THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2021

School of Mathematics and Statistics

MAST30001 Stochastic Modelling

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 16 pages (including this page) with 9 questions and 80 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).
- No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.

- You must not write your answers on an iPad or other electronic device.
- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.
- If you are writing answers on the exam or masked exam and need more space, use blank paper. Note this in the answer box, so the marker knows.
- If you are only writing on blank A4 paper, the first page must contain only your student number, subject code and subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning and Submitting

- **You must not leave Zoom supervision to scan your exam.** Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4.
- Submit your scanned exam as a single PDF file and carefully review the submission in Gradescope. Scan again and resubmit if necessary. Do not leave Zoom supervision until you have confirmed orally with the supervisor that you have received the Gradescope confirmation email.
- **You must not submit or resubmit after having left Zoom supervision.**

Question 1 (2 marks)

The Laplace transform of a non-negative random variable W is defined as

$$\mathcal{L}_W(s) = \mathbb{E}[e^{-sW}], \qquad s \geq 0.$$

Show that if $W \sim \text{Exp}(\lambda)$ with $\lambda > 0$ then $\mathcal{L}_W(s) = \lambda/(\lambda + s)$.

Question 2 (5 marks)

Let p_n be a sequence of numbers in $(0, 1)$ that are decreasing to 0, and let $G_n \sim$Geometric(p_n).

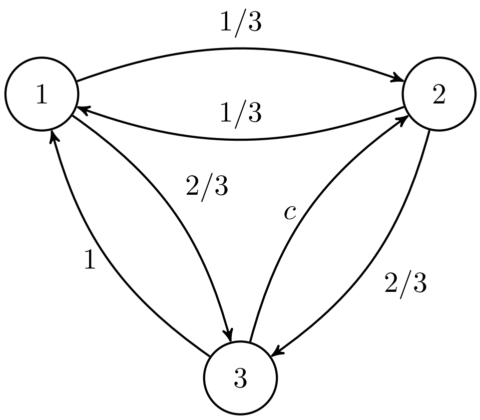
(a) Determine $\mathbb{P}(p_n G_n > x)$ for each $x < 0$.

(b) Determine $\mathbb{P}(p_n G_n > x)$ for each $x \geq 0$.

(c) Show that $p_n G_n$ converges in distribution (as $n \to \infty$) by taking the limit as $n \to \infty$ in your answers above, and specify the limiting distribution.

Question 3 (11 marks)

A simple DTMC $(X_n)_{n\in\mathbb{Z}_+}$ with state space $S = \{1, 2, 3\}$ has transition diagram:



(a) Determine the value of c and find the one-step transition matrix for the chain.

(b) Suppose that $\mathbb{P}(X_0 = 1) = 1$. Find $\mathbb{P}(X_4 = 1)$.

(c) Determine the communicating classes of the chain.

(d) Is this chain periodic or aperiodic? If periodic, specify the period.

(e) Suppose that $\mathbb{P}(X_0 = 1) = 1$, and let $T \geq 0$ be the time of the last visit to state 1 before the first visit to state 3. Is T a stopping time for this chain? Why or why not?

(f) Find the limiting distribution for this chain if $\mathbb{P}(X_0 = 1) = 1$.

(g) Find the limiting distribution of this chain if the initial distribution is uniform on S.

Question 4 (10 marks)

Let $(X_n)_{n\in\mathbb{Z}_+}$ be a DTMC with state space $S = \mathbb{Z}$ and transition probabilities $p_{i,i+1} = b$, $p_{i,i+2} = a$ and $p_{i,i-1} = 1 - (a + b)$, where $0 < a, b$ and $a + b < 1$. Suppose that $\mathbb{P}(X_0 = 0) = 1$ and let $h_{0,j}$ be the probability of ever hitting state j (starting from state 0).

(a) Find $\mathbb{E}[X_1]$.

(b) Explain why $h_{0,-1} < 1$ if $2b + 3a > 1$.

(c) Explain why $h_{0,-1} = 1$ if $2b + 3a < 1$.

(d) Show that $h_{0,-1}$ satisfies a cubic equation that has 1 as a solution.

(e) Find $h_{0,-1}$ when $a = b = 1/4$.

(f) Find $h_{0,-1}$ when $2b + 3a = 1$.

Question 5 (18 marks)

Let $(X_t)_{t\geq 0}$ be a CTMC with state space $\{1,2,3,4,5,6\}$ and corresponding generator

$$\begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 \\ 0 & b & 1/2 & 0 & 0 & 0 \\ 0 & 0 & c & 1/3 & 0 & 0 \\ 0 & 0 & 0 & d & 1/4 & 0 \\ 0 & 0 & 0 & 0 & e & 1/5 \\ 1/6 & 0 & 0 & 0 & 0 & f \end{pmatrix},$$

and suppose that $\mathbb{P}(X_0 = 1) = 1$.

