notes_2020_11_06

November 25, 2020

1 MATH 210 Introduction to Mathematical Computing

1.1 November 6, 2020

- Error formula for Euler's method
- Heun's method

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

1.2 Error formula for Euler's method

Consider $y' = f(t, y)$, $y(t_0) = y_0$ and let $y(t)$ be the solution. For step size h, we have the local (truncation) error

$$E_{local}(h) = |y(t_1) - y_1| \leq \frac{K_2 h^2}{2}$$

where $|y''(t)| \leq K_2$ for all $t \in [t_0, t_1]$ and $t_1 = t_0 + h$. In this case, we say that $E_{local}(h) = O(h^2)$ (using Big-O notiaton).

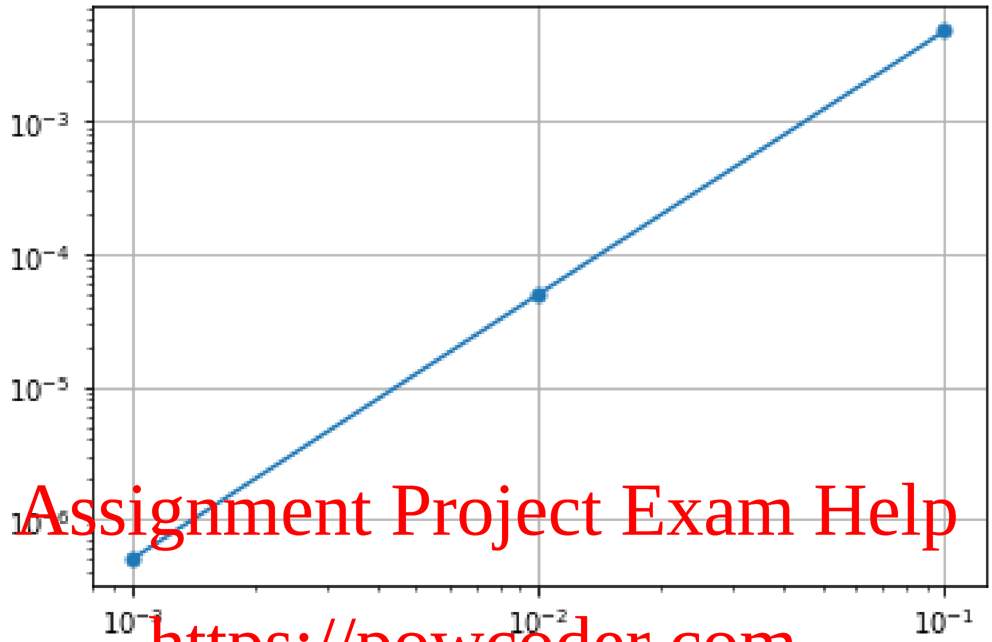
1.3 Loglog plot of error

[2]:
```
def odeEuler(f,t,y0):
    y = np.zeros(t.shape)
    y[0] = y0
    for n in range(0,len(t) - 1):
        y[n + 1] = y[n] + f(t[n],y[n])*(t[n + 1] - t[n])
    return y
```

[3]:
```
def f(t,y):
    return -y

y0 = 1
h = np.array([0.001,0.01,0.1])
E = np.zeros(len(h))
for n in range(0,len(h)):
    y = odeEuler(f,np.array([0,h[n]]),y0)
    E[n] = np.abs(np.exp(-h[n]) - y[1])
```

```python
plt.loglog(h,E,'.-',ms=10)
plt.grid(True)
plt.show()
```



The **order** of a numerical method for differential equations is m if

$$E_{local}(h) = O(h^{m+1})$$

Note that $m + 1$ is the slope in the plot $\log(E_{local})$ versus $\log(h)$.

