The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 3: Multinomials and Inclusion-Exclusion – Solutions

Q1: (a) This is this same as arranging 3 red blocks and 4 white blocks in a line so:

$$\binom{7}{3} \;=\; \frac{7!}{3!\,4!} \;=\; 7\cdot 5 \;=\; 35.$$

(b) There are $\binom{n}{k}$ ways to choose k t's from the factors in $(1-t)^n$. We are essentially asking for the number of ways to chooses a subset of size k from a set of size n. Thus

$$(1-t)^n \;=\; \sum_{k=0}^{n} \begin{pmatrix} \#\text{ subsets of size } k \\ \text{from a set of size } n \end{pmatrix} t^k \;=\; \sum_{k=0}^{n} \binom{n}{k} t^k.$$

(c) This is the same as arranging 4 red blocks, 3 green block, 1 blue block and 2 yellow blocks in a line so the answer is

$$\binom{10}{4,3,1,2} \;=\; \frac{10!}{4!\,3!\,1!\,2!} \;=\; 12600.$$

(d) Consider the set $N = \{1, 2, \ldots, n\}$. Associate with each element the symbol s if it belongs to subset N_1, the symbol t if it belongs to subset N_2 and the symbol 1 if it belongs to subset N_3. Let $|N_i| = k_i$ with $k_1 + k_2 + k_3 = n$ (each element must belong to one of the subsets). Then the coefficient of $s^{k_1} t^{k_2} 1^{k_3}$ in the expansion on the RHS is the number of ways to partition the set of size n into a subset of size k_1, a subset of size k_2 and a subset of size $k_3 = n - k_1 - k_2$. This is counted by the multinomial coefficient $\binom{n}{k_1, k_2, n-k_1-k_2}$ so

$$(1+s+t)^n \;=\; \sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2\leq n}} \binom{n}{k_1, k_2, n-k_1-k_2} s^{k_1} t^{k_2}.$$

Q2: (a) Make a bijection such that an element x_j from the set is associated with box j and place a block into box j each time the element x_j is selected.

(b) Here $n = 8$ and $r = 4$ so # ways is

$$\binom{8+4-1}{4} \;=\; \frac{11!}{4!\,7!} \;=\; \frac{11\times 10\times 9\times 8}{4!} \;=\; 11\times 10\times 3 \;=\; 330.$$

Q3: Let $L = \{\text{Latin readers}\}$, $G = \{\text{Greek readers}\}$ and $R = \{\text{Russian readers}\}$.

(a) Using the inclusion-exclusion principle we have

$$\begin{aligned}
\#\text{People} &= |L\cup G\cup R| \\
&= |L| + |G| + |R| - (|L\cap G| + |L\cap R| + |G\cap R|) + |L\cap G\cap R| \\
&= 6+6+7-(4+2-3)+1 \;=\; 11.
\end{aligned}$$

(b) Let $G_L = \{\text{Greek and Latin readers}\}$ and $R_L = \{\text{Russian and Latin readers}\}$. Then

$$\begin{aligned}
\#\text{Latin only readers} &= |L| - |G_L \cup R_L| \;=\; |L| - (|G_L| + |R_L| - |G_L \cap R_L| \\
&= |L| - (|G \cap L| + |R \cap L| - |L \cap G \cap R|) \\
&= 6 - (4 + 2 - 1) \;=\; 1.
\end{aligned}$$

(c) Similarly we get

$$\#\text{Russian only readers} \;=\; |R| - (|R \cap L| + |R \cap G| - |L \cap G \cap R|) \;=\; 7 - (3 + 2 - 1) \;=\; 3.$$

Q4: Think of the 3 pairs of socks as having different colours, say red, green and yellow.

(a) $\displaystyle\binom{6}{2,2,2} \;=\; \frac{6!}{2!2!2!} \;=\; 90.$

(b) We can regard a pair of socks always in succession as a single entity. Thus

(i) $\displaystyle\#\text{ arrangement with one pair always in succession} \;=\; \binom{5}{2,2,1} \;=\; \frac{5!}{2!2!1!} \;=\; 30.$

Note the remaining two pairs may or may not be in succession.

(ii) $\displaystyle\#\text{ arrangement with two pairs always in succession} \;=\; \binom{4}{2,1,1} \;=\; \frac{4!}{2!1!1!} \;=\; 12.$

(iii) $\displaystyle\#\text{ arrangement with three pairs always in succession} \;=\; \binom{3}{1,1,1} \;=\; \frac{3!}{1!1!1!} \;=\; 6.$

(c) From the complement and inclusion-exclusion principles we get with

$$\begin{aligned}
X_r &= \{\text{arrangements with a red pair of socks in succession}\} \\
X_g &= \{\text{arrangements with a green pair of socks in succession}\} \\
X_y &= \{\text{arrangements with a yellow pair of socks in succession}\}
\end{aligned}$$

that

$\#$ arrangements with a no pairs of socks in succession

$$\begin{aligned}
&= 90 - |X_r \cup X_g \cup X_y| \\
&= 90 - (|X_r| + |X_g| + |X_y| - |X_r \cap X_g| - |X_r \cap X_y| - |X_g \cap X_y| + |X_r \cap X_g \cap X_y|) \\
&= 90 - (3 \times 30 - 3 \times 12 + 6) \;=\; 30.
\end{aligned}$$

Q5: Here P_2, say, is the set of permutations where 2 stays in position 2, so $P_2 = \{123, 321\}$.

(a) $|P_k| \;=\; 2! \;=\; 2$.

(b) $P_j \cap P_k$ is the set of permutations where both j and k are fixed, so $|P_j \cap P_k| \;=\; 1$.

(c) $P_1 \cap P_2 \cap P_3$ is the set of permutations where all elements are fixed so $|P_1 \cap P_2 \cap P_3| \;=\; 1$.

(d)
$$\begin{aligned}
N(0) &= 3! - |P_1 \cup P_2 \cup P_3| \\
&= 3! - (|P_1| + |P_2| + |P_3| - |P_1 \cap P_2| - |P_1 \cap P_3| - |P_2 \cap P_3| + |P_1 \cap P_2 \cap P_3|) \\
&= 3! - (3 \times 2! - 3 + 1) \;=\; 3!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\Big) \;=\; 2.
\end{aligned}$$

(e) $\displaystyle\Pr\begin{pmatrix}\text{no one collects} \\ \text{correct umbrella}\end{pmatrix} \;=\; \Pr\begin{pmatrix}\text{permutation of } \{1,2,3\} \\ \text{leaves no number fixed}\end{pmatrix} \;=\; \frac{N(0)}{3!} \;=\; \frac{1}{3}.$