The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 1: Direct Enumeration – Solutions

Q1: Partition the problem on squared distance from origin:

$$S_i = \{\, (x,y) \,|\, x, y \in \mathbb{Z}, x^2 + y^2 = i \,\}$$

So there are six easy to check cases:

$$\begin{aligned}
&S_0 = \{\, (0,0) \,\} && |S_0| = 1\\
&S_1 = \{\, (1,0), (-1,0), (0,1), (0,-1) \,\} && |S_1| = 4\\
&S_2 = \{\, (1,1), (-1,1), (1,-1), (-1,-1) \,\} && |S_2| = 4\\
&S_3 = \emptyset && |S_3| = 0\\
&S_4 = \{\, (2,0), (-2,0), (0,2), (0,-2) \,\} && |S_4| = 4\\
&S_5 = \{\, (1,2), (2,1), \dots \,\} && |S_5| = 8
\end{aligned}$$

Since $S_i \cap S_j = \emptyset \quad \forall i \neq j$ and $S = S_1 \cup S_2 \cup \cdots \cup S_6$

$$\Rightarrow |S| = \sum_{i=0}^{5} |S_i| = 21 \text{ by the Addition Principle}$$

Q2: Partition S according to the value of $a \in \{1, 2, \dots, 100\}$.

For a given value of a the number of choices for b is $100 - k$
(if $a = k$ then $b \in A_k = \{k+1, k+2, \dots, 100\}$) and similarly for c

$$\Rightarrow |S| = |A_1|^2 + |A_2|^2 + \cdots + |A_{99}|^2 = 99^2 + 98^2 + \cdots + 1^2$$

Now use that $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ (Can be proved using induction)

$$\Rightarrow |S| = \frac{1}{6} 99 \cdot 100 \cdot 199.$$

We used the addition and multiplication principles.

Q3: The sum is even if both dice roll even or both roll odd. The are 3 even faces so the number of (even,even)-rolls is $3^2 = 9$. Similarly for (odd,odd)-rolls. So there are 18 possible rolls of two dice with an even sum. Hence the probability is $1/2$.

Q4: We use the complement principle. There are 36^8 possible passwords and 26^8 of these contain only letters. So there are $36^8 - 26^8$ passwords containing at least one digit.

Q5: Let $A = \{a, b, c, d, e, f, g\}$, $|A| = 7$

(a) $w \in A^5 \quad \Rightarrow \quad$ #words $= 7^5$.

(b) w is a 5-permutation of $A \quad \Rightarrow \quad$ #words $= (7)_5 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$.

(c) $7 \cdot 6^4$ since 7 choices for first letter then 6 choices for subsequent letters (next letter can't equal previous letter).

(d) This is more difficult. So lets start by doing smaller words:

	Starting Letter							
$\lvert w\rvert$	a	b	c	d	e	f	g	Total
1	1	1	1	1	1	1	1	7
2	7	6	5	4	3	2	1	28
3	28	21	15	10	6	3	1	84
4	84	56	35	20	10	4	1	210
5	210	126	70	35	15	5	1	462

The first two lines $|w| = 1, 2$ should be obvious. So lets look at $|w| = 3$. If we start with a then the next two letters can by any sequence of letters of length 2 so the sum of all entries in the row $|w| = 2$. If we start with b then we can take any sequence from the row above starting with a letter b or latter (c, d etc). This observation generalise. If $|w| = k$ then the number or words starting with α is the sum of the number of words of length $k-1$ starting with α or a letter following α alphabetically. So the value of any entry $C_{|w|,k}$ in the table is

$$C_{|w|,k} = \sum_{j=k}^{7} C_{|w|-1,j}$$

with initial condition $C_{1,k} = 1$. By the way the total is $\binom{|A|+|w|-1}{|w|}$ a result which we shall be able to prove by a simple argument later in the course. You may also note nice relations between the entries such as $C_{|w|,k} = C_{|w|,k-1} - C_{|w|-1,k-1}$. Indeed we have $C_{|w|,k} = \binom{|A|+|w|-k-1}{|w|-1}$

Q6: Let n be the number of digits in the bit sting. There are two cases:

n **even:** There are $2^{n/2}$ palindromes. The digits in the first half ($n/2$ of these) can be either 0 or 1 and then the digits in the second half are fixed by the palindrome constraint that the digit in position $n + 1 - k$ must equal that in position $k = 1, \ldots, n/2$.

n **odd:** There are $2^{(n+1)/2}$ palindromes. Similar arguments to previous case.

Q7: Let r indicate a red card and b a blue car. The restriction no red cars next to one another means we are looking for words of the form

$$w = b_0 r b_1 r b_1 r b_1 r b_0$$

where b_0 indicates possible blue cars and b_1 at least one blue car. So we use up 3 blue cars in the b_1 positions. The remaining 2 blue cars can then be placed in any of the five b positions.

There are two cases:

The two cars go into the same position in $\binom{5}{1} = 5$ possible ways.

The two cars go into two different positions in $\binom{5}{2} = 10$ possible ways.

So all in all there are 15 possible ways of parking the 9 cars.