Q7 a) Treat pair each as a single object giving 5 objects. Since left & right are distinct

$\Rightarrow 2\times 5!$ ways ie permute: $r_1\, r_2\, \underbrace{(b_1 b_2)}_{\times 2}\, g_1\, g_2$

b) • Choose ref first $\Rightarrow$ 23 ways

Choose team: $\binom{22}{11}$

$\Rightarrow 23\binom{22}{11}$ ways $(= 16\,224\,936)$

• Choose team: $\binom{23}{11}$ then ref $\binom{12}{1}$

$\Rightarrow 12\binom{23}{11}$ ways (= same)

c) Permute DIBT with gaps $\_\, l_1 \_\, l_2 \_\, l_3 \_\, l_4 \_$

(ADIABAT) $\Rightarrow 4!$ perms

Then choose 4 of 5 gaps

$\Rightarrow 4!\binom{5}{3}$ ways

d) Outcome set:

$d_1 \backslash d_2$	1	2	3	4	5	6
1	0	—	—	—	—	—
2	0	6	—	—	—	—
3	0	0	6	—	—	—
4	0	0	0	6	—	—
5	0	0	6	0	0	—
6	6	0	1	0	6	6

i) 21 events $T_{ij} \Rightarrow \text{Prob}(T_{ij}) = \frac{1}{21}$

ii) Prob $= \frac{3}{21}$

Q2 a) $k\binom{n}{k}$ ⇝ choose size k team from n-set, then choose captain from team

$n\binom{n-1}{k-1}$ ⇝ Choose captain from n-set then choose k-team from remaining (n-1)-set.

b) i) $\Omega_L = \bigcup_{k=0}^{n} \Omega_k$, Ω_k = set of n step binomial paths with k east steps

$\Omega_R = [2]^n$

Map: $T: \Omega_L \to \Omega_R$ $\quad T(p_1 \cdots p_n) = b_1 b_2 \cdots b_n$, $b_i = \begin{cases} 1 & \text{if } p_i = E \\ 0 & \text{otherwise} \end{cases}$

ii) $\Omega_L = \bigcup_{k=1}^{n} \Omega_k$, $\Omega_k = \dot{B}^n_k$

$\dot{B}^n_k$ = set of n step bin^l paths with k east steps, one marked

Ω_R = set of tuples $t \in [2]^n$ with an entry marked.

$T: \Omega_L \to \Omega_R$: $\quad T(p_1 \cdots p_n) = b_1 \ldots b_n$, $b_i = \begin{cases} 1 & \text{if } p_i = E \\ \dot{1} & \text{if } p_i = \dot{E} \\ 0 & \text{otherwise} \end{cases}$

Check it: $n=4$ $\quad 1\binom{4}{1} + 2\binom{4}{2} + 3\binom{4}{3} + 4\binom{4}{4}$

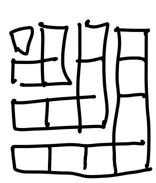
$= 4 + 12 + 12 + 4 = 32$ ✓ $\quad 4 \cdot 2^3 = 4 \cdot 8 = 32$ ✓

c) Check $n=4$

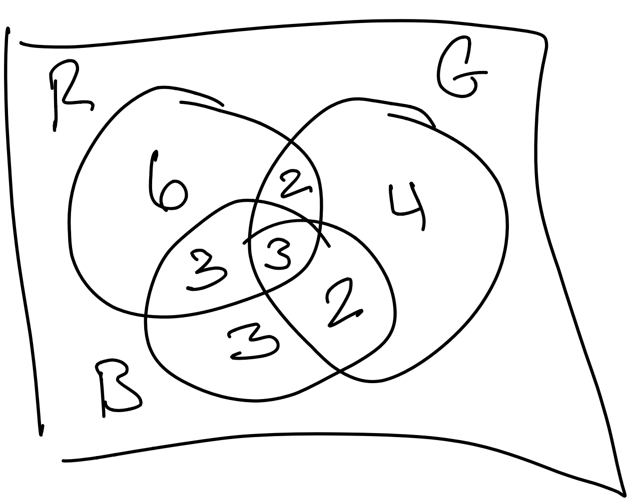
$k=0 \quad 1 \quad 2 \quad 3 = n-1$

$1 + 3 + 5 + 7 = 16$, 4^2

Proof:



