The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 7: Fibonacci – Answers

Q1: (a) $S_1 = 1$ and $S_2 = 2$.

(b) After a step of size 1, $n-1$ steps remain which can be climbed in S_{n-1} ways.
After a step of size 2, $n-2$ steps remain which can be climbed in S_{n-2} ways.
These are all the possibilities so

$$S_n = S_{n-1} + S_{n-2}.$$

This is the Fibonacci recurrence. Taking into account the initial conditions we have

$$S_n = F_{n+1}.$$

Q2: (a) $U_1 = 1$ and $U_2 = 1$.

(b) $U_n = F_{n+1}$.

Q3: Let L_n be any sequence of the required type with n letters. We can form sequences of n letters according to the concatenations:

$$BL_{n-1} \qquad ABL_{n-2}$$

All allowed sequences of n letters have this structure so $l_n = l_{n-1} + l_{n-2}$.

Now check initial conditions and verify.

Q4: (a) For each $(n+1)$-tiling of the board create the n-tuple $(b_0, b_1, \ldots, b_n)$ where $b_i = 0$ if cell i is covered by the *first* cell from a domino and $b_i = 1$ otherwise.

(b) For each arrangement of integers a_i according to the rules $a_1 a_2 \cdots a_n$ create the n-tiling where cells i is covered by a monomer if and only if $a_i = i$.

(c) Let T be a tiling of an n-board with tiles of odd length. Take each tile and break it into a monomer followed by dimers. Remove the first square in this tiling to get an tiling T' of an $(n-1)$-board with monomers and dimers.

Q5: The Fibonacci sequence $\{F_n\}$ (we adopt the convention that $F_0 = 0$) has generating function

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}.$$

(a) We observe that g_n consists of the n^{th} partial sum of the Fibonacci sequence. So

$$G(x) = \frac{F(x)}{1-x} = \frac{x}{(1-x)(1-x-x^2)}$$

(b) Multiply each side by x^n, sum over n and evaluate

(c) Simplify the expression for $H(x)$. Equating coefficients in the power series of gives

$$F_0 + F_1 + \cdots F_n \;=\; g_n \;=\; h_n \;=\; F_{n+2} - 1,$$

as required.

Q6: (a) F_{n_1} is the largest Fibonacci number not exceeding p so

$$0 \le p - F_{n_1} \qquad\qquad p \le F_{n_1+1} - 1$$

The second inequality can be written as $p - F_{n_1} \le F_{n_1+1} - F_{n_1} - 1$.
But $F_{n_1+1} - F_{n_1} = F_{n_1-1}$ so $0 \le p - F_{n_1} \le F_{n_1-1} - 1$.

(b) Repeating the process, select the largest Fibonacci number $\le p - F_{n_1}$. According to (a) the largest such Fibonacci number is no greater than F_{n_1-2}. This means that no two successive Fibonacci numbers occur. As $F_2 = 1$, the process must always terminate with a zero remainder, showing that all positive integers has the stated decomposition.

(c)

$1 = F_2$	$4 = F_4 + F_2$	$7 = F_5 + F_3$	$10 = F_6 + F_3$
$2 = F_3$	$5 = F_5$	$8 = F_6$	$11 = F_6 + F_4$
$3 = F_4$	$6 = F_5 + F_2$	$9 = F_6 + F_2$	$12 = F_6 + F_4 + F_2$

Let F_{n^*} be the smallest Fibonacci number in the decomposition and if

n^* even $p \leftrightarrow A$, $\qquad n^*$ odd $p \leftrightarrow a$

This prescription gives for the first 12 integers the letter sequence

$$AaAAaAaAAaAA$$

After making the replacements $A \mapsto Aa,\; a \mapsto A$ we get

$$AaAAaAaAAaAAaAaAAaAa$$

Note that the first 12 letters are unchanged. This is also a feature of the sequence of the Fibonacci rabbit problem and indeed for $n \to \infty$ the two sequences are identical.