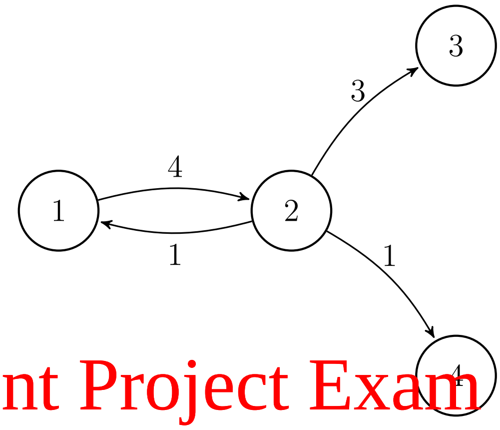
MAST30001 Stochastic Modelling

Tutorial Sheet 8

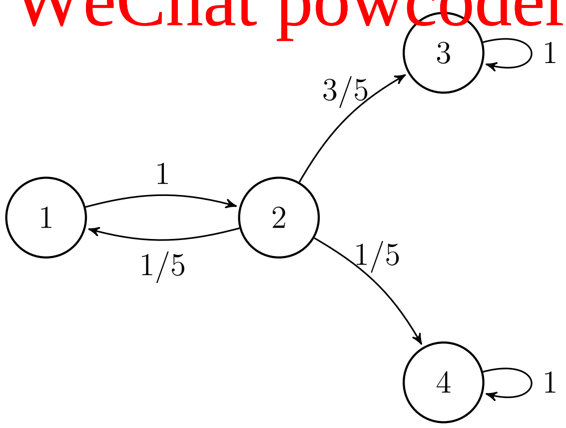
1. A CTMC $(X_t)_{t\geq 0}$ with state space $\mathcal{S} = \{1, 2, 3, 4\}$ has non-zero transition rates $q_{1,2} = 4$, $q_{2,1} = 1 = q_{2,4}$ and $q_{2,3} = 3$. Suppose that $\mathbb{P}(X_0 = 1) = 1$ (i.e. the chain starts in state 1), and let $T_1 = \inf\{t > 0 : X_t \neq X_0\}$ be the first jump time of $(X_t)_{t\geq 0}$, and $T_2 = \inf\{t > T_1 : X_t \neq X_{T_1}\}$ denote the time of the second jump.

   (a) Draw the transition diagram for the CTMC $(X_t)_{t\geq 0}$



   (b) Describe the communicating classes of $(X_t)_{t\geq 0}$
   3 and 4 are absorbing states (so their own communicating classes), while 1 *and* 2 *communicate with each other*

   (c) Find $h_{1,3}$, the probability of ever reaching state 3.
   This is the same as for the jump chain, which has transition diagram



   The hitting probability of state 3 is $\frac{3/5}{3/5+1/5} = 3/4$.

   (d) What is the distribution of T_1?
   exponential(4)

   (e) What is the distribution of X_{T_1}?
   $X_{T_1} = 2$ *with probability 1*

   (f) Find $\mathbb{E}[T_2]$.
   This is the expectation of the sum of the first two jump times which are respectively exponential(4) *and exponential*$(3+1+1)$ *random variables, so the expectation is* $\frac{1}{4} + \frac{1}{3+1+1} = 9/20$.

(g) What is the distribution of X_{T_2}?
$\mathbb{P}(X_{T_2} = 1) = \frac{1}{5} = \mathbb{P}(X_{T_2} = 4)$, *while* $\mathbb{P}(X_{T_2} = 3) = 3/5$

(h) Find $m_{1,\{3,4\}}$, which is the expected time until $(X_t)_{t\geq 0}$ reaches state 3 or 4.
Let $A = \{3, 4\}$ *then*

$$\begin{aligned} m_{1,A} &= \frac{1}{4} + m_{2,A} \\ m_{2,A} &= \frac{1}{5} + \frac{1}{5} m_{1,A} + 0 \end{aligned}$$

Solving gives $m_{1,A} = 9/16$.

(i) Find the limiting proportion of time spent in each state.
This is random. The limiting proportion of time spent in state 3 is 1 with probability $h_{1,3}$*, and zero with probability* $1 - h_{1,3}$*. The limiting proportion of time spent in state 4 is 1 with probability* $h_{1,4} = 1 - h_{1,3}$ *and zero with probability* $h_{1,3}$*. The limiting proportion of time spent in states 1 and 2 are zero respectively.*

2. Let $\lambda, \mu > 0$, and consider a CTMC with state space $\mathcal{S} = \mathbb{Z}_+$ whose non-zero transition rates are $q_{i,i+1} = \lambda$ and $q_{i+1,i} = (i + 1)\mu$ for each $i \in \mathbb{Z}_+$.

