THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 2 Assessment, 2019

School of Mathematics and Statistics

MAST30012 Discrete Mathematics

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 6 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Calculators, tablet devices or computers must not be used.
- No handwritten or print materials may be brought into the exam venue.
- No materials are authorised.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- Marks are allocated to working, mathematical accuracy and clarity of proofs.
- When appropriate answers may be given in terms of binomial or multinomial coefficients or other simple expressions.
- If you have any written or printed material related to the subject in your possession, you should immediately surrender it to an invigilator.
- The total number of marks available is 118.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.
- Mathematical tables, calculators or any other devices are NOT permitted.
- No written or printed material related to the subject may be brought into the exam.

Question 1 (12 marks)

For each part of this question give brief justifications for your answer.

(a) There are 19 distinct doors in a building. How many ways can the doors be opened if:

(i) At least one has to be open.

(ii) Exactly two have to be open.

(iii) At least 3 but no more than 10 doors must be open.

(b) Nine cars are parked in a row, four are red and five are blue. How many ways can the cars be parked so that no red cars are parked next to each other?

(c) How many distinct words of length eight, using letters taken from the alphabet $\{a, b, c, d, e\}$ are there?

Question 2 (12 marks)

(a) Given the three sets A, B and C, state the inclusion/exclusion formula for the size of $A \cup B \cup C$.

(b) Students in a school class all study at least one of the following subjects: mathematics, physics, and geography. Given that 14 students are studying mathematics, 11 are studying geography, 11 are studying physics, 6 are studying mathematics and geography, 5 are studying mathematics and physics, 5 are studying geography and physics, and 3 students are studying all three subjects:

(i) How many students are there in the class?

(ii) How many studied more than one subject?

(iii) How many studied exactly one subject?

Question 3 (12 marks)

(a) By using the interpretation of the binomial coefficient as the number of ways of choosing subsets from a set, show that the number of binomial paths which cross a rectangle from the point $(0,0)$ to the point (n,m) is

$$\binom{n+m}{n}.$$

(b) Write down an expression for the number of binomial paths which cross a rectangle from the point $(0,0)$ to the point $(5n,4m)$, with $n,m>1$, but which are constrained to have a vertex on the point $(2n,m)$. Justify your answer.

(c) State the bijection principle.

(d) Give a **bijective** proof of the formula

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$$

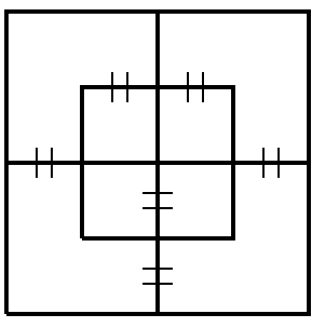
You should argue that your (bijective) function is well defined but you do *not* need to prove any other required properties of the function.

Question 4 (10 marks)

(a) State the floor plan lemma. Do *not* prove the lemma.

(b) Is it possible for a valid floor plan to be such that every room has exactly two doors and the plan has exactly one outside door? Justify your answer.

(c) Copy the following floor plan into your exam answer booklet and draw the paths which demonstrate that there must be an even number of rooms with a single door.



Question 5 (12 marks)

(a) State the binomial coefficient recurrence relation that gives rise to Pascal's triangle and thus (or otherwise) give a proof of the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \,.$$

Hint: Try induction.

(b) Prove the identity

$$\sum_{k=0}^{n} (-1)^k 2^{n-k} \binom{n}{k} = 1 \,.$$

Hint: Use part (a).

Question 6 (12 marks)

Consider the second order recurrence

$$a_n = 7a_{n-1} - 12a_{n-2}, \qquad n = 2, 3, 4, \ldots$$

with initial conditions $a_0 = 1$, $a_1 = 2$.

(a) Compute the coefficients a_2 and a_3.

(b) Introduce the generating function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and show that

$$G(x) = \frac{P(x)}{1 - 7x + 12x^2}.$$

where $P(x)$ is some polynomial.

