MAST30001 Stochastic Modelling – 2022

Assignment 1

Please complete the Plagiarism Declaration Form on the LMS before submitting this assignment.

Submission Instructions:

- Typed submissions (ideally using LaTeX) are preferred. For handwritten solutions:
  - Write your answers on blank paper. Write on one side of the paper only. Start each question on a new page. Write the question number at the top of each page.
  - Scan your solutions to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned or organised submissions may be impossible to mark or incur a penalty.
- Upload the PDF file to Assignment 1 in Gradescope via the LMS. Gradescope will ask you to identify on which of the uploaded pages your answers to each question are located.
- Upload R code from Question 3(d) as instructed there.

- The submission deadline is **5:00pm on Thursday, 8 September, 2020**.

There are 3 questions, all of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. [**13 marks**] A discrete time Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ has the following transition matrix.

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 11/12 & 0 & 1/12 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Write down the communication classes of the chain.

(b) Find the period of each communicating class.

(c) Determine which classes are essential.

(d) Classify each essential communicating class as transient or positive recurrent or null recurrent.

(e) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

(f) Given $X_0 = 4$, find the long run proportion of time the chain spends in state j for each $j \in S$.

(g) Find the expected number of steps taken for the chain to first reach state 4, given the chain starts at state 3.

2. **[10 marks]** For fixed $0 < \lambda \le 1$, an irreducible Markov chain on $\{0, 1, \ldots\}$ has transition probabilities

$$p_{i,i+1} = 1 - p_{i,i-1} = \frac{i\lambda + 1}{i(\lambda+1)+2}, \quad \text{for } i \ge 1,$$
$$p_{0,1} = 1 - p_{0,0} = 1/2.$$

   (a) Find the values of λ where the chain is positive recurrent, null recurrent, and transient.

   (b) Describe the long run behaviour of the chain.

3. **[7 marks]** A probability distribution supported on $S = \{-1,1\}^N$ with parameters $\beta \ge 0$ and $h \in \mathbb{R}$ is given by

$$\pi_x \propto \exp\left\{\beta\left(\sum_{j=1}^{N-1} x_j x_{j+1} + x_N x_1 + h\sum_{j=1}^{N} x_j\right)\right\} \quad \text{for } x = (x_1, \ldots, x_N) \in S.$$

Fix $N = 500$, let $X = (X_1, \ldots, X_{500})$ be distributed according to π, and set $\mu_{\beta,h} := \mathbb{E}[\frac{1}{500}\sum_{j=1}^{500} X_j]$, the mean of the magnetisation.

   (a) For $x = (x_1, \ldots, x_N) \in S$ and $u \in \{\pm 1\}$, denote $x^{(i,u)} := (x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_N)$. For $u \in \{\pm 1\}$, write a simple formula for the Gibbs sampler transition probability that the chain moves to $x^{(i,u)}$ at the next step, given it is currently in x and coordinate i is chosen to be resampled.

   (b) Use Gibbs sampling to approximately sample from π with $h = 0$ and a variety of $\beta > 0$. Do you find evidence of strongly different behaviour of π for larger versus smaller β, similar to the Curie-Weiss model? *[There is no need to include your code, but describe in a sentence or two what behaviour you saw to support your answer.]*

   (c) Using numerical experimentation, find a function $g(h)$ such that

$$\mu_{\beta,h} = \frac{\sinh(\beta g(h))}{\sqrt{\cosh^2(\beta h) - 2e^{-2\beta}\sinh(2\beta)}}.$$

   (d) Write an `R` script that simulates a vector named `mag` of length 10^6, where the jth entry is the magnetisation of the jth step of the Gibbs sampler for π with $\beta = 1$ and $h = 0.1$, started from i.i.d. coordinates uniformly distributed on $\{\pm 1\}$. Save the script as `assignment1.R` and submit it separately to the coding assignment in Gradescope.

   Warning! To get credit for this part of the assignment, your code must not load (or require) any packages, it must produce a vector of the correct name and length, and the file must be named `assignment1.R` (case sensitive). Once the "Autograder" finishes running after uploading, if you receive a mark of (1.0/1.0) on the "Length" test (see figure below), then this means your file is named correctly, and the script produces a vector of the correct name and length. If not, you can revise and resubmit as many times as necessary before the due date.