MAST30001 Stochastic Modelling

Tutorial Sheet 5

1. Fix $p \in (0,1)$. Consider the "Gambler's ruin" chain on $\mathcal{S} = \{0,1,\dots,k\}$ with $p_{i,i+1} = p$ and $p_{i,i-1} = 1-p$ for $1 \le i \le k-1$. Let $m_i := m_{i,\{0,k\}}$ denote the expected hitting time of $\{0,k\}$ starting from state i.

   (a) Show that if $p = 1/2$ then $m_i = i(k-i)$.
   Let $x_i = i(k-i)$. We show that these x_i do solve the equations (when $p = 1/2$). Firstly $x_0 = x_k = 0$ is trivial. Next we want (for $i \neq 0, k$)

   $$x_i = 1 + p x_{i+1} + (1-p) x_{i-1}.$$

   Substituting in the values for x_i (and using $p = 1/2$) the right hand side is

   $$1 + \frac{1}{2}(i+1)(k-(i+1)) + \frac{1}{2}(i-1)(k-(i-1)).$$

   Simplifying, we get $i(k-i)$ which is equal to x_i as required.

   (b) Show that if $p \neq 1/2$ then

   $$m_i = \frac{i}{1-2p} - \frac{k}{1-2p}\left[\frac{\alpha^i - 1}{\alpha^k - 1}\right],$$

   where $\alpha = (1-p)/p$.

   Again we show that $x_i = \frac{i}{1-2p} - \frac{k}{1-2p}\left[\frac{\alpha^i-1}{\alpha^k-1}\right]$ solve the equations. Clearly $x_0 = x_k = 0$. For $i \neq 0, k$ substituting in the values for x_i (and using $p \neq 1/2$) the right hand side of the equation

   $$x_i = 1 + p x_{i+1} + (1-p) x_{i-1}$$

   becomes

   $$1 + p\left(\frac{i+1}{1-2p} - \frac{k}{1-2p}\left[\frac{\alpha^{i+1}-1}{\alpha^k-1}\right]\right) + (1-p)\left(\frac{i-1}{1-2p} - \frac{k}{1-2p}\left[\frac{\alpha^{i-1}-1}{\alpha^k-1}\right]\right).$$

   Using the fact that $p\alpha^{i+1} = (1-p)\alpha^i$ and $(1-p)\alpha^{i-1} = p\alpha^i$, we can simplify this to get

   $$\frac{i}{1-2p} - \frac{k}{1-2p}\left[\frac{(1-p)\alpha^i - p + p\alpha^i - (1-p)}{\alpha^k - 1}\right],$$

   which is equal to

   $$\frac{i}{1-2p} - \frac{k}{1-2p}\left[\frac{\alpha^i - 1}{\alpha^k - 1}\right],$$

   as required.

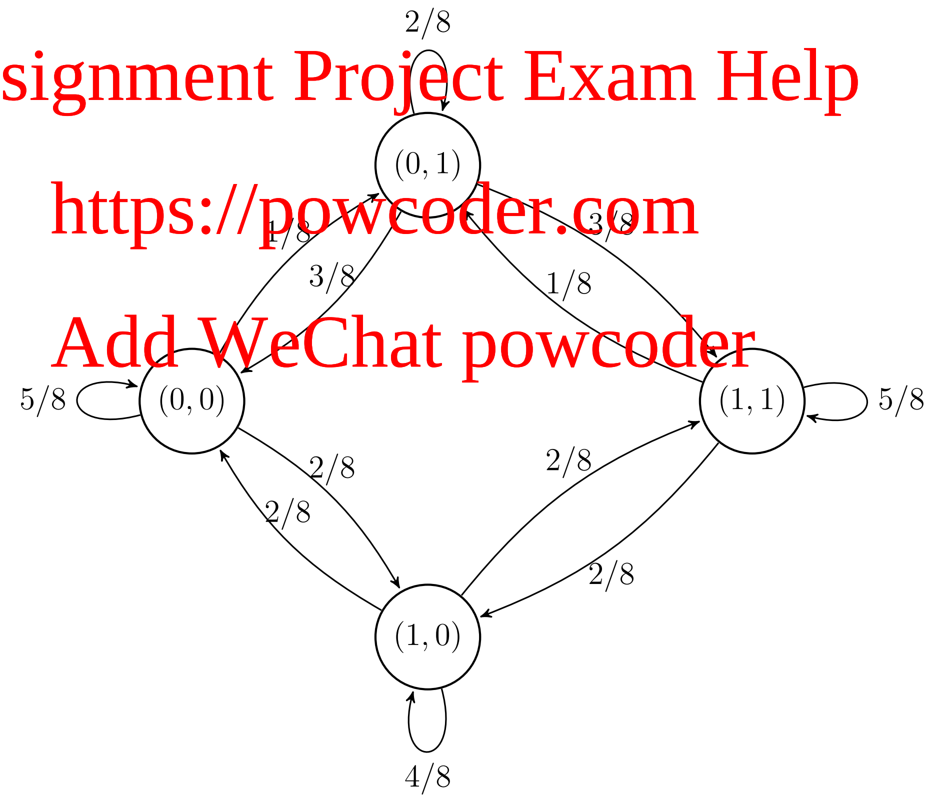
2. In a tiny politically divided village there are 12 villagers living in a 3×4 grid as shown below. On each border, 5 always support party 1, and 5 always support party 0 as shown below.

1	1	1	1
1	?	?	0
0	0	0	0

The opinion of each of the two interior villagers is influenced by their neighbours and fluctuates over time as follows: at each discrete time $n \in \mathbb{N}$, one of the two interior voters (chosen uniformly at random) adopts the political preference of one of its 4 immediate neighbours (also chosen uniformly at random). Let L_n and R_n be the opinions carried by the left and right interior villagers (as in the picture) respectively.

