(Continuous) Gaussian Processes

Multivariate normal

A random vector $\vec{X} = (X_1, \ldots, X_d)$ is *(multivariate) normal* or *(multivariate) Gaussian* if and only if every linear combination $\vec{x} \cdot \vec{X}$ is univariate normal (with variance in $[0, \infty)$).

The distribution of a (multivariate) normal vector can be specified by the *mean vector* and the *covariance matrix*.

A *Gaussian process* $(X_t)_{t \in I}$ is a random process for which the *finite-dimensional distributions* are all *multivariate normal*, i.e. $(X_{t_1}, \ldots, X_{t_r})$ is multivariate normal for every $r \in \mathbb{N}$, $t_1 < t_2 < \cdots < t_r$ all in I. Typically I is $[0, \infty)$ or $[0, 1]$.

Continuous Gaussian processes

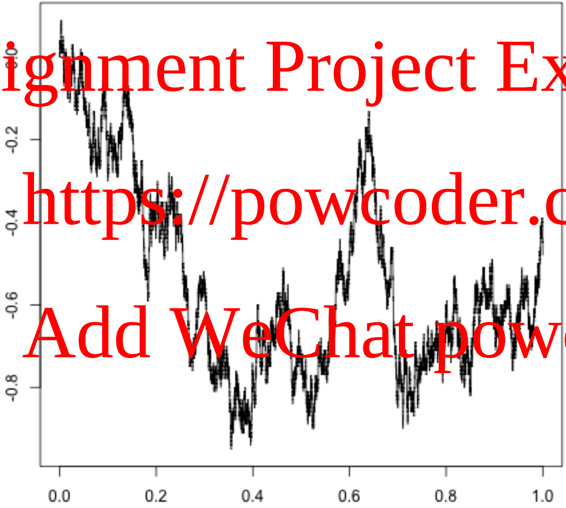
We'll restrict our attention to Gaussian processes $(X_t)_{t\in I}$ that are *continuous* in t with probability 1.

Recall that for continuous processes, the distribution of the process is determined by the finite-dimensional distributions.

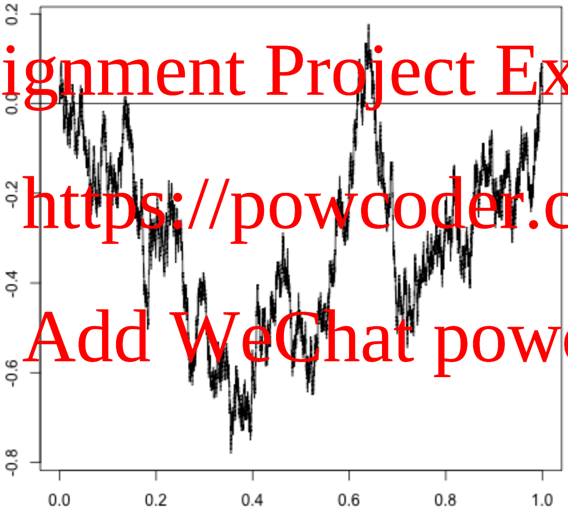
For a Gaussian process the f.d.d. are multivariate Gaussian, determined by the mean vectors and covariance matrices.

It follows that the distribution of a continuous Gaussian process $(X_t)_{t\in I}$ is determined by two functions:
the *mean function* $\mu(t) = \mathbb{E}[X_t]$ for $t \in I$ and the *covariance function* $\Sigma(s,t) = \mathrm{Cov}(X_s, X_t)$ for $s \leq t$ both in I.

Brownian motion simulation



Brownian bridge



Brownian motion and Brownian bridge

(standard) *Brownian motion* $(W_t)_{t\geq 0}$ is a continuous Gaussian process with $\mu(t) = 0$ and $\Sigma(s,t) = s$ for $s \leq t$.

(standard) *Brownian bridge* $(B_t)_{t\in[0,1]}$ is a continuous Gaussian process with $\mu(t) = 0$ and $\Sigma(s,t) = s(1-t)$ for $s \leq t$.

How do we know that such processes exist? We can construct them as limits of things that exist.

