MAST30001 Stochastic Modelling

Tutorial Sheet 3

1. Let $Y_1, Y_2, \dots$ be i.i.d. random variables with probability mass function given by the following table.

k	0	1	2	3
$\mathbb{P}(Y = k)$	0.2	0.2	0.3	0.3

Set $X_0 = 0$ and let $X_n = \max\{Y_1, \dots, Y_n\}$ be the largest Y_i observed to time n.

(a) Explain why (X_n) is a Markov chain and determine its transition matrix P.
We can compute the conditional probabilities given the entire history of the chain as

$$\mathbb{P}(X_{n+1} = k | X_n = j, X_{n-1} = x_{n-1}, \dots, X_1 = x_1, X_0 = 0) = \begin{cases} \mathbb{P}(Y_{n+1} = k), & k > j, \\ \mathbb{P}(Y_{n+1} \le j), & k = j, \end{cases}$$

which only depend on j (and not the x_i's) so the chain is Markov and with transitions given by the formula above:

$$P = \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.3 \\ 0 & 0.4 & 0.3 & 0.3 \\ 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Find all communicating classes for this chain and all absorbing states.
Each state is its own communicating class, and 3 is an absorbing state.

(c) Describe the long run behavior of the chain. In particular can you determine the matrix $\lim_{n\to\infty} P^n$?
The chain starts at 0 *and increases, eventually landing at the absorbing state* 3*. Since P^n is the n step transition matrix we expect*

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(d) Find the expected time to reach state 3, starting from state 0.
The time it takes to get to see a 3 in the sequence is Geometric$(.3)$*, so the expected time is* $10/3$

2. A game involves rolling a ball around a table, starting from a fixed position. Three players, Alex, Bobby and Célia, take turns attempting to grab the ball as it passes them. If Alex is unsuccessful then Bobby attempts to grab the ball and if he is also unsuccessful then Célia attempts to grab the ball. If they are all unsuccessful, the ball is rolled again from the starting position. The game stops as soon as any player is successful (and that player wins!). Suppose that on each attempt, Alex, Bobby and Célia have probabilities p_1, p_2 and p_3 (all $\in (0,1)$, with $p_1 > 0$) respectively, of grabbing the ball, independent of previous attempts.

(a) Using Markov chain theory, find the probability that Alex wins the game.
Consider a 6-state Markov chain, with states $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, with state $i \in \{1, 2, 3\}$ representing which player's turn it is and $i+3$ representing the state where player i wins the game. Then we are looking for the hitting probability $h_{1,4}$. The equations are

$$\begin{aligned} h_{1,4} &= p_1 + (1 - p_1)h_{2,4} \\ h_{2,4} &= (1 - p_2)h_{3,4} \\ h_{3,4} &= (1 - p_3)h_{1,4}. \end{aligned}$$

Therefore, letting $a = (1 - p_1)(1 - p_2)(1 - p_3)$ we see that

$$h_{1,4} = p_1 + ah_{1,4},$$

and solving gives $h_{1,4} = \frac{p_1}{1-a}$.
Note that, like most problems, there is more than one way to solve this problem. E.g. consider a different Markov chain with 4 states, which could roughly be described as 1="Alex wins", 2="Bobby wins", 3="Celia wins" and 0="noone wins" (representing what happens on each round). We start in state 0. Each round Alex wins with probability p_1 (transition $0 \to 1$), Bobby wins with probability $(1-p_1)p_2$ (transition $0 \to 2$), Celia wins with probability $(1-p_1)(1-p_2)p_3$ (transition $0 \to 3$) and noone wins with probability $(1-p_1)(1-p_2)(1-p_3)$ (transition $0 \to 0$). Thus, $p_{0,0} = (1 - p_1)(1 - p_2)(1 - p_3)$, $p_{0,1} = p_1$ etc.

