The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 9: Permutations – Solutions

Q1: (a) $264351 \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 3 & 5 & 1 \end{pmatrix} \;=\; (126)(34) \;=\; (16)(12)(34)$

(b) $315642 \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 6 & 4 & 2 \end{pmatrix} \;=\; (135462) \;=\; (12)(16)(14)(15)(13)$

Q2: (a) An r-cycle σ can be written as the product of $r-1$ 2-cycles τ_j. Hence

$$\mathrm{sgn}(\sigma) \;=\; \mathrm{sgn}(\tau_1 \circ \tau_2 \circ \cdots \circ \tau_{r-1}) \;=\; \mathrm{sgn}(\tau_1) \cdot \mathrm{sgn}(\tau_2) \cdots \mathrm{sgn}(\tau_{r-1}) \;=\; (-1)^{r-1},$$

since $\mathrm{sgn}\,((i\,j)) \;=\; -1$.

(b) $\sigma \;=\; 87654321 \;=\; (18)(27)(36)(45)$, $\qquad \tau \;=\; 46213875 \;=\; (14)(26853)$.

(c) $\mathrm{sgn}(\sigma) \;=\; (-1)^4 \;=\; +1$, $\qquad \mathrm{sgn}(\tau) \;=\; -1 \cdot (-1)^{5-1} \;=\; -1$.

(d) $(156)(247)(38) \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 8 & 7 & 6 & 1 & 2 & 3 \end{pmatrix}$

$$(1587)(23)(46) \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 3 & 2 & 6 & 8 & 4 & 1 & 7 \end{pmatrix}$$

Q3:

$$\begin{aligned} \sigma &= s_5 s_4 s_3 s_1 s_2 s_1 \\ &= (56)(45)(34)(12)(23)(12) \\ &= (16543)(2) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 3 & 4 & 5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \tau &= s_4 s_3 s_4 s_3 s_5 s_4 s_5 \\ &= (45)(34)(45)(34)(56)(45)(56) \\ &= (1)(2)(3465) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 6 & 3 & 5 \end{pmatrix} \end{aligned}$$

Q4: $(123)(234)(324) \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \qquad (213)(324)(324) \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$

Q5: (a)

$$\begin{aligned} A_n(x) &= \sum_{k=0}^{n} S_1(n,k)x^k \\ &= \sum_{k=0}^{n} \big((n-1)S_1(n-1,k) + S_1(n-1,k-1)\big)x^k \\ &= (n-1)\sum_{k=0}^{n} S_1(n-1,k)x^k \;+\; \sum_{k=0}^{n} S_1(n-1,k-1)x^k. \end{aligned}$$

Now

$$\begin{aligned} \sum_{k=0}^{n} S_1(n-1,k)x^k &= \sum_{k=0}^{n-1} S_1(n-1,k)x^k \qquad (\text{since } \; S_1(n-1,n)=0) \\ &= A_{n-1}(x), \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{n} S_1(n-1,k-1)x^k &= \sum_{k=1}^{n} S_1(n-1,k-1)x^k \qquad (\text{since } \; S_1(n-1,-1)=0) \\ &= S_1(n-1,0)x + S_1(n-1,1)x^2 + \cdots + S_1(n-1,n-1)x^n \\ &= xA_{n-1}(x). \end{aligned}$$

So

$$A_n(x) \;=\; (n-1)A_{n-1}(x) + xA_{n-1}(x) \;=\; (x+n-1)A_{n-1}(x).$$

(b)

$$\begin{aligned} A_0(x) &= S_1(0,0) \;=\; 1 \\ A_1(x) &= xA_0(x) = x \\ A_2(x) &= (x+1)A_1(x) = (x+1)x \\ A_3(x) &= (x+2)A_2(x) = (x+2)(x+1)x \\ &\;\vdots \\ A_n(x) &= (x+n-1)A_{n-1}(x) = (x+n-1)(x+n-2)\cdots(x+2)(x+1)x \end{aligned}$$

Q6: We have to seat n people at $n-2$ tables. There are two cases:

Case 1: Three people are seated at one table the remaining $n-3$ people must then be seated one each at the remaining $n-3$ tables. We can choose the 3 people in $\binom{n}{3}$ ways and there are $(3-1)! = 2$ ways to seat the 3 people (k people can be seated at a table in $(k-1)!$ ways since person 1 can be seated anywhere and then there are $(k-1)!$ arrangements of the remaining $k-1$ people). So all in all $2\binom{n}{3}$ seating arrangements for this case.

Case 2: Two people each are seated at two of the tables with the remaining $n-4$ seated one each at the remaining $n-4$ tables. The four people at the two tables can be chosen in $\binom{n}{2}\binom{n-2}{2}$ ways. However, we don't care about the ordering of the two tables so we divide by the 2! ways of arranging the two tables. So $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ seating arrangements for this case.

Putting everything together we have: $\quad S_1(n,n-2) = 2\dbinom{n}{3} + \dfrac{1}{2}\dbinom{n}{2}\dbinom{n-2}{2}.$