(c) Write $G(x)$ in partial fraction form and hence find an expression for a_n.

Question 7 (12 marks)

(a) Let $[4] = \{1, 2, 3, 4\}$ and $D = \{(d_1, d_2) \in [4] \times [4] \,:\, d_1 \geq d_2\}$ be the outcome set with all the events $\{(d_i, d_j)\}$ for all $(d_i, d_j) \in D$ equally probable.

(i) What is the probability of the event $\{(2, 1)\}$?

(ii) What is the probability of getting a pair (d_1, d_2) such that $d_1 + d_2 = 5$ (ie. the event $\{(d_1, d_2) \in D \,:\, d_1 + d_2 = 5\}$)?

(b) Consider Dyck paths with $2n$ steps and complete binary trees with $2n + 1$ nodes.

(i) Draw all possible Dyck paths with six steps.

(ii) Draw all possible complete binary trees with seven nodes.

(iii) Choose two of your Dyck paths from (i) and draw a complete binary tree "on top of" each path (ie. the nodes of the tree must be the vertices of the path) and thus illustrate how Dyck paths biject to complete binary trees.

Question 8 (12 marks)

(a) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 1 & 3 & 4 & 6 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 4 & 3 \end{pmatrix}$ be permutations.

(i) Write σ in standard cycle form and as a bipartite graph.

(ii) Compute the composition $\sigma \circ \tau$, writing it as a two line array.

(iii) Write down in standard cycle form the inverse permutation σ^{-1}.

(iv) Give the definition of an inversion in a permutation and hence compute the inversion set of σ.

(v) What is the sign of σ? Justify your answer.

(b) Give a definition of the Stirling numbers of the *first* kind $S_1(n, k)$. From the definition explain *briefly* how the recurrence

$$S_1(n, k) = (n - 1)S_1(n - 1, k) + S_1(n - 1, k - 1).$$

can be proved using a bijection.

Question 9 (12 marks)

(a) Let s_i, $i = 1 \ldots n{-}1$, be simple transpositions in the set of permutations, S_n of $\{1, 2 \cdots, n\}$.

(i) Compute the permutation $\pi \in S_5$ in two-line form, given by the composition of simple transpositions

$$\pi = s_1 s_2 s_1 s_3 s_4 s_2$$

and also write π as a composition of 3-cycles.

(ii) Write down the three basic algebraic relations that simple transpositions satisfy and hence show that

$$s_3 s_1 s_2 s_1 s_4 s_2 = s_3 s_2 s_1 s_4.$$

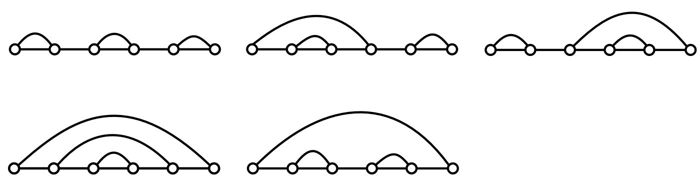
(b) In a 15-puzzle, are these board positions related by sliding moves? Justify your answer.

9	10	13	6
7	5	3	8
1	4		11
14	2	12	15

1	12	13	2
8	11	9	10
3	6	7	15
14	5		4

Question 10 (12 marks)

Link diagrams are constructed by taking $2n$ nodes in a line and joining all the nodes above the line by 'arches' or 'links'. A link connects exactly two distinct nodes and no two links intersect. There are five link diagrams on six nodes ($n = 3$):



The $n = 0$ case is an empty diagram.

(a) Draw all link diagrams for $n \in \{1, 2\}$.

(b) Give a schematic diagram (or otherwise) showing how a link diagram with $2n$ nodes can be uniquely factorised into two smaller link diagrams.

(c) Use the unique factorisation of link diagrams to give a bijective proof that the number of link diagrams with $2n$ nodes satisfies the Catalan recurrence

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-1} C_0$$

with $C_0 = 1$. In your proof you do not need to prove your function is a bijection.

End of Exam—Total Available Marks = 118