MAST30001 Stochastic Modelling

Tutorial Sheet 3

1. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with probability mass function given by the following table.

k	0	1	2	3
$\mathbb{P}(Y = k)$	0.2	0.2	0.3	0.3

   Set $X_0 = 0$ and let $X_n = \max\{Y_1, \ldots, Y_n\}$ be the largest Y_i observed to time n.

   (a) Explain why (X_n) is a Markov chain and determine its transition matrix P.

   (b) Find all communicating classes for this chain and all absorbing states.

   (c) Describe the long run behavior of the chain. In particular can you determine the matrix $\lim_{n\to\infty} P^n$?

   (d) Find the expected time to reach state 3, starting from state 0.

2. A game involves rolling a ball around a table, starting from a fixed position. Three players, Alex, Bobby and Célia, take turns attempting to grab the ball as it passes them. If Alex is unsuccessful then Bobby attempts to grab the ball and if he is also unsuccessful then Célia attempts to grab the ball. If they are all unsuccessful, the ball is rolled again from the starting position. The game stops as soon as any player is successful (and that player wins). Suppose that on each attempt, Alex, Bobby and Célia have probabilities p_1, p_2 and p_3 (all $\in (0, 1)$, with $p_1 > 0$) respectively, of grabbing the ball, independent of previous attempts.

   (a) Using Markov chain theory, find the probability that Alex wins the game.

   (b) What condition(s) do p_1, p_2 and p_3 have to satisfy so that each player is equally likely to win the game? (Include the largest possible values of p_1,p_2, and p_3 in your answer.)

3. Two dogs named Fluffy and Duffy have m fleas distributed between them. At discrete time steps a flea (chosen uniformly at random) jumps to the other dog. Let X_n be the number of fleas on Fluffly after the nth time step.

   (a) Is X_n a Markov chain? What are the transition probabilities?

   (b) If X_0 has a binomial distribution with parameters $(m, 1/2)$ (meaning that each flea tosses a fair coin to determine on which dog it starts), what is the distribution of X_{10}? *[Hint: First compute the distribution of X_1.]*

   (c) If $m = 3$, and $\mathbb{P}(X_0 = 1) = 1$, find the probability that Fluffy is free of fleas before Duffy.

4. Let $(X_n)_{n\geq 1}$ be a Markov chain with state space $\{1, \ldots, k\}$ for some $k \geq 2$. Show that if i and j communicate, then the probability that the chain started in state i reaches state j in fewer than k steps is greater than 0.