THE UNIVERSITY OF MELBOURNE

| Student number | |
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Semester 2 Assessment, 2021

School of Mathematics and Statistics

# MAST30001 Stochastic Modelling, Assignment 2

Submission deadline: 8 p.m. Friday 15 October

This assignment consists of 3 pages (including this page)

### Instructions to Students

- If you have a printer, print the assignment.

*Writing*

- There are 3 questions with marks as shown. The total number of marks available is 34.
- *** Write instructions you want that are not auto-loaded here, each preceded by \item
- Write your answers on A4 paper. Page 1 should only have your student number, the subject code and the subject name. Write on one side of each sheet only. Each question should be on a new page. The question number must be written at the top of each page.

*Scanning*



- Put the pages in question order and all the same way up. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4. Make sure that you upload the correct PDF file and that your PDF file is readable.



*Submitting*



- Go to the Gradescope window. Choose the Canvas assignment for this assignment. Submit your file. Get Gradescope confirmation on email.

**Question 1 (12 marks)**

Let $N_t$ be the number of earthquakes observed in New Zealand (NZ) up to time $t$ (time units in this question are always years). We will assume that $(N_t)_{t\geq 0}$ is a Poisson process with rate 5 (per year), and that the magnitude of each earthquake is Exponentially distributed with mean 2, independent of all other earthquakes.

(a) Let $A$ be the event that there are no earthquakes in the second half year. Express $A$ as an event in terms of $(N_t)_{t\geq 0}$ and find $\mathbb{P}(A)$.

(b) Let $T_1$ be the time of the first earthquake. Find $\mathbb{P}(T_1 > 1/2 | N_1 = 2)$.

(c) What is the distribution of the number of earthquakes in NZ of magnitude 6 or more in the first 10 years?

(d) Suppose that there are exactly 7 earthquakes in each of the first 3 years. What is the probability that there are also 7 earthquakes in the 4th year?

(e) Find the distribution function of the magnitude of the largest earthquake in the first year (if there is no earthquake this is defined to be 0).

(f) Give a reason why modelling earthquake occurrences as a Poisson process may be unrealistic.

**Question 2 (14 marks)**



In a small remote neighbourhood consisting of $k = 5$ individuals, one individual is infected with a virus at time 0. Suppose that each pair of individuals in this neighbourhood meets at times of a Poisson process of rate 1 (independent of other meetings). Meetings between an infected individual and an uninfected individual result in the uninfected individual being infected with probability $p \in (0, 1)$. Infected individuals remain infected for an Exponential amount of time with mean $\mu$, independent of everything else. Noone interacts with anyone from outside the town. Let $N_t$ denote the number of infected individuals at time $t$ (so $N_0 = 1$). For the CTMC $(N_t)_{t\geq 0}$:



(a) Draw the transition diagram.

(b) Find the generator matrix.

(c) Find all stationary distributions.

(d) Find the probability that at some time everyone in the neighbourhood is (simultaneously) infected if $p = \mu = 1$.

(e) Find the expected time until noone in the neighbourhood is infected by the virus when $\mu = p = 1$.

(f) How would your answers to (d) and (e) above change as $\mu \nearrow \infty$ or $\mu \searrow 0$?

(g) If $\mu = p = 1$, what happens to the answers to (d) and (e) as $k$ (the number of individuals in the neighbourhood) goes to infinity? (Note that for (d), the state $k$ that you are trying to reach is growing as $k \to \infty$, but so is e.g. $p_{1,2}$)

**Question 3 (8 marks)**

At a small airport, arriving passengers arrive as a Poisson process of rate 1 per minute and each such passenger is assigned uniformly at random (independent of the length of the queues) to one of $k$ servers (that each serve the customers in their queue in the order in which they arrived). Service times at each queue are Exponential random variables with mean 2.5 minutes (independent of everything else). Let $N_t^{(i)}$ denote the number of customers in queue $i = 1, \ldots, k$ at time $t$, and $N_t = \sum_{i=1}^{k} N_t^{(i)}$.

(a) Is this system an $M/M/k$ queue? Why or why not?

(b) How large does $k$ have to be to ensure that the queuing system is stable (i.e. that no matter how many customers are currently in the system, with probability 1 it will eventually become empty)?

(c) Assuming that $k$ is sufficiently large so that the system is stable, find the limiting distribution for $N_t$ as $t \to \infty$, e.g. find $\lim_{t\to\infty} \mathbb{P}(N_t = n)$ for each $n = 0, 1, 2, \ldots$.



**End of Assignment — Total Available Marks = 34**