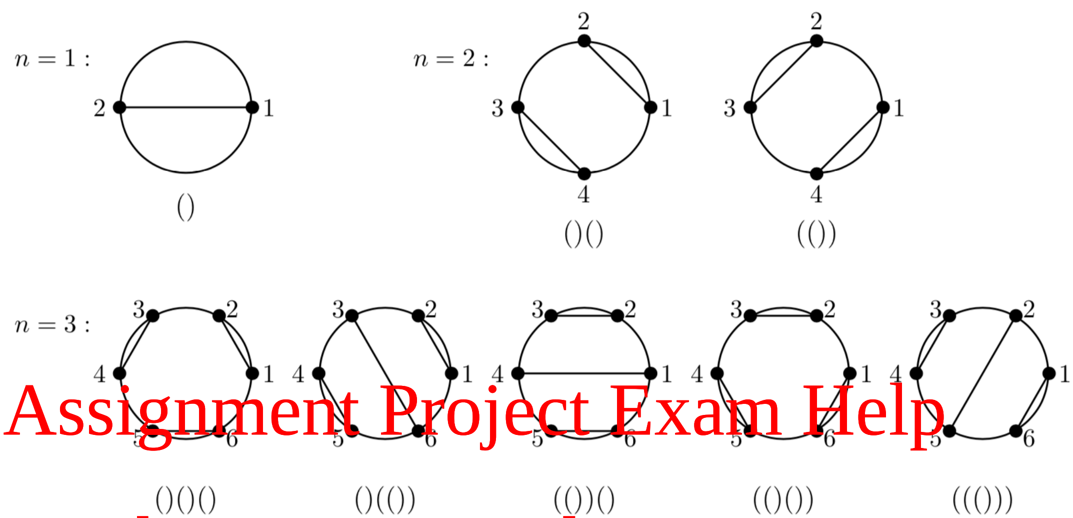
The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 8: Catalan and Functional Equations – Answers

Q1: Bijections to balanced parentheses or Dyck paths are clear.

Q2: (a) These are the possible diagrams on $2n$ points for $n = 1, 2, 3$:



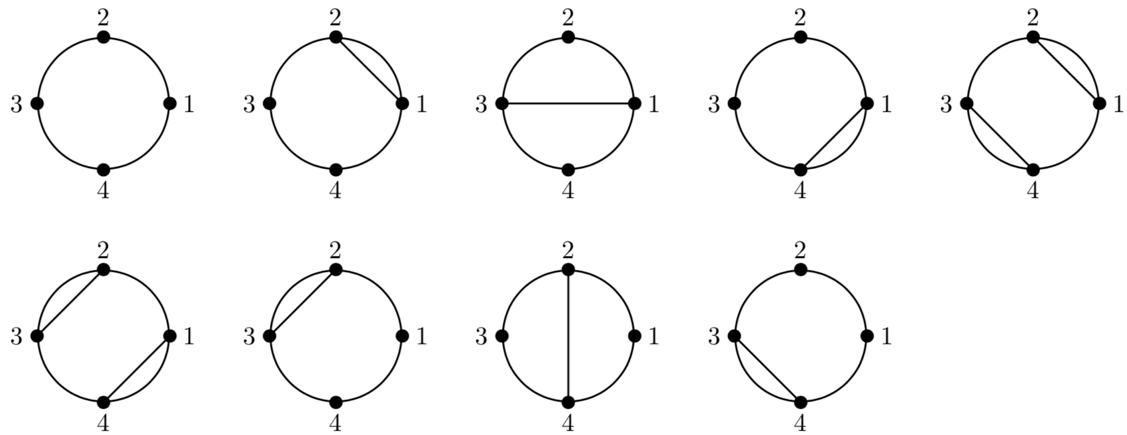
(b) Take points labelled i and $j > i$ and consider a chord between the points.

(c) Bijection between chord diagrams on $2n$ points and balanced parenthesis:
Go through the points in order of labelling and record:

- An opening parenthesis '(' iff the chord goes to a point of **higher** label.
- A closing parenthesis ')' iff the chord goes to a point of **lower** label.

(d) Label points 1 up to $2n$ counter-clockwise.Draw a chord from the point 1 to the point $2j$.
Show this gives rise to the Catalan recurrence.
Similar consideratiosn gives the Catalan functional equation.

Q3: (a) There are 9 possible diagrams on $n = 4$ points:



(b) Motzkin paths are lattice paths in $\mathbb{N}_0^2$ with step set $S = \{(1,0),(1,1),(1,-1)\}$, which we shall refer to as Right, Up and Down steps, respectively.

A bijection between chord diagrams on n points and Motzkin paths with n steps, is: Go through the points is order of labelling and record

- 'Right' step iff there is no chord from the point.
- 'Up' step iff there is a chord from the point to a point of **higher** label.
- 'Down' step iff there is a chord from the point to a point of **lower** label.

(c) There can be zero points on the circle giving a contribution 1.

Otherwise, we 'partition' on the point labelled 1. There are two cases.

Case 1: There is no chord from the first point.

Case 2: There is a chord from the first point to some other point on the circle.

This takes care of all the possible cases. Adding up all the contributions we have:

$$C(x) = 1 + xC(x) + x^2C(x)^2.$$

Q4: Differentiate term-by-term and consider the effect on x^n

$$x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} x^n = x^k n \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} x^{n-1} = x^k n(n-1)\cdots(n-k+1)x^{n-k} = n_k x^n.$$

The combinatorial interpretation is that $x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} L(x)$ counts lattice paths with k distinct steps marked **and** labelled (from 1 to k) in the order in which they were singled out.

Similarly

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k x^n = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-1}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) x^n = n\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-1} x^n = n^k x^n.$$

The combinatorial interpretation is that the generating function $(x\frac{\mathrm{d}}{\mathrm{d}x})^k L(x)$ counts lattice paths where we have marked k steps **and** labelled them (from 1 to k) in the order in which they were singled out but each step could be chosen (labelled) many times.

Now $k!$ counts the number of ways of arranging k elements in a line (permutations) so dividing by $k!$ 'gets rid of' the ordering of the marking of the k steps, i.e., $\frac{1}{k!}x^k\frac{\mathrm{d}^k}{\mathrm{d}x^k}L(x)$ counts lattice paths where we have simply marked k distinct steps.

Q5: Using our working (interpretations) from **Q4** we have

(a) $x\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{1-x-y}\right) = \dfrac{x}{(1-x-y)^2}$.

(b) $y\dfrac{\mathrm{d}}{\mathrm{d}y}\left(x\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{1-x-y}\right)\right) = y\dfrac{\mathrm{d}}{\mathrm{d}y}\left(\dfrac{x}{(1-x-y)^2}\right) = \dfrac{2xy}{(1-x-y)^3}$

(c) $\dfrac{1}{3!}x^3\dfrac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\dfrac{1}{1-x-y}\right) = \dfrac{x^3}{(1-x-y)^4}$

Q6: (a) Integrate term-by-term and and manipulate.

(b) Use convolution of generating functions.

$$a_n = \frac{1}{4^n}\binom{2n}{n}.$$