The University of Melbourne — School of Mathematics and Statistics
MAST30012 Discrete Mathematics — Semester 2, 2021

Practice Class 7: Fibonacci – Solutions

Q1: (a) $S_1 = 1$ (take one step) and $S_2 = 2$ (take $1+1$ steps or take 2 steps).

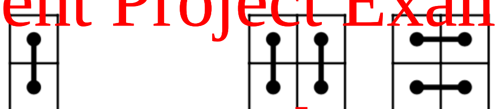
(b) After a step of size 1, $n-1$ steps remain which can be climbed in S_{n-1} ways. After a step of size 2, $n-2$ steps remain which can be climbed in S_{n-2} ways. These are all the possibilities so

$$S_n = S_{n-1} + S_{n-2}.$$

This is the Fibonacci recurrence. Taking into account the initial conditions we have

$$S_n = F_{n+1}.$$

Q2: (a)



$U_1 = 1$ $\qquad\qquad$ $U_2 = 2$

(b) Starting from the left, the 2×1 rectangle can either be placed vertically (leaving a $2 \times (n-1)$ grid to be tiled) or 2 tiles are placed horizontally (leaving a $2 \times (n-2)$ grid to be tiled). Hence

$$U_n = U_{n-1} + U_{n-2}.$$

The recurrence is the same as for the Fibonacci sequence,but with slightly 'shifted' initial conditions so

$$U_n = F_{n+1}.$$

Q3: Let L_n be any sequence of the required type with n letters. We can form sequences of n letters according to the concatenations:

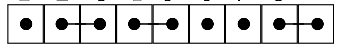
$$BL_{n-1} \qquad\qquad ABL_{n-2}$$

All allowed sequences of n letters have this structure so $l_n = l_{n-1} + l_{n-2}$. The initial conditions are $l_1 = 2$ (sequences A and B) and $l_2 = 3$ (sequences AB, BA and BB). It follows that

$$l_n = F_{n+2}.$$

Q4: (a) For each $(n+1)$-tiling of the board create the n-tuple $(b_0, b_1, \ldots, b_n)$ where $b_i = 0$ if cell i is covered by the *first* cell from a domino and $b_i = 1$ otherwise, e.g.,

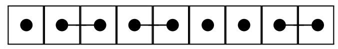
$i:$ 1 2 3 4 5 6 7 8



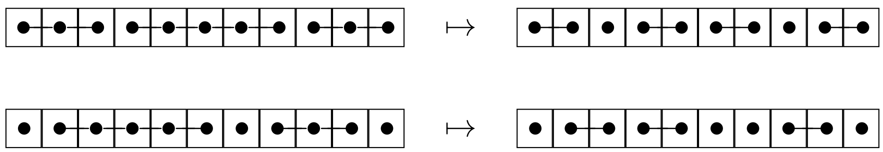
$b_i:$ 1 0 1 0 1 1 1 0

(b) For each arrangement of integers a_i according to the rules $a_1 a_2 \cdots a_n$ create the n-tiling where cells i is covered by a monomer if and only if $a_i = i$, e.g.,

$a_i:$ 1 3 2 5 4 6 7 9 8



(c) Let T be a tiling of an n-board with tiles of odd length. Take each tile and break it into a monomer followed by dimers. Remove the first square in this tiling to get an tiling T' of an $(n-1)$-board with monomers and dimers.



Q5: The Fibonacci sequence $\{F_n\}$ (we set $F_0 = 0$) has generating function

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}.$$

(a) Note that g_n is the n^{th} partial sum of the Fibonacci sequence and taking the partial sum of a sequence corresponds to dividing the generating function by $1-x$. So

$$G(x) = \frac{F(x)}{1-x} = \frac{x}{(1-x)(1-x-x^2)}$$

(b) From the given expression for h_n we get

$$\begin{aligned}
H(x) = \sum_{n=0}^{\infty} h_n x^n &= \sum_{n=0}^{\infty} F_{n+2} x^n - \sum_{n=0}^{\infty} x^n \\
&= \frac{1}{x^2} \sum_{n=0}^{\infty} F_{n+2} x^{n+2} - \frac{1}{1-x} \\
&= \frac{1}{x^2} \left(F(x) - F_0 - F_1 x\right) - \frac{1}{1-x} \\
&= \frac{1}{x^2} \left(\frac{x}{1-x-x^2} - x\right) - \frac{1}{1-x} \\
&= \frac{1}{x(1-x-x^2)} - \frac{1}{x} - \frac{1}{1-x}.
\end{aligned}$$

(c) Simplifying the expression for $H(x)$, we obtain

$$
\begin{aligned}
H(x) &= \frac{1}{x(1-x-x^2)} - \frac{1}{x} - \frac{1}{1-x} \\
&= \frac{1-(1-x-x^2)}{x(1-x-x^2)} - \frac{1}{1-x} \\
&= \frac{1+x}{1-x-x^2} - \frac{1}{1-x} \\
&= \frac{(1+x)(1-x)-(1-x-x^2)}{(1-x)(1-x-x^2)} \\
&= \frac{x}{(1-x)(1-x-x^2)} \quad = \quad G(x).
\end{aligned}
$$

Equating coefficients in the power series of $G(x)$ and $H(x)$ gives

$$F_0 + F_1 + \cdots F_n \;=\; g_n \;=\; h_n \;=\; F_{n+2} - 1.$$

Q6: (a) F_{n_1} is the largest Fibonacci number not exceeding p so

$$0 \le p - F_{n_1}, \qquad p \le F_{n_1+1} - 1$$

The second inequality can be written as $p - F_{n_1} \le F_{n_1+1} - F_{n_1} - 1$.
But $F_{n_1+1} - F_{n_1} = F_{n_1-1}$ so $0 \le p - F_{n_1} \le F_{n_1-1} - 1$.

(b) Repeating the process, select the largest Fibonacci number $\le p - F_{n_1}$. According to (a) the largest such Fibonacci number is no greater than F_{n_1-2}. This means that no two successive Fibonacci numbers occur. As $F_2 = 1$, the process must always terminate with a zero remainder, showing that all positive integers has the stated decomposition.

(c)

$1 = F_2$	$4 = F_4 + F_2$	$7 = F_5 + F_3$	$10 = F_6 + F_3$
$2 = F_3$	$5 = F_5$	$8 = F_6$	$11 = F_6 + F_4$
$3 = F_4$	$6 = F_5 + F_2$	$9 = F_6 + F_2$	$12 = F_6 + F_4 + F_2$

Let F_{n^*} be the smallest Fibonacci number in the decomposition and if

n^* even $p \leftrightarrow A$, $\qquad n^*$ odd $p \leftrightarrow a$

This prescription gives for the first 12 integers the letter sequence

$$AaAAaAaAAaAA$$

After making the replacements $A \mapsto Aa$, $a \mapsto A$ we get

$$AaAAaAaAAaAAaAaAAaAa$$

Note that the first 12 letters are unchanged. This is also a feature of the sequence of the Fibonacci rabbit problem and indeed for $n \to \infty$ the two sequences are identical.