Sketch construction of Brownian motion (on $[0,1]$)

Let $(Z_q)_{q \in \mathbb{Q} \cap [0,1]}$ be i.i.d. standard normal random variables.

Define a sequence of random functions $(W_t^{(n)})_{t \in [0,1]}$ for $n \in \mathbb{N}$ by:

$$W_t^{(1)} = tZ_1,$$

i.e. set $W_0^{(1)} = 0$ and $W_1^{(1)} = Z_1$, and then linearly interpolate.

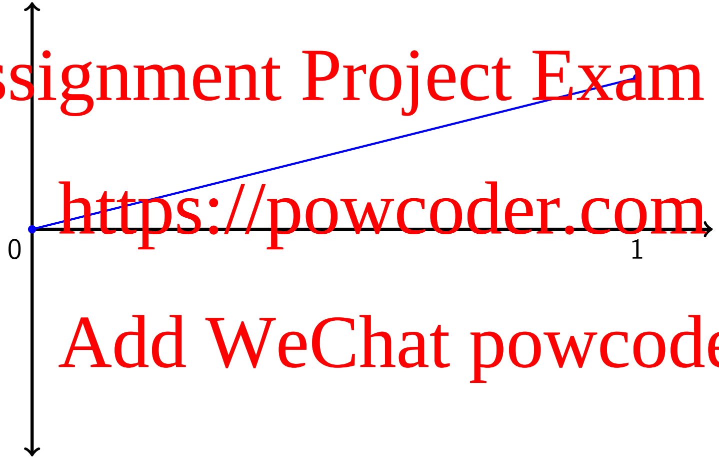
Set $W^{(2)}$ to be the same at 0 and 1 but set

$$W_{1/2}^{(2)} = W_{1/2}^{(1)} + \frac{1}{\sqrt{2^2}} Z_{1/2},$$

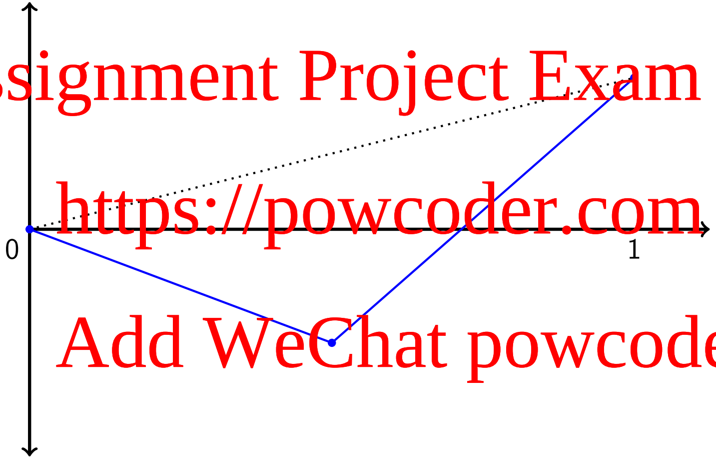
and then linearly interpolate in between.

More generally define $W^{(n+1)}$ to be equal to $W^{(n)}$ at points $2i/2^n$, and define $W^{(n+1)}$ at the points q of the form $(2i+1)/2^n$ by adding some extra randomness $\frac{1}{\sqrt{2^n}} Z_q$ to $W_q^{(n)}$ and then linearly interpolating.

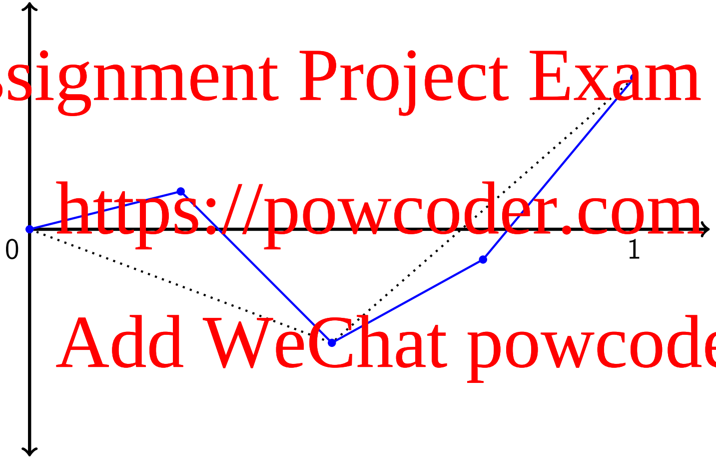
Sketch construction of Brownian motion (on $[0, 1]$)



Sketch construction of Brownian motion (on $[0, 1]$)



Sketch construction of Brownian motion (on $[0, 1]$)



Sketch construction of Brownian motion (on $[0, 1]$)

It's possible to check that $W^{(n+1)}$ has the claimed mean and covariance functions if we restrict to times of the form $i/2^n$.

