MAST30001 Stochastic Modelling

Tutorial Sheet 9

1. Show that in in $M/M/1$ queue with arrival rate λ and service rate $\mu > \lambda$, the expected lengths of the idle and busy periods are $1/\lambda$ and $1/(\mu - \lambda)$, respectively. *[Hint: the proportion of time the server is idle is equal to the stationary chance the system is empty.]*
   Since the arrivals follow a Poisson process (using in particular the memoryless property of the exponential), the time between the moment the system clears and the next arrival is exponential rate λ and so the expected length of an idle period is the expectation of this exponential, that is, $1/\lambda$.

   If b is the expected length of a busy period and $\pi_0 = 1-\lambda/\mu$ is the long run proportion of time the system is empty, then

   $$\pi_0 = \frac{1/\lambda}{1/\lambda + b},$$

   or $b = 1/(\mu - \lambda)$.

2. A rental car washing facility can wash one car at a time. Cars arrive to be washed according to a Poisson process with rate 3 per day and the service time to wash a car is exponential with mean 7/24 days. It costs the company \$150 per day to operate the facility and the company loses \$10 per day for each car tied up in the washing facility. The company can upgrade the facility to get down to a mean service time of 1/4 days at the cost of \$$C$ per day. What's the largest C can be for this upgrade to make economic sense?

   We can model the number of cars in the wash as an $M/M/1$ queue with arrival rate $\lambda = 3$ and current service rate $\mu = 24/7$ and so with stationary distribution geometric-1 with parameter $1 - 21/24 = 1/8$ having expectation 7. The companies current cost per day is

   $$150 + 10 \times 7 = 220.$$

   If the company pays C dollars per day to increase their service rate to 4, then similarly their new cost per day will be

   $$150 + C + 10 \times 3 = 180 + C.$$

   Thus they should spend no more than 40 dollars per day to increase their service rates.

3. (M/G/∞ queue) In a certain communications system, information packets arrive according to a Poisson process with rate λ per second and each packet is processed in one second with probability p and in two seconds with probability $1-p$, independent of the arrival times and other service times. Let N_t be the number of packets that have entered the system up to time t and X_t be the number of packets in the system (including those being served) at time t.

   (a) Is $(X_t)_{t\geq 0}$ a Markov chain? (No detailed argument is necessary here, just think about it heuristically.)
   X_t is not a Markov chain because the chance of the chain decreasing by one in the interval $(t, t+h)$ given the value of the chain at time t also depends on the times of the arrivals in the past.

(b) If $X_0 = 0$, what is the distribution of X_2?
If A_t are the arrivals that require one second of service, and B_t are the arrivals requiring two seconds of service, then A_t and B_t are independent Poisson processes with rates $p\lambda$ and $(1-p)\lambda$. And $X_2 = (N_2 - N_1) + B_1$; the sum of two independent Poisson variables (using independent increments) with respective means λ and $(1-p)\lambda$. So X_2 is Poisson with mean $\lambda(2-p)$.

(c) If $X_0 = 0$, is there a "stationary" limiting distribution $\pi_k = \lim_{t\to\infty} P(X_t = k)$? If so, what is it?
X_t only depends on the number of arrivals of the two different types in the interval $(t-2, t)$ since all arrivals previous to this time have left the system. As in part (b), we can write $X_t = (N_t - N_{t-1}) + (B_{t-1} - B_{t-2})$, and the two variables in parentheses are independent Poisson with respective means λ and $(1-p)\lambda$. So for $t \geq 2$, X_t is Poisson with mean $\lambda(2-p)$.

(d) If $X_0 = N_0 = 0$, what is the joint distribution of X_t and N_t?
When $0 < t \leq 1$, then $X_t = N_t$ and they're both distributed as Poisson mean t. The case $1 < t < 2$ is similar but easier than $t \geq 2$; the latter case we show here. Assuming $t \geq 2$, then as above we write $X_t = (N_t - N_{t-1}) + (B_{t-1} - B_{t-2})$ and also $N_t = X_t + (A_{t-1} - A_{t-2}) + N_{t-2}$, and note that by the comments of part (b), X_t is independent of $(A_{t-1} - A_{t-2})$ and these variables are both independent of N_{t-2}. So we can write $N_t = X_t + Y_t$, where Y_t is a Poisson variable with mean $\lambda(p + t - 2)$, independent of X_t which implies that for $0 \leq j \leq n$,

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\begin{aligned}
\mathbb{P}(X_t = j, N_t = n) &= \mathbb{P}(X_t = j, Y_t = n - j) \\
&= \mathbb{P}(X_t = j)\mathbb{P}(Y_t = n - j) \\
&= e^{-\lambda t} \frac{\lambda^n}{n!} \binom{n}{j} (2-p)^j (p + t - 2)^{n-j}.
\end{aligned}
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