(a) Find the values of a, b, c, d, e, f.

(b) Draw the transition diagram for this chain.

(c) Find all stationary distributions for the *jump chain* associated to this chain, and explain why the jump chain does not have a limiting distribution.

(d) Let $T = \inf\{t > 0 : X_t \neq 1\}$. Find the distribution of T.

(e) Is T a stopping time? Why or why not?

(f) Let $T' = \inf\{t > 0 : X_t = 6\}$. Find the expected value of T'.

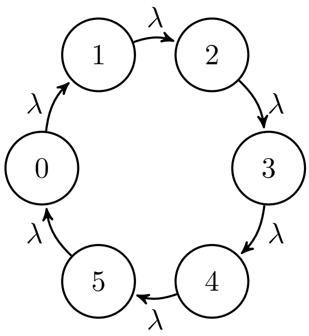
(g) Evaluate the Laplace transform of T'.

(h) Find the long run proportion of time spent in state 1.

(i) Draw the transition diagram for an irreducible, aperiodic DTMC $(Y_n)_{n\in\mathbb{Z}_+}$ with the same state space and stationary distribution as $(X_t)_{t\geq 0}$, and explain why your chosen DTMC has this property.

Question 6 (12 marks)

A CTMC $(X_t)_{t\geq 0}$ with the following transition diagram



starts in state 0, i.e. $\mathbb{P}(X_0 = 0) = 1$. Let $T_i = \inf\{t \geq 0 : X_t = i\}$ for $i \in \{0, 1, \ldots, 5\}$.

(a) Find the generator matrix Q for this chain.

(b) Find $\mathbb{P}(T_2 > 1)$.

(c) Find the cdf of T_2, conditional on the event $\{T_2 < t, T_3 > t\}$.

(d) Is this chain reversible? Why or why not?

(e) Give an infinite series expression for the quantity $P_{0,i}^{(t)}$ for each $i \in \{0, 1, \ldots, 5\}$.

(f) Show that your answers to part (e) satisfy the Kolmogorov forward equations for $P_{0,i}^{(t)}$, $i = 0, 1, \ldots, 5$.

Question 7 (5 marks)

At the Arrivals section of a small airport, passengers arrive as a Poisson process of rate 2 per minute and each such passenger is assigned at random (based on a fair coin toss, and independent of the length of the queues) to one of 2 servers (that each serve the customers in their queue in the order in which they arrived). Service times at each queue are Exponential random variables with mean ϕ, in minutes, and independent of everything else. Let N_t denote the total number of customers in this system at time t.

(a) How small does ϕ have to be to ensure that the queuing system is stable (i.e. that no matter how many customers are currently in the system, the time until it becomes empty has finite expectation)?

(b) Assuming that ϕ is sufficiently small so that the system is stable, find the limiting distribution for N_t as $t \to \infty$, i.e. find $\lim_{t\to\infty} \mathbb{P}(N_t = n)$ for each $n = 0, 1, 2, \ldots$.

Question 8 (7 marks)

In order to attend a large sports event at a stadium nearby, ticket holders park their cars (one behind the other) on a 1km stretch of road that lies between 2 barriers. A meticulous parking attendant makes sure that each car leaves exactly 1m space in front of it. As soon as a car arrives that cannot fit (with 1m space in front of it to the next car), the parking attendant sends that car away, puts up a "no parking" sign and forbids any further parking on the stretch of road. Assume that arriving cars have lengths that are independent and uniformly distributed between 2.5m and 5.5m, and that the demand for car park spaces on this road is high (so the no parking sign will eventually be put up).
Let N_{1000} denote the number of cars that park in this stretch of road for the event. Use your knowledge of renewal theory to answer the following.

(a) Give a point estimate and an approximate 95% confidence interval for N_{1000}.

(b) Estimate the cdf of the amount of space left behind the last parked car when the no parking sign is put up (you may leave your answer in integral form, but any integrand(s) should be explicit).

(c) Suppose that the expectation of the amount of space left behind the last parked car is z (metres). What is the (approximate) expected length of the first car that is sent away in terms of z?

Question 9 (10 marks)

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion.

(a) Find $\mathbb{E}[(W_3 - W_2)^2 W_2^2]$.

(b) Let $T_1 = \inf\{t \geq 0 : W_t = 1\}$ and $T_{\pm 1} = \inf\{t \geq 0 : |W_t| = 1\}$. Find $\mathbb{E}[W_{T_1}]$ and $\mathbb{E}[W_{T_{\pm 1}}]$.

(c) Fix $a > 0$, and for $t \geq 0$ let $Y_t = a^{-1/2} W_{at}$. Show that $(Y_t)_{t \geq 0}$ is a standard Brownian motion.

(d) For $t \in [0,1]$ let $Z_t = W_t - tW_1$. Show that $(Z_t)_{t \in [0,1]}$ is a standard Brownian bridge.

End of Exam — Total Available Marks = 80