Q3) a)



b)

$|R| = 14 \qquad |R \cap B| = 6$

$|G| = 11 \qquad |R \cap G| = 5 \qquad |R \cap G \cap B| = 3$

$|B| = 11 \qquad |B \cap G| = 5$

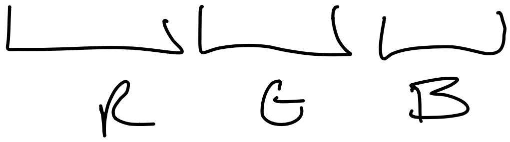
$\Rightarrow |R \cup B \cup G| = 14 + 11 + 11 - 6 - 5 - 5 + 3$

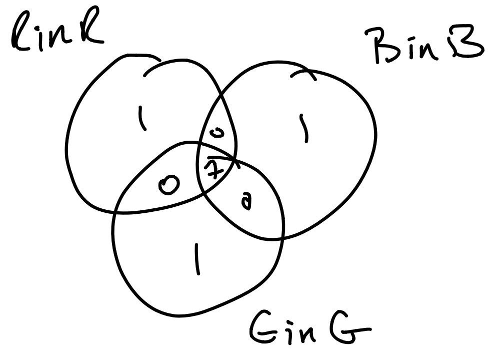
$= 36 - 16 + 3$

$= 23$

b) This Q might have been interpreted in two ways: Both i) & ii) hold simultaneously OR i) & ii) are separate questions.
— either interpretation is correct.

R, G, B





All ways = 3!

R in R: 2 ways

B in B: 2 way

G in G: 2 way

R in R, G in G: 1

R in R, B in B: 1

G in G; B in B: 1

R, G + B in same: 1

Let R = set of config with R in box R

B " B in box B

G = " G in box G

$\Rightarrow$ $|R| = 2$, $|B| = 2$, $|G| = 2$; $|R \cap B| = |R \cap G| = |B \cap G| = 1$, $|R \cap B \cap G| = 1$

$\Rightarrow$ $\Rightarrow$ #ways to have at least one ball in same colour =

$= |R| + |G| + |B| = 2+2+2-1-1-1+1 = 4.$

$\Rightarrow$ $3! - 4 = 2$ ways to have none in same colour

Check: BRG, GBR

Q4 a) $S_1(n,k)$ = # perms of [n] with exactly k-cycles

= # way to seat n people around k tables

(up to cyclic permutation) with at least 1 person per table

b) To get $S_1(n,k)$ either:

① Add person n between each of the n-1 seated people (for each possible seating)

$(n-1)$ ways to add to $S_1(n-1,k)$ seatings

② Add a new table with seating person 'n' alone.

1 way to add to $S_1(n-1,k)$ seatings.

③ Boundary : $S(n,0) = 0$ no ways to seat n at zero tables.

$S(n,n) = 1$ one way to seat n at n tables

c) $n=3$

123	132	312	213	231	321
(1)(2)(3)	(1)(23)	(132)	(12)(3)	(123)	(13)(2)

$\Rightarrow S(3,0)=1, \quad S(3,1)=2, \quad S(3,2)=3, \quad S(3,3)=1$

Q5 a) $R(a,b)$ = Smallest 2-coloured K_n such that there is certainly at least one with a red K_a subgraph and at least one blue K_b subgraph in every colouring

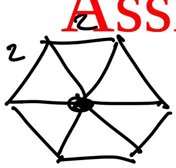
b) Th^m $R(n,2) = n$.

Proof: Case$_{n-1}$: Colour K_{n-1} blue $\Rightarrow$ no blue K_n or red K_2 $\Rightarrow R(2,n) > n-1$

Case n: Colour K_n blue $\Rightarrow$ blue K_n

Colour K_n with at least one red edge $\Rightarrow$ K_2 exist

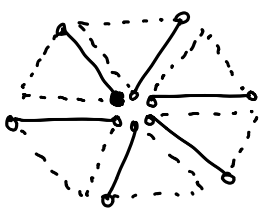
$\Rightarrow R(2,n) = n.$

c) 

6 triangles

$\Rightarrow$ all points inside △ are within distance 2.