(b) What condition(s) do p_1, p_2 and p_3 have to satisfy so that each player is equally likely to win the game? (Include the largest possible values of p_1,p_2, and p_3 in your answer.)
Proceeding as in the answer to the first part of this question (or using that answer and a rotational symmetry), the probabilities of Alex, Bobby and Célia winning are $\frac{p_1}{1-a}$, $\frac{(1-p_1)p_2}{1-a}$ and $\frac{(1-p_1)(1-p_2)p_3}{1-a}$ respectively.
These probabilities are equal if and only if $p_1 = (1-p_1)p_2$ and $p_2 = (1-p_2)p_3$, i.e. $p_2 = \frac{p_1}{1-p_1}$ and $p_3 = \frac{p_2}{1-p_2} = \frac{p_1}{1-2p_1}$. In particular this last equation requires (for p_3 to be a probability) that $\frac{p_1}{1-2p_1} \leq 1$ which can be rearranged to get $p_1 \leq \frac{1}{3}$. [This last inequality should be reduced to $p_1 < 1/3$ if we want to keep $p_3 < 1$ as stated in the preamble of the question].

3. Two dogs named Fluffy and Duffy have m fleas distributed between them. At discrete time steps a flea (chosen uniformly at random) jumps to the other dog. Let X_n be the number of fleas on Fluffy after the nth time step.

(a) Is X_n a Markov chain? What are the transition probabilities?
Yes, the one step transitions only depend on the current state. The transition probabilities are for $i = 0, \ldots, m$

$$p_{i,i+1} = 1 - \frac{i}{m} \qquad p_{i,i-1} = \frac{i}{m},$$

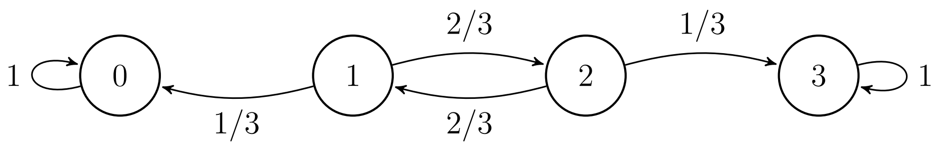
and zero otherwise.

(b) If X_0 has a binomial distribution with parameters $(m, 1/2)$ (meaning that each flea tosses a fair coin to determine on which dog it starts), what is the distribution of X_{10}? *[Hint: First compute the distribution of X_1.]*

A calculation (do it!) shows that if $\boldsymbol{x}$ *is the vector of binomial probabilities then* $\boldsymbol{x}P = \boldsymbol{x}$*. So inductively,* $\boldsymbol{x}P^n = \boldsymbol{x}$ *for all* $n = 0, 1, 2, \dots$ *so the distribution of* X_n *is the binomial distribution above.*

(c) If $m = 3$, and $\mathbb{P}(X_0 = 1) = 1$, find the probability that Fluffy is free of fleas before Duffy.

This is the hitting probability $h_{1,0}$ *for the chain with the following transition diagram*



Note that by symmetry, $h_{2,3} = h_{1,0}$*, so* $h_{2,0} = 1 - h_{2,3} = 1 - h_{1,0}$*. Also,* $h_{1,0} = \frac{1}{3} + \frac{2}{3} \cdot h_{2,0}$*. Solving gives* $h_{1,0} = 3/5$*.*

4. Let $(X_n)_{n \geq 1}$ be a Markov chain with state space $\{1, \dots, k\}$ for some $k \geq 2$. Show that if i and j communicate, then the probability that the chain started in state i reaches state j in fewer than k steps is greater than 0.

Since i *and* j *communicate,* $p_{i,j}^{(n)}$ *is positive for some* n*. Let* n_0 *be the smallest value of* n *for which* $p_{i,j}^{(n)} > 0$*. It is sufficient to prove that* $n_0 < k$*.*

Since $p_{i,j}^{(n_0)} > 0$ *there exists a path* $i = i_0 \to i_1 \to \cdots \to i_{n_0-1} \to i_{n_0} = j$ *such that* $p_{i_0,i_1} p_{i_1,i_2} \cdots p_{i_{n_0-1},i_{n_0}} > 0$*. Suppose that* $n_0 \geq k$*. Since the above path has* $n_0 + 1 > k$ *states appearing in it, and there are only* k *distinct states in* $\mathcal{S}$*, there must be a repeated state in the sequence. That means the path has a loop in it, and by removing this loop we have a shorter path whose product of transition probabilities is also positive (it is greater than the large product above). This contradicts the definition of* n_0*.*