MATH 210 Practice Questions

December 2018

INSTRUCTIONS

- Plain scientific calculators are allowed but not *graphing or programmable* calculators
- No cellphones, laptops or notes
- 120 minutes

Name: ____________________

Student #: ____________________

Signature: ____________________

1. (a) Start with the interval $[-2, 2]$ and implement the bisection method to find an interval of length 0.25 containing a solution of the equation

$$y = x^7 + x^3 - x + 2$$

(b) Approximate a solution of the equation above by executing 1 iteration of Newton's method with initial point equal to the midpoint of the interval found in part (a).

2. Consider the definite integral

$$I = \int_0^2 \sqrt{1+x^4}\,dx$$

The derivatives of $f(x) = \sqrt{1+x^4}$ are

$$f'(x) = \frac{2x^3}{\sqrt{x^4+1}} \qquad f''(x) = \frac{2x^2(x^4+3)}{\sqrt[3]{x^4+1}}$$

(a) Approximate the integral I using $N = 3$ subintervals of equal length with the trapezoid rule and find a bound on the error.

(b) Determine the length of partition N required to approximate the integral I with error less than 0.01 for each method:

i. Left (or right) Riemann sum

ii. Midpoint Riemann sum

iii. Trapezoid rule

3. Suppose the general solution of the system of equation $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} e^{4t} + C_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t}$$

Write Python code to find the coefficients C_1, C_2 and C_3 as a 2D NumPy array $C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$ given the initial condition $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$. (Do not solve the system of equations, write the Python code that would compute the coefficients when executed.)

4. Write Python code to find the coefficients of the unique quadratic polynomial interpolating $(-1, 2), (0, 2), (1, 3)$. (Do not solve the system of equations, write the Python code that would compute the coefficients when executed.)

5. Find the line $y = a_0 + a_1 x$ which best fits the points

$$(1.08, 13.63)\ ,\ (3.49, 6.08)\ ,\ (4.55, 2.93)\ ,\ (2.24, 9.15)$$

6. Canada's Gross Domestic Product (GDP) was \$742 billion USD (US dollars) in 2000, \$1169 billion USD in 2005, \$1613 billion USD in 2010 and \$1560 billion USD in 2015. Use linear regression to predict Canada's GDP in 2020.

7. (5 points) Consider the system of differential equations

$$\ddot{x}_1 = -2\omega_0^2 x_1 + \omega_0^2 x_2$$
$$\ddot{x}_2 = \omega_0^2 x_1 - 2\omega_0^2 x_2$$

The system describes a mass-spring system with two masses. The variables x_1 and x_2 are the positions of each mass with respect to their equilibrium position. Write Python code to plot the solutions $x_1(t)$ and $x_2(t)$ over time $0 \le t \le 2$ given $\omega_0 = 1$ and initial conditions $x_1(0) = 1$, $x_2(0) = -1$, $\dot{x}_1 = \dot{x}_2 = 0$.

```
def odefun(u,t):
    dudt = np.zeros(4)
    # Code required here





    return dudt





u = spi.odeint(odefun,u0,t)
# Code required here





plt.show()
```

8. Write a function called `elementary` which takes input parameters `i`, `j`, `k` and `n` and returns the elementary matrix E of size n which performs the operation: add k times row j to row i (using 0 indexing). For example, if $i = 1$, $j = 3$, $k = -2$ and $n = 5$ then

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Also, if $i = 2$, $j = 2$, $k = 7$ and $n = 4$ then

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

9. Write a function called `is_symmetric` which takes an input parameter `A` and returns `True` if A is a square symmetric matrix and `False` otherwise. For example,

```
is_symmetric(np.array([[1,2,3],[4,5,6]])
```

returns `False` and

```
is_symmetric(np.array([[1,2],[2,1]])
```

returns `True`.

10. Write a function called `almost_complete` which takes an input parameter `n` and returns the adjacency matrix of the *almost* complete graph on n vertices. The `almost` complete graph is constructed as follows: label the vertices $0, ..., n-1$ and then vertex i is connected to every other node except vertex $i+1$. This means vertex $n-1$ is not connected to vertex 0.

    For example, `almost_complete(4)` returns

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

11. Consider the initial value problem

$$y' = y + \sin(yt) + 1 \quad , \quad y(0) = 0$$

(a) Implement Euler's method to compute approximations for $y(0.2)$ and $y(0.4)$.

(b) Write Python code to compute approximations $y(t)$ for $t = 0.1, 0.2, \ldots, 2$.