The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 6: Difference Equations and Generating Functions — Solutions

Q1: (a) Consider a permutation of $\{1, 2, \cdots, n\}$.

For each permutation insert the element $n+1$ in any position. There are $n+1$ positions giving $n+1$ permutations of $\{1, 2, \cdots, n+1\}$.

For $n = 2$, a permutation of $\{1, 2\}$ is 21. There are 3 places to insert the element $n+1 = 3$:

321 231 213

With A_n denoting the number of permutations of $\{1, 2, \cdots, n\}$ it follows taht

$$A_{n+1} = (n+1)A_n$$

(b) We have $A_1 = 1$. To verify that $A_n = n!$ satisfies the recurrence and initial condition, note that the formula gives

$$A_1 = 1! = 1 \quad \text{and} \quad \text{LHS} = (n+1)! \quad \text{RHS} = (n+1)n! = (n+1)!$$

as required.

Q2: (a) Consider a k-tuple, $k < n$ of elements from $\{1, 2, \cdots, n\}$. This k-tuple does not include $n-k$ elements from the set. Any of these elements can be chosen to be the final entry in the $(k+1)$-tuple e.g. with $k = 1$ and $n = 3$ the 1-tuples are

(1) (2) (3)

These can be extended as follows:

$$\begin{array}{ccc} (1) & (2) & (3) \\ \swarrow\ \searrow & \swarrow\ \searrow & \swarrow\ \searrow \\ (1,2)\quad(1,3) & (2,1)\quad(2,3) & (3,1)\quad(3,2) \end{array}$$

Hence with $B_{n,k}$ denoting the number of ordered k-tuples which can be from n elements

$$B_{n,k+1} = (n-k)B_{n,k} \qquad (k = 1, \ldots, n-1).$$

(b) $B_{n,1} = \#$ 1-tuples from $\{1, 2, \cdots, n\} = n$.
With $B_{n,k} = \frac{n!}{(n-k)!}$, we have $B_{n,1} = \frac{n!}{(n-1)!} = n$, and for the recurrence

$$\text{LHS} = \frac{n!}{(n-k-1)!} \quad \text{RHS} = (n-k)\frac{n!}{(n-k)!} = \frac{n!}{(n-k-1)!}$$

as required.

Q3: (a) Setting $n = 2$ in the recurrence gives $a_2 = 5a_1 - 6a_0$. Since $a_2 = 5$ and $a_1 = 1$ it follows that $a_0 = 0$.

$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} a_n x^n \\
&= x + \sum_{n=2}^{\infty} (5a_{n-1} - 6a_{n-2}) x^n = x + 5x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 6x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= x + 5x \sum_{n=1}^{\infty} a_n x^n - 6x^2 \sum_{n=0}^{\infty} a_n x^n = x + (5x - 6x^2) A(x)
\end{aligned}$$

Hence

$$A(x) = \frac{x}{6x^2 - 5x + 1}$$

(b) Now $6x^2 - 5x + 1 = (1 - 3x)(1 - 2x)$ so we can write

$$A(x) = \frac{x}{6x^2 - 5x + 1} = \frac{a}{1 - 3x} + \frac{b}{1 - 2x}.$$

Put the RHS on a common denominator, equate terms in the numerator and solve for a and b, giving that $a = 1$ and $b = -1$. Thus using the geometric series

$$A(x) = \frac{1}{1 - 3x} - \frac{1}{1 - 2x} = \sum_{n=0}^{\infty} (3x)^n - \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} (3^n - 2^n) x^n.$$

Hence $a_n = [x^n] A(x) = 3^n - 2^n$.

Q4: (a)

$$\underbrace{(1 + x + x^2)}_{\substack{\text{Apples} \\ 1 \leftrightarrow \text{ no apples} \\ x \leftrightarrow \text{ 1 apple} \\ x^2 \leftrightarrow \text{ 2 apples}}} \underbrace{(1 + x)}_{\text{Pear}} \underbrace{(1 + x + x^2)}_{\text{Oranges}} \underbrace{(1 + x)}_{\text{Banana}}$$

(b)

$$\begin{aligned}
& (1 + x + x^2)(1 + x)(1 + x + x^2)(1 + x) \\
= \; & (1 + 2x + 2x^2 + x^3)^2 \\
= \; & 1 + 4x + 8x^2 + 10x^3 + 8x^4 + 4x^5 + x^6
\end{aligned}$$

We see that: $a_0 = 1, a_1 = 4, a_2 = 8, a_3 = 10, a_4 = 8, a_5 = 4, a_6 = 1$.

Q5: (a) In $(1+x+x^2+x^3+\cdots)^n$ each factor of $(1+x+x^2+x^3+\cdots)$ corresponds to the objects deposited in a given box with each term x^p corresponding to depositing p objects in the given box.

(b) The geometric series tells us that $1+x+x^2+x^3+\cdots = 1/(1-x)$, and hence

$$(1+x+x^2+x^3+\cdots)^n = \left(\frac{1}{1-x}\right)^n.$$

(c) A Taylor series for $f(x)$ about the origin is $f(x) = \sum_{p=0}^{\infty} \frac{f^{(p)}(0)}{p!} x^p$. Here

$$\begin{aligned}
f(x) &= (1+x)^\alpha & &\Rightarrow & f(0) &= 1 \\
f'(x) &= \alpha(1+x)^{\alpha-1} & &\Rightarrow & f'(0) &= \alpha \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} & &\Rightarrow & f''(0) &= \alpha(\alpha-1) \\
&\vdots \\
f^{(p)}(x) &= \alpha(\alpha-1)\cdots(\alpha-p+1)(1+x)^{\alpha-p} & &\Rightarrow & f^{(p)}(0) &= \alpha(\alpha-1)\cdots(\alpha-p+1)
\end{aligned}$$

Thus

$$(1+x)^\alpha = \sum_{p=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-p+1)}{p!} x^p.$$

(d) Making the replacement $x \mapsto -x$ and $\alpha = -n$ (n a positive integer) we see that

$$\begin{aligned}
\left(\frac{1}{1-x}\right)^n = (1-x)^{-n} &= \sum_{p=0}^{\infty} \frac{-n(-n-1)\cdots(-n-p+1)}{p!}(-x)^p \\
&= \sum_{p=0}^{\infty} \frac{n(n+1)\cdots(n+p-1)}{p!} x^p \\
&= \sum_{p=0}^{\infty} \binom{n+p-1}{p} x^p
\end{aligned}$$

The coefficient of x^r is thus $\binom{n+r-1}{r}$, which is the formula for the number of ways of distributing r identical objects into n distinct boxes.