MAST30001 Stochastic Modelling

Tutorial Sheet 6

You've probably already seen/done some of these before, but it's useful to do them yourselves/see them again!

1. Let X ~Exponential(λ), with $\lambda > 0$. Prove that $\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s)$ for every $s, t > 0$.

$$\mathbb{P}(X > s+t|X > t) = \frac{\mathbb{P}(X > s + t, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}(X > s).$$

2. Let $(X_i)_{i\in\mathbb{N}}$ be independent random variables with X_i ~Exponential(λ_i). Find the distribution of $Y_n = \min_{i\le n} X_i$.

$$\mathbb{P}(Y_n > y) = \mathbb{P}(\cap_{i=1}^n \{X_i > y\}) = \prod_{i=1}^n \mathbb{P}(X_i > y) = \prod_{i=1}^n e^{-\lambda_i y} = e^{-(\sum_{i=1}^n \lambda_i)y},$$

so Y_n *~Exponential*$(\sum_{i=1}^n \lambda_i)$.

3. Let $(T_i)_{i\in\mathbb{N}}$ be i.i.d. Exponential(λ) random variables, and let N be a Geometric(p) random variable that is independent of the other variables.

   (a) Find the moment generating function $\mathbb{E}[e^{tT_1}]$ of T_1.

$$\mathbb{E}[e^{tT_1}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \int_0^\infty e^{(t-\lambda)x}\lambda dx.$$

If $t \ge \lambda$ *then this is infinite. Otherwise it is* $\lambda/(\lambda - t)$.

   (b) Let $Y = \sum_{i=1}^N T_i$. Find the distribution of Y.
We'll find the MGF of Y *(which characterizes its distribution).*

$$\begin{aligned}\mathbb{E}[e^{tY}] &= \mathbb{E}[e^{t\sum_{i=1}^N T_i}] = \sum_{n=1}^\infty \mathbb{E}[e^{t\sum_{i=1}^N T_i}|N = n]\mathbb{P}(N = n)\\ &= \sum_{n=1}^\infty \mathbb{E}[e^{t\sum_{i=1}^n T_i}|N = n]\mathbb{P}(N = n) = \sum_{n=1}^\infty \mathbb{E}[e^{t\sum_{i=1}^n T_i}]\mathbb{P}(N = n),\end{aligned}$$

where we have used independence of N *from the* T_i *variables in the last line. Now use the fact that the* T_i *are i.i.d. as usual to see that for* $t < \lambda$ *the above is equal to*

$$\sum_{n=1}^\infty \left(\mathbb{E}[e^{tT_1}]\right)^n \mathbb{P}(N = n) = \sum_{n=1}^\infty \left(\frac{\lambda}{\lambda - t}\right)^n \mathbb{P}(N = n) = \frac{p}{1-p}\sum_{n=1}^\infty \left(\frac{\lambda(1-p)}{\lambda - t}\right)^n.$$

For $t < \lambda p$ *this is equal to*

$$\frac{\lambda p}{\lambda p - t}.$$

This is the MGF of an Exponential(λp) *distribution, hence* Y *~Exponential*(λp).

4. Let $X \geq 0$ be a random variable satisfying

$$(*) \quad \mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s), \qquad \text{for all } s, t \geq 0.$$

Show that $X \sim$Exponential(λ), for some $\lambda \geq 0$.

Let $G(s) = \mathbb{P}(X > s)$. [Note that $G(s) > 0$ for every s otherwise the conditioning in $(*)$ above is not well defined.] The property $(*)$ can be rewritten as $G(t + s) = G(t)G(s)$ for every $t, s \geq 0$. By induction $G(nt) = G(t)^n$ for each n and each $t \geq 0$. This shows that $G(n) = G(1)^n$ and $G(1/n) = G(1)^{1/n}$ for every $n \in \mathbb{N}$. It follows that for any non-negative rational number $r = m/n$,

$$G(r) = G(m/n) = G(1/n)^m = (G(1)^{1/n})^m = G(1)^r.$$

Now note that G is right continuous since $1-G$ is ($1-G$ is a cdf), so for any $t \geq 0$ and rationals $t_n \downarrow t$ we have

$$G(t) = \lim_{n\to\infty} G(t_n) = \lim_{n\to\infty} G(1)^{t_n} = G(1)^t.$$

If $G(1) = 1$ then $\mathbb{P}(X > n) = 1$ for all n so $\mathbb{P}(X = \infty) = 1$. This corresponds to the case of $\lambda = 0$. Otherwise $0 < G(1) < 1$ and we can write $G(1) = e^{-\lambda}$ for some $\lambda > 0$, and $G(t) = G(1)^t$ says that $\mathbb{P}(X > t) = e^{-\lambda t}$ for each $t \geq 0$, as required.