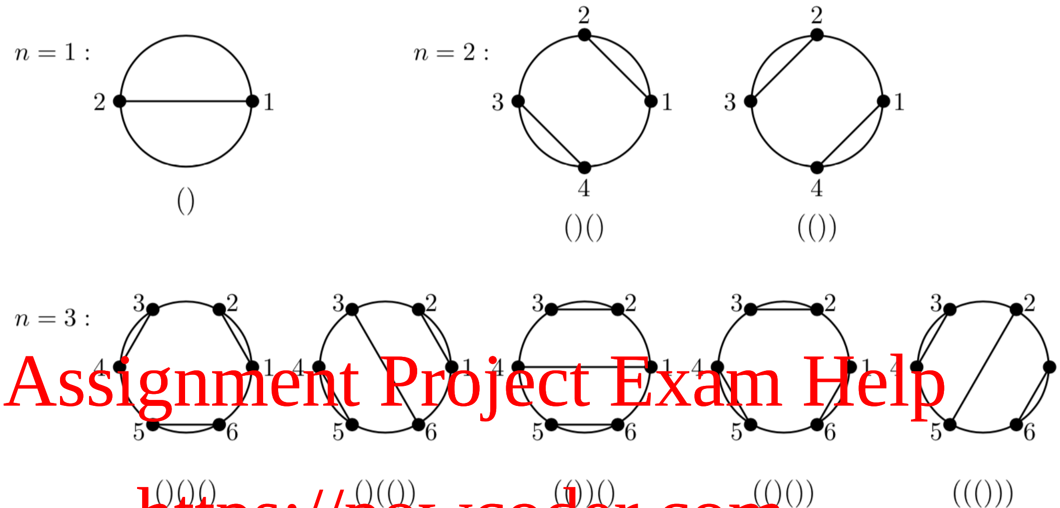
The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 8: Catalan and Functional Equations – Solutions

Q1: Bijections to balanced parentheses or Dyck paths are clear.

Q2: (a) These are the possible diagrams on $2n$ points for $n = 1, 2, 3$:

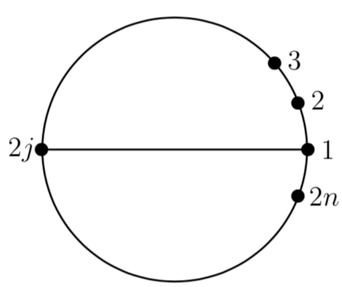


(b) Take points labelled i and $j > i$. A chord between the points creates an arc segment containing $j-i-1$ points (not counting the points labelled i and j) since the labels are consecutive around the circle. Since chords cannot cross these $j-i-1$ points must be mutually connected by chords so $j-i-1$ is even, therefore $j-i$ is odd and hence either i is even and j odd OR i is odd and j is even.

(c) Bijection between chord diagrams on $2n$ points and balanced parenthesis:
Go through the points is order of labelling and record

- An opening parenthesis '(' iff the chord goes to a point of **higher** label.
- A closing parenthesis ')' iff the chord goes to a point of **lower** label.

(d)



Label the points 1 up to $2n$ counter-clockwise.
First draw the chord from the point 1 to the point $2j$. This separates the points on the circle into two disjoint sets, one consisting of the points 2 to $2j-1$ and the second set the points $2j+1$ to $2n$. Let a_k denote the number of ways to join $2k$ points by non-intersecting chords. By the multiplication principle there is $a_{j-1}a_{n-j}$ ways to join the remaining points on the circle.
Summing over j shows that:

$$a_n = a_0a_{n-1} + a_1a_{n-2} + \cdots + a_{n-2}a_1 + a_{n-1}a_0.$$

This is the catalan recurrence and since $a_0 = 1$ if follows that $a_n = C_n$.

The **Catalan functional equation** is derived as follows:

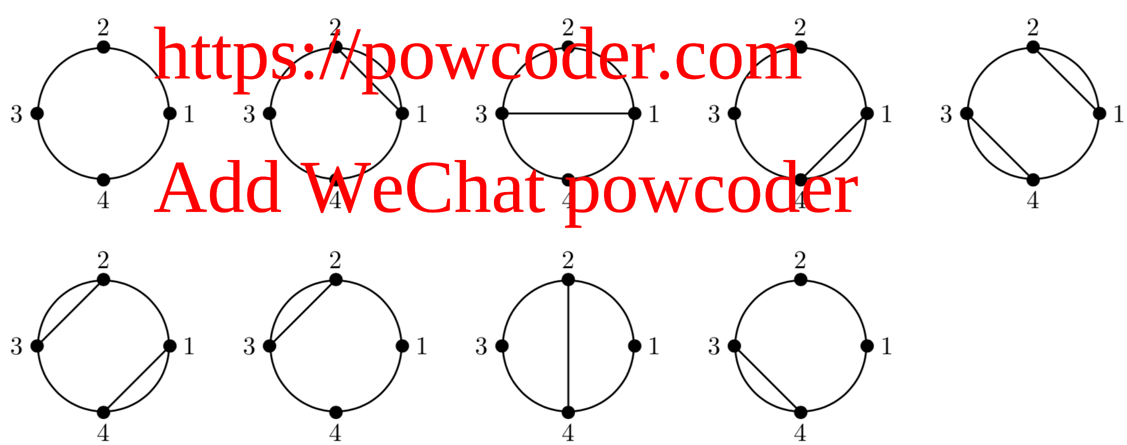
There can be zero points on the circle giving a contribution 1.

Otherwise, we 'partition' on the point labelled 1. There is a chord from the first point to some other point on the circle. Contract the chord between these two points. This leaves us with two circles and a 'double' point. On each of the two circles the remaining points form a chord diagram counted by $C(x)$ (since there is an arbitrary number of points on each circle). This gives a contribution $xC(x)^2$ (x from the double point).