(a) What is the state-space of the Markov chain $X_n = (L_n, R_n)$?
$\mathcal{S} = \{0,1\}^2 = \{(0,0),(1,0),(0,1),(1,1)\}$.

(b) Draw the transition diagram for the chain X_n.



(c) Is the chain X_n:

i. Irreducible?
Yes, since all states bi-communicate

ii. Aperiodic?
Yes, since there are self-loops

iii. Reversible?
Yes. The probability of traversing any loop is the same in the clockwise and anticlockwise directions (there is essentially only one that you need to

check, that is going a full cycle from state i to state i, which has probability $\frac{2}{8}\frac{3}{8}\frac{2}{8}\frac{2}{8}$ whether you go in the clockwise or anticlockwise directions).
Alternatively we can simply solve the detailed balance equations:

$$\frac{1}{8}\pi_{(1,1)} = \frac{3}{8}\pi_{(0,1)}, \quad \frac{2}{8}\pi_{(1,1)} = \frac{2}{8}\pi_{(1,0)}, \quad \frac{2}{8}\pi_{(0,0)} = \frac{2}{8}\pi_{(1,0)}, \frac{1}{8}\pi_{(0,0)} = \frac{3}{8}\pi_{(0,1)}.$$

This gives $\pi_{(1,1)} = \pi_{(1,0)} = \pi_{(0,0)}$ and $\pi_{(1,1)} = 3\pi_{(0,1)}$. Setting the sum equal to 1 gives $\pi_{(1,1)} = \pi_{(1,0)} = \pi_{(0,0)} = \frac{3}{10}$ and $\pi_{(0,1)} = \frac{1}{10}$. So the equations are solvable, hence the process is reversible.

(d) What is the limiting proportion of time that the left interior villager supports party 1?
This is given by $\pi_{(1,0)} + \pi_{(1,1)}$ where $\vec{\pi}$ is the (unique) stationary distribution. Thus, from our previous answer above,

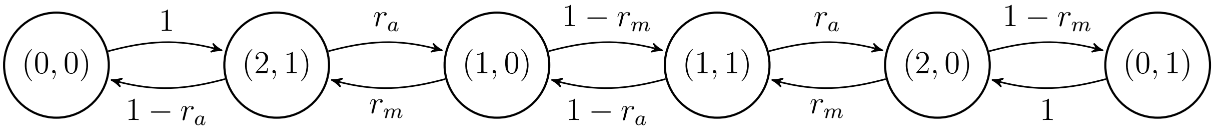
$$\pi_{(1,0)} + \pi_{(1,1)} = \frac{6}{10}.$$

(e) Suppose that at time $n = 1000000$ a vote is taken (and the whole village votes). Estimate the probability that party 1 wins the majority of votes, explaining your reasoning.

This is asking for the probability that the state at time 1000000 is $(1,1)$. This is approximately equal to the limiting probability of being in state $(1,1)$ which in this case (irreducible, aperiodic finite-state Markov chain) is the same as the equilibrium distribution. Therefore an approximate answer is $\pi_{(1,1)} = \frac{3}{10}$.

3. Each morning at 8am, it is raining with probability $r_m \in (0,1)$, and each afternoon at 4pm it is raining with probability $r_a \in (0,1)$ (both are independent of all previous weather conditions). Suppose that your MAST30001 lecturer has 2 umbrellas, and that he departs home for work at 8am each day and departs from work to home at 4pm each day (with a very short commute). Whenever it is raining at departure time s/he takes an umbrella on the trip if there was one available at the departure point. Find the long run proportion of trips for which it is raining on his departure but he has no umbrella available.

Let N_n denote the number of umbrellas at the lecturer's current location, and L_n denote the current location ($L_n = 0$ means at home, and $L_n = 1$ means at work). Then (N_n, L_n) is a discrete-time (time homogeneous) Markov chain with transition diagram:



This is a birth and death chain, so it is reversible. Therefore the (unique) stationary *distribution satisfies the detailed balance equations:*

$$\begin{aligned}
\pi_{(0,0)} &= (1 - r_a)\pi_{(2,1)} \\
\pi_{(2,1)} &= \frac{r_m}{r_a}\pi_{(1,0)} \\
\pi_{(1,0)} &= \frac{1 - r_a}{1 - r_m}\pi_{(1,1)} \\
\pi_{(1,1)} &= \frac{r_m}{r_a}\pi_{(2,0)} \\
\pi_{(2,0)} &= \frac{1}{1 - r_m}\pi_{(0,1)}.
\end{aligned}$$

Let $a_1 = 1/(1 - r_m)$, $a_2 = r_m/r_a$, $a_3 = (1 - r_a)/(1 - r_m)$, $a_4 = a_2$, $a_5 = 1 - r_a$.

Then $\pi_{(0,1)}[1 + \sum_{i=1}^{5} \prod_{j=1}^{i} a_j] = 1$, *so* $\pi_{(0,1)} = 1/[1 + \sum_{i=1}^{5} \prod_{j=1}^{i} a_j]$.

We are looking for $\pi_{(0,0)}r_m + \pi_{(0,1)}r_a$, *since these are the long run proportion of time that we are at home with no umbrella and it rains on departure, and we are at work with no umbrella and it rains on departure respectively.*

Since $\pi_{(0,0)} = \prod_{j=1}^{5} a_j \pi_{(0,1)}$ *we get that*

$$\pi_{(0,0)}r_m + \pi_{(0,1)}r_a = \frac{1}{1 + \sum_{i=0}^{5} \prod_{j=0}^{i} a_j}\left[r_a + r_m \prod_{j=1}^{5} a_j\right].$$