1.4 Example

$$y' = -\frac{y}{t} \;,\; y(1) = 1$$

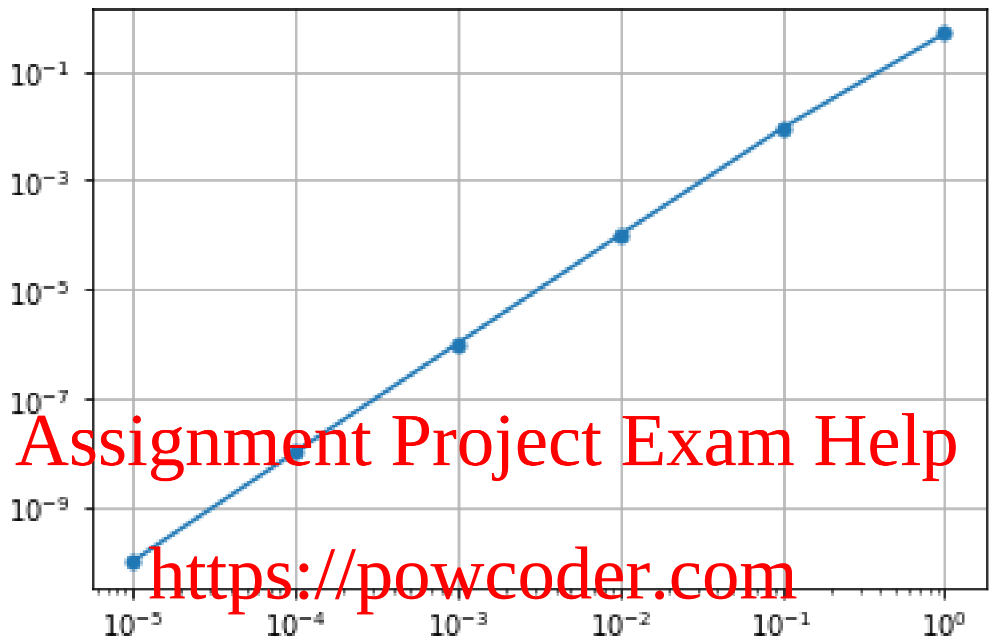
$$y(t) = \frac{1}{t}$$

[4]:
```python
def f(t,y):
    return -y/t

y0 = 1; t0 = 1;
h = np.logspace(0,-5,6)
E = np.zeros(len(h))
for n in range(0,len(h)):
```

```
    y = odeEuler(f,np.array([t0,t0 + h[n]]),y0)
    E[n] = np.abs(1/(t0 + h[n]) - y[1])

plt.loglog(h,E,'.-',ms=10)
plt.grid(True)
plt.show()
```



Note that the function `numpy.logspace` works as follows:

[5]:
```
np.logspace(0,-5,6)
```

[5]:
```
array([1.e+00, 1.e-01, 1.e-02, 1.e-03, 1.e-04, 1.e-05])
```

1.5 Heun's method

Euler's method is order 1 and uses linear approximation (degree 1 Taylor polynomial). What about higher order methods?

$$
\begin{aligned}
y(t+h) &= y(t) + y'(t)h + \frac{y''(t)h^2}{2} + \frac{y'''(c)h^3}{6} \\
y(t+h) &\approx y(t) + y'(t)h + \frac{y''(t)h^2}{2} \\
y(t+h) &\approx y(t) + \left(\frac{2y'(t) + y''(t)h}{2} \right) h \\
y(t+h) &\approx y(t) + \left(\frac{y'(t) + y'(t) + y''(t)h}{2} \right) h
\end{aligned}
$$

We know $y'(t) = f(t, y)$ and the degree 1 Taylor polynomial of $y'(t)$ is

$$
y'(t+h) \approx y'(t) + y''(t)h
$$

Therefore

$$
\begin{aligned}
y(t_1) &\approx y(t_0) + \left(\frac{y'(t_0) + y'(t_0) + y''(t_0)h}{2} \right) h \\
y(t_1) &\approx y(t_0) + \left(\frac{y'(t_0) + y'(t_1)}{2} \right) h \\
y(t_1) &\approx y(t_0) + \left(\frac{f(t_0, y_0) + f(t_1, y_1)}{2} \right) h
\end{aligned}
$$

