(b) Explain why this chain is positive recurrent.
The chain is irreducible. The mean return time to state 0 is the expected time until the next component fails, which is $\mathbb{E}[T_1] < \infty$. Therefore the mean return time is finite. By irreducibility the mean return time to every state is finite, and hence the chain is positive recurrent.

(c) Find all equilibrium distributions for this Markov chain.
This chain is (irreducible and) positive recurrent, so there is a unique stationary distribution π. The mean return time to state 0 is $\mathbb{E}[T_1]$, so $\pi_0 = 1/\mathbb{E}[T_1]$.
For $i \geq 0$ the full balance equations can be written as

$$\pi_{i+1} = p_i \pi_i.$$

So $\pi_i = \pi_0 \prod_{k=0}^{i-1} p_k$.