(1) (a) Let $G = D_{12}, H = \{g \in D_{12} \mid g^3 = R_0\}$. Decide if H is not a subgroup, a subgroup which is not normal or a normal subgroup of G.

Solution

Claim: H is a normal subgroup of G. First note that all reflection in D_{12} have order 2 so H consists of rotations only. hence $H = \{R_0, R_{120}, R_{-120}\}$ which is a cyclic subgroup of order 3. Next, since conjugation is an automorphism it follows that if $x \in D_{12}$ and $g \in H$ then $(xgx^{-1})^3 = xg^3x^{-1} = e$. Hence $xHx^{-1} \subset H$. Thus $H \triangleleft G$ is a normal subgroup by the normal subgroup test.

Answer: $H \triangleleft G$ is a normal subgroup.

(b) $G = S_{10}$, $H = \{\sigma \in S_{10}, \mid$ such that σ can be written as product of ≤ 3 disjoint cycles $\}$. Decide if H is not a subgroup, a subgroup which is not normal or a normal subgroup of G.

Solution

Let $\sigma = (12)(34), \tau = (56)(78)$. Then σ, and τ are in H but $\sigma\tau = (12)(34)(56)(78)$ is a product of 4 disjoint cycles. by uniqueness of disjoint cycle decomposition this means that $\sigma\tau \notin H$. Hence H is not a subgroup.

Answer: H is not a subgroup.

(c) Let G be an abelian group and let $H = \{g \in G \mid |g| < \infty\}$ Decide if H need not be a subgroup, must be a subgroup but it need not be normal, is always a normal subgroup of G.

Solution

Let us check that H is a subgroup. let $g, h \in H$ and $|g| = m, |h| = m$ are finite then Since H is abelian we have
$(gh)^{mn} = h^{mn}h^{mn} = (g^n)^m(h^m)^n = e \cdot e = e$, Hence gh has finite order and thus $gh \in H$. This shows that H is closed under the operation.
Next since $|g^{-1}| = |g|$ it follows that H is closed under inverses.
We have verified that H is a subgroup of G.
Since G is abelian every subgroup of G is normal.

Answer: $H \triangleleft G$ is a normal subgroup.

(d) Let G be a finite group, $a, b \in G$. Let $H = \langle a, b\rangle$.
then which of the following holds
a) $|H| = |a| \cdot |b|$
b) $|H| = lcm(|a|, |b|)$

c) $|H|$ is divisible by $lcm(|a|, |b|)$
d) H need not be divisible by $lcm(|a|, |b|)$.

Solution

Let $G = S_3, a = (12), b = (13)$. Then $|a| = |b| = 2$. But it's easy to check that $\langle a, b\rangle = S_3$ which has order 6. This means that a) and b) are false in general.
Next observe that since $a \in H$ by Lagrange's theorem we have that $|a|$ divides $|H|$. Similarly $|b|$ divides $|H|$. Hence $lcm(a, b)$ divides $|H|$ as well.

Answer: $|H|$ is divisible by $lcm(|a|, |b|)$

(e) How many normal subgroups of order 2 are there in D_{10}?

Solution

Subgroups of order 2 correspond to elements of order 2.

In D_{10} there are 11 elements of order 2: 10 reflections and R_{180}.
The subgroup $\langle R_{180}\rangle = \{R_0, R_{180}\}$ is equal to the center of D_{10} and hence is normal.
We claim that none of the subgroups generated by reflections are normal. Indeed, let F_l be a reflection, let $\alpha = 2\pi/10$. Then $R_\alpha \in D_{10}$.

But $R_\alpha F_l R_\alpha^{-1} = F_{R_\alpha(l)}$ by a formula from class. Note that $R_\alpha(l) \neq l$ and hence $R_\alpha F_l R_\alpha^{-1} = F_{R_\alpha(l)} \notin \langle F_l\rangle = \{R_0, F_l\}$. Hence $\langle F_l\rangle$ is not normal in D_{10}.

Therefore

Answer: There is exactly 1 normal subgroup of order 2 in D_{10}.

(2) Let G be a group and let $H, K \lhd G$ be normal subgroups.
Prove that HK is a normal subgroup of G.

Solution

Let us first verify that HK satisfies the subgroup test.

Let $hh' \in H, k, k' \in K$. We claim that $(hk)(h'k') \in HK$ also.

Since $H \lhd G$ we have that $kh'k^{-1} = h'' \in H$ and therefore $kh' = h''k$. Hence $hkh'k' = h(kh')k' = h(h''k)k' = (hh'')(kk') \in HK$ suince $hh'' \in H$ and $kk' \in K$. This shows that HK is closed under the noperation.

Next $(hk)^{-1} = k^{-1}h^{-1}$. We have that $x = k^{-1} \in K$ and $y = h^{-1} \in H$ since both H and K are subgroups. as before $xyx^{-1} = y' \in H$ since $H \lhd G$ and hence $xy = yx' \in HK$.

This verifies that HK is closed under inverses. Thus HK is a subgroup of G.

It remains to show that this subgroup is normal. Let $g \in g, h \in H, k \in K$ then $ghg^{-1} = h' \in H$ and $gkg^{-1} = k' \in K$. since H, K are normal. Therefore $g(hk)g^{-1} = ghg^{-1}gkg^{-1} = h'k' \in H$.

