notes_2020_10_26

October 30, 2020

1 MATH 210 Introduction to Mathematical Computing

1.1 October 26, 2020

- Riemann sums and error formulas
- Trapezoid rule and error formula
- `scipy.integrate`

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```


1.2 Riemann sums

The **right Riemann sum** for $f(x)$ over $[a,b]$ with N subintervals is

$$R_N(f) = \sum_{n=1}^{N} f(x_n)\Delta x$$

where $\Delta x = (b-a)/N$ and $x_n = a + n\Delta x$.

The **left Riemann sum** for $f(x)$ over $[a,b]$ with N subintervals is

$$L_N(f) = \sum_{n=1}^{N} f(x_{n-1})\Delta x$$

The **midpoint Riemann sum** for $f(x)$ over $[a,b]$ with N subintervals is

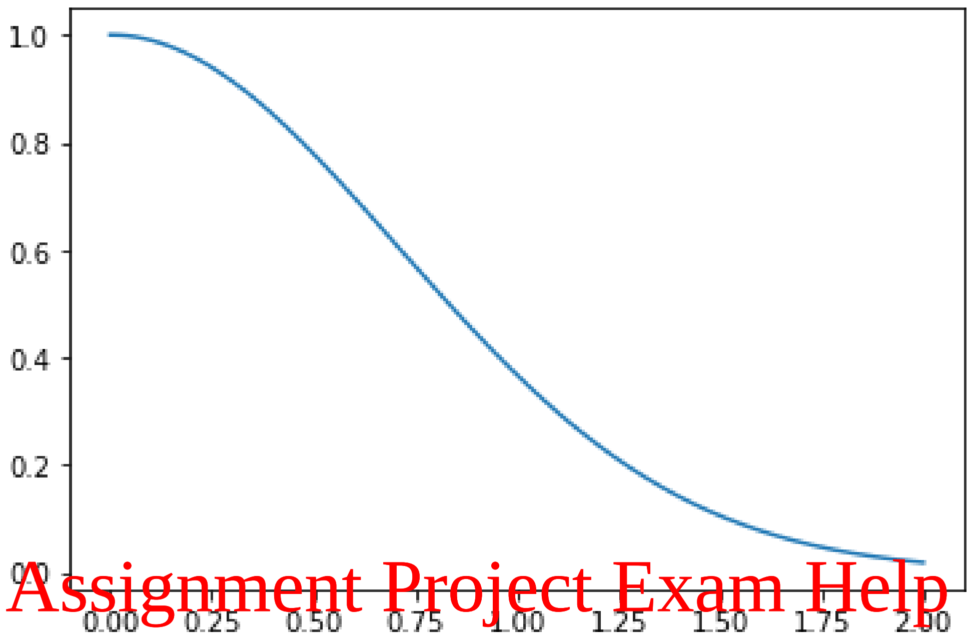
$$M_N(f) = \sum_{n=1}^{N} f((x_{n-1} + x_n)/2)\Delta x$$

1.2.1 Example

$$\int_0^1 e^{-x^2} dx$$

[2]:
```
x = np.linspace(0,2,200)
y = np.exp(-x**2)
```

```
plt.plot(x,y)
plt.show()
```



From last class, we need $N \geq 429$ such that $E_N^R(f) < 0.001$ (and also $E_N^L(f) < 0.001$).

But first let's look at how to create and slices arrays of x value to get left and right endpoints.

[3]:
```
N = 5
a = 0
b = 1
dx = (b - a)/N
x = np.linspace(a,b,N + 1)
x_right = x[1:]
x_left = x[:N]
x_midpoint = (x_right + x_left)/2
```

[4]:
```
x
```

[4]: array([0. , 0.2, 0.4, 0.6, 0.8, 1.])

[5]:
```
x_right
```

[5]: array([0.2, 0.4, 0.6, 0.8, 1.])