Partition hexagon



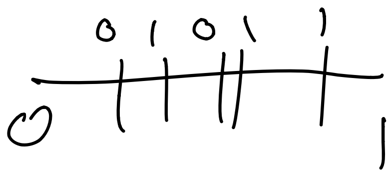
PHP $\Rightarrow$ at least one △ contains at least 2 pts

Inside polygon $\Rightarrow$ not on outside edges

$\Rightarrow$ two pts cannot be at two corners of same △

$\Rightarrow$ with distance 2.

d)



Sperner's Lemma: # different labelled subintervals is odd

$$\#(0,1) + \#(1,0) = 2k+1$$

Add n pts $\Rightarrow$ $n+1$ subintervals

$s_1 = \#(1,1)$, $s_0 = \#(0,0)$

$$\Rightarrow \quad s_1 + s_0 + (2k+1) = n+1$$

To show: $\#(0,1) - \#(1,0) = 1$

Induction: $0 - 1 - 1$ or $0 - 0 - 1$ true

Assume true for $n-1$ pts ($\Rightarrow$ n interval)

$\#(0,1) = k$, $\#(1,0) = k-1$

Add pt:

		$\#(0,1)$	$\#(1,0)$	
Case	$0 - 1 - 0$ — before	1	1	$\delta = \cancel{1}$
Case:	$0 - 1 - 1$	$\delta = 0$		
Case:	$1 - 1 - 0$	$\delta = 0$		
Case:	$1 - 1 - 1$	$\delta = 0$		
Case	$0 - 0 - 0$	$\delta = 0$		
	$0 - 0 - 1$ — before	δ		
	$1 - 0 - 0$	$\delta =$		
	$1 - 0 - 1$	$\delta = 0$		

True for all cases of n pts $\Rightarrow$ ...

Q6 i) $12a_{n+2} = 7a_{n+1} - a_n \; ; \quad a_1 = a_0 = 2$

$$12 \sum_{n=0} x^n a_{n+2} = \sum_{n=0} 7x^n a_{n+1} - \sum_{n=0} x^n a_n$$

$$12 \sum_2 x^n \; a_n = 7x \sum_1 \; x^n \; a_n \; - G(x)x^2$$

$$12\left(-x^0 a_0 - x a_1 + G\right) = -7x . x^0 a_0 + 7x\, G - G x^2$$

$$12\left(-2 - 2x + G\right) = -14x + 7x\,G - G\,x^2$$

$$G\left(12 - 7x + x^2\right) = 24 + 10x$$

$$\Rightarrow \quad G = \frac{24 + 10x}{12 - 7x + x^2}$$

ii) $12 - 7x + x^2 = (3-x)(4-x)$

$$\Rightarrow \frac{24+10x}{12-7x+x^2} = \frac{A}{3-x} + \frac{B}{4-x}$$

$$\Rightarrow 24 + 10x = A(4-x) + B(3-x)$$

$x = 4$: $\quad 64 = -B \quad , \quad x = 3$: $\quad 54 = A$

$$\Rightarrow \quad G = \frac{54}{3-x} - \frac{64}{4-x} = \frac{54}{3(1-x/3)} - \frac{64}{4(1-x/4)}$$

$$= \frac{54}{3} \sum_{n\geq 0} \left(\frac{x}{3}\right)^n - \frac{64}{4} \sum_{n \geq 0} \left(\frac{x}{4}\right)^n$$

$$\Rightarrow \quad a_n = \frac{54}{3}\frac{1}{3^n} - 16 \frac{1}{4^n} \qquad n \geq 0$$

