The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 10: Symmetric Group and Applications – Solutions

Q1: (a) For a permutation written in terms of 2-cycles

$$\text{Parity:} \qquad \text{sgn} \;=\; (-1)^{\#\text{(2-cycles)}}$$

On the RHS: $\#\,\text{(2-cycles)} \;=\; 2(j-i-1)+1$, so an odd number. i.e., parity is odd.

On the LHS only one 2-cycle so parity is odd.

(b) Note the action of a transposition is:

- Left action "swaps values".
- Right action "swaps positions".

So the left action on 213564 is

$$\begin{aligned}
(25)\circ 213564 &= 513264\\
s_4s_3s_2s_3s_4\circ 213564 &= s_4s_3s_2s_3\circ 213465\\
&= s_4s_3s_2\circ 214365\\
&= s_4s_3\circ 314265\\
&= s_4\circ 413265\\
&= 513264
\end{aligned}$$

Similarly for the right action

$$\begin{aligned}
213564\circ(25) &= 263514\\
213564\circ s_4s_3s_2s_3s_4 &= 213654\circ s_3s_2s_3s_4\\
&= 216354\circ s_2s_3s_4\\
&= 261354\circ s_3s_4\\
&= 263154\circ s_4\\
&= 263514
\end{aligned}$$

(c) To show that $(k+1\,k\,k+2) = (k\,k+1\,k+2)^{-1}$ it suffices to show

$$(k+1\,k\,k+2)\circ(k\,k+1\,k+2) = I.$$

The action on the LHS is $k \mapsto k+1 \mapsto k$; $k+1 \mapsto k+2 \mapsto k+1$; $k \mapsto k+1 \mapsto k$. Hence all elements are mapped to themselves as required.

Similarly, to show that $(k\,k+1\,k+2)^{-1} = (k\,k+1\,k+2)^2$ it suffices to show

$$= (k\,k+1\,k+2)^3 = I$$

We see that the action of the LHS is $k \mapsto k+1 \mapsto k+2 \mapsto k$. Showing that each element is mapped to itself as required.

(d) $A_3 = \{(123),(123)^2,(123)^3\}$.

Q2: To write these products of elementary transpositions as cycles or two-line arrays we just "follow the elements" under the mapping:

$$s_5 s_1 s_2 s_1 = (56)(12)(23)(12) = (13)(56) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 4 & 6 & 5 & 7 \end{pmatrix}$$

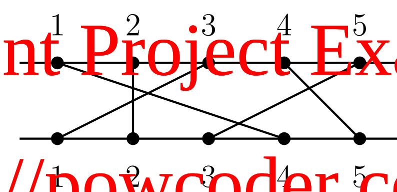
$$s_4 s_5 s_4 s_5 s_1 s_2 s_1 s_4 = (45)(56)(45)(56)(12)(23)(12)(45) = (13)(56) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 4 & 6 & 5 & 7 \end{pmatrix}$$

Use $(s_4 s_5 s_4) s_5 = (s_5 s_4 s_5) s_5 = s_5 s_4$, so

$$s_4 s_5 s_4 s_5 s_1 s_2 s_1 s_4 = s_5 s_4 s_1 s_2 s_1 s_4 = s_5 s_4 s_4 s_1 s_2 s_1 = s_5 s_1 s_2 s_1$$

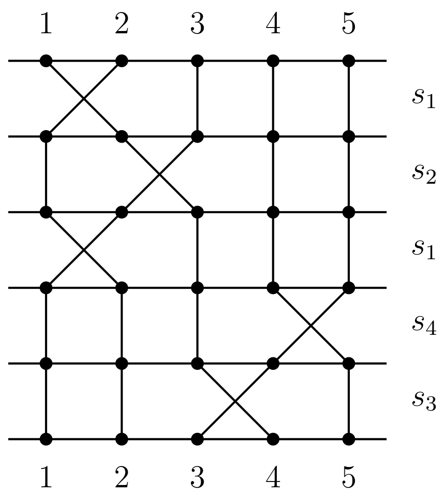
where the second equality follow since s_4 commutes with s_1 and s_2, and the third follows from $s_4^2 = 1$.

Q3: (a) The bipartite graph corresponding to $\sigma = 42153$ is



(b) Each crossing in (a) is an inversion. So the 5 crossings results in 5 inversions.

(c) $\sigma = s_3 s_4 s_1 s_2 s_1$, which has this bipartite graph:



(d) $I_\sigma = \{(1,2), (1,3), (1,5), (2,3), (4,5)\}$.

Q4:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \square \end{matrix} \quad \overset{\text{snake}}{\longmapsto} \quad \sigma = 1234765 = (57) \quad \Rightarrow \quad \text{odd parity.}$$

(a) We need to check the parity of the permutation

$$\begin{matrix} 6 & 5 & 1 & 2 \\ 4 & 7 & 3 & \square \end{matrix} \quad \overset{\text{snake}}{\longmapsto} \quad \sigma' = 6512374 = (1674253) \quad \Rightarrow \quad \text{even parity.}$$

Note parity $= (-1)^{7-1} = +1$. So the sought after ordering can't be achieved.

(b) $\sigma' = 4175236 = (1452)(376) \quad \Rightarrow \quad \text{parity } = (-1)^3(-1)^2 = -1.$

Since the parity is odd the sought after ordering can be achieved.

(c) $\sigma' = 7651234 = (174)(2635) \quad \Rightarrow \quad \text{parity} = (-1)^2(-1)^3 = -1.$

Since the parity is odd the sought after ordering can be achieved.

Q5: Two board positions are related if the snake-pattern permutations have the same parity. The standard snake-pattern permutations of the left and right boards are

$$\begin{aligned} \sigma_L &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 7 & 4 & 3 & 6 & 15 & 12 & 5 & 9 & 1 & 10 & 13 & 8 & 11 & 2 & 14 \end{pmatrix} \\ &= (1\,7\,5\,15\,14\,2\,4\,6\,12\,8\,9)(11\,13) \\ \Rightarrow \operatorname{sgn}(\sigma_L) &= (-1)^{10}(-1) = -1. \end{aligned}$$

$$\begin{aligned} \sigma_R &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 14 & 12 & 9 & 10 & 15 & 13 & 11 & 8 & 6 & 3 & 7 & 2 & 4 & 5 & 1 \end{pmatrix} \\ &= (1\,14\,5\,15)(2\,12)(3\,9\,6\,13\,4\,10)(7\,11) \\ \Rightarrow \operatorname{sgn}(\sigma_R) &= (-1)^3(-1)(-1)^5(-1) = +1. \end{aligned}$$

Permutations have different parity so board positions cannot be related by sliding moves.

Q6: Let us make the identifications

$$\begin{matrix} \mathrm{R}=1 & \mathrm{A}=2 & \mathrm{T}=3 & \mathrm{E}=4 & \mathrm{Y}=5 & \mathrm{O}=6 & \mathrm{U}=7 & \mathrm{R}=8 \\ \mathrm{M}=9 & \mathrm{I}=10 & \mathrm{N}=11 & \mathrm{D}=12 & \mathrm{P}=13 & \mathrm{A}=14 & \mathrm{L}=15 & \square=16 \end{matrix}$$

A key point here is that the 2 and 14 are indistinguishable in this particular version of the 15-puzzle. Consider then the even permutation

$$(2\,14) \circ (14\,15)$$

which first interchanges the two A's, then interchanges the A and L. Being even, it is attainable, and it leaves the last line reading PLA☐, and the other 3 lines reading as before.