This sequence of (random) continuous functions converges (as $n \to \infty$ (uniformly) to a random continuous function $(W_t)_{t\in[0,1]}$. This random function has the correct mean and covariance functions since it does at every dyadic rational point.

Brownian motion (BM)

BM has independent increments:
If $0 < s_1 < t_1 < s_2 < t_2, \ldots, < s_n < t_n$ then $(W_{t_i} - W_{s_i})_{i \le n}$ are independent random variables.

e.g. $(W_2 - W_1, W_4 - W_2)$ is bivariate normal (why?), and

$$\begin{aligned}
&\mathbb{E}[(W_4 - W_2)(W_2 - W_1)] \\
&= \mathbb{E}[W_4 W_2] - \mathbb{E}[W_2^2] - \mathbb{E}[W_4 W_1] + \mathbb{E}[W_2 W_1] \\
&= 2 - 2 - 1 + 1 = 0.
\end{aligned}$$

Definition of BM is equivalent to saying that $(W_t)_{t \ge 0}$ is a continuous process with:

(i) $W_0 = 0$ and,

(ii) with independent increments (if they are disjoint)

(iii) and $W_t - W_s \sim \mathcal{N}(0, t - s)$ for every $t > s \ge 0$.

Exercises

Suppose that $(W_t)_{t\geq 0}$ is a BM.

(a) If $s > 0$ and $X_t = W_{t+s} - W_s$ then $(X_t)_{t\geq 0}$ is a BM (and in fact it is independent of $(W_u)_{u<s}$).

(b) If $X_0 = 0$ and $X_t = tW_{1/t}$ for $t > 0$ then $(X_t)_{t\geq 0}$ is a BM.

(c) If $c > 0$ and $X_t = W_{ct}/\sqrt{c}$ then $(X_t)_{t\geq 0}$ is a BM.

(d) If $X_t = W_t - tW_1$ then $(X_t)_{t\in[0,1]}$ is a Brownian Bridge.

Path properties of BM

Brownian motion is *recurrent*: for every $t > 0$ there exists $T > t$ such that $W_T = 0$.

Sketch proof: Note that $W_n - W_{n-1}$ for $n \in \mathbb{N}$ are i.i.d. $\sim \mathcal{N}(0, 1)$. Eventually one of these (say $W_N - W_{N-1}$) has size greater than 2. Thus either $|W_N| > 1$ or $|W_{N-1}| > 1$. Since W is continuous this shows that $T_1 = \inf\{t : |W_t| = 1\}$ is finite.

Similarly we can define $T_j = \inf\{t > T_{j-1} : |W_t - W_{T_{j-1}}| = 1\}$. One can show that $(S_i)_{i \in \mathbb{Z}_+}$ defined by $S_0 = 0$ and $S_j = W_{T_j}$ for $j \geq 1$ is a simple (unbiased) random walk. This simple random walk visits 0 infinitely often.... In fact, something stronger is true, e.g. in any interval of time $[0, \varepsilon]$ where $\varepsilon > 0$, BM visits 0 infinitely often.

Since this simple random walk also visits every integer infinitely often this shows that BM visits every point in $\mathbb{R}$ infinitely often as well.

Properties of BM

BM is both a *Markov process* (with state space $\mathbb{R}$), and a *Martingale*.

Brownian motion is not differentiable at any point.

E.g. $\frac{W_h}{h}$ for small h is like $tW_{1/t}$ for large t, which as a function of t has the same law as $(W_t)_{t>0}$ so it oscillates (does not converge) as $t \to \infty$.

