School of Mathematics and Statistics

MAST30012 Discrete Mathematics 2021

Assignment 1 – Solutions

Q1: We first state an equivalent counting problem and then give the answer.

(a) This is equivalent to choosing a subset of size 5 (the people given apples) from a set of size 12. The answer is:

$$\binom{12}{5} = \frac{12!}{7!\,5!}.$$

(b) This is like distributing 16 balls (people) into 3 boxes (oranges, pears and bananas) with 7 balls in box 1, 4 balls in box 2, and 5 balls in box 3. The answer is

$$\binom{16}{7,\,4,\,5} = \frac{16!}{7!\,4!\,5!}.$$

(c) The distribution of each type of fruit (oranges or apples) is independent.

The distribution of a particular type of fruit is the problem of sampling with replacement (think of people as labelled boxes and fruit as identical balls) so the oranges can be distributed in:

$$\binom{20+6-1}{6} = \binom{25}{6}$$

ways, while the apples can be distributed in

$$\binom{20+12-1}{12} = \binom{31}{12}$$

ways. By the multiplication principle the oranges and apples can then be distributed in

$$\binom{25}{6}\binom{31}{12}$$

ways.

(d) Each integer can be represented as an n-tuple or as a word of length n with entries from $\{0, 1, \ldots, 9\}$. In the case $n = 3$ we would have $7 = (0, 0, 7), 389 = (3, 8, 9), 1000 = (0, 0, 0)$ etc. with the entries being the coefficients in the base 10 expansion of the integer. There are 10^n tuples and in each position (column) of the n-tuples there will be 10^{n-1} occurrences of the digit 3. Since there are n positions we would write down the digit 3 a total $n10^{n-1}$ times.

Q2: In each case we make use of the probability axiom. Let T be the set of possible outcomes and F the set of favourable outcomes. Assuming each outcome is equally likely, the probability of F is:

$$\Pr(F) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} = \frac{|\,F\,|}{|\,T\,|}.$$

Now each roll of five dice can be represented as a five tuple (we distinguish between the dice) with entries 1, 2, 3, 4, 5 or 6. Hence $|\,T\,| = 6^5 = 7776$.

(a) Five of a kind can be achieved in 6 ways. Hence

$$\Pr(\text{Five of a kind}) \;=\; \frac{6}{6^5} \;=\; \frac{1}{6^4} \;\approx\; 0.0771\%.$$

(b) The required dice values can appear in any order. Hence $|\,F\,| = 5!$ is the number of permutations of $2, 3, 4, 5, 6$, and

$$\Pr(\text{A large straight}) \;=\; \frac{5!}{6^5} \;\approx\; 1.543\%.$$

(c) The number of different full-houses is $6 \times 5 = 30$ (6 choices for three of a kind and 5 for two of a kind). The three dice of the same value can be placed in $\binom{5}{3} = 10$ positions. Hence

$$\Pr(\text{Full-house}) \;=\; \frac{300}{6^5} \;\approx\; 3.858\%.$$

(d) We partition F into 4 cases: Five of a kind, Four of kind and one other, Full-house, and Three of a kind and two **distinct** other dice.
Four of kind and one other can occur in $6 \times 5 \times 5 = 150$ ways (30 combinations of dice and 5 configuration in each case).
Three of kind and two **distinct** other dice can occur in $6 \times 5 \times 4 \times \binom{5}{3} = 1200$ ways.
Hence $|\,F\,| = 6 + 150 + 300 + 1200 = 1656$ and

$$\Pr(\text{Three of a kind}) \;=\; \frac{1656}{6^5} \;\approx\; 21.296\%.$$

(e) The only way two of a kind **cannot** occur is that all dice show distinct values. This can happen in $(6)_5 = 6!$ ways. So $|\,F\,| = 6^5 - 6! = 7056$ by the complement principle and hence

$$\Pr(\text{Two of a kind}) \;=\; \frac{7056}{6^5} \;\approx\; 90.74\%.$$

Alternative Solution: Two of a kind obviously appear in any configuration with three of a kind. There are two further cases two pairs and a distinct third (as in 22445) or two of a kind and three distinct others as in (55136).

Two pairs and a distinct third: $6 \times 5 \times 4/2$ possible combinations (divide by 2 to avoid counting combinations of pairs twice) and for each combination $\binom{5}{2}\binom{3}{2}$ configurations. In total there are 1800 possibilities.

Two of a kind and three distinct others: $6 \times 5 \times 4 \times 3$ possible combinations and for each combination $\binom{5}{2}$ configurations (note configurations of the distinct 3 are already counted via the combinations). In total there are 3600 possibilities.

