The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 6: Difference Equations and Generating Functions – Answers

Q1: (a) For each permutation of $\{1, 2, \ldots, n\}$ insert the element $n + 1$ in any position

(b) We have $A_1 = 1$. To verify that $A_n = n!$ satisfies the recurrence and initial condition, note that the formula gives

$$A_1 = 1! = 1 \quad \text{and} \quad \text{LHS} = (n+1)! \quad \text{RHS} = (n+1)n! = (n+1)!$$

as required.

Q2: (a) Consider a k-tuple, $k < n$ of elements from $\{1, 2, \cdots, n\}$. This k-tuple does not include $n - k$ elements from the set. Any of these elements can be chosen to be the final entry in the $(k+1)$-tuple

(b) $B_{n,1} = n$. With $B_{n,k} = \frac{n!}{(n-k)!}$, we have $B_{n,1} = \frac{n!}{(n-1)!} = n$, and for the recurrence

$$\text{LHS} = \frac{n!}{(n-k-1)!} \quad \text{RHS} = (n-k)\frac{n!}{(n-k)!} = \frac{n!}{(n-k-1)!}$$

as required.

Q3: Consider the recurrence

$$a_n = 5a_{n-1} - 6a_{n-2}, \qquad n = 3, 4, \ldots$$

with the initial conditions $a_1 = 1$, $a_2 = 5$.

(a) Introduce generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ take out first two terms then use $a_n = 5a_{n-1} - 6a_{n-2}$, change summation index and solve for $A(x)$.

(b) $6x^2 - 5x + 1 = (1 - 3x)(1 - 2x)$ so we can write

$$A(x) = \frac{x}{6x^2 - 5x + 1} = \frac{a}{1 - 3x} + \frac{b}{1 - 2x}.$$

Q4: (a)

$$\underbrace{(1 + x + x^2)}_{\substack{\text{Apples} \\ 1 \leftrightarrow \text{ no apples} \\ x \leftrightarrow \text{ 1 apple} \\ x^2 \leftrightarrow \text{ 2 apples}}} \underbrace{(1 + x)}_{\text{Pear}} \underbrace{(1 + x + x^2)}_{\text{Oranges}} \underbrace{(1 + x)}_{\text{Banana}}$$

(b) $a_0 = 1, a_1 = 4, a_2 = 8, a_3 = 10, a_4 = 8, a_5 = 4, a_6 = 1.$

Q5: (a) In $(1+x+x^2+x^3+\cdots)^n$ each factor corresponds to the objects deposited in a given box.

(b) The geometric series tells us that $1+x+x^2+x^3+\cdots = 1/(1-x)$.

(c) Taylor series for $f(x)$ at $x=0$ is $f(x) = \sum_{p=0}^{\infty} \frac{f^{(p)}(0)}{p!} x^p$. Now set $f(x) = (1+x)^\alpha$.

(d) Make the replacement $x \mapsto -x$ and $\alpha = -n$ (n a positive integer).