(a) Explain intuitively why this CTMC is positive recurrent.
Let $i_0 = \min\{i : i\mu > \lambda\}$*. Then at each state* $i \geq i_0$ *we wait for less than a exponential*$(\lambda + \mu)$ *amount of time before jumping, and we jump down with probability*

$$\frac{i\mu}{i\mu + \lambda} \geq \frac{i_0\mu}{i_0\mu + \lambda} =: \kappa > 1/2.$$

Therefore for all states $i \geq i_0$ *(i.e. all but finitely many of the states) we have a bias (that is bounded below) to the left. Thus,* $m_{i,i_0} < \infty$ *for each* $i \geq i_0$*. On the other hand* $m_{i,i_0} < \infty$ *for each* $i < i_0$ *since this is the same as the expected hitting time of* i_0 *starting from* i *for a finite-state Markov chain with state space* $\{0, 1, \ldots, i_0\}$*. This shows that* m_{i,i_0} *is finite for every* i*, and thus the expected return time to* i_0 *is finite. This means that the expected return time to every state is finite, so the chain is positive recurrent.*

(b) Find the stationary distribution.
This is a birth and death chain so it satisfies the detailed balance equations. Thus, for each $i \in \mathbb{N}$,

$$\pi_i i\mu = \pi_{i-1}\lambda.$$

Solving gives $\pi_i = \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}\pi_0$*. Summing over* $i \in \mathbb{Z}_+$ *gives*

$$1 = \pi_0 \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} = \pi_0 e^{\lambda/\mu},$$

so $\pi_0 = e^{-\lambda/\mu}$ *and* $\pi_i = e^{-\lambda/\mu}\left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}$*. So the stationary distribution is Poisson with parameter* λ/μ*.*

(c) Find the limiting distribution, starting from initial distribution a.
This is an irreducible positive recurrent Markov chain, so the limiting distribution is π *regardless of the starting distribution*

(d) Find the limiting proportion of time spent in each state.
As above, this is π

3. (CTMCs as limits of DTMCs) Let P be a stochastic matrix with i,j-th entry $p_{i,j}$, and such that $p_{i,i} = 0$ for all i. For $(\lambda_i)_{i\in\mathcal{S}}$ and for each integer $m > \sup_{i\in\mathcal{S}} \lambda_i$, define a DTMC $(Y_n^{(m)})_{n\in\mathbb{Z}_+}$ by

$$\mathbb{P}(Y_{n+1}^{(m)} = i | Y_n^{(m)} = i) = \left(1 - \frac{\lambda_i}{m}\right),$$

and for $i \neq j$

$$\mathbb{P}(Y_{n+1}^{(m)} = j | Y_n^{(m)} = i) = \frac{\lambda_i}{m} p_{ij}.$$

Define a continuous time process (not a CTMC though) by

$$X_t^{(m)} = Y_{\lfloor mt \rfloor}^{(m)},$$

where $\lfloor a \rfloor$ is the greatest integer not bigger than a.

(a) What does a typical trajectory of $X^{(m)}$ look like? At what times does it jump?
The chain only has jumps at times k/m for k an integer. Given the chain is in state i, it stays there for a geometric λ_i/m (> 0) number of $1/m$ time units and then jumps according to the one step transition matrix P. The number of these time units are the number of integer time units between jumps in the $Y^{(m)}$ chain.

(b) Given $X_0^{(m)} = i$, what is the distribution of the random time

$$T^{(m)}(i) = \min\{t \geq 0 : X_t^{(m)} \neq i\}$$

As mentioned in the previous problem, the $Y^{(m)}$ chain stays at state i for a geometric (λ_i/m) number of time units before jumping. Then considering the time change to get from $Y^{(m)}$ to $X^{(m)}$, the variable $mT^{(m)}(i)$ is geometric λ_i/m (> 0); that is for $k = 1, 2, \ldots$

$$\mathbb{P}(T^{(m)}(i) = k/m) = \frac{\lambda_i}{m}\left(1 - \frac{\lambda_i}{m}\right)^{k-1}.$$

(c) As $m \to \infty$, to what distribution does that of the previous item converge?
A standard calculation (do it!) shows that if Z_p is geometric p, then pZ_p converges in distribution to an exponential distribution with mean 1 as $p \to 0$. Since $mT^{(m)}(i)$ is geometric λ_i/m, $T^{(m)}(i)$ converges to an exponential variable with rate λ_i.

(d) It turns out that $X^{(m)}$ converges (in a certain sense) as $m \to \infty$ to a continuous time Markov chain. What are its transition rates?
Since the holding times converge to exponential variables as m goes to infinity, the description of the limiting chain is as follows. Given $X_0 = i$ the chain waits an exponential with rate λ_i time and then jumps to state j with probability $p_{i,j}$ (the state jumped to is independent of the time of the jump). Then the chain stays in state j and exponential λ_j amount of time and jumps to state k with probability $p_{j,k}$, and so on. Thus the transition rates are $\lambda_i p_{i,j}$ for $j \neq i$.