Adding it all up we get $1656 + 1800 + 3600 = 7056$ as before.

Note that had we not already done the hard work with three of a kind the complement way is so much easier.

Q3: A configuration of leaders is an m-tuple with n possible distinct 'values' (students) per entry.

(a) With no constraint there are n^m ways of choosing the leaders.

(b) There are $(n-k)^m$ ways of choosing the leaders if a group a size k are excluded from leading.

(c) Let $L = \{1, 2, \ldots, n\}$ denote the set of possible leaders and $L_j = L \backslash \{j\}$ the set of leaders with student j excluded. The set of configurations in which every student gets to lead at least once is the complement of L^m with $\bigcup_{j=1}^n L_j^m$ since L_j^m is the set of configurations where student j never gets to lead on any of the m days. So

$$S(m,n) = |L^m| - \left|\bigcup_{j=1}^n L_j^m\right|$$

Use the inclusion-exclusion theorem

$$= n^m - \sum_{k=1}^n (-1)^{k-1} \sum_{\{s_1,\ldots,s_k\} \subset \{1,2,\ldots,n\}} |\, L_{s_1}^m \cap L_{s_2}^m \cap \cdots \cap L_{s_k}^m \,|$$

$L_{s_1}^m \cap L_{s_2}^m \cap \cdots \cap L_{s_k}^m$ is a set with k students $s_1, s_2, \ldots, s_k$ excluded so $|\, L_{s_1}^m \cap L_{s_2}^m \cap \cdots \cap L_{s_k}^m \,| = (n-k)^m$ and there are $\binom{n}{k}$ such subsets.

$$= n^m - \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (n-k)^m$$

$$= \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m.$$

(d) You didn't try to show this using the formula, did you? Combinatorially, this is easy!

If there are fewer days than students, then clearly it is impossible to have a configuration in which every students leads at least once. Hence $S(m, n) = 0,\ m < n$.

When $m = n$ there are n days and n students. Every permutation of the students is a valid configuration and hence $S(n, n) = n!$.

Note: This is a famous result since $S(m, n)$ counts the number of **onto** functions from a set of size m to a set of size n.

Q4: (a) Given a group of 33 students choose a committee of size 12 with a subcommittee of size 6 and from these six an executive committee of size 3.

LHS: Choose the 12 students for the committee in $\binom{33}{12}$ ways. From the committee of 12 choose a subcommittee of size 6 in $\binom{12}{6}$, and finally choose the executive committee of size 3 in $\binom{6}{3}$ ways. The choices at each step are independent and the result follows.

RHS: Choose the executive committee of size 3 from all students in $\binom{33}{3}$ ways. Then choose the remaining 3 members of the subcommittee from the remaining 30 students in $\binom{30}{3}$ ways. Finally, choose the remaining 6 members of the committee in $\binom{27}{6}$ ways.

Note how you can now derive endless lists of binomial identities such as in this case

$$\binom{33}{12}\binom{12}{6}\binom{6}{3} = \binom{33}{6}\binom{27}{6}\binom{6}{3} = \binom{33}{9}\binom{9}{3}\binom{24}{3} = \cdots$$

(b) This is a generalisation of the 'Vandermonde at the ice cream parlour' example from lectures.

An ice cream parlour serves n flavours. How many different ice cream platters are there with k scoops of ice cream.

LHS: $\left(\!\binom{n}{k}\!\right)$ by definition using sampling with replacement or multi-choice.

RHS: We partition on the number $m \leq k$ of distinct flavours. The m flavours can be chosen in $\binom{n}{m}$ ways and we take one scoop of each. Then we multi-choose the remaining $k - m$ scoops from the m flavours in $\left(\!\binom{m}{k-m}\!\right)$ ways. We then sum over the number of chosen flavours while appealing to the addition and multiplication principles.

Alternatively:

Distribute k oranges to n children with no constraint on how many oranges a child may get.

LHS: $\left(\!\binom{n}{k}\!\right)$ by definition.

RHS: Condition on the number, m, of children who gets at least one orange. We choose m children in $\binom{n}{m}$ ways and give each child an orange. Next distribute the remaining $k - m$ oranges among the m children in any way. Sum over the number of chosen children.

(c) Choose subsets of size $k + 1$ from the set $\{1, 2, \dots, n, n + 1\}$.

LHS: By definition, $\binom{n+1}{k+1}$ is the number of subsets.

RHS: Condition on the largest number $m + 1$ in the subset. The remaining k elements can be chosen in $\binom{m}{k}$ ways from $\{1, 2, \dots, m\}$. Now $m + 1$ can range from $k + 1$ to $n + 1$ and the result follows by the addition principle.