School of Mathematics and Statistics
MAST30012 Discrete Mathematics 2021
Assignment 2 – Solutions

Q1: (a) This is true. Place the numbers into n pigeonholes as $(2k-1, 2k)$, $k = 1, \dots, n$. Thus each pair of numbers have greatest common divisor 1. Since we select $n+1$ numbers, by the PHP at least one pigeonhole contains two numbers which then have greatest common divisor 1.

(b) This is not true. Counterexamples include choosing $1, 3, 4$ from $\{1, 2, 3, 4\}$ or choosing $1, 4, 5, 6$ from $\{1, 2, 3, 4, 5, 6\}$ (of course one counterexample is enough).

(c) This is true. Every number $a_i \in \{1, 2, 3, \dots, 2n\}$ can be expressed uniquely as $a_i = 2^k b_i$, for some integer k and odd integer b_i. There are n odd integers in $\{1, 2, 3, \dots, 2n\}$ which form our pigeonholes. Since we select $n+1$ numbers, by the PHP at least one pigeonhole contains two numbers, say, $a_1 = 2^j b_i$ and $a_2 = 2^k b_i$. If $j > k$, then $a_1/a_2 = 2^{j-k} > 1$ and so a_2 divides a_1. Similarly, if $k > j$, then $a_2/a_1 = 2^{k-j} > 1$ and so a_1 divides a_2. In any case there must be a pair of integers such that one divides the other.

Q2: (a) Consider two cases:

Case 1. All points have the same label (either A or B). In this case there are no neighbouring points labelled either AB or BA. Zero is an even number so the parity is even.

Case 2. Some points are labelled A some B. At least one pair of neighbouring points are labelled BA. Take such a pair and remove the arc connecting them. This reduces the number of neighbouring points labelled either AB or BA by one. The labelling of the remaining points is a Sperner labelling on an interval. Hence by Sperner's lemma for an interval there is an odd number of neighbouring points labelled either AB or BA. Hence the original labelled circle had an even number of neighbouring points labelled either AB or BA.

In either case the number of neighbouring points labelled either AB or BA is even.

Alternative proof. Consider adding a single new point labelled A or B. Repeat arguments from lectures to show change in the number of neighbouring points labelled either AB or BA is even. Any configuration can be obtained by adding labelled points one at a time. We start with a single point labelled A or B, which can be viewed as an AA or BB interval. In either case there are no neighbouring points labelled either AB or BA and adding new points does not change the parity. Hence the total number of neighbouring points labelled either AB or BA is even.

(b) Make use of the floor plan lemma: In a floor plan with all rooms having 0, 1 or 2 doors, the parity of the number of rooms with 1 door equals the parity of the number of outside doors.

Here the segments AB (or BA) are considered as doors.

The triangle labelled ABC has one door. Any other triangle has either 0 or 2 doors (show this by drawing the labelled triangles).

Hence the floor plan lemma applies to this case and the sought after result follows.

Q3: (a) Take the derivative on both sides of the given generating function:

$$n(1+x)^{n-1} = \frac{d}{dx}(1+x)^n = \frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}x^k = \sum_{k=1}^{n}k\binom{n}{k}x^{k-1}.$$

Setting $x = 1$, we obtain

$$\sum_{k=1}^{n}k\binom{n}{k} = n2^{n-1}.$$

(b) Integrate on both sides from 0 to 1:

$$\int_0^1 (1+x)^n dx = \left[\frac{1}{n+1}(1+x)^{n+1}\right]_0^1 = \frac{2^{n+1}-1}{n+1}.$$

$$\int_0^1 \sum_{k=0}^{n}\binom{n}{k}x^k dx = \left[\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}x^{k+1}\right]_0^1 = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}.$$

Hence

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}.$$

(c) Integrate on both sides from -1 to 0:

$$\int_{-1}^0 (1+x)^n dx = \left[\frac{1}{n+1}(1+x)^{n+1}\right]_{-1}^0 = \frac{1}{n+1}.$$

$$\int_{-1}^0 \sum_{k=0}^{n}\binom{n}{k}x^k dx = \left[\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}x^{k+1}\right]_{-1}^0 = -\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k}.$$

Hence

$$1 - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k} = -\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k} = \frac{1}{n+1}.$$

That is,

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k} = \frac{n}{n+1}.$$

Q4: (a) Partition on the number of dimers in a paving. If there are k dimers there are $n-2k$ monomers and a total of $n-k$ pavers. The positions of the k dimers can then be chosen in $\binom{n-k}{k}$ ways. The result follows from the addition principle by summing over the values of k from 0 to $\lfloor n/2 \rfloor$.

(b) Take a board of size $2n$. There are F_{2n+1} pavings in total. Consider the configuration at the centre of the board at positions $(n, n+1)$. These cells are either covered by a dimer or they are not. In the first case there are boards of length $n-1$ to the left and right of the dimer each of which can be paved independently in F_n ways for a total number of pavings F_n^2. In the second case the pavings can be obtained from concatenating any two boards of length n each of which can be paved in F_{n+1} ways for a total number of pavings F_{n+1}^2.

Q5: (a) The five successive sequences which result by the application of the substitution rule are:

$$A \mapsto B \mapsto AAB \mapsto BBAAB \mapsto AABAABBBAAB.$$

(b) From the substitution rule we see that every A in the previous sequence gives rise to a B in the next sequence and every B in the previous sequence gives rise to a two A's and a B in the next sequence. Hence

$$A_n = 2B_{n-1}, B_n = A_{n-1} + B_{n-1}.$$

Therefore

$$\begin{aligned} S_n &= A_n + B_n \\ &= 2B_{n-1} + A_{n-1} + B_{n-1} \\ &= S_n + 2B_{n-1} \\ &= S_n + 2(A_{n-2} + B_{n-2}) \\ &= S_{n-1} + 2S_{n-2} \end{aligned}$$

The boundary condition is $S_0 = 1$ and $S_1 = 1$, since initially we have the string A which after one application of the substitution rule becomes B.

(c) We have

$$\sum_{n=2}^{\infty} S_n x^n = \sum_{n=2}^{\infty} S_{n-1} x^n + 2 \sum_{n=2}^{\infty} S_{n-2} x^n \quad \Rightarrow$$

$$\sum_{n=0}^{\infty} S_n x^n - S_1 x - S_0 = x \left(\sum_{n=0}^{\infty} S_n x^n - S_0 \right) + 2x^2 \sum_{n=0}^{\infty} S_n x^n \quad \Rightarrow$$

$$S(x) - x - 1 = x(S(x) - 1) + 2x^2 S(x) \quad \Rightarrow$$

$$S(x)(1 - x - 2x^2) = 1$$

(d) We have $1 - x - 2x^2 = (1+x)(1-2x)$. So

$$\begin{aligned}
\frac{1}{1-x-2x^2} = \frac{A}{1+x} + \frac{B}{1-2x} \quad &\Rightarrow \\
1 = A(1-2x) + B(1+x) \quad &\Rightarrow \\
A + B = 1, \quad B - 2A = 0 \quad &\Rightarrow \\
A = \frac{1}{3}, \quad B = \frac{2}{3}
\end{aligned}$$

Using the geometric series, we then have

$$S(x) = \sum_{n=0}^{\infty} S_n x^n = \frac{2}{3} \sum_{n=0}^{\infty} 2^n x^n + \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n x^n$$

so that

$$S_n = \frac{2^{n+1} + (-1)^n}{3}.$$