b) $\left(x^2 - 3x + 2\right) \sum_{n\geq 0} x^n b_n = 1 + x$

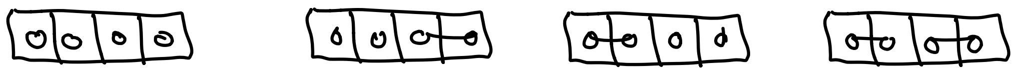
$$\sum_{n\geq 0} x^{n+2} b_n - 3 \sum_0 x^{n+1} b_n + 2 \sum_0 x^n b_n = 1 + x$$

$$\sum_2 x^n b_{n-2} - 3 \sum_1 x^n b_{n-1} + 2 \sum x^n b_n = 1 + x$$

$[x^0]$: $\quad 2b_0 = 1 \quad \Rightarrow b_0 = 1/2$

$[x^1]$: $\quad -3b_0 + 2b_1 = 1 \quad \Rightarrow \quad b_1 = \frac{1}{2}\left(1 + 3 . \frac{1}{2}\right) = \frac{5}{4}$

$[x^n]$: $\quad b_{n-2} - 3b_{n-1} + 2b_n = 0$

Q7 a) 

b) Let $\mathbb{P}_n$ = set of all pavings of n-board

$\Rightarrow |\mathbb{P}_n| = P_n$

$\Omega_L = \mathbb{P}_{n+2} \qquad \Omega_R = \mathbb{P}_{n+1} \cup \mathbb{P}_n$

$T: \Omega_R \Rightarrow \Omega_L$

$$T(b) = \begin{cases} b\boxed{\circ} & \text{if } b \in \mathbb{P}_{n+1} \quad (1) \\ b\boxed{\circ\!-\!\circ} & \text{if } b \in \mathbb{P}_n \quad (2) \end{cases}$$

T well define: $T(b) \in \mathbb{P}_{n+2}$: case (1) adds cell to n+1 $\rightarrow$ n+2 paving

case (2) add dimer to n $\rightarrow$ n+2 board.

c) $\sum_1 x^n P_{n+2} = \sum_1 x^n P_{n+1} + \sum_1 x^n P_n$

$$\sum_3 x^n P_n = x \sum_2 x^n P_n + x^2 P(x)$$

$$-xP_1 - x^2P_2 + P = x(-xP_1 + P) + x^2 P$$

$$\Rightarrow \quad -x - 2x^2 + P = x(-x + P) + x^2 P$$

$$P(x) = \frac{x + x^2}{1 - x - x^2} = -1 + 1 + \frac{x + x^2}{1 - x - x^2}$$

$$= -1 + \frac{1 - x - x^2 + x + x^2}{1 - x - x^2}$$

$$\Rightarrow \quad 1 + P(x) = + \frac{1}{1 - x - x^2}$$

d) $1 + P = \sum_{n\geq 0} \sum_{r=0}^{n} \binom{n}{r} x^r (x^2)^{n-r}$ $\quad r + 2n - 2r$

$\vdots$ see lecture notes.

Q8

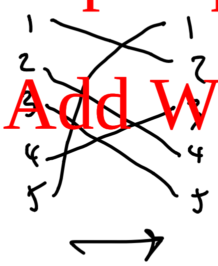
$\sigma = 23145, \qquad \tau = 24531$

a) i) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}, \quad \sigma = (12435)$

ii) $\sigma^{-1} = (31245)$

iii) $23145 \circ 24531 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$

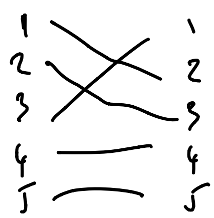
iv) $\tau = 24531$



$\Rightarrow I_\tau = \{(1,5), (2,5), (2,4), (3,5), (3,4), (4,5)\}$

v) $\mathrm{sgn}(\tau) = (-1)^6 = +1$

vi)



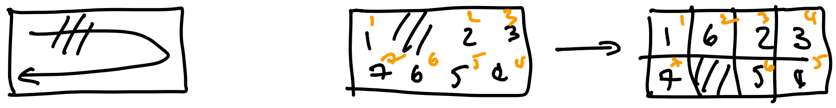
$\Rightarrow \sigma = s_1 s_2$

b). $\mathrm{sgn}(\alpha) =$ parity of # trans^{k}. If $\alpha = t_1 \circ \cdots \circ t_k$ & $\beta = r_1 \circ \cdots \circ r_m$