Functional CLT

Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. random variables with mean 0 and variance 1. Let

$$Z_t^{(n)} = \frac{\sum_{i=1}^{\lfloor nt \rfloor} X_i}{\sqrt{n}}.$$

Then $(Z_t^{(n)})_{t\geq 0} \xrightarrow{\mathcal{D}} (W_t)_{t\geq 0}$.

(If $Z^{(n)}, Z$ are random objects taking values in some space E we write $Z^{(n)} \xrightarrow{\mathcal{D}} Z$ if $\mathbb{E}[f(Z^{(n)})] \to \mathbb{E}[f(Z)]$ as $n \to \infty$ for every bounded continuous function $f : E \to \mathbb{R}$)

Brownian bridge does not have independent increments.
E.g. $B_1 - B_{1/2} = -(B_{1/2} - B_0)$.

Exercise: Suppose that $(B_t)_{t\in[0,1]}$ is a BB, and $Z \sim \mathcal{N}(0,1)$ is independent of $(B_t)_{t\in[0,1]}$. Then $X_t = B_t + tZ$ is a BM on $[0,1]$.

FCLT for empirical processes

Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. with cdf F, and let $F^{(n)}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i \le x\}}$.
Then

$$\left(\sqrt{n}(F^{(n)}(x) - F(x))\right)_{x\in\mathbb{R}} \xrightarrow{\mathcal{D}} \left(B_{F(x)}\right)_{x\in\mathbb{R}}.$$

Note that $F(x) \in [0,1]$ so the right hand side is well defined.

If $X_i \sim U(0,1)$ then $B_{F(t)} = B_t$.

Exercise: Compute the mean of the left hand side at the point x.
Calculate the covariance of the left hand side evaluated at the points x and y where $x \le y$.

More amazing facts

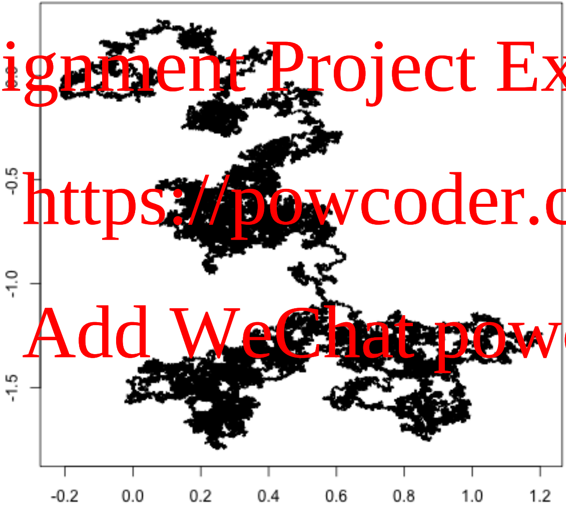
$$\frac{W_t}{\sqrt{t}} \sim \mathcal{N}(0,1), \quad \text{for every } t \geq 0,$$

but $\dfrac{W_t}{\sqrt{t}}$ oscillates unboundedly (see the *Law of the Iterated Logarithm*).

(There is a similar result for simple random walk: as $n \to \infty$ the distribution of $n^{-1/2}S_n$ converges to $\mathcal{N}(0,1)$ but as a random sequence $n^{-1/2}S_n$ does not converge.)

Let $(W_t^{[i]})_{t\geq 0}$ be independent BM for $i \in \mathbb{N}$. Then $\left((W_t^{[1]}, W_t^{[2]})\right)_{t\geq 0}$ is a 2-dimensional BM, $\left((W_t^{[1]}, W_t^{[2]}, W_t^{[3]})\right)_{t\geq 0}$ is a 3-dimensional BM, etc.

2-dimensional BM



More amazing facts

- (1-dimensional) BM visits every point in $\mathbb{R}$ infinitely often.
- for 2-dimensional BM, for every $k \in \mathbb{Z}_+$ there are (random) points in $\mathbb{R}^2$ visited exactly k times. Every neighbourhood of every point is visited infinitely often.
- for 3-dimensional BM there are (random) points visited exactly twice, and no point in $\mathbb{R}^3$ is visited 3 or more times, $|B_t| \to \infty$ as $t \to \infty$.
- for 4-dimensional BM no point in $\mathbb{R}^4$ is hit more than once.