The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 9: Permutations – Answers

Q1: (i) $264351 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 3 & 5 & 1 \end{pmatrix} = (126)(34) = (16)(12)(34)$

(ii) $315642 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 6 & 4 & 2 \end{pmatrix} = (135462) = (12)(16)(14)(15)(13)$

Q2: (i) (i) An r-cycle σ can be written as the product of $r-1$ 2-cycles so $\mathrm{sgn}(\sigma) = (-1)^{r-1}$.

(ii) $\sigma = 87654321 = (18)(27)(36)(45), \qquad \tau = 46213875 = (14)(26853)$.

(iii) $\mathrm{sgn}(\sigma) = +1, \qquad \mathrm{sgn}(\tau) = ;= -1$.

(iv) $(156)(247)(38) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 8 & 7 & 6 & 1 & 2 & 3 \end{pmatrix}$

$(1587)(23)(46) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 3 & 2 & 6 & 8 & 4 & 1 & 7 \end{pmatrix}$

Q3: $\sigma = s_5 s_4 s_3 s_1 s_2 s_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 6 & 3 & 5 \end{pmatrix}$

Q4: $(123)(234)(324) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \quad (213)(324)(324) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$.

Q5: (i) $A_n(x) = \sum_{k=0}^{n} S_1(n,k)x^k = (n-1)\sum_{k=0}^{n} S_1(n-1,k)x^k + \sum_{k=0}^{n} S_1(n-1,k-1)x^k$.

Now rewrite each term and prove

$$A_n(x) = (n-1)A_{n-1}(x) + xA_{n-1}(x) = (x+n-1)A_{n-1}(x).$$

(ii)
$$\begin{aligned}
A_0(x) &= S_1(0,0) = 1 \\
A_1(x) &= xA_0(x) = x \\
A_2(x) &= (x+1)A_1(x) = (x+1)x \\
A_3(x) &= (x+2)A_2(x) = (x+2)(x+1)x \\
&\vdots \\
A_n(x) &= (x+n-1)A_{n-1}(x) = (x+n-1)(x+n-2)\cdots(x+2)(x+1)x
\end{aligned}$$

Q6: We have to seat n people at $n-2$ tables. There are two cases:

Case 1: Three people are seated at one table the remaining $n-3$ people must then be seated one each at the remaining $n-3$ tables. There are $2\binom{n}{3}$ seating arrangements for this case.

Case 2: Two people each are seated at two of the tables with the remaining $n-4$ seated one each at the remaining $n-4$ tables. There are $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ seating arrangements for this case.

Putting everything together we have: $S_1(n, n-2) = 2\binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$.