This takes care of all the possible cases. Adding up all the contributions we get the Catalan functional equation:

$$C(x) = 1 + xC(x)^2.$$

Q3: (a) There are 9 possible diagrams on $n = 4$ points:



(b) Motzkin paths are lattice paths in $\mathbb{N}_0^2$ with step set $S = \{(1,0),(1,1),(1,-1)\}$, which we shall refer to as Right, Up and Down steps, respectively.

A bijection between chord diagrams on n points and Motzkin paths with n steps, is:

Go through the points is order of labelling and record

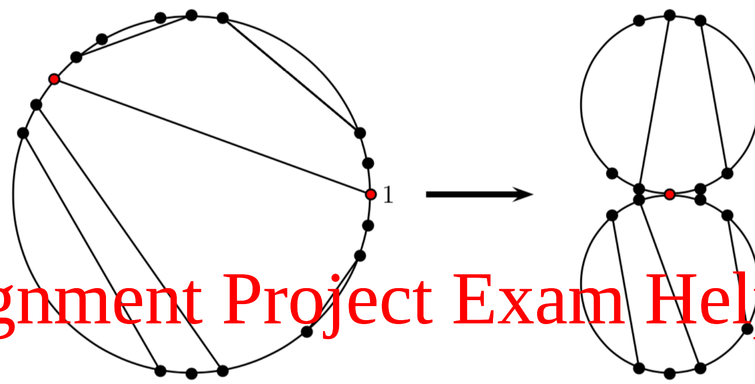
- 'Right' step iff there is no chord from the point.
- 'Up' step iff there is a chord from the point to a point of **higher** label.
- 'Down' step iff there is a chord from the point to a point of **lower** label.

(c) There can be zero points on the circle giving a contribution 1.

Otherwise, we ‘partition’ on the point labelled 1. There are two cases.

Case 1: There is no chord from the first point. What follows this point is a chord diagram on one less point. However, since there can by an **arbitrary** number of points the generating function for such chord diagrams is still $C(x)$ so we get a contribution $xC(x)$ where the factor x is ‘counting’ point 1.

Case 2: There is a chord from the first point to some other point on the circle. In the figure below we show a typical chord diagram (arbitrary number of points so only a few shown). Contract the chord connecting the red points. This leaves us with two circles and a red ‘double’ point. On each of the two circles the black points form a chord diagram counted by $C(x)$ (since there are an arbitrary number of poins on each circle). This gives a contribution $x^2C(x)^2$ (x^2 from the red double point).



This takes care of all the possible cases. Adding up all the contributions we have:

$$C(x) = 1 + xC(x) + x^2C(x)^2.$$

Q4: We differentiate term-by-term and consider the effect on x^n

$$x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} x^n = x^k n \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} x^{n-1} = x^k n(n-1)\cdots(n-k+1)x^{n-k} = n_k x^n.$$

n_k counts the number of ways to choose an **ordered** sample of k **distinct** elements from n elements. So the combinatorial interpretation is that the generating function $x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} L(x)$ counts lattice paths where we have marked k distinct steps **and** labelled them (from 1 to k) in the order in which they were singled out.

Similarly

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k x^n = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-1}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)x^n = n\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-1} x^n = n^k x^n.$$

This is like choosing an ordered sample of k elements from a set with n elements where each element can be chosen repeatedly. So the combinatorial interpretation is that the generating function $(x\frac{\mathrm{d}}{\mathrm{d}x})^k L(x)$ counts lattice paths where we have marked k steps **and** labelled them (from 1 to k) in the order in which they were singled out but each step could be chosen (labelled) many times.

Now $k!$ counts the number of ways of arranging k elements in a line (permutations) so dividing by $k!$ ‘gets rid of’ the ordering of the marking of the k steps, i.e., $\frac{1}{k!} x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} L(x)$ counts lattice paths where we have simply marked k distinct steps.

Q5: Using our working (interpretations) from **Q4** we have

(a) $x\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{1-x-y}\right) = \dfrac{x}{(1-x-y)^2}$.

(b) $y\dfrac{\mathrm{d}}{\mathrm{d}y}\left(x\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{1-x-y}\right)\right) = y\dfrac{\mathrm{d}}{\mathrm{d}y}\left(\dfrac{x}{(1-x-y)^2}\right) = \dfrac{2xy}{(1-x-y)^3}$

(c) We don't care about the order of the marking and each step is marked only once.

$$\frac{1}{3!}x^3\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{1}{1-x-y}\right) = \frac{x^3}{(1-x-y)^4}$$

Q6: (a) Let

$$D(x) = \sum_{n=0}^{\infty}\binom{2n}{n}x^n$$

$$\Downarrow$$

$$\int D(x)\,dx = \int\left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\right)dx = \sum_{n=0}^{\infty}\int\binom{2n}{n}x^n dx = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}$$

$$= xC(x) = \frac{1}{2}\left(1-\sqrt{1-4x}\,\right).$$

$$\Downarrow$$

$$D(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\left(1-\sqrt{1-4x}\,\right)\right) = \frac{1}{\sqrt{1-4x}}.$$

(b) Let $b_n = \sum_{k=0}^{n} a_k a_{n-k}$ and let $A(x)$ and $B(x)$ be the generating functions of $\{a_n\}$ and $\{b_n\}$, respectively. Now $B(x) = 1/(1-x)$ and since b_n is the convolution of $\{a_n\}$ with itself we have $(A(x))^2 = B(x)$ so that

$$A(x) = \frac{1}{\sqrt{1-x}}.$$

Comparing to $D(x/4)$ from (a) we get

$$a_n = \frac{1}{4^n}\binom{2n}{n}.$$