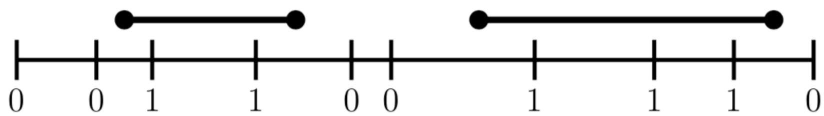
The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 5: Parity and Lattice Paths – Solutions

Q1: (a) We have for example:



We see that:

$$\#\text{ subintervals with different labelling at each endpoint} = 2\times\ (\#\text{ paths})$$

Hence the parity of this number is even.

(b) Deleting the rightmost string of 0's reduces by one the number of $(1,0)$–subintervals and makes the endpoint have label 1. Subtracting one from an even number gives an odd number and we can thus deduce the result in the box from (a).

Q2: (a) Suppose the top left corner is white. Then the bottom right corner is also white. After their removal there are thus 32 black and 30 white squares left.

(b) A 2×1 rectangular tile must always cover 1 white and 1 black square.

(c) From (b) is follows that any tiling with 2×1 rectangular tiles must cover an equal number of black and white squares. From (a) we know there are more black than white squares. It follows that we cannot tile an 8×8 board with 2×1 rectangular tiles if the top left and bottom right squares have been removed.

Q3: (a) Suppose that in some row (respectively column) there are k black squares and thus $8 - k$ white squares. Reversing the colours gives $8 - k$ black and k white squares. The change in the number of black squares is thus $(8 - k) - k = 8 - 2k$. This is an even number. So there is always an even number of black squares in total

(b) Starting with an even number of black squares the stated 'reversing operation' always leaves an even number of black squares, so we can never end up with a single black square.

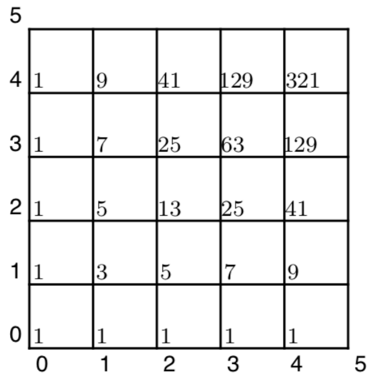
Q4: Colouring the 5×5 square board as for a chess board gives 12 white squares and 13 black squares (or vice versa). The neighbours of white squares are black squares. The 12 spiders on the white squares must move to the 13 black squares leaving at least one black square empty. By the way the 13 spiders on the black squares move to the 12 white squares leaving at least one white square with at least 2 spiders by the pigeonhole principle.

Q5: (a) Given the step set S one can reach a point $(m,n) \in \mathbb{N}_0^2$ via paths from points to the left $(m-1,n)$, below $(m,n-1)$ or diagonally to the left and below $(m-1,n-1)$. This construction of paths to (m,n) is clearly unique and exhaustive. So

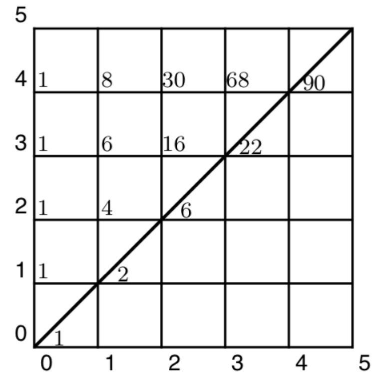
$$D_{m,n} = D_{m-1,n} + D_{m,n-1} + D_{m-1,n-1}, \quad D_{0,0} = 1,\ D_{m,n} = 0 \ \ m,n < 0.$$

(b,c) Grid counts:

Grid count for $D_{m,n}$.



Grid count for ballot version.



The 'boundary conditions' in ballot case: $B_{0,0} = 1$, $B_{0,n} = 1$, $B_{m,n} = 0$ if $m > n$

(d) From the two grids we see that the 'ballot' counts $B_{m,n} = D_{m,n} - D_{n+1,m-1}$, e.g,

$$\begin{aligned} B_{1,3} &= 6 = 7 - 1 = D_{1,3} - D_{4,0}, \\ B_{3,3} &= 22 = 63 - 41 = D_{3,3} - D_{4,2}, \\ B_{2,4} &= 30 = 41 - 11 = D_{2,4} - D_{5,1}. \end{aligned}$$

(e) Take a Delannoy path ending at (m,n) with k diagonal step. There are then $m-k$ East steps and $n-k$ North steps. The total number number of steps is $m+n-k$. Forming a Delannoy path is like putting $m-k$ elements into box marked 'East', $n-k$ elements into box marked 'North' and k elements into box marked 'Diagonal'. The number of such paths are therefore given by the multinomial coefficient

$$\binom{m+n-k}{k, m-k, n-k} = \binom{m+n-k}{n-k}\binom{m}{k}\binom{m-k}{m-k} = \binom{m+n-k}{n-k}\binom{m}{k},$$

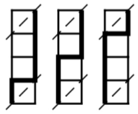
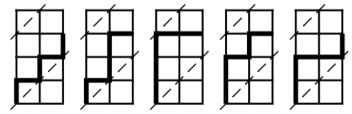
where we used a result from Practice Class 2. Summing over k we get

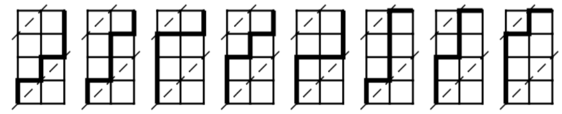
$$D_{m,n} = \sum_{k\geq 0}\binom{m}{k}\binom{m+n-k}{n-k} = \sum_{k\geq 0}\binom{m}{m-k}\binom{m+n-k}{m} = \sum_{k\geq 0}\binom{m}{k}\binom{n+k}{m}.$$

We used that $\binom{a}{b} = \binom{a}{a-b}$ and then changed summation variable from k to $m-k$.

