The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 5: Parity and Lattice Paths – Answers

Q1: (a) Draw a picture and consider #(subintervals with different labelling at each endpoint).

(b) Deleting the rightmost string of 0's reduces by one the number of $(1, 0)$–subintervals.

Q2: (a) Suppose the top left corner is white. Then the bottom right corner is also white. After their removal there are thus 32 black and 30 white squares left.

(b) A 2×1 rectangular tile must always cover 1 white and 1 black square.

(c) Now consider what happens in a tiling with 2×1 rectangles.

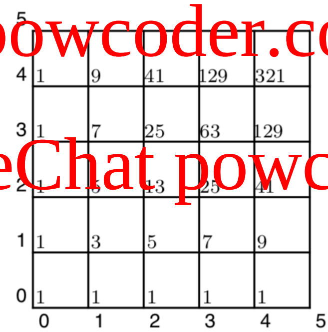
Q3: (a) Consider a row (column) with k black squares the change in the number of black squares as colours are reversed.

(b) Now use parity to complete the argument.

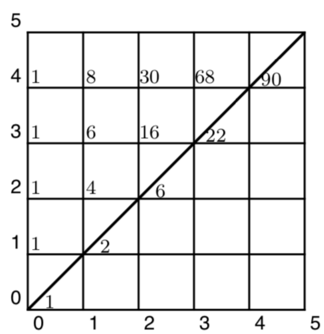
Q4: Colour the 5×5 square board as for a chess board and consider the number of white and black squares

Q5: (a) $D_{m,n} = D_{m-1,n} + D_{m,n-1} + D_{m-1,n-1}$, $D_{0,0} = 1$; $D_{m,n} = 0$ $m, n < 0$.

(b) The grid count for $D_{m,n}$. By symmetry $D_{m,n} = D_{n,m}$



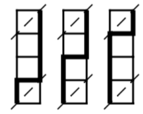
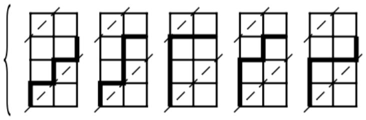
(c) Repeat with 'boundary conditions' $B_{0,0} = 1$, $B_{0,n} = 1$, $B_{m,n} = 0$ if $m > n$

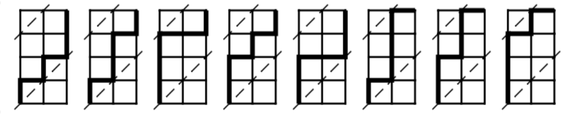


(d) $B_{m,n} = D_{m,n} - D_{n+1,m-1}$.

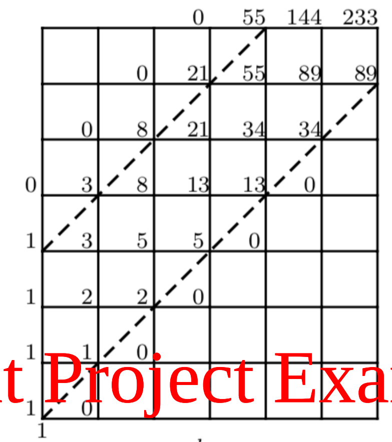
(e) Proof required.

Q6: (a) The paths in $B_1^3(3)$, $B_2^1(3)$ and $B_2^2(3)$ are

$B_1^3(3) = \{$  $\}$ $|B_1^3(3)| = 3$. $B_2^1(3) = \{$  $\}$ $|B_2^1(3)| = 5$.

$B_2^2(3) = \{$  $\}$ $|B_2^2(3)| = 8$.

(b) $$B_n^h(c) = B_{n-1}^{h+1}(c) + B_n^{h-1}(c), B_n^{-1}(c) = 0, B_n^{c+1}(c) = 0, B_0^0(c) = 1.$$



$|B_n^h(3)|$

Conjecture $|B_n^0(3)| = F_{2n-1}, n \geq 1$.

(c) $v_m = T^m v_0 = T \cdot T^{m-1} v_0 = T v_{m-1}$.

$$v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix},$$

$$v_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 13 \end{pmatrix} = \begin{pmatrix} 13 \\ 21 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 13 \\ 21 \end{pmatrix} = \begin{pmatrix} 21 \\ 34 \end{pmatrix}$$

Conjecture :

$$v_m = \begin{pmatrix} x_m \\ y_m \end{pmatrix} \text{ then } x_m = y_{m-1} \quad \text{and} \quad x_m = \begin{cases} \left|B^0_{\frac{m+3}{2}}(3)\right| & \text{if } m \text{ is odd} \\ \left|B^3_{\frac{m}{2}}(3)\right| & \text{if } m \text{ is even} \end{cases}$$

(d) The eigenvalues λ_1 and λ_2 of the matrix T are $\lambda_1 = \frac{1}{2}(1+\sqrt{5})$, $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$. With this we can then show that the entries of T^m are Fibonacci numbers.