The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 4: Pigeonholes and Ramsey Numbers – Solutions

Q1: (a) Taking into account the number of days in each month and the fact that 28 days equals 4 weeks we find that the 13th day of each month corresponds to days of the week as

$$\text{Jan} \leftrightarrow 0, \text{Feb} \leftrightarrow 3, \text{Mar} \leftrightarrow 3, \text{Apr} \leftrightarrow 6, \text{May} \leftrightarrow 1, \text{Jun} \leftrightarrow 4,$$

$$\text{Jul} \leftrightarrow 6, \text{Aug} \leftrightarrow 2, \text{Sep} \leftrightarrow 5, \text{Oct} \leftrightarrow 0, \text{Nov} \leftrightarrow 3, \text{Dec} \leftrightarrow 5.$$

So

0	1	2	3	4	5	6
Oct Jan	May	Aug	Nov Mar Feb	Jun	Dec Sep	Jul Apr

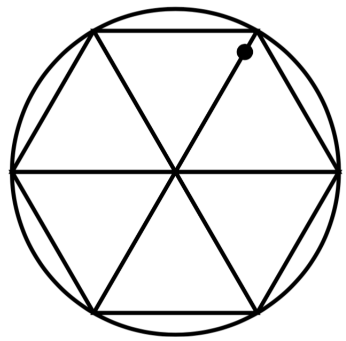
(b) Since the pigeonholes contain at least one and no more than three months it follows that there must be at least one, and no more than 3, Friday the 13ths in any one year.

Q2: First suppose there is at least one person who doesn't know any one else. Then any one person can have $0, 1, 2 \ldots n-2$ friends ($n-1$ is not possible since this person would be friends with all other people in the group).

So there are n people and $n-1$ 'pigeonholes' $\Rightarrow$ at least one pigeonhole contains at least two people who then have the same number of friends.

Next consider the case where no one has 0 friends. Then any one person can have $1, 2 \ldots n-1$ friends. Again there are n people and $n-1$ pigeonholes and at least one pigeonhole contains at least two people who then have the same number of friends.

Q3: Following the hint we divide the circle into 6 sectors of equal size and choose one of the boundaries to pass through one the points. Each sector contains an equilateral triangle.



Two points inside a sector are at most a unit distance apart. There are two possibilities to consider. Either one of the two neighbouring sectors (of the point on the boundary) contains a point. In this case we are done. Otherwise, we have to put 5 points inside the remaining 4 sectors. By the pigeonhole principle at least one sector contains at least two points and again we are done.

Q4: (a) The number of games played equals the number of subsets of size 2 from a set of size 10 so $\binom{10}{2} = 45$.

(b) The number of draws is at least $45 \times 0.7 = 31.5$ so not less than 32. Hence no more than 13 games did not end in a draw.

(c) Suppose all players have different scores. At most one player has a score of 0. So we have 9 players with either a positive or negative score. By the generalised pigeonhole principle there are at least 5 players with a positive score or at least 5 players with a negative score.

(d) Suppose at least 5 players have positive scores which are all different. Then there must be a minimum of $1 + 2 + 3 + 4 + 5 = 15$ wins. This contradicts the result from (b). Hence our assumption that all scores are different is not correct.

Q5: $R(a, b)$ is the minimum number of points, which when connected with red or blue lines, will always produce a complete graph K_a in red or a complete graph K_b in blue.

Q6: Ramsey theory tells us that among any group of 6 people, there must be a group of 3 mutual friends or a group of 3 mutual strangers.

Identify sending a message about relationship gossip as 'friend' and sending a message about what's on next week as 'stranger'.

Q7: (a) After singling out person 1 we are left with 9 people to distribute among the two pigeonholes 'friend of 1' or 'stranger to 1'. By the generalised pigeonhole principle one or the other contains at least 5 people.

(b) Suppose there are 5 or more mutual strangers to person 1. If 4 or more of these are mutual friends we are done. Otherwise there must be at least one pair of mutual strangers. Combining such a pair with person 1 produces a group of 3 mutual strangers.

(c) The above argument still holds with 4 mutual strangers to person 1.

(d) If there are 3 or less strangers to person 1 then there must be 6 or more mutual friends of person 1. But we know that in any group of 6 people there are at least 3 mutual friends or 3 mutual strangers. In the latter case we are done, while in the former case combining with person 1 we have a group of 4 mutual friends.

Q8: (a) K_9 consists of 9 points (vertices) each connected to all the other points by an edge.

(b) The number of edges is $\binom{9}{2} = 36$ (choose 2-sets from a 9-set)

(c) Each edge can be coloured independently in either of two colours. By the multiplication principle the number of possible colourings is 2^{36}.

(d) We know that $R(3,4) = 9$. This means that in any two-colouring of K_9 there must be a K_3 in one of the colours or a K_4 in the other colour.

The Ramsey number is the *smallest* value of n such that K_n has this property. Thus it is not true for K_8.