But we don't know what y_1 is! What do we do? Use Euler's method to approximate y_1 to plug into $f(t_1, y_1)$.

$$
\begin{aligned}
\tilde{y}_1 &= y_0 + f(t_0, y_0)h \\
y_1 &= y_0 + \left(\frac{f(t_0, y_0) + f(t_1, \tilde{y}_1)}{2} \right) h
\end{aligned}
$$

This is called Heun's method:

$$
\begin{aligned}
k_1 &= f(t_0, y_0) \\
k_2 &= f(t_1, y_0 + k_1 h) \\
y_1 &= y_0 + \frac{k_1 + k_2}{2} h
\end{aligned}
$$

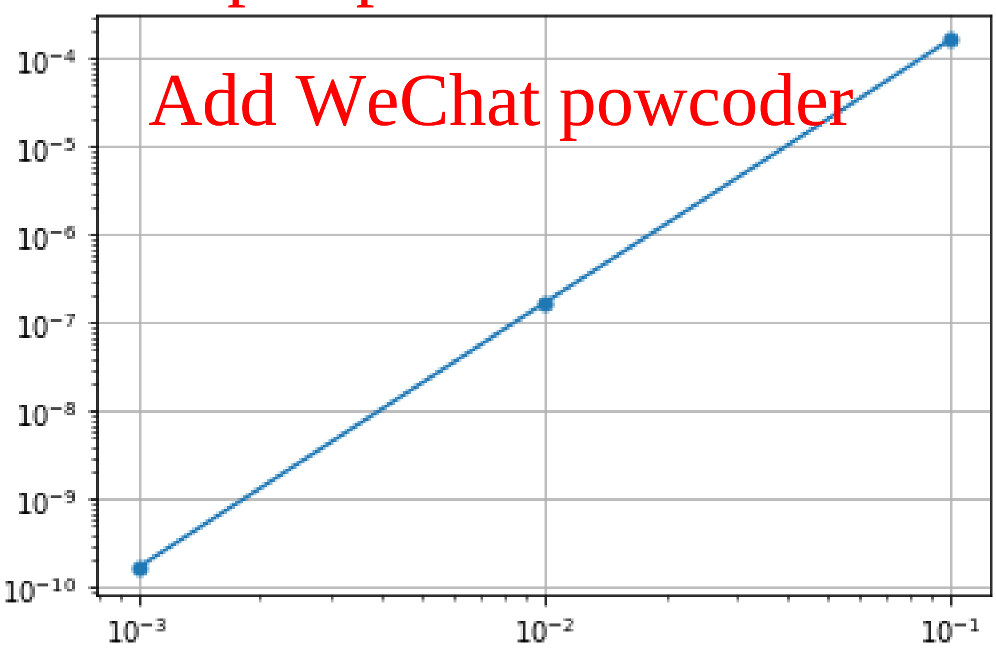
1.6 Implementation

[6]:
```python
def odeHeun(f,t,y0):
    y = np.zeros(t.shape)
    y[0] = y0
    for n in range(0,len(t) - 1):
        h = t[n + 1] - t[n]
        k1 = f(t[n],y[n])
        k2 = f(t[n + 1], y[n] + k1*h)
        y[n + 1] = y[n] + (k1 + k2)/2*h
    return y
```

[7]:
```python
def f(t,y):
    return -y

y0 = 1
h = np.array([0.001,0.01,0.1])
E = np.zeros(len(h))
for n in range(0,len(h)):
    y = odeHeun(f,np.array([0,h[n]]),y0)
    E[n] = np.abs(np.exp(-h[n]) - y[1])

plt.loglog(h,E,'.-',ms=10)
plt.grid(True)
plt.show()
```

The sloped of the loglog plot is 3 as expected since Heun's method is order 2.