Alternatively: Take a Delannoy path ending at (m,n) with k diagonal step. First form an ED path with $m-k$ East and k Diagonal steps. There are $\binom{m}{k}$ of these. We then add $n-k$ North steps to this path. The North steps can be inserted in $m+1$ positions and we may place from 0 to $n-k$ North steps in a given positions. This is a problem of sampling with **replacement**. The number of choices for each ED-path is therefore $\binom{m+1+n-k-1}{n-k} = \binom{m+n-k}{n-k}$. The final result follows as per above.

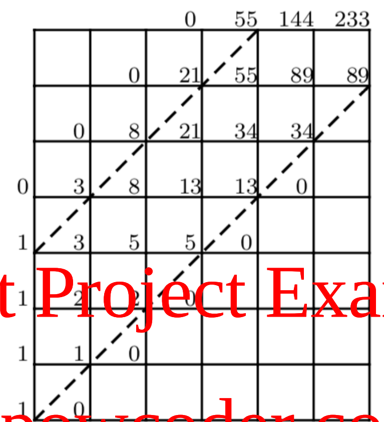
Q6: (a) The paths in $B_1^3(3)$, $B_2^1(3)$ and $B_2^2(3)$ are

$B_1^3(3) = \Big\{$  $\Big\}$ $|B_1^3(3)| = 3$. $B_2^1(3) = \Big\{$  $\Big\}$ $|B_2^1(3)| = 5$.

$B_2^2(3) = \Big\{$  $\Big\}$ $|B_2^2(3)| = 8$.

(b) The basic recurrence is as for binomial and ballot paths but with more restrictions:

$$B_n^h(c) = B_{n-1}^{h+1}(c) + B_n^{h-1}(c), B_n^{-1}(c) = 0, B_n^{c+1}(c) = 0, B_0^0(c) = 1.$$



$|B_n^h(3)|$

Now that looks rather Fibonacci like and indeed we have

$$\{F_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \cdots\}$$

So we conjecture that $|B_n^0(3)| = F_{2n-1}, n \geq 1$.
We also have $|B_n^1(3)| = F_{2n+1}, n \geq 0$; $|B_n^2(3)| = |B_n^3(3)| = F_{2n+2}, n \geq 0$.

(c) $v_m \ = \ T^m v_0 \ = \ T \cdot T^{m-1} v_0 \ = \ T v_{m-1}$.

$$v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix},$$

$$v_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 13 \end{pmatrix} = \begin{pmatrix} 13 \\ 21 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 13 \\ 21 \end{pmatrix} = \begin{pmatrix} 21 \\ 34 \end{pmatrix}$$

Comparing with (b) we conjecture that the components of v_m are related to the pair of numbers $|B_n^h(3)|$ and $|B_n^{h+2}(3)|$. Explicitly we conjecture that:

$$v_m \ = \ \begin{pmatrix} x_m \\ y_m \end{pmatrix} \text{ then } x_m = y_{m-1} \quad \text{and} \quad x_m = \begin{cases} \left| B^0_{\frac{m+3}{2}}(3) \right| & \text{if } m \text{ is odd} \\ \left| B^3_{\frac{m}{2}}(3) \right| & \text{if } m \text{ is even} \end{cases}$$

(d) If D is the diagonal matrix of eigenvalues of T then $\exists Q$ so $T = QDQ^{-1}$ and hence $T^m = QD^mQ^{-1} \Rightarrow v_m = QD^mQ^{-1}v_0$. The eigenvalues λ_1 and λ_2 of the matrix T, are obtained from the characteristic equation:

$$\det\,(T - \lambda I) \;=\; \det\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \;=\; 0$$

$$\Rightarrow \quad \lambda^2 - \lambda - 1 = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{2}(1+\sqrt{5}),\;\; \lambda_2 = \frac{1}{2}(1-\sqrt{5}).$$

With this we can then show that

$$Q = \frac{1}{5^{1/4}} \begin{pmatrix} 1 & -1 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \qquad \text{and} \qquad Q^{-1} = \frac{1}{5^{1/4}} \begin{pmatrix} -\lambda_2 & 1 \\ -\lambda_1 & 1 \end{pmatrix}$$

and thus

$$\begin{aligned} QD^mQ^{-1} &= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -1 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{pmatrix} \begin{pmatrix} -\lambda_2 & 1 \\ -\lambda_1 & 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -1 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} -\lambda_2\lambda_1^m & \lambda_1^m \\ -\lambda_1\lambda_2^m & \lambda_2^m \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{m-1} - \lambda_2^{m-1} & \lambda_1^m - \lambda_2^m \\ & \\ \lambda_1^m - \lambda_2^m & \lambda_1^{m+1} - \lambda_2^{m+1} \end{pmatrix}, \end{aligned}$$

where we used $\lambda_1\lambda_2 = -1$. We recognise the entries as Fibonacci numbers.