**The University of Melbourne — School of Mathematics and Statistics**
**MAST30012 Discrete Mathematics — Semester 2, 2021**

# Practice Class 2: Arrangements and Combinations – Solutions

**Q1**: (a) We have to choose 3 out of 20, where order doesn't matter. The number of choices is

$$\binom{20}{3} = \frac{20!}{3!\,17!} = 1140.$$

(b) We think of the people as standing in a line from left to right. The first person can by paired with any of the 7 remaining people. 6 people remain and the leftmost person can be paired with any of 5 people etc. The number of different pairs is

$$7 \cdot 5 \cdot 3 \cdot 1 = 105$$

(c) Out of 8 people a pair can be chosen in $\binom{8}{2}$ ways and put into box 1. This leaves 6 people from which to choose the second pair in $\binom{6}{2}$ ways to put into box 2. Now there are $\binom{4}{2}$ ways to choose a pair that goes into box 3. The last pair (one choice) is then put into box 4. The number of different pairs is



$$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!} = \frac{8!}{2^4\,4!} = \frac{8!}{8 \cdot 6 \cdot 4 \cdot 2} = 7 \cdot 5 \cdot 3 = 105$$

(d) Total number of possible configurations is $2^{10}$. 5 heads can be chosen in $\binom{10}{5}$ ways so



$$\Pr(5 \text{ heads}) = \frac{\binom{10}{5}}{2^{10}} = \frac{63}{256} = 0.246073\ldots$$



Let H indicate a string of 5 heads. Note that hH and Hh is the same 6 heads in a row. For uniqueness we distinguish two cases: Start with H followed by 5 heads or tails or H is preceded by a tail. i.e., tH, while the remaining 4 tosses are heads or tails. There are $2^5$ configurations in the first case and $5 \times 2^4$ is the second case (the tH can be placed in any of 5 positions in the string of 4 heads/tails). The number of favourable outcomes is $2^5 + 5 \times 2^4 = 112$ and hence

$$\Pr(\text{at least 5 heads in a row}) = \frac{112}{1024} = \frac{7}{64} = 0.109375.$$

Exactly 5 heads in a row is similar to previous case except the positions before and after the 5 heads must be either a tail or empty, e.g., we have

Htxxxx, tHtxxx, xtHtxx, xxtHtx, xxxtHt, xxxxtH

where x indicates a position in which we could have a head or a tail. So the number of favourable outcomes is $16 + 8 + 8 + 8 + 8 + 16 = 64$ and therefore

$$\Pr(\text{exactly 5 heads in a row}) = \frac{1}{16} = 0.0625.$$

**Note:** Precious problem of at least 5 heads in a row can be done by counting exactly $k$ heads in a row $k = 5, 6, \ldots, 10$ for which we get using the argument above $64, 28, 12, 5, 2, 1$ configurations, respectively. They sum to 112.

**Q2**: (a) 3 elements with repetition from $\{a, b\}$ (order does not matter)

$$aaa,\ aab,\ abb,\ bbb \qquad \text{total of} \quad 4 = \binom{2+3-1}{3} = \binom{4}{3}.$$

(b) We have to choose 3 out of 20 with repetition allowed (replacement). The number of choices is

$$\binom{20+3-1}{3} = \binom{22}{3} = \frac{22!}{3!\,19!} = 1540.$$

(c) The number of ways to arrange $n-1$ symbols $I$, and $r$ symbols in a line is

$$\binom{n+r-1}{r}$$

Each choice of $r$ symbols from $n$ (with replacement) can be written as a an arrangement of $n-1$ $I$'s and $r$ $x$'s with the number of $x$'s between the $(j-1)$th $I$ and the $j$th $I$ being the number of times the $j$th elements occurs in the sample. From **Q2**(a)

$$aaa \leftrightarrow xxxI \qquad aab \leftrightarrow xxIx \qquad abb \leftrightarrow xIxx \qquad bbb \leftrightarrow Ixxx$$



(d) Identify the $r$ identical objects as $x$'s and insert the $x$'s between the $(j-1)$th $I$ and the $j$th $I$ to indicate the number of $x$'s which go to person $I$. This is the same counting problem as **Q2**(c) so the number of ways of doing this is



$$\binom{n+r-1}{r}$$



**Q3**: (a) The number of different ways of putting $r_1$ labelled blocks into box $B_1$ is $\binom{n}{r_1}$. This leaves us with $n - r_1$ blocks. We can but $r_2$ of these in box $B_2$ in $\binom{n-r_1}{r_2}$ different ways. Continuing along these lines of reasoning we get (using the multiplication principle) that

$$\binom{n}{r_1, r_2, \cdots, r_p} = \binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-\cdots-r_{p-1}}{r_p}$$

Using the formula $\displaystyle\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ this simplifies to

$$\binom{n}{r_1, r_2, \cdots, r_p} = \frac{n!}{r_1! r_2! \cdots r_p!}$$

(b)

$$\binom{n}{r_1, n-r_1} = \frac{n!}{r_1!(n-r_1)!} = \binom{n}{r_1}.$$

(c) Think of labelled blocks as elements of a set and boxes as subsets of prescribed sizes.

**Q4**: (a) From the recurrence we have

$$\binom{n+r+1}{r} = \binom{n+r}{r} + \binom{n+r}{r-1}$$

But the recurrence also tells us that

$$\binom{n+r}{r-1} = \binom{n+r-1}{r-1} + \binom{n+r-1}{r-2}$$

so that

$$\binom{n+r+1}{r} = \binom{n+r}{r} + \binom{n+r-1}{r-1} + \binom{n+r-1}{r-2}$$

Now apply the recurrence to the term $\displaystyle\binom{n+r-1}{r-2}$ etc. to get the stated formula.

(b) LHS of identity is the number of subsets of size $r$ from a set $\{1, 2, \ldots, n+r+1\}$.

These subsets can be partitioned as follows: subsets which don't contain the element '1', of which there are $\binom{n+r}{r}$; subsets which contain '1' but not '2', of which there are $\binom{n+r-1}{r-1}$; subsets which contain the elements '1' and '2' but not '3' , of which there are $\binom{n+r-2}{r-2}$; etc. etc. Adding all these up (using the addition principle) we get the RHS.



**Q5**: (a) Let the set of $n$ elements be $\{1, 2, \ldots, n\}$. From this set choose a subset of size $r$. Now form an ordering of $r$ '1's and $n-r$ '2's by putting a '1' in position $j$ of the line if $j$ is in the subset and '2' otherwise. For example with $n = 4$, $r = 2$ the subset $\{2, 4\}$ is identified with 2121. Hence the total numbers are the same for both problems giving $\binom{n}{r}$ ways to order $r$ '1's and $n-r$ '2's in a line.



(b) By the stated correspondence (bijection) the two counting problems are the same. Thus the number of orderings of $r_1$ lots of 1's, $r_2$ lots of 2's, $\ldots$ $r_p$ lots of $p$'s in a line is the same as the number of ways to partition a set of $r_1 + r_2 + \cdots + r_p = n$ symbols into subsets of size $r_1, r_2, \ldots, r_p$. From **Q3**(c) this number is the multinomial coefficient

$$\binom{n}{r_1, r_2, \ldots, r_p}$$