The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Problems on Proof and Logic – Solutions

Q1: (a) Statement A: $n+1$ blocks are placed in n boxes.

Statement B: At least one box contains 2 or more blocks.

(b) Statement A: $f : \Delta \to \Delta$ is a mapping from a triangular shaped region Δ into the same region and f is continuous.

Statement B: There exist a fixed point (i.e. a point (x, y) such that $f((x, y)) = (x, y)$).

Q2: (a) The converse is: If a square matrix is invertible then the rows are linearly independent.

Yes the converse statement is true. Consequently we have

The rows of a square matrix are linearly independent if and only if the matrix is invertible.

(b) If $x \in \mathbb{Z}$, then $\sin \pi x = 0$.

If $\sin \pi x = 0$, then $x \in \mathbb{Z}$.

Q3: (a) If no box contains 2 or more blocks, then there were not $n+1$ blocks placed in the n boxes.

(b) If box 1 contains less than a_1 blocks and box 2 contains less than a_2 blocks then the two boxes do not contain $a_1 + a_2 - 1$ blocks.

To prove this proposition note from the assumptions that

total # of blocks $\le (a_1 - 1) + (a_2 - 1) = a_1 + a_2 - 2$

and hence the two boxes do not contain $a_1 + a_2 - 1$ blocks.

(c) If for real numbers $x_1, \ldots, x_n$ there is no value of i for which $x_i > i$, then $x_1 + \cdots + x_n \le n(n+1)/2$.

Proof: From the assumptions we have

$$x_1 + \cdots + x_n \le 1 + 2 + \cdots + n = n(n+1)/2.$$

Q4: (a) Assuming each box contains one or less blocks means that the maximum number of blocks is n. This contradicts the statement that $n+1$ blocks are places in the n boxes.

(b) Suppose that neither box 1 contains a_1 or more blocks nor box 2 contains a_2 or more blocks and that the boxes contains $a_1 + a_2 - 1$ blocks.

By the assumptions

total # of blocks $\leq (a_1 - 1) + (a_2 - 1) = a_1 + a_2 - 2$

This contradicts the statement that the boxes contains $a_1 + a_2 - 1$ blocks.

(c) Suppose for real numbers $x_1, \dots, x_n,\ x_i \leq i$ for all i and $x_1 + \cdots + x_n > n(n+1)/2$.

By the assumptions

$$x_1 + \cdots + x_n \leq 1 + 2 + \cdots + n = n(n+1)/2.$$

This contradicts the statement that $x_1 + \cdots + x_n > n(n+1)/2$.

Q5: (a) LHS $= \sum_{p=1}^{1} p = 1$ RHS $= \frac{1(1+1)}{2} = 1$.

Hence the statement is true for $n = 1$

(b) Must verify statement is true for $n = k$ assuming it is true for $n < k$. Now

$$\begin{aligned}
\sum_{p=1}^{k} p &= \sum_{p=1}^{k-1} p + k \\
&= \frac{(k-1)k}{2} + k \qquad \text{(by induction hypothesis)} \\
&= \frac{(k-1)k + 2k}{2} \\
&= \frac{k(k+1)}{2}. \qquad \text{(as required)}
\end{aligned}$$

Q6: (a) $F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8.$

(b) First check the case $n = 1$:

LHS $= \sum_{p=1}^{1} (F_p)^2 = 1^2 = 1,$

RHS $= F_1 F_2 = 1 \cdot 1 = 1$, so true.

Must verify statement is true for $n = k$ assuming it is true for $n < k$. Now

$$
\begin{aligned}
\sum_{p=1}^{k}(F_p)^2 &= \sum_{p=1}^{k-1}(F_p)^2 + (F_k)^2 \\
&= F_{k-1}F_k + (F_k)^2 \qquad \text{(by induction hypothesis)} \\
&= F_k(F_{k-1} + F_k) \\
&= F_k F_{k+1} \qquad \text{(by the basic recurrence relation)}
\end{aligned}
$$