The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 10: Symmetric Group and Applications – Answers

Q1: (a) For a permutation written in terms of 2-cycles: $\quad \text{sgn} \;=\; (-1)^{\#(\text{2-cycles})}$.
Count 2-cycles on LHS and RHS and show there is an odd number of these.

(b) Note the action of a transposition is:

- Left action "swaps values".
- Right action "swaps positions".

So the left action on 213564 is

$$(25) \circ 213564 \;=\; 513264 \;=\; s_4 s_3 s_2 s_3 s_4 \circ 213564$$

for the right action

$$213564 \circ (25) \;=\; 263514 \;=\; 213564 \circ s_4 s_3 s_2 s_3 s_4$$

(c) To show that $(k+1\, k\, k+2) = (k\, k+1\, k+2)^{-1}$ it suffices to show

$$(k+1\, k\, k+2) \circ (k\, k+1\, k+2) = I$$

Similarly, to show that $(k\, k+1\, k+2)^{-1} = (k\, k+1\, k+2)^2$ it suffices to show

$$(k\, k+1\, k+2)^3 = I$$

(d) $A_3 = \{(123), (123)^2, (123)^3\}$.

Q2:

$$s_5 s_1 s_2 s_1 = (56)(12)(23)(12) = (13)(56) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 4 & 6 & 5 & 7 \end{pmatrix}$$

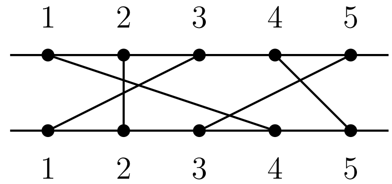
$$s_4 s_5 s_4 s_5 s_1 s_2 s_1 s_4 = (45)(56)(45)(56)(12)(23)(12)(45) = (13)(56) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 4 & 6 & 5 & 7 \end{pmatrix}$$

Use $(s_4 s_5 s_4) s_5 = (s_5 s_4 s_5) s_5 = s_5 s_4$, so

$$s_4 s_5 s_4 s_5 s_1 s_2 s_1 s_4 = s_5 s_4 s_1 s_2 s_1 s_4 = s_5 s_4 s_4 s_1 s_2 s_1 = s_5 s_1 s_2 s_1$$

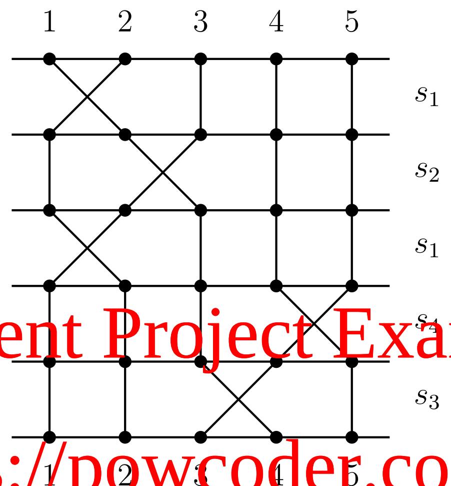
where the second equality follow since s_4 commutes with s_1 and s_2, and the third follows from $s_4^2 = 1$.

Q3: (a) The bipartite graph corresponding to $\sigma = 42153$ is



(b) Each crossing in (a) is an inversion. So the 5 crossings results in 5 inversions.

(c) $\sigma = s_3 s_4 s_1 s_2 s_1$, which has this bipartite graph:



(d) $I_\sigma = \{(1,2),(1,3),(1,5),(2,3),(4,5)\}$.

Q4: The desired board position has a snake-pattern permutations of odd parity.

(a) Here the parity is even so the sought after ordering can't be achieved.

(b) Since the parity is odd the sought after ordering can be achieved.

(c) Since the parity is odd the sought after ordering can be achieved.

Q5: The two snake-pattern permutations have different parity so board positions cannot be related by sliding moves.

Q6: Let us make the identifications

R = 1	A = 2	T = 3	E = 4	Y = 5	O = 6	U = 7	R = 8
M = 9	I = 10	N = 11	D = 12	P = 13	A = 14	L = 15	□ = 16

A key point here is that the 2 and 14 are indistinguishable in this particular version of the 15-puzzle. Consider then the even permutation

$$(2\,14) \circ (14\,15)$$

which first interchanges the two A's, then interchanges the A and L. Being even, it is attainable, and it leaves the last line reading PLA□, and the other 3 lines reading as before.