MAST30001 Stochastic Modelling – 2022

Assignment 2

Submission Instructions:

- Typed submissions (ideally using LaTeX) are preferred. For handwritten solutions:
  - Write your answers on blank paper. Write on one side of the paper only. Start each question on a new page. Write the question number at the top of each page.
  - Scan your solutions to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Upload the PDF file to Assignment 2 in Gradescope via the LMS. Gradescope will ask you to identify on which of the uploaded pages your answers to each question are located.
- The submission deadline is **5:00pm on Thursday, 20 October**.

There are 2 questions, and both will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. **[16 marks]** At the lunch rush of a hawker-style food hall, customers walk by a certain stall with one server according to a Poisson process with rate 20 per minute. If there is either only one person being served or no one being served, then a customer goes to the stall with probability $p/2$ for some $p \in (0, 1]$. Once there are $k \geq 1$ people waiting to be served (so $(k + 1)$ in the system), then arriving customers join the queue with probability $pk/(k + 1)$. (Everyone knows a long queue signifies a good stall.) The time from when a customer orders to when they receive their food is exponentially distributed with mean 90 seconds.

   For $t \geq 0$, let X_t be the number of customers in the system (that is, those waiting for service or in service) t minutes into the lunch rush.

   (a) Model $(X_t)_{t\geq0}$ as a continuous time Markov chain and specify its generator.

   (b) Find the values of p where $(X_t)_{t\geq0}$ has a stationary distribution $\pi = (\pi_n)_{n\geq0}$, and for those values of p, write down a simple formula (i.e., no infinite sums) for π_n.

   For the remainder of the problem, assume that $p = 1/60$.

   (c) What is the expected number of customers in the system in stationary?

   (d) What proportion of customers walking by the stall decide to eat there?

   (e) Given a customer decides to eat at the stall, what is their expected waiting time until they are served?

2. **[14 marks]** A counting process $(N_t)_{t\geq0}$ is an *inhomogeneous* Poisson process with intensity function $\lambda : [0, \infty) \to (0, \infty)$ satisfying $\int_0^t \lambda(s)ds < \infty$ for all $t > 0$ and $\int_0^\infty \lambda(s)ds = \infty$, if it has

(i) independent increments, and

(ii) $N_t \sim \text{Poisson}\big(\int_0^t \lambda(s)ds\big)$ for each $t \geq 0$.

For $k = 0, 1, \ldots$, define $T_k := \inf\{t > 0 : N_t = k\}$, the time of the kth arrival.

(a) Derive the distribution of $N_t - N_s$ for $0 \leq s < t$.

(b) For each $k \geq 1$, derive the density of T_k.

(c) For each $k \geq 2$, derive the density of T_k given $(T_1, \ldots, T_{k-1}) = (t_1, \ldots, t_{k-1})$.

(d) For each $k \geq 2$, derive the joint density of $(T_1, \ldots, T_k)$.

(e) Given $T_{k+1} = k$, the distribution of $(T_1, \ldots, T_k)$ are the same as k i.i.d. order statistics for some distribution. Find the density of that distribution.

(f) Let $(L_t)_{t\geq 0}$ be a homogeneous Poisson process with rate one, and for $k = 0, 1, \ldots$, define $S_k := \inf\{t > 0 : L_t = k\}$. Assume that F is a strictly increasing function with $F(0) = 0$, $\lim_{x\to\infty} F(x) = \infty$, and derivative $F'(x) = f(x)$. Define a new counting process $(M_t)_{t\geq 0}$ by

$$M_t := \sup\{k \geq 0 : F^{-1}(S_k) \leq t\}.$$

That is, the "arrivals" of $(M_t)_{t\geq 0}$ occur at the times $F^{-1}(S_k)$, $k \geq 0$. Show that $(M_t)_{t\geq 0}$ is an inhomogeneous Poisson process, and find its intensity.

[For parts (a)-(e), you should check that your answers reduce to lecture results in the homogeneous case.]