[6]:
```
x_left
```

```
[6]: array([0. , 0.2, 0.4, 0.6, 0.8])
```

```
[7]: x_midpoint
```

```
[7]: array([0.1, 0.3, 0.5, 0.7, 0.9])
```

Now compute the approximations for $N = 429$.

```
[8]: N = 429
     a = 0
     b = 1
     dx = (b - a)/N
     x = np.linspace(a,b,N + 1)
     x_right = x[1:]
     x_left = x[:N]
     x_midpoint = (x_right + x_left)/2
     R = np.sum(np.exp(-x_right**2))*dx
     L = np.sum(np.exp(-x_left**2))*dx
     M = np.sum(np.exp(-x_midpoint**2))*dx
     print("Right Riemann sum:", R)
     print("Midpoint Riemann sum:", M)
     print("Left Riemann sum:", L)
```



```
Right Riemann sum: 0.7460370624144178
Midpoint Riemann sum: 0.7468242993874284
Left Riemann sum: 0.7475605369105216
```

Why is $R_N(f) < M_N(f) < L_N(f)$? Because $f(x) = e^{-x^2}$ is decreasing on $[0, 1]$.

Also from last class, we need only $N \geq 10$ for $E_N^M(f) < 0.001$. The approximation with $N = 429$ above gives a much smaller error:

```
[9]: K2 = 2
     E = (b - a)**3/(24*N**2)*K2
     print("N =",N)
     print("Midpoint Riemann sum error:", E)
```

```
N = 429
Midpoint Riemann sum error: 4.5279765559485837e-07
```

1.3 Trapezoid rule

The **trapezoid rule** for $f(x)$ over $[a, b]$ with N subintervals is

$$T_N(f) = \sum_{n=1}^{N} \frac{f(x_{n-1}) + f(x_n)}{2} \Delta x = \frac{R_N(f) + L_N(f)}{2}$$

Now let's try the trapezoid rule. We saw last time that $N \geq 13$ then $E_N^T(f) < 0.001$.

[10]:
```python
N = 13
a = 0
b = 1
dx = (b - a)/N
x = np.linspace(a,b,N + 1)
x_right = x[1:]
x_left = x[:N]
T = (np.sum(np.exp(-x_right**2)) + np.sum(np.exp(-x_left**2)))*dx/2
print("Trapezoid rule:", T)
```

```
Trapezoid rule: 0.7464612610366896
```

1.4 Infinite Gaussian integral

Let's look at the infinite integral

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Let's approximate the infinite integral using the trapezoid rule on the interval $[0, 10]$ with $N = 1000$.

[11]:
```python
N = 1000
a = 0
b = 10
dx = (b - a)/N
x = np.linspace(a,b,N + 1)
x_right = x[1:]
x_left = x[:N]
T = (np.sum(np.exp(-x_right**2)) + np.sum(np.exp(-x_left**2)))*dx/2
print("Trapezoid rule:", T)
```



```
Trapezoid rule: 0.8862269254527582
```

[12]:
```python
np.sqrt(np.pi)/2
```

[12]: 0.8862269254527579

1.5 scipy.integrate

The subpackage `scipy.integrate` has many functions for approximating integrals.

[13]:
```python
import scipy.integrate as spi
```

[14]:
```python
spi.trapz?
```

[15]:
```python
N = 13
a = 0
b = 1
dx = (b - a)/N
```

```
x = np.linspace(a,b,N + 1)
x_right = x[1:]
x_left = x[:N]
T = (np.sum(np.exp(-x_right**2)) + np.sum(np.exp(-x_left**2)))*dx/2
print(T)
```

```
0.7464612610366896
```

[16]:
```
N = 13
a = 0
b = 1
dx = (b - a)/N
x = np.linspace(a,b,N + 1)
y = np.exp(-x**2)
T = spi.trapz(y,x)
print(T)
```

```
0.7464612610366896
```