$\Rightarrow \alpha \circ \beta = t_1 \circ \cdots \circ t_k \circ r_1 \circ \cdots \circ r_m \Rightarrow$ parity of $(\alpha \circ \beta) =$

parity (α) + parity (β)

Q9



a) $1\ 2\ 3\ 4\ 5\ 6\ 7 \rightarrow 1\ 6\ 2\ 3\ 4\ 5\ 7$

$1\ 2\ 3\ 4\ \underbrace{5\ 6}\ 7 \rightarrow (56)$

$1\ 2\ 3\ \underbrace{4\ 6}\ 5\ 7 \rightarrow (45)$

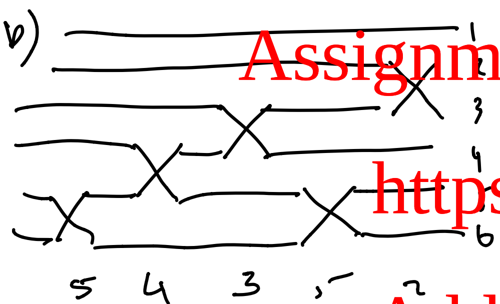
$1\ 2\ \underbrace{3\ 6}\ 4\ 5\ 7$
$1\ \underbrace{2\ 6}\ 3\ 4\ 5\ 7 \rightarrow (34)$

$1\ 6\ 2\ 3\ 4\ 5\ 7 \rightarrow (23)$

$\Rightarrow 1623457 = 1234567 \circ (56) \circ (45) \circ (34) \circ (23)$

$\Rightarrow \text{parity}(\mu_2) = \text{even}$

b)



$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 5 & 4 \end{pmatrix}$$

c)

$s_2 s_4 s_3 s_2 s_1$

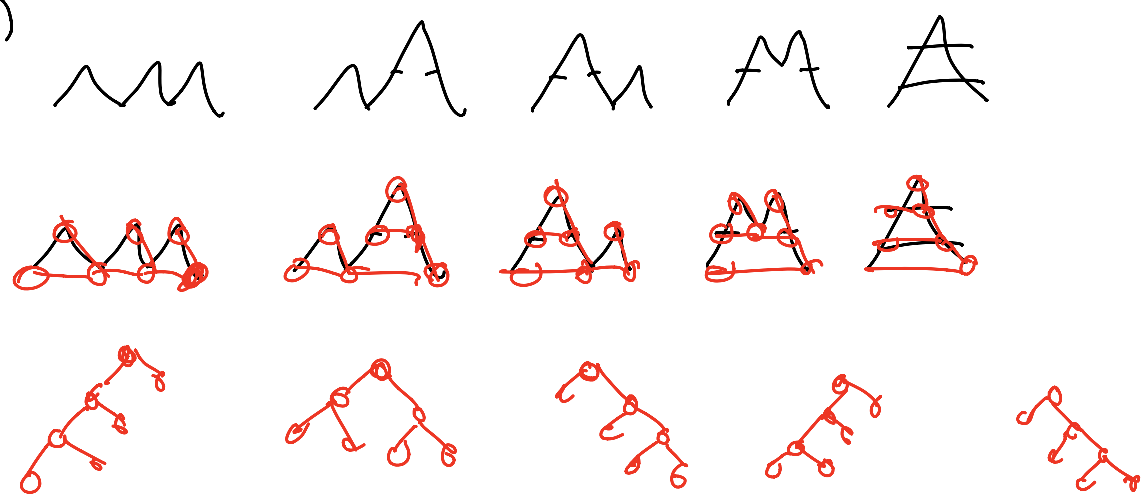
$s_4\ \underline{s_2 s_3 s_2} s_1$

$\underline{s_4\ s_3}\ s_2 s_3 s_1$

$s_3 s_4 s_2 s_3 s_1 \qquad s_4 s_3 s_2\ s_1 s_3$

Q 10

a)



b) bijection