School of Mathematics and Statistics

MAST30012 Discrete Mathematics 2021

Assignment 2

Due 23:59pm Monday 27 September 2021

Student Name	*Student Number*
Tutor's Name	*Practice Class Day/Time*

Submit your assignment online in Canvas LMS.

Please attach this cover sheet to your assignment or use a blank sheet of paper as the first page of your assignment with Student Name, Student Number, Tutor's Name, Practice Class Day/Time clearly stated. .

- Late submission will not be accepted unless accompanied by a medical certificate (or a similar special consideration). If there are extenuating circumstances you need to contact your lecturer, preferably prior to the submission deadline. Medical certificates are usually required.
- Information on how to submit assignments can be found in the Canvas LMS.
- Full working must be shown in your solutions.
- Marks will be deducted for incomplete working, insufficient justification or incorrect notation.
- Unless otherwise stated, proofs of identities etc. must use combinatorial arguments.
- There are 5 problems (on three pages) each worth 6 marks.

Q1: Choose $n+1$ different integers from the set $\{1, 2, 3, \ldots, 2n\}$. Are the following statements true or false? For a true statement, give a rigorous proof of it; for a false statement, give a counterexample to disprove it.

(a) At least two of the chosen integers have greatest common divisor 1.

(b) Among the chosen integers there must be a pair such that one is twice the other.

(c) Among the chosen integers there must be a pair such that one divides the other.

Q2: Consider the following variations on Sperner labelling problems.

(a) Suppose that an arbitrary number of distinct points are placed along the circumference of a circle, and each point is labelled by the symbol A or B in an arbitrary manner.

Prove that the number of pairs of neighbouring points labelled AB or BA is even.

(b) Consider a triangulated triangle. Let each vertex of all the triangles be labelled A, B or C arbitrarily (including the vertices on the outside perimeter).

As in the proof of Sperner's lemma place doors on each AB edge.

Show that the number of segments on the outside perimeter labelled AB has the same parity as the number of triangles in the triangulation labelled ABC. Note that the order of the labels is not relevant, so AB and BA are equivalent labellings.

Q3: Using the generating function, $(1+x)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k$, show that

(a) $$\sum_{k=1}^{n} k\binom{n}{k} \;=\; n2^{n-1};$$

(b) $$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} \;=\; \frac{2^{n+1}-1}{n+1};$$

(c) $$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+1}\binom{n}{k} \;=\; \frac{n}{n+1}.$$

Q4: Let F_n denote the nth Fibonacci number. Recall that F_{n+1} counts the number of pavings of an n-board using monomers and dimers. Hence give combinatorial proofs of the following identities:

(a)

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \;=\; F_{n+1};$$

(b)

$$F_n^2 + F_{n+1}^2 \;=\; F_{2n+1}.$$

Q5: Starting with the letter A, make sequences of symbols by the substitution rule

$$\mathrm{A} \mapsto \mathrm{B}, \quad \mathrm{B} \mapsto \mathrm{AAB}.$$

(a) Write out a list of the five successive sequences which are resulted from applications of this rule (including the initial sequence A).

(b) Let A_n, B_n be, respectively, the number of letters A, B in the sequence after n applications of the substitution rule. Let S_n denote the total number of letters in the sequence after n applications of the substitution rule. Find recurrences for A_n, B_n, and use these to show that

$$S_n = S_{n-1} + 2S_{n-2}, \qquad (n \geq 2)$$

with $S_0 = 1$, $S_1 = 1$.

(c) Introduce the generating function

$$S(x) = \sum_{n=0}^{\infty} S_n x^n$$

and show that

$$S(x) = \frac{1}{1 - x - 2x^2}.$$

(d) Use a partial fraction expansion of $S(x)$ to find an exact expression for S_n.