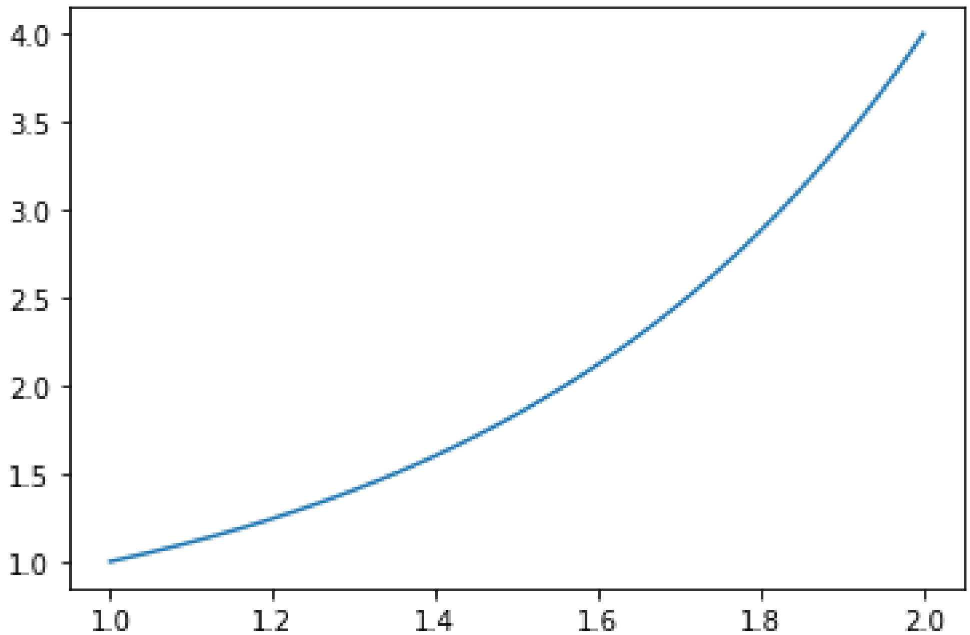
1.6 Example

Approximate the integral

$$\int_1^2 x^x \, dx$$

using the trapezoid rule with error less than 0.0001.

[17]:
```
x = np.linspace(1,2,200)
y = x**x
plt.plot(x,y)
plt.show()
```



The error formula is

$$E_N^T(f) \leq \frac{(b-a)^3}{12N^2} K_2$$

where $|f''(x)| \leq K_2$ for $x \in [a,b]$.

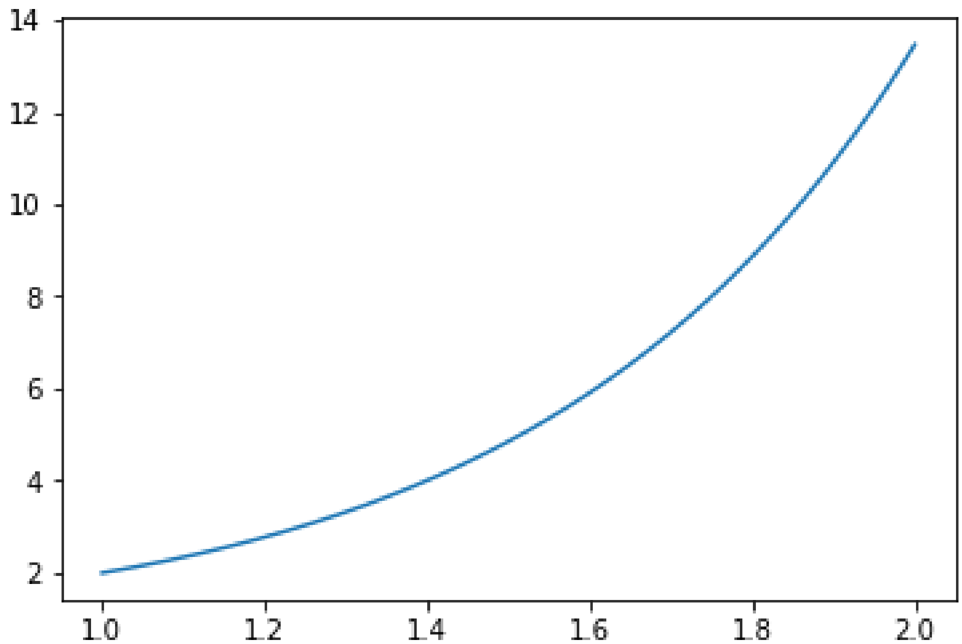
We have

$$f'(x) = x^x(\log(x) + 1)$$

$$f''(x) = x^x(\log(x) + 1)^2 + x^{x-1}$$

Plot $f''(x)$ and find a bound K_2.

[18]:
```
x = np.linspace(1,2,200)
d2f = x**x*(np.log(x) + 1)**2 + x**(x - 1)
plt.plot(x,d2f)
plt.show()
```



[19]:
```
K2 = 2**2*(np.log(2) + 1)**2 + 2**(2 - 1)
K2
```

[19]: 13.46698950015237

[20]:
```
N = np.sqrt(K2/12/0.0001)
N
```

[20]: 105.93626031782968

[21]:
```
N = 106
a = 1
b = 2
dx = (b - a)/N
x = np.linspace(a,b,N + 1)
y = x**x
I = spi.trapz(y,x)
print(I)
```

2.0504890474484894