MAST30001 Stochastic Modelling

Tutorial Sheet 2

1. Show that the Markov property does not in general imply that for any events A, B and C,
$$\mathbb{P}(X_{n+1} \in C | X_n \in B, X_{n-1} \in A) = \mathbb{P}(X_{n+1} \in C | X_n \in B).$$
(That is, define a Markov chain and events A, B, C where the equality doesn't hold.)

2. Let $(X_n)_{n \in \mathbb{Z}_+}$ be a Markov chain with state space $\{1, 2, 3\}$ and transition matrix
$$\begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/4 & 3/4 & 0 \\ 2/5 & 0 & 3/5 \end{pmatrix}$$
   (a) Draw the transition diagram corresponding to this chain.
   (b) Compute $\mathbb{P}(X_3 = 1, X_2 = 2, X_1 = 2 | X_0 = 1)$.
   (c) If X_0 is uniformly distributed on $\{1, 2, 3\}$, compute $\mathbb{P}(X_3 = 1, X_2 = 2, X_1 = 2)$.
   (d) Now assuming that $\mathbb{P}(X_0 = 1) = \mathbb{P}(X_0 = 2) = 1/2$, compute $\mathbb{P}(X_1 = 1, X_4 = 1, X_7 = 1)$.

3. A simplified model for the spread of a contagion in a small population of size 4 is as follows. At each discrete time unit, two individuals in the population are chosen uniformly at random to meet. If one of these persons is healthy and the other has the contagion, then with probability 1/4 the healthy person becomes sick. Otherwise the system stays the same.

   (a) If X_n is the number of healthy people at step n, then explain why $X_0, X_1 \ldots$ is a Markov chain.

   (b) Specify the transition probabilities of X_n.
   (c) Draw the transition diagram for this chain.
   (d) If initially the chance that a given person in the population has the disease equals 1/2, determined independently, then what is the chance everyone has the disease after two steps in the process?
   (e) Now suppose that exactly one person is infected at time 0. Find the expected time until everyone is infected.

4. Let $(Y_n)_{n \geq 0}$ be i.i.d. random variables with $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2$ and let $X_n = (Y_{n+1} + Y_n)/2$.
   (a) Find the transition probabilities $\mathbb{P}(X_{n+m} = k | X_n = j)$ for $m = 1, 2, \ldots$ and $j, k = 0, \pm 1$.
   (b) Show that $(X_n)_{n \geq 0}$ is *not* a Markov chain.