This verifies that $g(HK)g^{-1} \subset HK$.

Therefore $HK \lhd G$ by the normal subgroup test. $\square$

(3) Let $n > 2$.

Prove that $|U(n)|$ is even.

Hint: Use Lagrange's theorem.

Solution

Note that $gcd(n-1, n) = 1$ since if d divides both n and $n-1$ then it also divides $n - (n-1) = 1$.

Therefore $\overline{n-1} \in U(n)$.

Next $(n-1) = -1 \mod n$ and hence $(n-1)^2 = (-1)^2 = 1 \mod n$. This can also be seen more directly since $(n-1)^2 = n^2 - 2n + 1 = 1 \mod n$.

Since $n > 2$ we have that $n - 1 > 1$ and hence $\overline{n-1} \neq \bar{1}$.

This means that $\overline{n-1}$ has order 2 in $U(n)$.

By a corollary to Lagrange's theorem we have that $2 = |\overline{n-1}|$ divides $|U(n)|$. $\square$.

(4) Let $m, n > 1$ be relatively prime. Let $\varphi\colon \mathbb{Z}_{mn} \to \mathbb{Z}_m \oplus \mathbb{Z}_n$ be given by

$$\varphi(k \mod mn) = (k \mod m, k \mod n)$$

Prove that φ is an isomorphism.

Solution

Let us first check that φ is well defined and preserves the operation.

if $k_1 = k_2 \mod mn$ then mn divides $k_1 - k_2$. Hence both m, n divide $k_1 - k_2$ and therefore $k_1 = k_2 \mod m$ and $k_1 = k_2 \mod n$.

This shows that φ is well defined.

Next, $\varphi(\bar{k}_1 + \bar{k}_2) = ((k_1 + k_2) \mod m, (k_1 + k_2) \mod n)) = (k_1 \mod m + k_2 \mod m, k_1 \mod n + k_2 \mod n) = (k_1 \mod m, k_1 \mod n) + (k_2 \mod m, k_2 \mod n) = \varphi(\bar{k}_1) + \varphi(\bar{k}_2)$. This shows that φ preserves operation.

Let us show that φ is $1-1$.

Suppose $\varphi(\bar{k}_1) = \varphi(\bar{k}_2)$. This means that $k_1 = k_2 \mod m$ and $k_1 = k_2 \mod m$. Therefore both m and n divide $k_1 - k_2$.

Since $gcd(m, n) = 1$ this implies that mn divides $k_1 - k_2$ as well, i.e. $\bar{k}_1 = \bar{k}_2$ in $\mathbb{Z}_{mn}$.

This proves that φ is 1-1.

Since $|\mathbb{Z}_{mn}| = mn = |\mathbb{Z}_m \oplus \mathbb{Z}_n|$ this means that φ is an injective map between two finite sets with the same number of elements. Hence φ must be onto.

This verifies that φ is a bijection. $\square$

(5) Let G be a group such that $G/Z(G)$ is abelian. Prove that for any $a, b, c \in G$ it holds that $[[a, b], c] = e$.

Solution

Let $H = Z(G)$. Since G/H is abelian we have that $[aH, bH] = H$. Recall that $(aH)^1 = a^{-1}H$ and $(bH)^{-1} = b^{-1}H$.

Therefore $H = [aH, bH] = (aH)(bH)(aH)^{-1}(bH)^{-1} = (aH)(bH)(a^{-1}H)(b^{-1}H) = aba^{-1}b^{-1}H = [a, b]H$. Therefore $z = [a, b] \in Z(G)$. This means that z commutes with any element of G and hence $[z, c] = e$ i.e. $[[a, b], c] = e$. $\square$

(6) Let $G = U(20), H = \langle\bar{9}\rangle, K = \langle\overline{11}\rangle$ be subgroups of G.

Are G/H and G/K isomorphic?

Solution

Recall that in general $|aH|$ is the smallest positive n such that $(aH)^n = H$ which is equivalent to $a^nH = H$ or $a^n \in H$.

Let us compute orders of various elements of G/H and G/K.

First, we have that $U(20) = \{\bar{1}, \bar{3}, \bar{7}, \bar{9}, \overline{11}, \overline{13}, \overline{17}, \overline{19}\} \subset \mathbb{Z}_{20}$.

We have that $\bar{3}^2 = \bar{9} \notin K, \bar{3}^3 = 27 = 7 \mod 20 \notin K, \bar{3}^4 = 81 = 1 \mod 20$. Hence $|\bar{3}K| = 4$.

On the other hand $\bar{3}^2 = \bar{9} \in H, \bar{7}^2 = 49 \mod 20 = 9 \mod 20 \in H, \bar{9}^2 = \bar{1} \in H, \overline{11}^2 = \overline{-9}^2 = \bar{1} \in H, \overline{13}^2 = \overline{-7}^2 = \bar{9} \in H, \overline{17}^2 = \overline{-3}^2 = \bar{9} \in H, \overline{19}^2 = \overline{-1}^2 = \bar{1} \in H$. This shows that every element of G/H has order at most 2.

Therefore $G/H \ncong G/K$.

Answer: $G/H \ncong G/K$.